L^2-cohomology and automorphic representations

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Abstract. This article is an introduction to the L^2-cohomology of arithmetic symmetric spaces. We shall describe properties of automorphic representations which lead to a remarkable interplay among Hodge structures, Lefschetz structures, and Hecke operators. In order to make the discussion slightly more concrete, we shall focus on the special case of Siegel moduli space.

Résumé. Cet article est une introduction à la cohomologie L^2 des espaces symétriques arithmétiques. On décrit des propriétés des représentations automorphes où interagissent de façon remarquable les structures de Hodge et de Lefschetz et les opérateurs de Hecke. Pour fixer les idées, on se restreint au cas particulier des espaces de modules de Siegel.

I was asked by the editors to submit a general article on the trace formula. There are several such papers already [4], [22], [6], [23], [15], and I was not sure I could add anything. I decided to write a survey article on a different topic, one to which the trace formula has been applied and will certainly be applied further.

The article is on the relationship between L^2-cohomology and automorphic representations. It is an introduction of sorts to the papers [3] and [8], especially [8, §9]. I shall try to illustrate some general phenomena by looking at a basic case—Siegel moduli space $S(N)$ of level N. The phenomena, partly known (Theorem 1) and partly conjectural (Conjecture 2), concern the interplay among Hodge structures, Lefschetz structures and Hecke operators. Taken together, they may be regarded as a reciprocity law for $S(N)$ at the Archimedean place. For a general discussion of reciprocity laws at the unramified finite places, see the papers [26] and [21].

1. L^2-cohomology of $S(N)$

Fix a positive integer n and let $\mathcal{H}$ be the Siegel upper half space of genus n. Then $\mathcal{H}$ is the space of $(n \times n)$-complex matrices of the form

$$Z = X + iY,$$

where X is a real symmetric matrix and Y is a real positive-definite matrix. Observe that $\mathcal{H}$ is an open subset of the complex vector space of complex

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symmetric matrices, and is therefore a complex manifold. Let G be the symplectic group $\mathrm{Sp}(2n)$ of rank n. Then $G(\mathbb{R})$ acts on $\mathcal{H}$ by

$$Z \longrightarrow gZ = (AZ+B)(CZ+D)^{-1}, \qquad Z \in \mathcal{H},$$

for any element $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $G(\mathbb{R})$. The isotropy group of the matrix iI is a maximal compact subgroup K of $G(\mathbb{R})$, which is isomorphic to the unitary group $U(n, \mathbb{C})$. We obtain a canonical diffeomorphism

$$G(\mathbb{R})/K \xrightarrow{\sim} \mathcal{H}.$$

If N is a positive integer,

$$\Gamma(N) = \{\gamma \in \mathrm{Sp}(2n, \mathbb{Z}) : \gamma \equiv I (\mathrm{mod}\ N)\}$$

is a discrete subgroup of $G(\mathbb{R})$. We shall assume that N is large enough so that $\Gamma(N)$ acts properly discontinuously on $\mathcal{H}$. The quotient

$$S(N) = \Gamma(N)\backslash\mathcal{H} \cong \Gamma(N)\backslash G(\mathbb{R})/K$$

is called the *Siegel moduli space* of level N. The space is a compact manifold which is known to be a quasi-projective algebraic variety [11], and it comes with a complete Hermitian metric. Many of its most interesting features can be seen in its cohomology. The space is noncompact, however, so it is best to take the L^2-cohomology

$$H^*_{(2)}(S(N)) = \bigoplus_{m=0}^{2\dim S(N)} H^m_{(2)}(S(N)),$$

with respect to the given complete metric. The L^2-cohomology of such a space behaves exactly like the ordinary cohomology of a nonsingular projective variety [14]. It is a finite dimensional, graded, complex vector space which satisfies Poincaré duality, it has a Hodge decomposition, and it satisfies the hard Lefschetz theorem. We would like to describe these things explicitly.

One of the reasons that $S(N)$ is interesting is that it also comes with a large family of correspondences, which show up as Hecke operators on the cohomology. Before we describe these operators, however, we need to recall the adèlic description of $S(N)$.

From our given N we can construct an open compact subgroup

$$K_f(N) = \{x \in G(\hat{\mathbb{Z}}) : x \equiv I (\mathrm{mod}\ N)\}$$

of $G(\mathbb{A}_f)$, the group of points with values in the ring $\mathbb{A}_f$ of finite adèles. Recall also that we have a diagonal embedding

$$G(\mathbb{Q}) \subset G(\mathbb{A}) = G(\mathbb{R}) \times G(\mathbb{A}_f)$$

of $G(\mathbb{Q})$ as a discrete subgroup of the full adèlic group. The group G is simply connected. The strong approximation theorem [19] then tells us that

$$G(\mathbb{A}) = G(\mathbb{Q}) \cdot G(\mathbb{R}) K_f(N).$$

Since

$$G(\mathbb{Q}) \cap G(\mathbb{R}) K_f(N) = \Gamma(N),$$

we obtain a diffeomorphism

$$\Gamma(N) \backslash G(\mathbb{R}) \xrightarrow{\sim} G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_f(N).$$

Dividing on the right by the group $K \subset G(R)$ (which of course commutes with $K_f(N)$), we obtain the adèlic representation

$$S(N) = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K K_f(N) = G(\mathbb{Q}) \backslash (\mathcal{H} \times G(\mathbb{A}_f) / K_f)$$

of $S(N)$. This looks more cumbersome than the original description, but it has some distinct advantages. For example, one could treat all the varieties $S(N)$ simultaneously by taking the topological inverse limit

$$S = \varprojlim_N S_N = G(\mathbb{Q}) \backslash (\mathcal{H} \times G(\mathbb{A}_f)).$$

The Hecke operators come from the action of $G(\mathbb{A}_f)$ on S by right translation.

The L^2-cohomology of S is a direct limit

$$H^*_{(2)}(S) = \varinjlim_N H^*_{(2)}(S(N))$$

of finite dimensional spaces. The action of an element $g \in G(\mathbb{A}_f)$ on S determines an endomorphism $H^*_{(2)}(g)$ on the infinite dimensional space $H^*_{(2)}(S)$, but the endomorphism does not generally leave invariant the subspace

$$H^*_{(2)}(S(N)) = H^*_{(2)}(S)^{K_f(N)}.$$

However, if we average over the points in the open compact subset $K_f(N) \cdot g \cdot K_f(N)$ of $G(\mathbb{A}_f)$, the resulting operator will leave $H^*_{(2)}(S(N))$ invariant. More generally, if h is an element in the space

$$\mathcal{A}_f(N) = C_c(K_f(N) \backslash G(\mathbb{A}_f) / K_f(N))$$

of compactly supported, $K_f(N)$-bi-invariant functions on $G(\mathbb{A}_f)$, the endomorphism

$$H^*_{(2)}(h) = \int_{G(\mathbb{A}_f)} h(g_1) H^*_{(2)}(g_1)\,dg_1$$

acts on the finite dimensional vector space $H^*_{(2)}(S(N))$. In fact, $\mathcal{A}_f(N)$ is an algebra under convolution on $G(\mathbb{A}_f)$, and as such, operates on $H^*_{(2)}(S(N))$. It is these endomorphisms which are the Hecke operators. They carry fundamental arithmetic information, in the form of numerical data obtained from their eigenvalues on the various spaces $H^m_{(2)}(S(N))$.

Using the trace formula, I proved a closed formula for the Lefschetz number

$$\mathcal{L}(h) = \sum_m (-1)^m \mathrm{tr}(H^m_{(2)}(h))$$

of any Hecke operator [7]. The formula has a finite number of terms, each of which can be evaluated explicitly in terms of the characters of discrete series representations of $G(\mathbb{R})$. In principle it could be used to calculate the eigenvalues of Hecke operators. However, this is not really the role of the formula, and it would not be practical in any case. For one thing, the number of terms increases with the height

$$\|g\|, \qquad g \in G(\mathbb{A}_f),$$

of a Hecke operator $K_f(N) \cdot g \cdot K_f(N)$. The formula should rather be used in conjunction with some arithmetic Lefschetz fixed point formula, which displays the arithmetic information in terms of its origins in algebraic geometry. An important step in this direction was taken by Goresky and Macpherson (in collaboration with Harder and Kottwitz), who established a topological proof of the formula for $\mathcal{L}(h)$. (See [16].) They obtained a formula for the intersection cohomology Lefschetz number of a Hecke correspondence on the Bailey-Borel compactification $\overline{S(N)}$ of $S(N)$. It is a difficult business since $\overline{S(N)}$ is highly singular, and the Hecke correspondences intersect the diagonal in a complicated way. One hopes that their methods will eventually lead to similar formulas for the ℓ-adic version of intersection cohomology.

We cannot discuss such things here. Our purpose rather is to describe the interplay on L^2-cohomology of the Hecke operators, the Hodge structure, and the Lefschetz structure.

The Hodge and Lefschetz structures are decompositions of $H^*_{(2)}(S(N))$ into irreducible subspaces, relative to the actions of two different groups. For the Hodge structure, the relevant group is $\mathbb{C}^*$. The irreducible constituents are one-dimensional representations of the form

$$z \longrightarrow z^p \bar{z}^q, \qquad z \in \mathbb{C}^*,$$

with p and q nonnegative integers. This gives the Hodge decomposition

$$H^*_{(2)}(S(N)) = \bigoplus_{m=0}^{2\dim S(N)} \Big(\bigoplus_{p+q=m} H^{p,q}_{(2)}(S(N)) \Big)$$

of the cohomology into isotypical subspaces. For the Lefschetz structure the group is $\mathrm{SL}(2,\mathbb{C})$. The differential of its representation on $H^*_{(2)}(S(N))$, evaluated at the element $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in the Lie algebra, is the nilpotent action by cup product of a Kähler form. If $d = \dim(S(N))$, the element $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$ acts on $H^{d+k}_{(2)}(S(N))$ by z^k. As our third ingredient we have the Hecke operators, acting as a representation of the convolution algebra $\mathcal{A}_f(N)$ on $H^*_{(2)}(S(N))$.

The action of $\mathcal{A}_f(N)$ on $H^*_{(2)}(S(N))$ commutes with that of both $\mathbb{C}^*$ and $\mathrm{SL}(2,\mathbb{C})$. We can also make the two group actions commute, if we replace the Hodge structure by a Tate twist. More precisely, if μ' and ν are the representations of $\mathbb{C}^*$ and $\mathrm{SL}(2,\mathbb{C})$ just described, we replace μ' by

$$\mu(z) = \mu'(z)\nu\begin{pmatrix} (z\bar{z})^{-\frac{1}{2}} & 0 \\ 0 & (z\bar{z})^{\frac{1}{2}} \end{pmatrix}, \qquad z \in \mathbb{C}^*.$$

Then the three actions all commute. In other words the Hecke operators, the Hodge structure, and the Lefschetz decomposition can all be put together as a single representation of

$$\mathbb{C}^* \times \mathrm{SL}(2,\mathbb{C}) \times \mathcal{A}_f(N)$$

on the cohomology $H^*_{(2)}(S(N))$. We would like to describe this representation more explicitly.

2. The spectral decomposition of cohomology

The first step is a qualitative description of the cohomology in terms of the spectral decomposition of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. It is not difficult to derive this decomposition, at least in the case of compact quotient. Suppose for the moment that $S(N) = \Gamma(N)\backslash\mathcal{H}$ is replaced by $\Gamma\backslash\mathcal{H}$, where $\Gamma \subset G(\mathbb{R})$ is a discrete cocompact subgroup which operates properly discontinuously on $\mathcal{H}$. We shall sketch the arguments from [13, VII, §1-3].

The de Rham cohomology $H^*(\Gamma\backslash\mathcal{H})$ is the cohomology of the complex $A^*(\Gamma\backslash\mathcal{H})$ of differential forms on $\Gamma\backslash\mathcal{H}$. Thus, $A^m(\Gamma\backslash\mathcal{H})$ is the space of complex m-forms

$$\omega\colon \Lambda^m T(\Gamma\backslash\mathcal{H}) \longrightarrow \mathbb{C}$$

with the boundary operator

$$d\omega(X_0,\dots,X_m) = \sum_i (-1)^i X_i \omega(X_0,\dots,\hat{X}_i,\dots,X_m) + \sum_{i<j} (-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_m).$$

Here $T(\Gamma\backslash\mathcal{H})$ denotes the complexified tangent bundle, and $X_0,\dots,X_m$ are complex vector fields.

A preliminary observation is that $A^m(\Gamma\backslash\mathcal{H})$ can be identified with the space $A^m(\mathcal{H})^\Gamma$ of Γ-invariant forms on $\mathcal{H}$. Next, we recall that the complexified tangent bundle on $\mathcal{H}$ is homogeneous. More precisely,

$$T(\mathcal{H}) = T(G(\mathbb{R})/K) = G(\mathbb{R})\times_K(\mathfrak{g}/\mathfrak{k}),$$

where $\mathfrak{g}$ and $\mathfrak{k}$ denote the complexified Lie algebras of $G(\mathbb{R})$ and K, and K acts by the adjoint representations on $\mathfrak{g}/\mathfrak{k}$. A similar assertion applies to any exterior power $\Lambda^m T(\mathcal{H})$ of the tangent bundle, so we may identify $A^m(\mathcal{H})^\Gamma$ with

$$\{f \in C^\infty(\Gamma\backslash G(\mathbb{R}),\Lambda^m(\mathfrak{g}/\mathfrak{k})^*) : f(gk) = \mathrm{Ad}(k)^* f(g)\},$$

a space of functions on $\Gamma\backslash G(\mathbb{R})$ with values in the dual space of $\Lambda^m(\mathfrak{g}/\mathfrak{k})$. This in turn can be written as

$$C^m\Big(\mathfrak{g},K;C^\infty(\Gamma\backslash G(\mathbb{R}))\Big) = \mathrm{Hom}_{\mathbb{C}}\Big(\Lambda^m(\mathfrak{g}/\mathfrak{k}),C^\infty(\Gamma\backslash G(\mathbb{R}))\Big)^K,$$

the space of K-invariant linear transformations between two complex vector spaces. The boundary operator above then has an immediate interpretation as a map

$$d: C^m\Big(\mathfrak{g},K;C^\infty(\Gamma\backslash G(\mathbb{R}))\Big) \longrightarrow C^{m+1}\Big(\mathfrak{g},K;C^\infty(\Gamma\backslash G(\mathbb{R}))\Big).$$

The cohomology of the resulting complex is, by definition, the relative Lie algebra cohomology $H^*(\mathfrak{g},K;C^\infty(\Gamma\backslash G(\mathbb{R})))$ of $(\mathfrak{g},K)$ with values in the $(\mathfrak{g},K)$-module $C^\infty(\Gamma\backslash G(\mathbb{R}))$. In other words

$$H^*_{(2)}(\Gamma\backslash\mathcal{H}) \cong H^*\Big(\mathfrak{g},K;C^\infty(\Gamma\backslash G(\mathbb{R}))\Big).$$

The third observation is that we can exploit the spectral decomposition

$$L^2(\Gamma\backslash G(\mathbb{R})) = \bigoplus_{\pi_{\mathbb{R}}} m(\pi_{\mathbb{R}},\Gamma)\pi_{\mathbb{R}}$$

of $L^2(\Gamma\backslash G(\mathbb{R}))$ into irreducible representations $\pi_{\mathbb{R}}$ of $G(\mathbb{R})$. We obtain

$$H^*(\mathfrak{g},K;C^\infty(\Gamma\backslash G(\mathbb{R}))) = \bigoplus_{\pi_{\mathbb{R}}} m(\pi_{\mathbb{R}},\Gamma)H^*(\mathfrak{g},K;\pi_{\mathbb{R}}^\infty),$$

where $\pi_{\mathbb{R}}^\infty$ is the dense subspace of smooth vectors for the representation $\pi_{\mathbb{R}}$.

If Γ is replaced by the group $\Gamma(N)$ we started with, $\Gamma(N)\backslash G(\mathbb{R})$ is no longer compact. Then $L^2(\Gamma(N)\backslash G(\mathbb{R}))$ has a continuous as well as a discrete spectrum, and one must use the arguments above with caution. However, Borel and Casselman [12] have shown that they remain valid. The upshot is a decomposition

$$H^*_{(2)}(\Gamma(N)\backslash\mathcal{H}) = \bigoplus_{\pi_{\mathbb{R}}} m_{\text{disc}}(\pi_{\mathbb{R}},\Gamma(N))H^*(\mathfrak{g},K;\pi_{\mathbb{R}}^\infty),$$

where $m_{\text{disc}}(\pi_{\mathbb{R}},\Gamma(N))$ is the multiplicity of $\pi_{\mathbb{R}}$ in the subspace $L^2_{\text{disc}}(\Gamma(N)\backslash G(\mathbb{R}))$ of $L^2(\Gamma(N)\backslash G(\mathbb{R}))$ which decomposes discretely.

The diffeomorphism

$$\Gamma(N)\backslash G(\mathbb{R}) \cong G(\mathbb{Q})\backslash G(\mathbb{A})/K_f(N)$$

is compatible with right translation by $G(\mathbb{R})$ on both spaces. We therefore have a $G(\mathbb{R})$-isomorphism

$$L^2_{\text{disc}}(\Gamma(N)\backslash G(\mathbb{R})) \cong L^2_{\text{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f(N)).$$

To analyze the right hand space more fully, it makes sense to consider the action of the larger group $G(\mathbb{A})$ on

$$L^2_{\text{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A})) = \varinjlim_N L^2_{\text{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f(N)).$$

As a representation of $G(\mathbb{A})$, this space is a direct sum of irreducible representations π of $G(\mathbb{A})$, with finite multiplicities $m_{\text{disc}}(\pi)$. Each such π is a tensor product

$$\pi = \pi_{\mathbb{R}} \otimes \pi_f, \qquad \pi_{\mathbb{R}} \in \Pi(G(\mathbb{R})),\ \pi_f \in \Pi(G(\mathbb{A}_f)),$$

of irreducible representations of the groups $G(\mathbb{R})$ and $G(\mathbb{A}_f)$. Since $L^2_{\text{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f(N))$ is the subspace of vectors in $L^2_{\text{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ which are fixed by $K_f(N)$ under right translation by $G(\mathbb{A}_f)$, we obtain decompositions

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f(N)) = \bigoplus_{\pi} m_{\text{disc}}(\pi)(\pi_{\mathbb{R}} \otimes \pi_f^{K_f(N)})$$

and

$$m_{\mathrm{disc}}(\pi_{\mathbb{R}},\Gamma(N)) = \sum_{\pi=\pi_{\mathbb{R}}\otimes\pi_f} m_{\mathrm{disc}}(\pi)\dim(\pi_f^{K_f(N)}),$$

where $\pi_f^{K_f(N)}$ is the finite dimensional subspace of vectors in the $G(\mathbb{A}_f)$-module π_f fixed by $K_f(N)$. It follows that

$$H^*_{(2)}(S(N)) \cong \bigoplus_{\pi=\pi_{\mathbb{R}}\otimes\pi_f} m_{\mathrm{disc}}(\pi)\,(H^*(\mathfrak{g},K;\pi^\infty_{\mathbb{R}})\otimes\pi_f^{K_f(N)}).$$

This is the spectral decomposition of cohomology in the form we want.

The Hodge and Lefschetz groups act on the cohomology through the graded vector spaces $H^*(\mathfrak{g},K;\pi^\infty_{\mathbb{R}})$. Indeed, the complex structure on $G(\mathbb{R})/K$ comes from a K-equivariant decomposition

$$\mathfrak{g}/\mathfrak{k} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

of the complex tangent space at 1, and this determines a Hodge decomposition

$$H^*(\mathfrak{g},K;\pi^\infty_{\mathbb{R}}) = \bigoplus_{p,q\geq 0} H^{p,q}(\mathfrak{g},K;\pi^\infty_{\mathbb{R}})$$

of the $(\mathfrak{g},K)$-cohomology. (See for example [8, p. 60].) The Killing form provides a canonical element in $H^{1,1}(\mathfrak{g},K;\mathbb{C})$, whose image in $H^{1,1}_{(2)}(S(N))$ is just the Kähler class. In particular, the cup product with this element gives a map

$$H^{p,q}(\mathfrak{g},K;\pi^\infty_{\mathbb{R}}) \longrightarrow H^{p+1,q+1}(\mathfrak{g},K;\pi^\infty_{\mathbb{R}}),$$

which in turn (by the Jacobson-Morosov theorem) provides a representation of $SL(2,\mathbb{C})$ on $H^*_{(2)}(\mathfrak{g},K;\pi^\infty_{\mathbb{R}})$. Combining the two structures as we did earlier (with a Tate twist), we obtain a representation of the group $C^* \times SL(2,\mathbb{C})$ on each of the graded vector spaces $H^*(\mathfrak{g},K;\pi^\infty_{\mathbb{R}})$. The Hecke algebra, on the other hand, acts through the spaces $\pi_f^{K_f(N)}$. The action is simply given by the operators

$$\pi_f(h) = \int_{G(\mathbb{A}_f)} h(g_1)\pi_f(g_1)\,dg_1, \qquad h \in \mathcal{A}_f(N),$$

on the finite dimensional spaces $\pi_f^{K_f(N)}$. The spectral decomposition thus brings the essential questions into focus. It conveniently separates the further investigation of the cohomology into the study of three distinct objects - the spaces $H^*(\mathfrak{g},K;\pi^\infty_{\mathbb{R}})$, the spaces $\pi_f^{K_f(N)}$, and the multiplicities $m_{\mathrm{disc}}(\pi)$.

The first objects are the best understood. Vogan and Zuckerman [31] have classified the irreducible unitary representations $\pi_{\mathbb{R}}$ with $(\mathfrak{g}, K)$-cohomology, and for each such $\pi_{\mathbb{R}}$ they construct the graded vector space $H^*(\mathfrak{g}, K; \pi_{\mathbb{R}}^{\infty})$. Less is known about the spaces $\pi_f^{K_f(N)}$, although there has been considerable progress, especially in the case that N is a product of distinct primes ([17], [29], [28]). The multiplicities $m_{\text{disc}}(\pi)$ are the deepest part of the problem. The nonvanishing of $m_{\text{disc}}(\pi_{\mathbb{R}} \otimes \pi_f)$ is a very strong condition, that is to be regarded as a subtle link between two otherwise unrelated representations $\pi_{\mathbb{R}}$ and π_f. The conjectural theory of endoscopy ([24], [20], [8]) suggests a simple formula for $m_{\text{disc}}(\pi)$ in terms of objects which classify the representations $\{\pi_{\mathbb{R}}\}$ and $\{\pi_f\}$. One hopes eventually to establish the formula for $m_{\text{disc}}(\pi)$ by comparing trace formulas for G ([5], [9]) with those of related groups.

3. Parameters over $\mathbb{R}$ and Hodge-Lefschetz structures

Our goal has been to describe the representation of $\mathbb{C}^* \times \mathrm{SL}(2,\mathbb{C}) \times \mathcal{A}_f(N)$ on $H^*_{(2)}(S(N))$. Towards this end, let us look more closely at the constituents of the spectral decomposition. We shall consider the groups $H^*(\mathfrak{g}, K; \pi_{\mathbb{R}}^{\infty})$ in this section and the multiplicities $m_{\text{disc}}(\pi)$ in the next.

We first recall the Langlands classification ([25], [18], [30]) of the irreducible representations $\{\pi_{\mathbb{R}}\}$ of $G(\mathbb{R})$. The necessary ingredients are the Weil group of $\mathbb{R}$ and the dual group of G. The Weil group $W_{\mathbb{R}}$ is generated by the group $\mathbb{C}^*$ and an element σ, subject to conditions $\sigma z \sigma^{-1} = \bar{z}$ for any $z \in \mathbb{C}^*$, and $\sigma^2 = -1$. It is the unique nontrivial extension

$$1 \longrightarrow C^* \longrightarrow W_{\mathbb{R}} \longrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \longrightarrow 1$$

(with the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acting in the obvious way on $\mathbb{C}^*$), and its commutator quotient $W_{\mathbb{R}}^{ab}$ is canonically isomorphic to $\mathbb{R}^*$. The Weil group is thus a pretty straightforward object. So is the dual group. It is simply the complex Lie group $\hat{G} = \mathrm{SO}(2n+1,\mathbb{C})$. Langlands classified the irreducible representations $\pi_{\mathbb{R}}$ as a disjoint union of finite packets $\Pi_{\phi_{\mathbb{R}}}$, parametrized by the $\hat{G}$-conjugacy classes of continuous homomorphisms

$$\phi_{\mathbb{R}}: W_{\mathbb{R}} \longrightarrow \hat{G}.$$

For a given $\phi_{\mathbb{R}}$, take the centralizer

$$S_{\phi_{\mathbb{R}}} = \mathrm{Cent}(\mathrm{Image}(\phi), \hat{G})$$

of the image of $\phi_{\mathbb{R}}$, and its group

$$\mathcal{S}_{\phi_{\mathbb{R}}} = S_{\phi_{\mathbb{R}}} / S^0_{\phi_{\mathbb{R}}}$$

of connected components. Then there is an injective map

$$\pi_{\mathbb{R}} \longrightarrow \langle s, \pi_{\mathbb{R}} \rangle, \qquad \pi_{\mathbb{R}} \in \Pi_{\phi_{\mathbb{R}}}, \ s \in S_{\phi_{\mathbb{R}}},$$

from the finite packet $\Pi_{\phi_{\mathbb{R}}}$ into the set of irreducible characters on the finite group $S_{\phi_{\mathbb{R}}}$, which is defined by some natural character identities. In this way, the irreducible representations $\pi_{\mathbb{R}}$ are classified by some rather concrete objects—maps of $W_{\mathbb{R}}$ into $\widehat{G}$, and irreducible characters on the finite groups $S_{\phi_{\mathbb{R}}}$.

Not all representations $\pi_{\mathbb{R}}$ will be the $\mathbb{R}$-components of automorphic representations π of $G(\mathbb{A})$. I later introduced a family of parameters which I conjectured would determine the $\mathbb{R}$-components of automorphic representations [3]. These parameters are the $\widehat{G}$-conjugacy classes of maps

$$\psi_{\mathbb{R}}: W_{\mathbb{R}} \times \mathrm{SL}(2,\mathbb{C}) \longrightarrow \widehat{G}$$

such that the image of $W_{\mathbb{R}}$ is bounded. For a given $\psi_{\mathbb{R}}$, the map

$$\phi_{\psi_{\mathbb{R}}}(w) = \psi_{\mathbb{R}}\left(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right), \qquad w \in W_{\mathbb{R}},$$

is a parameter of the first kind. In fact $\psi_{\mathbb{R}}$ is uniquely determined by $\phi_{\psi_{\mathbb{R}}}$, so the parameters $\{\psi_{\mathbb{R}}\}$ can be regarded as a subset of the parameters $\{\phi_{\mathbb{R}}\}$. (The absolute value $|w|$ comes from $\mathbb{R}^*$ via the isomorphism $W_{\mathbb{R}}^{ab} \cong \mathbb{R}^*$.) For a given $\psi_{\mathbb{R}}$ set

$$S_{\psi_{\mathbb{R}}} = \mathrm{Cent}(\mathrm{Image}(\psi_{\mathbb{R}}), \widehat{G})$$

and

$$\mathcal{S}_{\psi_{\mathbb{R}}} = S_{\psi_{\mathbb{R}}} / S_{\psi_{\mathbb{R}}}^0.$$

I conjectured the existence of a finite packet $\Pi_{\psi_{\mathbb{R}}}$ of irreducible representations of $G(\mathbb{R})$, and a map

$$\pi_{\mathbb{R}} \longrightarrow \langle s, \pi_{\mathbb{R}} \rangle, \qquad \pi_{\mathbb{R}} \in \Pi_{\psi_{\mathbb{R}}}, \ s \in \mathcal{S}_{\psi_{\mathbb{R}}},$$

from $\Pi_{\psi_{\mathbb{R}}}$ into the set of characters on the finite group $\mathcal{S}_{\psi_{\mathbb{R}}}$ satisfying natural character identities. (I actually conjectured that the map was injective and that the characters on $\mathcal{S}_{\psi_{\mathbb{R}}}$ in the image were irreducible, in analogy with the Langlands classification. The first condition, at least, turned out to be too strong.)

If $\phi_{\mathbb{R}} = \phi_{\psi_{\mathbb{R}}}$, there is a surjective map $\mathcal{S}_{\psi_{\mathbb{R}}} \longrightarrow \mathcal{S}_{\phi_{\mathbb{R}}}$, and hence an injective map $\widehat{\mathcal{S}}_{\phi_{\mathbb{R}}} \hookrightarrow \widehat{\mathcal{S}}_{\phi_{\mathbb{R}}}$ from irreducible characters on $\mathcal{S}_{\phi_{\mathbb{R}}}$ to those on $\mathcal{S}_{\psi_{\mathbb{R}}}$. It was natural then to conjecture that the packet $\Pi_{\phi_{\mathbb{R}}}$ was a subset of $\Pi_{\psi_{\mathbb{R}}}$. The complement of $\Pi_{\phi_{\mathbb{R}}}$ in $\Pi_{\psi_{\mathbb{R}}}$, however, need not be a union of other Langlands packets $\Pi_{\phi'_{\mathbb{R}}}$. Nor do the packets $\{\Pi_{\psi_{\mathbb{R}}}\}$ have to be disjoint. Their role is not so much one of classification, as in the description of how representations of $G(\mathbb{R})$ contribute to autormorphic forms.

In the paper [2], Adams and Johnson constructed packets $\Pi_{\psi_{\mathbb{R}}}$ of representations of $G(\mathbb{R})$ for a finite set of parameters $\psi_{\mathbb{R}} \in \Psi_{\text{cohom}}$. They showed that the irreducible constituents of the elements in those packets were precisely the representations with $(\mathfrak{g}, K)$-cohomology, classified earlier by Vogan and Zuckerman. (See also [8, §5].) The parameters $\psi_{\mathbb{R}}$ in the set Ψ_{cohom} all have the property that the group $S_{\psi_{\mathbb{R}}}$ is finite, so that $\mathcal{S}_{\psi_{\mathbb{R}}} = S_{\psi_{\mathbb{R}}}$.

More recently, Adams, Barbasch and Vogan [1] constructed the packets $\Pi_{\psi_{\mathbb{R}}}$ for all of the parameters $\psi_{\mathbb{R}}$. They also established the pairing

$$\langle s, \pi_{\mathbb{R}} \rangle, \qquad s \in \mathcal{S}_{\psi_{\mathbb{R}}},\ \pi_{\mathbb{R}} \in \Pi_{\psi_{\mathbb{R}}}.$$

by proving all the relevant character identities. The Langlands packets $\Pi_{\phi_{\mathbb{R}}}$, with $\phi_{\mathbb{R}} = \phi_{\psi_{\mathbb{R}}}$, were built naturally into the construction as subsets *of the corresponding packets* $\Pi_{\psi_{\mathbb{R}}}$. *Their work is likely to play an important* role in the future study of automorphic representations. Here we are only discussing representations with cohomology, so it will be enough to consider the finite set Ψ_{cohom} of parameter studied by Adams and Johnson.

Let us fix a parameter $\psi_{\mathbb{R}}$ in Ψ_{cohom}. We shall consider the finite dimensional graded vector space

$$V_{\psi_{\mathbb{R}}} = \bigoplus_{\pi_{\mathbb{R}}} H^*(\mathfrak{g}, K; \pi_{\mathbb{R}}^{\infty}), \qquad \pi_{\mathbb{R}} \in \Pi_{\psi_{\mathbb{R}}}.$$

As we have observed, there is a representation of $\mathbb{C}^* \times \mathrm{SL}(2,\mathbb{C})$ on each space $H^*(\mathfrak{g}, K; \pi_{\mathbb{R}}^{\infty})$, so the direct sum is a representation on $V_{\psi_{\mathbb{R}}}$. The group $\mathcal{S}_{\psi_{\mathbb{R}}}$ is abelian, and the functions $s \longrightarrow \langle s, \pi_{\mathbb{R}} \rangle$ are one-dimensional linear characters. They provide us with a representation

$$s \longrightarrow \bigoplus_{\pi_{\mathbb{R}}} \langle s, \pi_{\mathbb{R}} \rangle, \qquad s \in \mathcal{S}_{\psi_{\mathbb{R}}},\ \pi_{\mathbb{R}} \in \Pi_{\psi_{\mathbb{R}}},$$

of $\mathcal{S}_{\psi_{\mathbb{R}}}$ on $V_{\psi_{\mathbb{R}}}$ which commutes with $\mathbb{C}^* \times \mathrm{SL}(2,\mathbb{C})$. We thus have a representation

$$\rho_{\psi_{\mathbb{R}}} : \mathcal{S}_{\psi_{\mathbb{R}}} \times \mathbb{C}^* \times \mathrm{SL}(2,\mathbb{C}) \longrightarrow GL(V_{\psi_{\mathbb{R}}})$$

of the product of three groups. Is it possible to describe this representation more explicitly?

The question has a nice answer if we are willing to work in a slightly different context. We therefore make a change of notation that will be in effect from now on. We take G to be the group $G\,\mathrm{Sp}(2n)$ of symplectic similitudes, instead of $\mathrm{Sp}(2n)$. The dual $\hat{G}$ becomes the group

$$G\,\mathrm{Spin}(2n+1,\mathbb{C}) = (\mathrm{Spin}(2n+1,\mathbb{C})\times\mathbb{C}^*)/\{\pm 1\},$$

an extension of $\mathrm{SO}(2n+1,\mathbb{C})$ by $\mathbb{C}^*$. The corresponding variety $S(N)$ is no longer connected, but it can be identified with a disjoint union of several Siegel modular varieties. Everything else carries over as above. In particular, for any $\psi_{\mathbb{R}} \in \Psi_{\mathrm{cohom}}$, we have the finite dimensional vector space $V_{\psi_{\mathbb{R}}}$, and the representation $\rho_{\psi_{\mathbb{R}}}$ of $S_{\psi_{\mathbb{R}}} \times C^* \times \mathrm{SL}(2,\mathbb{C})$ on $V_{\psi_{\mathbb{R}}}$.

From our point of view, the advantage of the change is that $\hat{G} = G\,\mathrm{Spin}(2n+1,\mathbb{C})$ has a spin representation

$$r_{\mathrm{spin}}: \hat{G} \longrightarrow GL(V_{\mathrm{spin}})$$

(of dimensional 2^n). The representation is not unique (since $\hat{G}$ has a one dimensional center), so we have first to fix a fundamental dominant weight μ_{spin} to pin it down. Given μ_{spin}, there is a natural representative of $\psi_{\mathbb{R}}$ (within its $\hat{G}$-orbit) such that $S_{\psi_{\mathbb{R}}}$ lies in the maximal torus on which μ_{spin} is defined. In particular, μ_{spin} determines a linear character on the group $S_{\psi_{\mathbb{R}}}$. Let

$$\sigma_{\psi_{\mathbb{R}}}: S_{\psi_{\mathbb{R}}} \times \mathbb{C}^* \times \mathrm{SL}(2,\mathbb{C}) \longrightarrow GL(V_{\mathrm{spin}})$$

be the representation defined by

$$\sigma_{\psi_{\mathbb{R}}}(s,z,u) = r_{\mathrm{spin}}(s\psi_{\mathbb{R}}(z,u))\mu_{\mathrm{spin}}(s)^{-1}.$$

Since the connected component $S^0_{\psi_{\mathbb{R}}}$ equals the center of $\hat{G}$, the left hand side really is a function of s in the quotient $S_{\psi_{\mathbb{R}}} = S_{\psi_{\mathbb{R}}}/S^0_{\psi_{\mathbb{R}}}$, even though the individual terms on the right are not.

Theorem 1. *For each $\psi_{\mathbb{R}} \in \Psi_{\mathrm{cohom}}$, the representations $\rho_{\psi_{\mathbb{R}}}$ and $\sigma_{\psi_{\mathbb{R}}}$ of $S_\psi \times \mathbb{C}^* \times \mathrm{SL}(2,\mathbb{C})$ are equivalent.*

This result was motivated by the lemma on p. 240 of [27], and was proved in [8, Proposition 9.1]. It provides insight into the Hodge and Lefschetz structures on $H^*_{(2)}(S(N))$. For a complete picture, however, we need to incorporate the action of the Hecke algebra. Again there is a rather striking answer, but it rests on the conjectural formulas for the multiplicities $m_{\mathrm{disc}}(\pi)$.

4. Global conjectures and Hecke operators

The global conjectures begin with the understanding that automorphic representations π should also occur in packets. They are to be parametrized by $\hat{G}$-conjugacy classes of homomorphisms

$$\psi: L_{\mathbb{Q}} \times \mathrm{SL}(2,\mathbb{C}) \longrightarrow \hat{G},$$

such that the image of $L_{\mathbb{Q}}$ is bounded. The Langlands group $L_{\mathbb{Q}}$ here is a hypothetical locally compact extension of the Galois group of $\bar{\mathbb{Q}}/\mathbb{Q}$ [27], [20]. Among its various properties there should be an injection $W_{\mathbb{R}} \to L_{\mathbb{Q}}$. Therefore any ψ defines a local parameter

$$\psi_{\mathbb{R}}: W_{\mathbb{R}} \times \mathrm{SL}(2,\mathbb{C}) \longrightarrow \hat{G}$$

by restriction, and hence a finite packet $\Pi_{\psi_{\mathbb{R}}}$ of irreducible representations of $G(\mathbb{R})$. It is conjectured that there is also a packet Π_{ψ_f} (often infinite) of representations of $G(\mathbb{A}_f)$ attached to ψ. (These in fact would be tensor products $\pi_f = \bigotimes_p \pi_p$, where each π_p lies in a local packet Π_{ψ_p} attached to ψ.) The global packet is the set of tensor products

$$\Pi_\psi = \{\pi_{\mathbb{R}} \otimes \pi_f : \pi_{\mathbb{R}} \in \Pi_{\psi_{\mathbb{R}}}, \pi_f \in \Pi_{\psi_f}\}.$$

As in the local case, we set

$$S_\psi = \mathrm{Cent}(\mathrm{Image}(\psi), \hat{G})$$

and

$$S_\psi = S_\psi / S_\psi^0.$$

There is an obvious map $S_\psi \longrightarrow S_{\psi_{\mathbb{R}}}$, so we obtain a character

$$s \longrightarrow \langle s, \pi_{\mathbb{R}} \rangle, \qquad s \in S_\psi,$$

on S_ψ by restriction, for each $\pi_{\mathbb{R}} \bullet \Pi_{\psi_{\mathbb{R}}}$. It is conjectured that the representations $\pi_f \bullet \Pi_{\psi_f}$ also determine characters $\langle s, \pi_f \rangle$ on S_ψ. This provides a character

$$\langle s, \pi \rangle = \langle s, \pi_{\mathbb{R}} \rangle \langle s, \pi_f \rangle, \qquad s \in S_\psi,$$

on S_ψ for every representation $\pi = \pi_{\mathbb{R}} \otimes \pi_f$ in the global packet Π_ψ. Finally, there is a canonical sign character

$$\varepsilon_\psi : S_\psi \longrightarrow \{\pm 1\}$$

on S_ψ, whose definition we shall recall in a moment. With these objects we can state the conjectural formula for $m_{\mathrm{disc}}(\pi)$ [8, §8].

The packet Π_ψ should contribute representations to the discrete spectrum (taken modulo the center of $G = G\,\mathrm{Sp}(2n)$) if and only if S_ψ is finite (modulo the center of $\hat{G} = G\,\mathrm{Spin}(2n+1)$). Suppose this is so. The multiplicity formula is then

$$m_{\mathrm{disc}}(\pi) = |S_\psi|^{-1} \sum_{s \in S_\psi} \varepsilon_\psi(s) \langle s, \pi \rangle, \qquad \pi \in \Pi_\psi.$$

This is just the multiplicity of ε_ψ in the character $\langle\cdot,\pi\rangle$ on S_ψ. I am anticipating here that the different global packets Π_ψ are actually disjoint. Otherwise the multiplicity $m_{\mathrm{disc}}(\pi)$ would have to be expressed as a sum, over all ψ with $\pi\in\Pi_\psi$, of the expression above.

The sign character ε_ψ is defined in terms of symplectic root numbers. For any ψ there is a finite dimensional representation

$$R_\psi(s,\ell,u)=\mathrm{Ad}(s\psi(\ell,u)),\qquad (s,\ell,u)\in S_\psi\times L_{\mathbb{Q}}\times \mathrm{SL}(2,\mathbb{C}),$$

of $S_\psi\times L_{\mathbb{Q}}\times\mathrm{SL}(2,\mathbb{C})$ on the Lie algebra of $\hat{G}$. (Notice the formal similarity of R_ψ with the representation $\sigma_{\psi_{\mathbb{R}}}$ defined earlier.) Let

$$R_\psi=\bigoplus_{i\in I_\psi}(\lambda_i\otimes\mu_i\otimes\nu_i)$$

be the decomposition of R_ψ into irreducible representations. The Langlands group $L_{\mathbb{Q}}$ is a generalization of the Galois group of $\bar{\mathbb{Q}}/\mathbb{Q}$, and should have similar properties. In particular, we can expect each representation μ_i to have an L-function $L(s,\mu_i)$ with analytic continuation and functional equation

$$L(s,\mu_i)=\varepsilon(s,\mu_i)L(1-s,\tilde{\mu}_i).$$

Observe that if μ_i equals its contragredient $\tilde{\mu}_i$, the number $\varepsilon\left(\frac{1}{2},\mu_i\right)$ equals 1 or -1. Let J_ψ be the set of indices $i\in I_\psi$ such that $\tilde{\mu}_i=\mu_i$ and $\varepsilon\left(\frac{1}{2},\mu_i\right)=-1$. (If the analogy with Galois representations carries over, each such μ_i will be symplectic.) We define the sign character as

$$\varepsilon_\psi(s)=\prod_{i\in J_\psi}\det(\lambda_i(s)),\qquad s\in S_\psi.$$

We can now give a solution to the original problem, based on the global conjectures. Suppose that ψ is a global parameter such that $\psi_{\mathbb{R}}$ belongs to the finite set Ψ_{cohom} of $\mathbb{R}$-parameters with cohomology. Then the group $S_{\psi_{\mathbb{R}}}$ is finite modulo the center of $\hat{G}=G\mathrm{Spin}(2n+1,\mathbb{C})$, and S_ψ is actually a subgroup of $S_{\psi_{\mathbb{R}}}$. Let ρ_ψ and σ_ψ be the restrictions of the corresponding two representations of $S_{\psi_{\mathbb{R}}}\times\mathbb{C}^*\times\mathrm{SL}(2,\mathbb{C})$ on $V_\psi=V_{\psi_{\mathbb{R}}}$ to the subgroup $S_\psi\times\mathbb{C}^*\times\mathrm{SL}(2,\mathbb{C})$. By Theorem 1, the two are equivalent. On the other hand, for each representation π_f in Π_{ψ_f} we have the character $\langle\cdot,\pi_f\rangle$ on S_ψ. Let $U(\pi_f)$ be a corresponding (finite-dimensional) S_ψ-module. Then

$$U_\psi^N=\bigoplus_{\pi_f\in\Pi_{\psi_f}}(U(\pi_f)\otimes\pi_f^{K_f(N)})$$

is a finite-dimensional complex vector space, equipped with an action of $S_\psi\times\mathcal{A}_f(N)$. (It is implicit in the conjectures above that for any N, only finitely many spaces $\pi_f^{K_f(N)}$ in this sum are nonzero.) We combine the two spaces as a tensor product $V_\psi \otimes U_\psi^N$, with the tensor product action of S_ψ. Each of the other three objects acts on its own factor, and we obtain a module over

$$S_\psi \times \mathbb{C}^* \times \mathrm{SL}(2,\mathbb{C}) \times \mathcal{A}_f(N).$$

Let

$$(V_\psi \otimes U_\psi^N)_{\varepsilon_\psi}$$

be the subspace of $V_\psi \otimes U_\psi^N$ which transforms under S_ψ according to the sign character ε_ψ. It is a $(\mathbb{C}^* \times \mathrm{SL}(2,\mathbb{C}) \times \mathcal{A}_f(N))$-module.

If we substitute the conjectural formula for $m_{\mathrm{disc}}(\pi)$ into the spectral decomposition, we obtain the formula we want.

Conjecture 2. There is a canonical isomorphism

$$H^*_{(2)}(S(N)) \cong \bigoplus_{\{\psi\,:\,\psi_{\mathbb{R}} \in \Psi_{\mathrm{cohom}}\}} (V_\psi \otimes U_\psi^N)_{\varepsilon_\psi}$$

of $(\mathbb{C}^* \times \mathrm{SL}(2,\mathbb{C}) \times \mathcal{A}_f(N))$-modules.

Let us summarize the main features of this formula. The group $\mathbb{C}^* \times \mathrm{SL}(2,\mathbb{C})$ acts through the factors V_ψ, according to the representations σ_ψ obtained from the parameters. The algebra $\mathcal{A}_f(N)$ acts through the other factors U_f^N, although we do not have an analogue of Theorem 1 to describe its action in terms of parameters. The two actions are linked by the sign character ε_ψ, through the representation of S_ψ on each factor. It is noteworthy that ε_ψ is defined in terms of symplectic root numbers. These rather subtle arithmetic invariants have thus a direct bearing on the cohomology of $S(N)$. They determine how irreducible representations of $\mathcal{A}_f(N)$ in U_ψ^N are paired with irreducible representations of $\mathbb{C}^* \times \mathrm{SL}(2,\mathbb{C})$ in V_ψ.

An apparent weakness of the construction is its dependence on the hypothetical Langlands group. This is actually not as serious as it seems. For many classical groups at least, one could set things up in a way that is more elementary, if perhaps harder to motivate. The global parameters ψ would be replaced by self-contragredient automorphic representations of a general linear group. The groups S_ψ could then be constructed directly from these objects. In fact, all aspects of the global conjectures could be formulated in similar terms. To prove them, or even formulate them in this way, one would have to classify automorphic representations of G in terms of those of $GL(N)$. This is a serious problem, but it is certainly more accessible than that of classifying automorphic representations by parameters on the hypothetical group $L_{\mathbb{Q}}$. For an elementary introduction to this problem, see [10, especially §3].

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