Contemporary Mathematics

Report on the Trace Formula

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This paper is dedicated to Steve Gelbart on the occasion of his sixtieth birthday.

Abstract. We report briefly on the present state of the trace formula and some of its applications.

This article is a summary of the two hour presentation/discussion on the trace formula. The proposed topic was very broad. It included a recapitulation of the trace formula, past and present, as well as an outlook for its future. The article will treat these matters in only the most concise terms.

I include just two references,

J. Arthur, *An introduction to the trace formula,* in *Harmonic Analysis, the Trace Formula and Shimura Varieties,* Clay Mathematics Proceedings, Volume 4, 2005, American Mathematical Society, p. 1–263.

and

R. Langlands, *Un nouveau point de repère dans la théorie des formes automorphes,* to appear in Canad. Math. Bull.

The first of these is a general (and detailed) introduction to the trace formula and related topics. It contains references to just about everything discussed in this article. The second is a review by Langlands of his ideas for possible application of the trace formula to the general principle of functoriality. We shall discuss this topic at the end of the article.

1. Invariant trace formula

Let G be a connected reductive algebraic group over a global field F of characteristic 0. Then $G(F)$ embeds as a discrete subgroup of the locally compact adelic group $G(\mathbb{A})$. We write R for the unitary representation of $G(\mathbb{A})$ on $L^2\big(G(F)\backslash G(\mathbb{A})\big)$ by right translation. For any function f in the global Hecke algebra $\mathcal{H}(G)$ (with respect to a suitable maximal compact subgroup $K \subset G(\mathbb{A})$), the average

$$R(f) = \int_{G(\mathbb{A})} f(y)R(y)\mathrm{d}y$$

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is an integral operator on $G(F)\backslash G(\mathbb{A})$, with kernel

$$K(x,y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y).$$

Suppose for a moment that $G(F)\backslash G(\mathbb{A})$ is compact. Then R decomposes discretely into a direct sum of irreducible representations, each occuring with finite multiplicity. The operator $R(f)$ in this case is of trace class, and

$$\mathrm{tr}\big(R(f)\big) = \int_{G(F)\backslash G(\mathbb{A})} K(x,x)\mathrm{d}x.$$

In addition, any element $\gamma \in G(F)$ is semisimple. Let G_γ denote the identity component of its centralizer in G. Then the quotient of $G_\gamma(\mathbb{A})$ by $G_\gamma(F)$ is compact, and $f(x^{-1}\gamma x)$ is integrable as a function of x in $G_\gamma(\mathbb{A})\backslash G(\mathbb{A})$. These facts are all closely related. Taken together, they lead to an identity

$$\sum_{\gamma \in \Gamma(G)} a^G(\gamma) f_G(\gamma) = \sum_{\pi \in \Pi(G)} a^G(\pi) f_G(\pi), \tag{1.1}$$

where $\Gamma(G)$ denotes the set of conjugacy classes $G(F)$, and $\Pi(G)$ is a set of equivalence classes of irreducible unitary representations of $G(\mathbb{A})$. For any γ and π,

$$a^G(\gamma) = \mathrm{vol}\big(G_\gamma(F)\backslash G_\gamma(\mathbb{A})\big)$$

and

$$a^G(\pi) = \mathrm{mult}(\pi, R),$$

while

$$f_G(\gamma) = \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\gamma x)\mathrm{d}x$$

is the *invariant orbital integral* of f at γ, and

$$f_G(\pi) = \mathrm{tr}\big(\pi(f)\big)$$

is the *irreducible character* of f at π. This identity is known as the *Selberg trace formula* for compact quotient. It was apparently introduced by Selberg only after he had established his considerably more sophisticated trace formula for noncompact arithmetic quotients of $SL(2,\mathbb{R})$.

In general, $G(F)\backslash G(\mathbb{A})$ is not compact. Then the properties on which the proof of (1.1) rests break down. In particular, R has a continuous spectrum, and $R(f)$ is not of trace class. Moreover, elements $\gamma \in G(F)$ may not be semisimple, $G_\gamma(F)\backslash G_\gamma(\mathbb{A})$ need not be compact or even have finite volume, and $f(x^{-1}\gamma x)$ need not be integrable over x in $G_\gamma(\mathbb{A})\backslash G(\mathbb{A})$. It thus becomes much more difficult to establish a trace formula in general. The failure of the various properties leads to several kinds of divergence, in integrals of terms in both the geometric and spectral expansions of $K(x,x)$. However, it turns out that the geometric and spectral sources of divergence are parallel. To make a long story short, one finds that they cancel each other, in some natural sense. The final result is an explicit trace formula, whose terms are parametrized by Levi subgroups M of G (taken up to conjugacy).

Theorem (Invariant trace formula). *There is an identity*

$$\sum_M |W(M)|^{-1} \sum_{\gamma\in\Gamma(M)} a^M(\gamma) I_M(\gamma, f) = \sum_M |W(M)|^{-1} \int_{\Pi(M)} a^M(\pi) I_M(\pi, f) \mathrm{d}\pi, \tag{1.2}$$

for invariant linear forms $I_M(\gamma, f)$ and $I_M(\pi, f)$ in $f \in \mathcal{H}(G)$, and coefficients $a^M(\gamma)$ and $a^M(\pi)$.

The set $W(M)$ here is the Weyl group of G with respect to the split part A_M of the center of M, while $\mathrm{d}\pi$ is a natural (but rather complicated) measure on $\Pi(M)$, which has both a continuous and a discrete part. If $M = G$, $I_M(*, f)$ equals the linear form $f_G(*)$, which we recall is either an invariant orbital integral or an irreducible (invariant) character. If $M \neq G$, however, $I_M(*, f)$ is a more complicated invariant linear form, built out of a combination of weighted orbital integrals and weighted characters. (We recall that a linear form I on $\mathcal{H}(G)$ is *invariant* if $I(f_1 * f_2)$ equals $I(f_2 * f_1)$ for every f_1 and f_2.) The coefficients $a^M(\gamma)$ and $a^M(\pi)$ depend only on M. They are essentially as before (in case $M = G$) if γ is an elliptic semisimple class in $G(F)$ or π is an irreducible representation of $G(\mathbb{A})$ that occurs in the discrete spectrum. However, they are more elaborate for general γ and π.

In the interest of simplicity, we have suppressed two technical matters from the notation (1.2). The left hand side really depends implicitly on a large finite set V of valuations of F. This reflects the lack of a theory for (invariant) unipotent orbital integrals over $G(\mathbb{A})$. In addition, the convergence of the sum-integral on the right hand side is conditional, at least insofar as matters are presently understood. These difficulties are in some sense parallel to each other. It would be interesting to resolve then, but they are not an impediment to present day applications of the trace formula.

There is one part of the invariant trace formula (1.2) that is particularly relevant to applications. It is the *discrete part*, defined as the contribution of the discrete part of the measure $\mathrm{d}\pi$ to the term with $M = G$ on the spectral side. It satisfies the explicit formula

$$\sum_M |W(M)|^{-1} \sum_{w\in W(M)_{\mathrm{reg}}} |\det(1-w)_{\mathfrak{a}_M/\mathfrak{a}_G}|^{-1} \mathrm{tr}\big(M_P(w)\mathcal{I}_P(f)\big), \tag{1.3}$$

expressed in standard notation. In particular, $\mathcal{I}_P$ is the representation of $G(\mathbb{A})$ induced parabolically from the discrete spectrum of $L^2\big(M(F)A^+_{M,\infty}\backslash M(\mathbb{A})\big)$, while $M_P(w)$ is the global intertwining operator attached to the Weyl element w. The sum over M in (1.3) is of course different from that of (1.2), since it represents only a piece of the term with $M = G$ in (1.2). The term with $M = G$ in (1.3) gives the discrete spectrum for G, which is of course where the applications are aimed. However, in the comparison of trace formulas, one cannot separate this term from the larger sum over M.

2. Stable trace formula

For the comparison of trace formulas on different groups, one needs a refinement of the invariant trace formula, known as the stable trace formula. Stability is a local concept, which was introduced by Langlands. It is based on the three basic notions of *stable* conjugacy class, *stable* orbital integral, and *stable* linear form.

Suppose that v is a valuation of F. We consider elements $\gamma_v \in G(F_v)$ that are strongly G-regular, in the sense that their centralizers in G are tori. Recall two such elements are said to be *stably conjugate* if they are conjugate over $G(\overline{F}_v)$. Any strongly G-regular stable conjugacy class $\delta_v \in \Delta_{G\text{-reg}}(G_v)$ is a finite union of $G(F_v)$-conjugacy classes $\{\gamma_v\}$. The *stable orbital integral* of a function $f_v \in \mathcal{H}(G_v)$ at δ_v is the corresponding sum

$$f_v^G(\delta_v) = \sum_{\gamma_v} f_{v,G}(\gamma_v)$$

of invariant orbital integrals. Lastly, a linear form S_v on the local Hecke algebra $\mathcal{H}(G_v)$ of $G_v = G/F_v$ is said to be *stable* if $S_v(f_v)$ depends only on the function

$$f_v^G : \ \Delta_{G\text{-reg}}(G_v) \longrightarrow \mathbb{C}$$

defined by the stable orbital integrals of f_v. In other words,

$$S_v(f_v) = \widehat{S}_v(f_v^G), \qquad f_v \in \mathcal{H}(G_v),$$

for a linear form $\widehat{S}_v$ on the space

$$\mathcal{S}(G_v) = \big\{ f_v^G : \ f_v \in \mathcal{H}(G_v) \big\}.$$

Suppose that G'_v is an endoscopic datum for G over F_v, a notion we shall recall presently (but only in the briefest of terms). We assume for simplicity that G'_v comes with an L-embedding ${}^L G'_v \subset {}^L G_v$ of its L-group into that at G_v. This is something that can always be arranged if, for example, the derived group of G is simply connected.

Given G'_v, Langlands and Shelstad have introduced a transfer mapping $f_v \to f'_v$ from functions $f_v \in \mathcal{H}(G_v)$ to functions f'_v on $\Delta_{G\text{-reg}}(G'_v)$. It is defined by a sum

$$f'_v(\delta'_v) = \sum_{\gamma_v} \Delta(\delta'_v, \gamma_v) f_{v,G}(\gamma_v), \qquad \delta'_v \in \Delta_{G\text{-reg}}(G'_v),$$

where γ_v ranges over the set $\Gamma_{G\text{-reg}}(G_v)$ of strongly G-regular conjugacy classes, and

$$\Delta : \ \Delta_{G\text{-reg}}(G'_v) \times \Gamma_{G\text{-reg}}(G_v) \longrightarrow \mathbb{C}$$

is a Langlands-Shelstad transfer factor. We recall that $\Delta(\delta'_v, \gamma_v)$ is a complicated but ultimately quite explicit function, which for any δ'_v vanishes for all but finitely many γ_v.

Conjecture (Langlands, Shelstad). *For any $f_v \in \mathcal{H}(G_v)$, the function $f'_v = f_v^{G'_v}$ lies in the space $\mathcal{S}(G'_v)$.*

There is a famous (even notorious) variant of the Langlands-Shelstad conjecture, known as the fundamental lemma. It applies to the case that G_v is unramified, which is to say that v is p-adic, and that the group $G_v = G/F_v$ is quasisplit and split over an unramified extension of F_v.

Variant (Fundamental lemma). *Assume that G_v is unramified, and that f_v is the characteristic function of a hyperspecial maximal compact subgroup $K_v \subset G(F_v)$. Then f'_v equals $h_v^{G'_v}$, where h_v is the characteristic function of a hyperspecial maximal compact subgroup $K'_v \subset G'(F_v)$.*

Theorem (Shelstad). *The Langlands-Shelstad transfer conjecture holds if v is archimedean.*

Theorem (Waldspurger). *The fundamental lemma implies the Langlands-Shelstad transfer conjecture for any p-adic v.*

Assume that the fundamental lemma is valid, and that G' is an endoscopic datum for G over F. Then the correspondence

$$f = \prod_v f_v \longrightarrow f' = \prod_v f'_v$$

extends to a global transfer mapping from $\mathcal{H}\big(G(\mathbb{A})\big)$ to the global stable Hecke space $\mathcal{S}\big(G'(\mathbb{A})\big)$. Notice that the fundamental lemma has a dual role here. It is the required hypothesis for Waldspurger's theorem. But it also tells us that f' is globally smooth, in the sense that at almost all places v, it is the image of the characteristic function of a hyperspecial maximal compact subgroup of $G'(F_v)$.

As promised, we include a few remarks on the notion of endoscopic datum. We confine these comments to the global case, in which we regard G as a group over the global field F. Recall first that the L-group ${}^LG$ of G is a semidirect product $\widehat{G} \rtimes \Gamma$ of the complex dual group $\widehat{G}$ of G with the Galois group $\Gamma = \Gamma_F$ of $\overline{F}/F$. An *endoscopic datum* for G over F is a quasisplit group G' over F, together with a semisimple element $s' \in \widehat{G}$ such that

(i) $\widehat{G}' = \mathrm{Cent}(s', \widehat{G})^0$

and

(ii) ${}^LG' \subset \mathrm{Cent}(s', {}^LG)$.

We retain here our simplifying convention that G' comes with an L-embedding of ${}^LG'$ into ${}^LG$. This embedding has to satisfy (ii), a constraint that still leaves room for a choice beyond that of the semisimple element s'. Recall also that G' is *elliptic* if the image of ${}^LG'$ is not contained in any proper Levi subgroup ${}^LM$ of ${}^LG$. There is a natural notion of isomorphism of endoscopic data, and we write $\mathcal{E}_{\mathrm{ell}}(G)$ for the set of isomorphism classes of elliptic endoscopic data for G.

Examples (Quasi-split orthogonal and symplectic groups).

(i)
$$G = SO(2n+1), \qquad \widehat{G} = Sp(2n, \mathbb{C}),$$
$$\widehat{G}' = Sp(2m, \mathbb{C}) \times Sp(2n-2m, \mathbb{C}),$$
$$G' = SO(2m+1) \times SO(2n-2m+1).$$

(ii)
$$G = Sp(2n), \qquad \widehat{G} = SO(2n+1, \mathbb{C}),$$
$$\widehat{G}' = SO(2m+1, \mathbb{C}) \times SO(2n-2m, \mathbb{C}),$$
$$G' = Sp(2m) \times SO(2n-2m).$$

(iii)
$$G = SO(2n), \qquad \widehat{G} = SO(2n, \mathbb{C}),$$
$$\widehat{G}' = SO(2m, \mathbb{C}) \times SO(2n-2m, \mathbb{C}),$$
$$G' = SO(2m) \times SO(2n-2m).$$

In each case, s' is an element in $\widehat{G}$ with $(s')^2 = 1$. In (i), its centralizer in $\widehat{G}$ is connected, and both G and G' are split. In (ii) and (iii), however, the centralizer of s' has two connected components (except when s' is central). There is consequently a further choice to be made in that of the group ${}^LG'$. This amounts to a choice of an automorphic character η' for F with $(\eta')^2 = 1$, which specifies G' as a quasisplit group over F. In cases that $\widehat{G}'$ has a factor $SO(2,\mathbb{C})$, one must in fact take a nontrivial outer twist in order for G' to be elliptic. With this proviso, the list of G' in each case gives a complete set of representatives of $\mathcal{E}_{\mathrm{ell}}(G)$.

There is a generalization of the fundamental lemma, which applies to *weighted* orbital integrals of the characteristic function of a hyperspecial maximal compact subgroup. We assume it, without giving the precise statement, in what follows.

THEOREM 2.1 (Stable trace formula). (a) *There is a decomposition*

$$I_{\mathrm{disc}}(f) = \sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G)} \iota(G,G')\widehat{S}'_{\mathrm{disc}}(f') \tag{2.1}$$

of $I_{\mathrm{disc}}(f)$*, for stable linear forms* $S'_{\mathrm{disc}} = S^{G'}_{\mathrm{disc}}$ *on* $\mathcal{H}(G')$*, and explicit coefficients* $\iota(G,G')$.

(b) *If G is quasisplit (which is to say that G itself represents an element in* $\mathcal{E}_{\mathrm{ell}}(G)$*),* $S^G_{\mathrm{disc}}(f)$ *is the discrete part of a stable trace formula*

$$\begin{aligned} &\sum_M |W(M)|^{-1} \sum_{\delta \in \Delta(M)} b^M(\delta)S_M(\delta,f) \\ &= \sum_M |W(M)|^{-1} \int_{\Phi(M)} b^M(\phi)S_M(\phi,f)\mathrm{d}\phi, \end{aligned} \tag{2.2}$$

an identity that is parallel to the invariant trace formula, and whose terms are stable linear forms.

The proof of (b) comes first. It is very elaborate. All of the terms in (2.2) are defined inductively by setting up analogues of (2.1) for the corresponding terms in the invariant trace formula (1.2). The identity (2.1) in (a) comes at the very end of the process, as a consequence of the corresponding identities for all of the other terms, and the invariant trace formula.

The identity (2.1) is what one brings to applications. How useful is it? Well, taken on its own, it has definite limitations. Suppose for example that G is quasisplit (such as one of the groups $SO(2n+1)$, $Sp(2n)$ and $SO(2n)$ whose endoscopic data we described above). Then (2.1) represents *only* an inductive definition of the summands on the right hand side, in terms of the explicit formula (1.3) for the left hand side. All it says is that the term $S^G_{\mathrm{disc}}(f)$ with $G' = G$ in (2.1), expressed by means of $I_{\mathrm{disc}}(f)$ and the other terms on the right hand side, is stable. An interesting result, no doubt, but certainly not enough to classify the representations that make up the terms in $I_{\mathrm{disc}}(f)$.

The solution, at least for many classical groups, is to combine (2.1) with a similar identity that applies to twisted groups. By a *twisted group*, we shall mean a pair $G = (G^0, \theta)$, where θ is an automorphism of G^0 over F. In this case, we take f to be an element in the Hecke space $\mathcal{H}(G)$ of functions on $G(\mathbb{A}) = G^0(\mathbb{A}) \rtimes \theta$.

Much of the discussion above carries over to twisted groups. For example, the twisted version of the invariant trace formula (1.2) has been established. Its discrete part $I_{\mathrm{disc}}(f)$ takes the form (1.3), with the terms interpreted as twisted induced representations and twisted intertwining operators. Twisted versions of endoscopic data also make sense. Given the twisted analogue of our earlier simplifying convention, a *twisted endoscopic datum* for G over F is a quasisplit group G' over F, together with a semisimple element s' in the set $\widehat{G} = \widehat{G}^0 \rtimes \widehat{\theta}$, such that

(i) $\widehat{G}' = \mathrm{Cent}(s', \widehat{G}^0)^0$
and
(ii) ${}^L G' \subset \mathrm{Cent}(s', {}^L G^0)$.

Kottwitz and Shelstad have constructed twisted transfer factors, which they use to define a local correspondence $f_v \to f'_v$ from $\mathcal{H}(G_v)$ to functions on $\Delta_{G\text{-reg}}(G'_v)$.

It is expected that the identity (2.1) will remain valid as stated for a general twisted group $G = (G^0, \theta)$. The proof will require a twisted fundamental lemma, and its generalization to twisted weighted orbital integrals. It also calls for twisted versions of the theorems of Shelstad and Waldspurger stated above. Finally, it will require a stabilization of the twisted trace formula for G. This has not been done, although many of the techniques that lead to the stabilization of the standard invariant trace formula should carry over in some form.

We note that there has been much recent progress on the fundamental lemma. Laumon and Ngo are now working from a very broad perspective, following geometric ideas introduced by Goresky, Kottwitz and MacPherson. This has lead to a proof of the standard fundamental lemma for the group $G = U(n)$, and will probably go considerably further. D. Whitehouse has used special methods to establish all forms of the fundamental lemma for endoscopic data of the twisted form of $GL(4)$.

3. Classical groups

We describe work in progress on the automorphic representations of quasisplit orthogonal and symplectic groups. These are the groups whose endoscopic data we described in the three examples above. We first look at a fourth example, that of twisted endoscopic data G for general linear groups $\widetilde{G}$.

Example. $\widetilde{G} = (\widetilde{G}^0, \theta)$, $\widetilde{G}^0 = GL(N)$, $\theta(x) = {}^t x^{-1}$,

$$\widehat{\widetilde{G}} = \widehat{\widetilde{G}}^0 \rtimes \widehat{\theta} = GL(N, \mathbb{C}) \rtimes \widehat{\theta},$$

$$\widehat{G} = SO(N_+, \mathbb{C}) \times Sp(N_-, \mathbb{C}), \qquad N = N_+ N_-$$

$$G = \begin{cases} SO(N_+) \times SO(N_- + 1), & \text{if } N_+ \text{ is even,} \\ Sp(N_+ - 1) \times SO(N_- + 1), & \text{if } N_+ \text{ is odd.} \end{cases}$$

We take

$$\widehat{\theta}(x) = \widetilde{J}^{-1}\, {}^t x^{-1} \widetilde{J}, \qquad \widetilde{J} = \begin{pmatrix} 0 & & & 1 \\ & & -1 & \\ & \iddots & & \\ (-1)^N & & & 0 \end{pmatrix},$$

for the dual automorphism, since it stabilizes the standard splitting of $GL(N)$. The semisimple element attached to G is of the form

$$s = \begin{pmatrix} \pm 1 & & 0 \\ & \ddots & \\ 0 & & \pm 1 \end{pmatrix} \rtimes \widehat{\theta}.$$

The centralizer of s in $\widehat{\widetilde{G}}^0$ has two connected components (unless $N_+ = 0$), so there is a further choice to be made in that of the subgroup ${}^L G$ of ${}^L\widetilde{G}^0$. If N_+ is even, this serves to define the factor $SO(N_+)$ of G as quasisplit group over F. If N_+ is odd, it serves only to define the embedding of ${}^L G$ into ${}^L\widetilde{G}^0$, since G must be split. In either case, the supplementary choice is tantamount to that an automorphic character η for F with $\eta^2 = 1$. Like in the earlier examples, η must be nontrivial if $N_+ = 2$ if G is to be elliptic. With this proviso, our list of G gives a complete set of representatives of the set $\mathcal{E}_{\text{ell}}(\widetilde{G})$ of isomorphism classes of elliptic (twisted) endoscopic data for $\widetilde{G}$. We shall say that $G \in \mathcal{E}_{\text{ell}}(\widetilde{G})$ is *simple* if it has only one factor, which is to say that N equals either N_+ or N_-, in the notation above. In the first case, $\widehat{G}$ equals $SO(N, \mathbb{C})$ and G equals $SO(N)$ or $Sp(N-1)$, according to whether N is even or odd. In the second case, $\widehat{G}$ equals $Sp(N, \mathbb{C})$ and G equals $SO(N+1)$. Simple endoscopic data play a special role, since one would expect to apply induction arguments to the factors of any $G \in \mathcal{E}_{\text{ell}}(\widetilde{G})$ that is not simple.

The problem, then, is to try to classify the automorphic representations of a group G that represents a simple endoscopic datum for $\widetilde{G} = GL(N) \rtimes \theta$. We have at our disposal the identity

$$\text{(i)} \qquad I_{\text{disc}}(f) = \sum_{G' \in \mathcal{E}_{\text{ell}}(G)} \iota(G, G') \widehat{S}^G_{\text{disc}}(f'), \qquad f \in \mathcal{H}(G),$$

for any $G \in \mathcal{E}_{\text{ell}}(G)$, and its twisted analogue

$$\text{(ii)} \qquad I_{\text{disc}}(\widetilde{f}) = \sum_{G \in \mathcal{E}_{\text{ell}}(\tilde{G})} \iota(\widetilde{G}, G) \widehat{S}^G_{\text{disc}}(\widetilde{f}^G), \qquad \widetilde{f} \in \mathcal{H}(\widetilde{G}),$$

for $\widetilde{G}$. This is the raw material we have to work with. It consists of the original explicit formulas for the left hand sides of (i) and (ii), the inductive definition of $S^G_{\text{disc}}(f)$ provided by the right hand side of (i), and the explicit identity among these distributions provided by the right hand side of (ii).

The goal is to describe representations of G in terms of the self dual representations of $\widetilde{G}^0 = GL(N)$. Since the argument is based on the trace formula, it is focused on all of the automorphic representations in the spectral decomposition. This means that generic representations will have no special role in the proof. In general, both the trace formula and the endoscopic transfer of functions are theories that are founded on characters. Any classification to which they might lead has also to be characer theoretic. This is probably a necessary condition for a proper understanding of the zeta functions and cohomology of Shimura varieties.

The argument is long. However, it also seems to be very natural. Here are some fundamental properties of representations that must be brought to bear on the identities (i) and (ii).

(1) *The classification of isobaric representations of $GL(N)$ (Jacquet-Shalika), which generalizes the theorem of strong multiplicity one.*
(2) *The classification of automorphic representations that occur in the spectral decomposition of $GL(N)$ (Moeglin-Waldspurger).*
(3) *The local Langlands classification for $GL(N)$ (Harris-Taylor, Henniart).*
(4) *Trace identities for normalized intertwining operators (beginning with work of Shahidi).*

(5) *Twisted orthogonality relations, which follow from the twisted form of the local trace formula.*
(6) *Duality for representations of p-adic groups.*

To this mix, we must also add the indisputable (but critical) fact that an irreducible representation in the automorphic discrete spectrum occurs with *positive, integral* multiplicty!

I will not state the theorems that are likely to follow from this analysis. Let me just say that for a quasisplit orthogonal or symplectic group G, they include the following results.

(1) *A description of local and global representations of G in terms of packets (L-packets, A-packets).*
(2) *A classification of the expected counterexamples of the analogue of Ramanujan's conjecture for G.*
(3) *A formula for the multiplicity of an irreducible representation in the automorphic discrete spectrum of G.*
(4) *The local Langlands correspondence for G (up to automorphisms in the case $G = SO(2n)$).*
(5) *Proof of functoriality for the L-embeddings ${}^LG' \subset {}^LG$ and ${}^LG \subset {}^L\widetilde{G}^0$. This in turn implies basic properties of Rankin-Selberg L-functions for representations of G.*
(6) *Proof of conjectural properties of symmetric square L-functions $L(s, \pi, S^2)$ (and skew-symmetric square L-functions $L(s, \pi, \Lambda^2)$), and of orthogonal root numbers $\varepsilon(\frac{1}{2}, \pi_1 \times \pi_2)$.*

Finally, let me add the likelihood of establishing the conjectured existence of Whitaker models for certain representations of G. That this should then follow from the work of Cogdell, Kim, Piatetskii-Shapiro and Shahidi, and of Ginzburg, Rallis and Soudry, has been pointed out by Rallis and Shahidi. It thus appears that the two general approaches to the study of automorphic forms, L-functions and the trace formula, might in fact be complementary.

4. Beyond endoscopy

I was asked to include some discussion of Langlands' recent ideas for a general study of the principle of functoriality. The conjectural theory of endoscopy, represented in small part by our discussion above, is really aimed at the internal structure of representations of a given group. Its application to the principle of functoriality is incidental, and quite limited. In it most general form, the theory applies only to an endoscopic embedding

$$\xi' : \ {}^LG' \longrightarrow {}^LG$$

of L-groups, where G' represents a (twisted) endoscopic datum for G (relative to an outer automorphism θ). One would hope to compare the (twisted) trace formula for G with stable trace formulas for groups G', using the Langlands-Shelstad-Kottwitz transfer $f \to f'$ of functions.

Suppose now that G and G' are arbitrary reductive groups over F, and that

$$\rho : \ {}^LG' \longrightarrow {}^LG$$

is an arbitrary embedding of their L-groups. Are there trace formulas for G and G' that one can compare? How might one transfer a function $f \in \mathcal{H}(G)$ from G to G'?

What is needed is some sort of trace formula for G that applies only to a part of the discrete spectrum. One would like a trace formula that counts only those automorphic representations π of G that are tempered and cuspidal, and more to the point, are functorial transfers from G'. Now, the question of whether π is as a functorial transfer should be reflected in the analytic behaviour of its automorphic L-functions $L(s, \pi, r)$, for finite dimensional representations

$$r : \ {}^L G \longrightarrow GL(N, \mathbb{C}).$$

Specifically, one should be able to characterize those π that come from G', perhaps up to some measurable obstruction, in terms of the orders of poles of L-functions $L(s, \pi, r)$ at $s = 1$. One can thus pose an alternate problem as follows. For a given r, find a trace formula in which the contribution of π is weighted by the order of the pole of $L(s, \pi, r)$ at $s = 1$. This is still a very tall order. For among other things, we are far from knowing even that $L(s, \pi, r)$ has meromorphic continuation.

In any case, suppose that r is fixed, and that π is a tempered, cuspidal automorphic representation of G. The partial Euler product

$$L^V(s, \pi, r) = \prod_{v \notin V} \det\big(1 - r\big(c(\pi_v)\big) q_v^{-s}\big)^{-1},$$

defined for any finite set V of valuations of F that contains the set $S_{\mathrm{ram}}(\pi, r)$ at which either π or r ramify, converges if $\mathrm{Re}(s) > 1$. Suppose that this function also has meromorphic continuation to the line $\mathrm{Re}(s) = 1$. Then the nonnegative integer

$$n(\pi, r) = \operatorname*{res}_{s=1} \Big(-\frac{\mathrm{d}}{\mathrm{d}s} \log L^V(s, \pi, r)\Big)$$

is defined, and is equal to the order of the pole at $s = 1$ of $L^V(s, \pi, r)$. If $\mathrm{Re}(s) > 1$, we have

$$\begin{aligned}
& -\frac{\mathrm{d}}{\mathrm{d}s} \log L^V(s, \pi, r) \\
& = \sum_{v \notin V} \frac{\mathrm{d}}{\mathrm{d}s} \log\Big(\det\big(1 - r\big(c(\pi_v)\big) q_v^{-s}\big)\Big) \\
& = \sum_{v \notin V} \sum_{k=1}^{\infty} \log(q_v) \mathrm{tr}\Big(r\big(c(\pi_v)\big)^k\Big) q_v^{-ks}.
\end{aligned}$$

It then follows from the Wiener-Ikehara tauberian theorem that

$$n(\pi, r) = \lim_{N \to \infty} \Big(\frac{1}{N} \sum_{\{v \notin V : q_v \le N\}} \log(q_v) \mathrm{tr}\Big(r\big(c(\pi_v)\big)\Big)\Big).$$

Suppose that $f \in \mathcal{H}(G)$ is fixed, and is unramified outside of V. For any N, define a function h_N^V in the unramified Hecke algebra $\mathcal{H}(G^V, K^V)$ for $G(\mathbb{A}^V)$ by setting

$$\mathrm{tr}\big(h_N^V(\pi^V)\big) = \sum_{\{v \notin V : q_v \le N\}} \log(q_v) \mathrm{tr}\Big(r\big(c(\pi_v)\big)\Big),$$

for any unramified representation

$$\pi^V = \bigotimes_{v \notin V} \pi_v$$

of $G(\mathbb{A}^V)$. We use this to form a new function f_N^r in $\mathcal{H}(G)$ by setting

$$f_N^r(x) = f(x) h_N^V(x^V), \qquad x \in G(\mathbb{A}).$$

Here x^V is the projection of x onto $G(\mathbb{A}^V)$. We then have a limit formula

$$\sum_\pi n(\pi, r) m_{\mathrm{temp,cusp}}(\pi) f_G(\pi) = \lim_{N \to \infty} \Big(\frac{1}{N} \mathrm{tr}\big(R_{\mathrm{temp,cusp}}(f_N^r)\big)\Big), \tag{4.1}$$

where $R_{\mathrm{temp,cusp}}$ is the regular representation of $G(\mathbb{A})$ on the tempered, cuspidal part of the discrete spectrum of $L^2\big(G(F)A^+_{G,\infty} \backslash G(\mathbb{A})\big)$, and $m_{\mathrm{temp,cusp}}(\pi)$ is the multiplicity of π in $R_{\mathrm{temp,cusp}}$.

The formula (4.1) holds under the assumption that for each π with $m_{\mathrm{temp,cusp}}(\pi)$ positive, $L^V(s, \pi, r)$ has meromorphic continuation to the line $\mathrm{Re}(s) = 1$. Langlands' proposal, which he has called a "pipe dream", is to try to show that the limit exists without this assumption. The linear form

$$I_{\mathrm{temp,cusp}}(f_N^r) = \mathrm{tr}\big(R_{\mathrm{temp,cusp}}(f_N^r)\big)$$

can be regarded as a piece of $I_{\mathrm{disc}}(f_N^r)$, and hence as a part of the invariant trace formula. The idea would be to prove that the limit

$$I^r_{\mathrm{temp,cusp}}(f) = \lim_{N \to \infty} \Big(\frac{1}{N} I_{\mathrm{temp,cusp}}(f_N^r) \Big)$$

exists, by establishing corresponding limits for all of the other terms in the invariant trace formula. The resulting formula $I^r_{\mathrm{temp,cusp}}(f)$ would then be a trace formula for those π with $n(\pi, r) > 0$.

It is better to think of these ideas in the context of the stable trace formula. Let $S_{\mathrm{temp,cusp}}(f_N^r)$ be the tempered, cuspidal part of the stable trace formula (evaluated at f_N^r). By this, I mean the contribution to $S_{\mathrm{disc}}(f_N^r)$ from global L-packets of tempered cuspidal representations.

I take the liberty of dividing the implications of Langlands' proposal, as they apply here, into three parts.

Pipe Dream (a). *Prove that the limit*

$$S^r_{\mathrm{temp,cusp}}(f) = \lim_{N \to \infty} \Big(\frac{1}{N} S_{\mathrm{temp,cusp}}(f_N^r) \Big)$$

exists, by establishing corresponding limits for all of the other terms in the stable trace formula.

A solution of (a) would give a stable trace formula for $S^r_{\mathrm{temp,cusp}}(f)$, though it would undoubtedly be very complicated. Whatever its nature, such a formula is unlikely to be of much use in isolation. One would also need something with which to compare it.

Assume that the local Langlands classification holds for G. This means (among other things) that for any v, the stable Hecke algebra $\mathcal{S}(G_v)$ may be regarded as a Paley-Wiener space on the set $\Phi_{\mathrm{temp}}(G_v)$ of tempered Langlands parameters ϕ_v for G_v. Given an L-embedding ρ, whose domain G' also satisfies this assumption, we define local mappings

$$f_v \longrightarrow f_v^\rho$$

from $\mathcal{H}(G_v)$ to $\mathcal{S}(G'_v)$ by setting

$$f_v^\rho(\phi'_v) = f_v^G(\rho \circ \phi'_v), \qquad \phi'_v \in \Phi_{\mathrm{temp}}(G'_v).$$

We can then form the global mapping

$$f = \prod_v f_v \longrightarrow f^\rho = \prod_v f_v^\rho$$

from $\mathcal{H}(G)^v$ to $\mathcal{S}(G')$. It is appropriate to call this mapping *functorial transfer* of functions, since it is quite different from endoscopic transfer $f \to f'$, even when ρ happens to be an endoscopic embedding.

Pipe dream (b). *Given r, prove that*

$$S^r_{\mathrm{temp,cusp}}(f) = \sum_\rho \sigma(r,\rho)\widehat{S}_{\mathrm{temp,cusp}}(f^\rho),$$

for $\widehat{G}$-conjugacy classes of elliptic embeddings ρ, with coefficients $\sigma(r,\rho)$.

The focus is here slightly at odds with that of Langlands, insofar as r is fixed. It has the attraction of showing off some formal similarities with the theory of endoscopy, even if they may not be entirely appropriate. In the end, however, one will have to try to invert the identity of (b).

Pipe dream (c). *Establish the principle of functoriality from (b) by allowing r to vary.*

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