A STABLE TRACE FORMULA II. GLOBAL DESCENT

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Introduction

This paper is the second of three articles designed to stabilize the trace formula. The goal is to stabilize the global trace formula for a general connected group, subject to a condition on the fundamental lemma that has been established in some special cases. In the first article [I], we laid out the foundations of the process. We also stated a series of local and global theorems, which together amount to a stabilization of each of the terms in the trace formula. In this paper, we shall take a significant step towards the proof of the theorems. We shall reduce the proof of Global Theorem 1 of [I, §7] to the special case of unipotent elements. This reduction will play a key role in the proof of all of the theorems, which will be carried out in the last of the three articles.

We refer the reader to the introduction of [I] for a general discussion of the problem of stabilization. We begin the discussion here by recalling that the theorems stated in [I] apply to the four kinds of terms that occur in the global trace formula. Let G be a connected reductive group over a number field F. The trace formula for G is the identity obtained from two different expansions of a certain linear form $I(f)$. The geometric expansion

$$I(f) = \sum_{M} |W_0^M||W_0^G|^{-1} \sum_{\delta} a^M(\gamma) I_M(\gamma, f) \tag{1}$$

is a linear combination of distributions parametrized by conjugacy classes γ in Levi subgroups M. The spectral expansion

$$I(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \int a^M(\pi) I_M(\pi, f) d\pi \tag{2}$$

is a linear combination of distributions parametrized by representations π of Levi subgroups M. The local theorems stated in [I, §6] apply to the distributions $I_M(\gamma, f)$ and $I_M(\pi, f)$. The global theorems stated in [I, §7] apply to the coefficients $a^M(\gamma)$ and $a^M(\pi)$.

The objects of study in this paper will be the geometric coefficients $a^M(\gamma)$. We are interested in the general case, in which γ is not required to be semisimple. We should note in this connection that unipotent classes over the global field F lead directly to convergence problems that have never been solved. One avoids them by working with the subgroup

$$G_V = G(F_V) = \prod_{v \in V} G(F_v)$$

of $G(\mathbb{A})$, where V is a finite set of valuations of F outside of which G is unramified. In particular, the test function f in (1) and (2) is defined on G_V, and the elements γ in (1) represent conjugacy classes in M_V. The geometric coefficients can be studied in terms of the linear form

$$I_{\mathrm{orb}}(f) = \sum_{\gamma} a^G(\gamma) f_G(\gamma), \qquad f \in C_c^\infty(G_V), \tag{3}$$

that represents the purely "orbital" part of the trace formula. The sum is taken over the conjugacy classes in G_V, while $f_G(\gamma)$ denotes Harish-Chandra's invariant orbital integral

$$|D(\gamma)|^{\frac{1}{2}} \int_{G_{\gamma,V} \backslash G_V} f(x^{-1}\gamma x) dx.$$

The problem of stabilizing the geometric coefficients amounts to decomposing $I_{\mathrm{orb}}(f)$ in terms of endoscopic groups. Let $\mathcal{E}_{\mathrm{ell}}(G, V)$ denote the set of elliptic endoscopic data for G that are unramified outside of V. We assume for the introduction that the derived group of G is simply connected. The problem is then to establish a decomposition

$$I_{\mathrm{ell}}(f) = \sum_{G' in \mathcal{E}_{\mathrm{ell}}(G,V)} \iota(G, G') \widehat{S}'(f'), \tag{4}$$

for stable distributions $S' = S^{G'}$ on the endoscopic groups G'_V. For any G', f' denotes the transfer

$$f'(\delta') = \sum_{\gamma} \Delta(\delta', \gamma) f_G(\gamma)$$

of f that is defined by the absolute transfer factor $\Delta(\delta', \gamma)$ of Langlands and Shelstad. Langlands's monograph [L2] included a solution of this problem in the special case that f is supported on the strongly regular set in G_V. Kottwitz later established a simple formula for the coefficients $\iota(G, G')$ [K2], and extended Langlands's results to singular semisimple elements γ [K3]. The purpose of this paper is to solve the problem in the more general case that f vanishes on an invariant neighbourhood of the center of G_V. This represents a stabilization of the coefficients $a^G(\gamma)$ for elements γ whose semisimple parts are not central.

The existence of a decomposition (4) is a simple reformulation of Global Theorem 1′, which was stated in [I, §7] as a series of identities for the coefficients $a^G(\gamma)$. For most of the paper, we shall work directly with the coefficients. We shall in fact work exclusively with the more fundamental "elliptic" coefficients $a^G_{\text{ell}}(\dot{\gamma}_S)$, in terms of which the coefficients $a^G(\gamma)$ are defined [I, (2.8)]. The subscript S here denotes a large finite set of valuations containing V, while $\dot{\gamma}_S$ stands for a conjugacy class in G_S that intersects the product of γ with a compact subgroup of $G^V_S = \prod_{v \in S - V} G_v$. The class $\dot{\gamma}_S$ is of course allowed to have a unipotent part; our use of the term "elliptic" refers to the semisimple part of $\dot{\gamma}_S$, or rather, elements in $G(F)$ that project onto the semisimple part of $\dot{\gamma}_S$. Global Theorem 1 applies to the coefficients $a^G_{\text{ell}}(\dot{\gamma}_S)$, and is parallel to Global Theorem 1′. In [I, Proposition 10.3] it was shown that Global Theorem 1 implies Global Theorem 1′. It would therefore be enough to prove Global Theorem 1 in order to stabilize the coefficients in (1), and to establish a decomposition (4).

Global Theorem 1 was stated in terms of two other families of elliptic coefficients $a^{G,\mathcal{E}}_{\text{ell}}(\dot{\gamma}_S)$ and $b^G_{\text{ell}}(\dot{\delta}_S)$. These are to be regarded as "endoscopic" and "stable" variants of the original elliptic coefficients $a^G_{\text{ell}}(\dot{\gamma}_S)$. The assertions of Global Theorem 1 are that $a^{G,\mathcal{E}}_{\text{ell}}(\dot{\gamma}_S)$ equals $a^G_{\text{ell}}(\dot{\gamma}_S)$, and that $b^G_{\text{ell}}(\dot{\delta}_S)$ vanishes unless $\dot{\delta}_S$ lies in the "stable" subset $\Delta_{\text{ell}}(G, S)$ of its domain $\Delta^{\mathcal{E}}_{\text{ell}}(G, S)$. Our goal is to reduce these assertions to the case that the elements $\dot{\gamma}_S$ and $\dot{\delta}_S$ are unipotent. We shall do so by establishing descent formulas for the three families of coefficients.

The elements $\dot{\gamma}_S$ and $\dot{\delta}_S$ that index the elliptic coefficients are actually more general than just conjugacy classes. This is because the theory of endoscopy for real groups, implicit in the work of Shelstad, does not transfer unipotent classes to unipotent classes. It applies rather to the larger space of invariant distributions that are supported on the unipotent variety. Since the two supplementary families $a_{\mathrm{ell}}^{G,\mathcal{E}}(\dot{\gamma}_S)$ and $b_{\mathrm{ell}}^{G}(\dot{\delta}_S)$ are defined by relations of endoscopic transfer, the elements $\dot{\gamma}_S$ and $\dot{\delta}_S$ have to be taken to be invariant distributions on G_S. They belong to the bases $\Gamma(G_S)$ and $\Delta^{\mathcal{E}}(G_S)$ of distributions fixed as in [I, §1 and §5]. The coefficients $a_{\mathrm{ell}}^{G}(\dot{\gamma}_S)$ and $a_{\mathrm{ell}}^{G,\mathcal{E}}(\dot{\gamma}_S)$ are actually supported on a respective pair of discrete subsets

$$\Gamma_{\mathrm{ell}}(G,S) \subset \Gamma_{\mathrm{ell}}^{\mathcal{E}}(G,S)$$

of $\Gamma(G_S)$, which are defined by global conditions. The coefficient $b_{\mathrm{ell}}^{G}(\dot{\delta}_S)$ is defined only when G is quasisplit. It is supported on a discrete subset $\Delta^{\mathcal{E}}(G,S)$ of $\Delta^{\mathcal{E}}(G_S)$, which is also defined by global conditions.

The elements $\dot{\gamma}_S$ and $\dot{\delta}_S$ do have Jordan decompositions, even though they are more general than conjugacy classes. This is implicit in the conditions imposed on the choice of bases $\Gamma(G_S)$ and $\Delta^{\mathcal{E}}(G_S)$ in [I]. Any element in $\Gamma(G_S)$ can be written as a formal product

$$\dot{\gamma}_S = c_S \dot{\alpha}_S,$$

where c_S is a semisimple conjugacy class in G_S. The unipotent part $\dot{\alpha}_S \in \Gamma_{\mathrm{unip}}(G_{c_S})$ is an invariant distribution on the connected centralizer of (a representative of) c_S in G_S that is supported on the unipotent set. Similar decompositions are valid for the elements in the endoscopic basis $\Delta^{\mathcal{E}}(G_S)$. For example, any element in the "stable" subset $\Delta(G_S)$ of $\Delta^{\mathcal{E}}(G_S)$ can be written as a formal product

$$\dot{\delta}_S = d_S \dot{\beta}_S,$$

where d_S is a semisimple stable conjugacy class in a quasisplit inner form G_S^* of G_S. The unipotent part $\dot{\beta}_S \in \Delta_{\mathrm{unip}}(G_{d_S}^*)$ is a stable distribution on the connected centralizer of (a suitable representative of) d_S in G_S^* that is supported on the unipotent set.

We shall review these notions in §1. We shall then describe a descent formula

$$a_{\mathrm{ell}}^{G}(\dot{\gamma}_S) = \sum_c i^G(S,c) a_{\mathrm{ell}}^{G_c}(\dot{\alpha})$$

for the original elliptic coefficients in terms of the Jordan decomposition $c_S\dot{\alpha}_S$ of $\dot{\gamma}_S$. The sum is over semisimple conjugacy classes c in $G(F)$ that project onto c_S, while for any such c, $\dot{\alpha}$ is the image of $\dot{\alpha}_S$ in $\Gamma_{\mathrm{unip}}(G_{c,S})$. This formula follows immediately from the definition [I, (2.6)] of the coefficients. Our aim is to establish similar descent formulas for the coefficients $a_{\mathrm{ell}}^{G,\mathcal{E}}(\dot{\gamma}_S)$ and $b_{\mathrm{ell}}^{G}(\dot{\delta}_S)$. We shall state the formulas in Theorem 1.1. This theorem applies under the condition we have imposed for the introduction that the derived group be simply connected. However, we shall see in §2 that the condition can be relaxed. Theorem 1.1 is the main result of the paper. It implies the reductions of Global Theorems 1 and 1′ and the special case of the stabilization of the distribution $I_{\mathrm{orb}}(f)$.

We shall prove Theorem 1.1 in the remaining sections 3 to 6. The basic argument will be carried out in §6. The problem is to compare the expansion that goes into the definition [I, (7.3)] of $a_{\mathrm{ell}}^{G,\mathcal{E}}(\dot{\gamma}_S)$ with the appropriate linear combination of expansions that define the coefficients $a_{\mathrm{ell}}^{G_c,\mathcal{E}}(\dot{\alpha})$ of descent. Near the end of §6, we shall find that the two expressions match. We will then be able to establish the required formulas for $a_{\mathrm{ell}}^{G,\mathcal{E}}(\dot{\gamma}_S)$ and $\beta_{\mathrm{ell}}^{G}(\dot{\delta}_S)$ by standard means.

Sections 3 and 4 contain some preparations for the discussion in §6. The essential ingredient is the descent theorem [LS2] of Langlands and Shelstad for local transfer factors. In §3, we shall investigate a descent mapping for endoscopic data that was a starting point for the local results in [LS2]. Proposition 3.1 gives some properties of the mapping that are particular to the global setting at hand. The main result of Langlands and Shelstad is Theorem 1.6.A of [LS2]. It asserts that for any semisimple element c_S of G_S, the quotient of an absolute transfer factor for G_S by the corresponding transfer factor for G_{c_S} approaches a limit at c_S. In §4, we shall establish some simple properties of this limit (Lemma 4.1). We shall also observe that the Langlands-Shelstad descent theorem applies to the generalized transfer factors that relate elements $\dot{\gamma}_S$ and $\dot{\delta}'_S$ (Lemma 4.2).

The global descent mapping of §3 is not surjective, in contrast to its local counterpart. The points that lie in the complement of its image appear capable of causing trouble. The purpose of §5 is to show that they do not. We shall establish a simple relationship among the transfer factors of descent, which is nontrivial for points outside the image (Lemma 5.1). We shall use this result in §6 to show that the contributions from the extraneous terms cancel. The remaining terms will be attached to points in the image of the mapping, and will be seen to correspond with a parallel set of terms attached to points in the domain of the mapping. This observation comes near the end of §6, but is really the logical heart of the argument. It allows us to deduce that two complicated expressions are equal, and leads readily to a conclusion of the proof.

§1. Statement of a theorem.

The basic objects of study in this paper will be the "elliptic" coefficients associated with the geometric side of the global trace formula. We are referring to the coefficients $a^G_{\text{ell}}(\dot{\gamma}_S)$ obtained from [I, (2.6)], together with their endoscopic and stable counterparts $a^{G,\mathcal{E}}_{\text{ell}}(\dot{\gamma}_S)$ and $b^G_{\text{ell}}(\dot{\delta}_S)$ introduced in §7 of [I]. We shall recall a few of the definitions in a moment. For the most part, however, we are going to take the various constructions from [I] for granted. We shall follow the notational conventions of [I], often without comment. For example, the notation $\dot{\gamma}_S$ and $\dot{\delta}_S$, while more complicated than necessary, is meant to draw attention to the global role of these objects.

Let F be a fixed global field of characteristic 0. As in [I, §4], we take G to be a global K-group over F. Then G is a disjoint union $\coprod\limits_{\alpha} G_\alpha$ of connected reductive algebraic groups over F, together with some extra structure that includes a compatible family $\psi_{\alpha\beta}\colon\ G_\beta \to G_\alpha$ of inner twists. The disconnected K-group G is a convenient device for treating trace formulas for several connected groups at the same time. We shall often use implicit extensions to G of notions that apply to connected groups, when the meaning is clear. For example, in this paper Z will denote a central induced torus in G over F. In other words, Z is an induced torus over F, together with a compatible family of central embeddings $Z \subset G_\alpha$ over F. We fix Z, and also a character ζ on $Z(\mathbb{A})/Z(F)$.

Suppose that S is a finite set of valuations of F that contains the set $V_{\text{ram}}(G,\zeta)$ of ramified places for (G,ζ), and that $\dot{\gamma}_S$ belongs to the set

$$\Gamma(G_S,\zeta_S) = \coprod_{\alpha} \Gamma(G_{S,\alpha_S},\zeta_S)$$

defined in [I, §1 and §4]. (The elements in $\Gamma(G_{S,\alpha_S},\zeta_S)$ can be regarded as generalizations of the conjugacy classes in the group $G_{S,\alpha_S} = G_{S,\alpha_S}(F_S)$. They are elements in some fixed basis of the space of G_{S,α_S}-invariant, ζ_S-equivariant distributions on G_{S,α_S}.) We assume that $\dot{\gamma}_S$ is admissible in the sense of [I, §1]. Roughly speaking, this means that for most nonarchimedean places v in S, the local component $\dot{\gamma}_v$ is bounded, in the sense that the projection of its support onto the quotient

$$\overline{G}_v = (G/Z)_v = G_v/Z_v = G_v(F_v)/Z(F_v)$$

intersects a compact subgroup. We can then form the coefficient $a^G_{\text{ell}}(\dot{\gamma}_S) = a^{G_\alpha}_{\text{ell}}(\dot{\gamma}_S)$, where α indexes the component of G that supports $\dot{\gamma}_S$ [I, (2.6)]. Recall that $a^G_{\text{ell}}(\dot{\gamma}_S)$ depends on a choice of open, hyperspecial maximal compact subgroup $K^S = \prod_{v \notin S} K_v$ of $G^S(\mathbb{A}^S)$, and vanishes unless $\dot{\gamma}_S$ belongs to the subset

$$\Gamma_{\text{ell}}(G, S, \zeta) = \coprod_\alpha \Gamma_{\text{ell}}(G_\alpha, S, \zeta)$$

of $\Gamma(G_S, \zeta_S)$ [I, §2].

The endoscopic and stable analogues of $a^G_{\text{ell}}(\dot{\gamma}_S)$ depend on a choice of quasisplit inner twist

$$\psi = \coprod_\alpha \psi_\alpha : \ G \to G^*$$

of G. They are related by an expression

$$a^{G,\mathcal{E}}_{\text{ell}}(\dot{\gamma}_S) = \sum_{G'} \sum_{\dot{\delta}'_S} \iota(G, G') b^{\widetilde{G}'}_{\text{ell}}(\dot{\delta}'_S) \Delta_G(\dot{\delta}'_S, \dot{\gamma}_S) + \varepsilon(G) \sum_{\dot{\delta}_S} b^G_{\text{ell}}(\dot{\delta}_S) \Delta_G(\dot{\delta}_S, \dot{\gamma}_S), \tag{1.1}$$

where $\dot{\gamma}_S$ is an admissible element in the set $\Gamma^{\mathcal{E}}_{\text{ell}}(G, S, \zeta) \supset \Gamma_{\text{ell}}(G, S, \zeta)$, and G', $\dot{\delta}'_S$, and $\dot{\delta}_S$ are summed over sets $\mathcal{E}^0_{\text{ell}}(G, S)$, $\Delta_{\text{ell}}(\widetilde{G}', S, \widetilde{\zeta}')$ and $\Delta^{\mathcal{E}}_{\text{ell}}(G, S, \zeta)$ [I, (7.3)]. We recall here that $\iota(G, G') = \iota(G_\alpha, G')$ is the constant from [L2] and [K2, §8], and that $\Delta_G(\dot{\delta}'_S, \dot{\gamma}_S)$ is the extended transfer factor of [I, §5]. Moreover,

$$\varepsilon(G) = \begin{cases} 1, & \text{if } G \text{ is quasisplit,} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathcal{E}^0_{\text{ell}}(G, S) = \begin{cases} \mathcal{E}_{\text{ell}}(G, S) - \{G^*\}, & \text{if } G \text{ is quasisplit,} \\ \mathcal{E}_{\text{ell}}(G, S), & \text{otherwise,} \end{cases}$$

where G is said to be quasisplit if it has a connected component that is quasisplit. The coefficients $b^G_{\text{ell}}(\dot{\delta}_S)$ exist only if $\varepsilon(G) = 1$. They are defined inductively in this case by the further requirements that

$$a^{G,\mathcal{E}}_{\text{ell}}(\dot{\gamma}_S) = a^G_{\text{ell}}(\dot{\gamma}_S), \qquad \dot{\gamma}_S \in \Gamma^{\mathcal{E}}_{\text{ell}}(G, S, \zeta), \tag{1.2}$$

and

$$(1.3) \qquad b_{\mathrm{ell}}^{G^*}(\dot\delta_S^*) = b_{\mathrm{ell}}^G(\dot\delta_S), \qquad \dot\delta_S \in \Delta_{\mathrm{ell}}(G,S,\zeta).$$

(See [I, (7.4)].)

We recall that $\Delta_{\mathrm{ell}}^{\mathcal{E}}(G,S,\zeta)$ is a subset of the general endoscopic basis $\Delta^{\mathcal{E}}(G_S,\zeta_S)$. The latter is a purely local object, whose definition requires the Langlands-Shelstad transfer conjecture [I, §5]. The former is a global object, which is defined in terms of $\Delta^{\mathcal{E}}(G_S,\zeta_S)$ and the subset $\Gamma_{\mathrm{ell}}(G,S,\zeta)$ of $\Gamma(G_S,\zeta_S)$ [I, §7]. It supports the coefficient b_{ell}^G, and contains the subset

$$\Delta_{\mathrm{ell}}(G,S,\zeta) = \Delta_{\mathrm{ell}}^{\mathcal{E}}(G,S,\zeta) \cap \Delta(G_S,\zeta_S)$$

of stable basis elements. Similarly, $\Gamma_{\mathrm{ell}}^{\mathcal{E}}(G,S,\zeta)$ is a subset of $\Gamma(G_S,\zeta_S)$. It is a global object that supports the coefficient $a_{\mathrm{ell}}^{G,\mathcal{E}}$, and contains the set $\Gamma_{\mathrm{ell}}(G_S,\zeta_S)$. It would be quite possible to get by with just the local sets $\Gamma(G_S,\zeta_S)$, $\Delta^{\mathcal{E}}(G_S,\zeta_S)$ and $\Delta(G_S,\zeta_S)$. We use the global subsets, at the risk of overloading the notation, in order to emphasize the global nature of the coefficients.

One of our long-term goals is to establish Global Theorem 1, stated in [I, §7]. This theorem asserts that $a_{\mathrm{ell}}^{G,\mathcal{E}}(\dot\gamma_S)$ equals $a_{\mathrm{ell}}^G(\dot\gamma_S)$ in general, and that b_{ell}^G vanishes on the complement of $\Delta_{\mathrm{ell}}(G,S,\zeta)$ in $\Delta_{\mathrm{ell}}^{\mathcal{E}}(G,S,\zeta)$, if G is quasisplit. An obvious implication of the theorem is that $a_{\mathrm{ell}}^{G,\mathcal{E}}$ and b_{ell}^G are supported on the respective sets $\Gamma_{\mathrm{ell}}(G,S,\zeta)$ and $\Delta_{\mathrm{ell}}(G,S,\zeta)$. This perhaps makes the notation seem more natural. If the second assertion of the theorem is valid, the definition (1.1) simplifies to

$$(1.1^*) \qquad a_{\mathrm{ell}}^{G,\mathcal{E}}(\dot\gamma_S) = \sum_{G'} \sum_{\dot\delta'_S} \iota(G,G') b_{\mathrm{ell}}^{\widetilde{G}'}(\dot\delta'_S) \Delta_G(\dot\delta'_S,\gamma_S),$$

where G' is summed over the full set $\mathcal{E}_{\mathrm{ell}}(G,S)$, and $\dot\delta'_S$ is summed over $\Delta_{\mathrm{ell}}(\widetilde{G}',S,\widetilde{\zeta}')$. We shall use this streamlined form of (1.1) in future induction arguments.

One reason for reviewing these definitions is to point out that the coefficients can actually be defined for a connected component G_α of G. Suppose that $\dot\gamma_S$ belongs to the subset $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G_\alpha, S, \zeta)$ of $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G, S, \zeta)$. There is nothing to say about the coefficient $a^G_{\mathrm{ell}}(\dot\gamma_S)$, since it is defined a priori in terms of G_α. For the endoscopic coefficient, we set $a^{G_\alpha,\mathcal{E}}_{\mathrm{ell}}(\dot\gamma_S) = a^{G,\mathcal{E}}_{\mathrm{ell}}(\dot\gamma_S)$. Then $a^{G_\alpha,\mathcal{E}}_{\mathrm{ell}}(\dot\gamma_S)$ depends only on the connected group G_α and the inner twist ψ_α: $G_\alpha \to G^*$ (rather than the larger K-group G of which G_α is a component). This follows from the definition (1.2) if $\varepsilon(G) = 1$, and from the fact that $\Delta_G(\dot\delta'_S, \dot\gamma_S)$ depends only on G_α [I, Corollary 4.4] in case $\varepsilon(G) = 0$. As for the stable coefficients, the relation (1.3) provides a definition of $b^{G^*}(\dot\delta^*_S)$ for the connected quasisplit group G^* and the element $\dot\delta^*_S$ in $\Delta_{\mathrm{ell}}(G^*, S, \zeta)$. It is only this case that we shall need. We shall generally treat the coefficients as objects attached to the K-group G, but we shall rely on the remarks above to state the descent formulas.

Our aim is to establish formulas of descent for the coefficients. The starting point will be the descent formula [I, (2.4)] for the original coefficients $a^G(S, \dot\gamma)$ of [I, §2], which we are going to transform into a corresponding formula for the coefficients $a^G_{\mathrm{ell}}(\dot\gamma_S)$. This requires a preliminary word about the Jordan decomposition for elements in $\Gamma(G_S, \zeta_S)$.

For the moment, we can take S to be any finite set of valuations. Suppose that $\dot\gamma_S$ belongs to $\Gamma(G_S, \zeta_S)$. The semisimple part of $\dot\gamma_S$ is defined as a semisimple conjugacy class $c_S \in \Gamma_{\mathrm{ss}}(\overline{G}_S)$ in (one of the components of) the quotient

$$\overline{G}_S = (G/Z)_S = G_S/Z_S = \coprod_\alpha (G_{\alpha,S}/Z_S).$$

If c_S is contained in the component $G_{\alpha,S}/Z_S = \overline{G}_{\alpha,S}$, we write

$$\overline{G}_{c_S,+} = \prod_{v\in S} \overline{G}_{c_v,+}$$

for the centralizer of c_S in the component $\overline{G}_\alpha$, and we write $\overline{G}_{c_S}$ for the connected component of 1 in this group. Similarly, we write G_{c_S} for the preimage of $\overline{G}_{c_S}$ in G_α. We shall frequently take the liberty of letting G_{c_S} stand also for the group $G_{c_S,S} = G_{c_S}(F_S)$ of points with values in F_S. The unipotent part $\dot\alpha_S$ of $\dot\gamma_S$ is defined to be an element in the subset $\Gamma_{\mathrm{unip}}(G_{c_S}, \zeta_S)$ of distributions in the basis $\Gamma(G_{c_S}, \zeta_S)$ with semisimple part equal to 1. By assumption, the elements in the basis $\Gamma(G_S, \zeta_S)$ are constructed in a canonical way from their semisimple and unipotent components. We write

$$\dot{\gamma}_S = c_S \dot{\alpha}_S,$$

and refer to this formal product as the Jordan decomposition of $\dot{\gamma}_S$. (See [I, §1].) As in the usual case of conjugacy classes, the distribution $\dot{\alpha}_S$ is determined by $\dot{\gamma}_S$ only up to the action of the finite group

$$\overline{G}_{c_S,+}(F_S)/\overline{G}_{c_S}(F_S) = \prod_{v \in S} \big(\overline{G}_{c_v,+}(F_v)/\overline{G}_{c_v}(F_v)\big)$$

on $\Gamma_{\mathrm{unip}}(G_{c_S}, \zeta_S)$.

Suppose now that S contains $V_{\mathrm{ram}}(G, \zeta)$, and that $\dot{\gamma}_S$ is an admissible element in $\Gamma_{\mathrm{ell}}(G, S, \zeta)$. The general descent formula for $a^G_{\mathrm{ell}}(\dot{\gamma}_S)$ is stated in terms of the Jordan decomposition $\dot{\gamma}_S = c_S \dot{\alpha}_S$. It takes the form

$$a^G_{\mathrm{ell}}(\dot{\gamma}_S) = \sum_{c} \sum_{\dot{\alpha}} i^{\overline{G}}(S, c) \big|\overline{G}_{c,+}(F)/\overline{G}_c(F)\big|^{-1} a^{G_c}_{\mathrm{ell}}(\dot{\alpha}), \tag{1.4}$$

where c is summed over those elements in the set $\Gamma_{\mathrm{ss}}(\overline{G})$ of semisimple conjugacy classes in $\overline{G}(F)$ whose image in $\Gamma_{\mathrm{ss}}(\overline{G}_S)$ equals c_S, and $\dot{\alpha}$ is summed over the orbit of $\overline{G}_{c,+}(F_S)/\overline{G}_c(F_S)$ in $\Gamma_{\mathrm{unip}}(G_{c,S}, \zeta_S)$ determined by $\dot{\alpha}_S$. The symbol $\overline{G}$ stands for the quotient G/Z as above, while G_c denotes the preimage of $\overline{G}_c$ in G. The symbol $i^{\overline{G}}(S, c)$ is defined as in [I, (2.4)]. It equals 1 if c is an F-elliptic element in $\overline{G}$ whose $G^S(\mathbb{A}^S)$-conjugacy class meets the maximal compact subgroup $\overline{K}^S = K^S Z(\mathbb{A}^S)/Z(\mathbb{A}^S)$, and equals 0 otherwise. If $i^{\overline{G}}(S, c)$ is nonzero, as we may assume, we choose an element $g^S \in G(\mathbb{A}^S)$ that conjugates c to $\overline{K}^S$, and use it to form the subgroup

$$K^S_c = (g^S)^{-1} K^S g^S \cap G_c(\mathbb{A}^S) \tag{1.5}$$

of $G_c(\mathbb{A}^S)$. Since c_S is admissible, Proposition 7.1 of [K3] applies to the components $K_{c,v}$ of K^S_c, for places $v \notin S$. It tells us that $K_{c,v}$ is a hyperspecial maximal compact subgroup of $G_c(F_v)$ whose conjugacy class is independent of the choice of g^S. The coefficient $a_{\mathrm{ell}}^{G_c}$ in (1.4) is defined relative to K_c^S. The argument $\dot{\alpha}$ in $a_{\mathrm{ell}}^{G_c}(\dot{\alpha})$ is to be understood as the element in $\Gamma(G_{c,S}, \zeta_S)$ with Jordan decomposition $1 \cdot \dot{\alpha}$.

We leave the reader to derive (1.4) from [I, (2.4) and (2.6)]. The main point is to note that the quotient

$$\left|\overline{G}_{c,+}(F)/\overline{G}_c(F)\right| \left|G_{c,+}(F)/G_c(F)\right|^{-1}$$

equals the order of the stabilizer

$$Z(F, \widetilde{c}) = \{z \in Z(F) : \ z\widetilde{c} = \widetilde{c}\}$$

in $Z(F)$ of any conjugacy class $\widetilde{c} \in \Gamma_{\mathrm{ss}}(G)$ in the preimage of c.

We shall be mainly concerned with the special case of (1.4) in which the derived (multiple) group $G_{\mathrm{der}} = \coprod_{\alpha} G_{\alpha,\mathrm{der}}$ is simply connected, in the sense that it equals $G_{\mathrm{sc}} = \coprod_{\alpha} G_{\alpha,\mathrm{sc}}$, and Z equals $\{1\}$. The first condition implies that $G_{c,+} = G_c$, for any $c \in \Gamma_{\mathrm{ss}}(G)$. The second condition is that $\overline{G} = G$. The descent formula in this case reduces to

$$a_{\mathrm{ell}}^G(\dot{\gamma}_S) = \sum_c i^G(S, c) a_{\mathrm{ell}}^{G_c}(\dot{\alpha}), \tag{1.6}$$

where c is summed over the elements in $\Gamma_{\mathrm{ss}}(G)$ whose image in $\Gamma_{\mathrm{ss}}(G_S)$ equals c_S, and $\dot{\alpha}$ is the image of $\dot{\alpha}_S$ in $\Gamma_{\mathrm{unip}}(G_{c,S})$. This special case in fact follows immediately from the corresponding special cases of [I, (2.4) and (2.6)].

One could also define a Jordan decomposition for elements in $\Delta^{\mathcal{E}}(G_S, \zeta_S)$, by using constructions from the paper [A5]. For example, the semisimple part of any element $\dot{\delta}_S \in \Delta^{\mathcal{E}}(G_S, \zeta_S)$ would be a semisimple stable conjugacy class $d_S \in \Delta_{\mathrm{ss}}(\overline{G}_S^*)$ in the group $\overline{G}_S^* = G_S^*/Z_S$, together with some extra structure. (The inner twist ψ: $G \to G^*$ of course allows us to identity Z with a central subgroup of G^*.) In the present paper, we shall be concerned with the case in which G is quasisplit, and $\dot{\delta}_S$ belongs to the subset $\Delta(G_S, \zeta_S)$ of $\Delta^{\mathcal{E}}(G_S, \zeta_S)$. Assume that this is so. The extra structure for d_S is then trivial. We choose a representative of the class d_S (which we continue to denote by d_S) such that the connected centralizer $G^*_{d_S}$ is quasisplit [K1, Lemma 3.3]. The unipotent part of $\dot{\delta}_S$ can then be defined as an element $\dot{\beta}_S$ in the subset $\Delta_{\mathrm{unip}}(G^*_{d_S}, \zeta_S)$ of elements in $\Delta(G^*_{d_S}, \zeta_S)$ with semisimple part equal to 1. This gives a Jordan decomposition of any element $\dot{\delta}_S$ in $\Delta(G_S, \zeta_S)$, which we again denote by a formal product

$$\dot{\delta}_S = d_S \dot{\beta}_S.$$

In this case, $\dot{\beta}_S$ is determined only up to the action of the finite group

$$(\overline{G}^*_{d_S,+}/\overline{G}^*_{d_S})(F_S) = \prod_{v \in S} (\overline{G}^*_{d_v,+}/\overline{G}^*_{d_v})(F_v)$$

on $\Delta_{\mathrm{unip}}(G^*_{d_S}, \zeta_S)$.

The descent formula for $b^G_{\mathrm{ell}}(\dot{\delta}_S)$ will be stated in terms of an element d in the set $\Delta_{\mathrm{ss}}(\overline{G}^*)$ of semisimple stable conjugacy classes in $\overline{G}^*(F)$, whose image in $\Delta_{\mathrm{ss}}(\overline{G}^*_S)$ equals d_S. If d exists, it is uniquely determined by d_S. Assuming that it does exist, we define $i^{\overline{G}^*}(S, d)$ to be 1 if d is F-elliptic and bounded at each place $v \notin S$, and to be 0 otherwise. We then define a coefficient

$$j^{\overline{G}^*}(S, d) = i^{\overline{G}^*}(S, d) \tau(\overline{G}^*) \tau(\overline{G}^*_d)^{-1}, \tag{1.7}$$

where G^*_d stands for a quasisplit connected centralizer of an appropriate representative of the class d, and $\tau(\cdot)$ denotes the absolute Tamagawa number. The absolute Tamagawa number equals the relative Tamagawa number $\tau_1(\cdot)$ of [K2, §5], by virtue of the proof [K4] of Weil's conjecture and the proof [C] of the Hasse principle.

We are now ready to state the main result of the paper. It concerns the special case to which (1.5) applies. It also relies on an induction hypothesis, which we shall describe after stating the theorem.

Theorem 1.1. *Assume that G_{der} is simply connected and that $Z = 1$.*

(a) Suppose that $\dot{\gamma}_S$ is an admissible element in $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G,S)$ with Jordan decomposition $\dot{\gamma}_S = c_S\dot{\alpha}_S$. Then

$$a^{G,\mathcal{E}}_{\mathrm{ell}}(\dot{\gamma}_S) = \sum_c i^G(S,c)a^{G_c,\mathcal{E}}_{\mathrm{ell}}(\dot{\alpha}), \tag{1.8}$$

with c and $\dot{\alpha}$ being as in (1.5). That is, c is summed over the elements in $\Gamma_{\mathrm{ss}}(G)$ that map to c_S, and $\dot{\alpha}$ is the image of $\dot{\alpha}_S$ in $\Gamma_{\mathrm{unip}}(G_{c,S})$.

(b) Suppose that G is quasisplit, and that $\dot{\delta}_S$ is an admissible element in $\Delta_{\mathrm{ell}}(G,S)$ with Jordan decomposition $\dot{\delta}_S = d_S\dot{\beta}_S$. Then

$$b^G_{\mathrm{ell}}(\dot{\delta}_S) = \sum_d j^{G^*}(S,d)b^{G^*_d}_{\mathrm{ell}}(\dot{\beta}), \tag{1.9}$$

where d is summed over the set of elements in $\Delta_{\mathrm{ss}}(G^*)$ whose image is $\Delta_{\mathrm{ss}}(G^*_S)$ equals d_S (a set of order 0 to 1), and $\dot{\beta}$ is the image of $\dot{\beta}_S$ in $\Delta_{\mathrm{unip}}(G^*_{d,S})$. Moreover, b^G_{ell} vanishes on the complement of $\Delta_{\mathrm{ell}}(G,S)$ in the set of admissible elements in $\Delta^{\mathcal{E}}_{\mathrm{ell}}(G,S)$ whose semisimple part is not central in G^*_S.

Theorem 1.1 reduces the study of the global coefficients to the study of their values at unipotent elements. It is an important step towards the proof of the general theorems stated in [I, §6-7]. In particular, it will provide the main reduction in the proof of Global Theorem 1 [I, §7]. As we have noted, the latter asserts that $a^{G,\mathcal{E}}_{\mathrm{ell}}(\dot{\gamma}_S)$ equals $a^G_{\mathrm{ell}}(\dot{\gamma}_S)$ in general, and that b^G_{ell} is supported on the subset $\Delta_{\mathrm{ell}}(G,S,\zeta)$ of $\Delta^{\mathcal{E}}_{\mathrm{ell}}(G,S,\zeta)$. We shall assume inductively that these two assertions hold for various supplementary groups attached to G. More precisely, we assume that Global Theorem 1 holds if G is replaced by any group H over F such that *either* $\dim(H_{\mathrm{der}}) < \dim(G_{\mathrm{der}})$, *or* $\varepsilon(G) = 0$ and $H = G^*$. In particular, we assume that the streamlined form of the definition (1.1*) is valid for any such H.

We shall carry the induction hypothesis throughout the rest of the paper. In the next section, we shall establish some consequences of Theorem 1.1, including its application to the proof of Global Theorem 1 for G. In the remaining sections 3 to 6, we shall prove

Theorem 1.1. We shall appeal to the induction hypothesis several times, but we shall always apply it to groups H obtained from G by operations that are consistent with our condition [I, Assumption 5.2] on the fundamental lemma. (See [I, Lemma 5.3].) The resolution of the induction argument will have to wait until the next paper, in which we shall complete the proof of the theorems stated in [I, §6-7] for K-groups G that satisfy Assumption 5.2 of [I].

Before we proceed with the main part of the paper, we need to say something about the values taken by the global coefficients at unipotent elements. Consider a *connected* group H over F, with central character data (Z, ζ). In practice, we would take H to be one of the groups G_c or G_d^* obtained from G as above. We write $\mathcal{D}_{\text{unip}}(H_S, \zeta_S)$, as in [I, §1], for the space of distributions on H_S spanned by the basis $\Gamma_{\text{unip}}(H_S, \zeta_S)$. For example, in the special case that Z is trivial, the corresponding space $\Gamma_{\text{unip}}(H_S)$ consists of the invariant distributions on H_S that are supported on the unipotent variety. It is generally larger than the space spanned by the unipotent orbital integrals on H_S. Our use of the larger space is necessitated by questions of endoscopic transfer. Now the commutator quotient

$$H_S^{ab} = H_S/H_{\text{der},S} = \prod_{v \in V} \big(H(F_v)/H_{\text{der}}(F_v)\big)$$

acts by conjugation on the space $\mathcal{D}_{\text{unip}}(H_{\text{der},S})$. We define a linear map from $\mathcal{D}_{\text{unip}}(H_{\text{der},S})$ to $\mathcal{D}_{\text{unip}}(H_S, \zeta_S)$ by sending any $D \in \mathcal{D}_{\text{unip}}(H_{\text{der},S})$ to the linear form

$$f \longrightarrow \sum_{a \in H_S^{ab}} (aD)(f_{\text{der}}), \qquad f \in \mathcal{H}(G_S, \zeta_S),$$

where f_{der} denotes the restriction of f to $H_{\text{der},S}$. The map is not generally injective. It is also not generally surjective, since there can be distributions in $\mathcal{D}_{\text{unip}}(H_S, \zeta_S)$ with derivatives in the direction of the center of H_{S_∞}. We shall write $\mathcal{D}_{\text{unip,der}}(H_S, \zeta_S)$ for the image of $\mathcal{D}_{\text{unip}}(H_{\text{der},S})$ in $\mathcal{D}_{\text{unip}}(H_S, \zeta_S)$.

There are two points to this definition. The first is that there is a canonical isomorphism between $\mathcal{D}_{\text{unip,der}}(H_S, \zeta_S)$ and the corresponding space $\mathcal{D}_{\text{unip,der}}(H_S)$ with trivial central character data. It is defined by sending an element $D \in \mathcal{D}_{\text{unip,der}}(H_S)$ to the linear form

$$f \longrightarrow D(f_c), \qquad\qquad f \in \mathcal{H}(H_S, \zeta_S),$$

where f_c is any function in $C_c^\infty(H_S)$ that equals f on an invariant neighbourhood of 1. We assume implicitly that the basis $\Gamma_{\text{unip}}(H_S, \zeta_S)$ has been chosen so that the subset

$$\Gamma_{\text{unip,der}}(H_S, \zeta_S) = \Gamma_{\text{unip}}(H_S, \zeta_S) \cap \mathcal{D}_{\text{unip,der}}(H_S, \zeta_S)$$

is a basis of $\mathcal{D}_{\text{unip,der}}(H_S, \zeta_S)$, and is in bijection with the corresponding basis $\Gamma_{\text{unip,der}}(H_S)$ of $\mathcal{D}_{\text{unip,der}}(H_S)$ under the isomorphism. Similar remarks apply to the subspace $S\mathcal{D}_{\text{unip,der}}(H_S, \zeta_S)$ of stable distributions in $\mathcal{D}_{\text{unip,der}}(H_S, \zeta_S)$. It is clear that the isomorphism maps $S\mathcal{D}_{\text{unip,der}}(H_S)$ onto $S\mathcal{D}_{\text{unip,der}}(H_S, \zeta_S)$. We assume implicitly that the set

$$\Delta_{\text{unip,der}}(H_S, \zeta_S) = \Delta_{\text{unip}}(H_S, \zeta_S) \cap S\mathcal{D}_{\text{unip,der}}(H_S, \zeta_S)$$

is a basis of $S\mathcal{D}_{\text{unip,der}}(H_S, \zeta_S)$, and is in bijection with the corresponding basis $\Delta_{\text{unip,der}}(H_S)$ of $S\mathcal{D}_{\text{unip,der}}(H_S)$ under the isomorphism.

The second point is that the unipotent global coefficients a_{ell}^H and $a_{\text{ell}}^{H,\mathcal{E}}$ are supported on the subset $\Gamma_{\text{unip,der}}(H_S, \zeta_S)$ of $\Gamma_{\text{unip}}(H_S, \zeta_S)$. Similarly, if H is quasisplit, the unipotent coefficient b_{ell}^H is supported on the subset $\Delta_{\text{unip,der}}(H_S, \zeta_S)$ of $\Delta_{\text{unip}}(H_S, \zeta_S)$. To see this, one first recalls from [I, §2] that a_{disc}^H is actually supported on the subset of elements in $\Gamma_{\text{unip,ell}}(H_S, \zeta_S)$ that come from the unipotent orbital integrals. One can then establish the assertions for $a_{\text{ell}}^{H,\mathcal{E}}$ and b_{ell}^H from the definitions (1.1) and (1.2), and the adjoint relations [I, (5.4), (5.5)]. We shall be most concerned with the case of the stable coefficients b_{ell}^H. There are actually two such coefficients, one defined on $\Delta_{\text{unip,der}}(H_S, \zeta_S)$, and the other on $\Delta_{\text{unip,der}}(H_S)$. We have just noted that there is a canonical bijection between the two domains. At the end of §2, we shall verify that the corresponding values of the two coefficients are equal.

§2. Extensions and ramifications

For the rest of the paper, we fix a large finite set of valuations $S \supset V_{\mathrm{ram}}(G,\zeta)$ of F. We shall begin the proof of Theorem 1.1 in the next section, at which point we will adopt the proposed restrictions on G and Z. In the meantime, we take G, Z, and ζ to be arbitrary. The main purpose of this section is to extend Theorem 1.1 to the general case. The arguments, which are largely formal, are based on some of the simpler constructions of [I].

We shall first show that Theorem 1.1 implies a major reduction in the proof of Global Theorem 1 of [I, §7].

Proposition 2.1. *Assume that Theorem 1.1 has been proved for some z-extension $\widetilde{G}$ of G.*

(a) Suppose that $\dot{\gamma}_S$ is an admissible element in $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G,S,\zeta)$ whose semisimple part is not central in $\overline{G}_S$. Then

$$a^{G,\mathcal{E}}_{\mathrm{ell}}(\dot{\gamma}_S) = a^G_{\mathrm{ell}}(\dot{\gamma}_S).$$

(b) Suppose that G is quasisplit, and that $\dot{\delta}_S$ is an admissible element in $\Delta^{\mathcal{E}}_{\mathrm{ell}}(G,S,\zeta)$ whose semisimple part is not central in $\overline{G}^*_S$. Then $b^G_{\mathrm{ell}}(\dot{\delta}_S)$ vanishes unless $\dot{\delta}_S$ lies in the subset $\Delta_{\mathrm{ell}}(G,S,\zeta)$ of $\Delta^{\mathcal{E}}_{\mathrm{ell}}(G,S,\zeta)$.

Proof. Let $\widetilde{G}$ be the given z-extension. Then $\widetilde{G}$ is a central extension of G by an induced torus $\widetilde{C}$ over F such that $\widetilde{G}_{\mathrm{der}}$ is simply connected. We write $\widetilde{Z}$ for the preimage of Z in $\widetilde{G}$, and $\widetilde{\zeta}$ for the pullback of ζ to $\widetilde{Z}(\mathbb{A})/\widetilde{Z}(F)$. We have of course to choose S so that $\widetilde{G}$ and $\widetilde{\zeta}$ are unramified for each v outside of S.

Recall [K1, Lemma 1.1 (3)] that

$$G_S = G(F_S) \cong \widetilde{G}(F_S)/\widetilde{C}(F_S) = \widetilde{G}_S/\widetilde{C}_S.$$

We can therefore identity functions (or distributions) on G_S with functions (or distributions) on $\widetilde{G}_S$ that are invariant under translations by $\widetilde{C}_S$. In particular, there is a canonical isomorphism $\dot{f}_S \to \widetilde{\dot{f}}_S$ from the space $\mathcal{H}_{\mathrm{adm}}(G,S,\zeta)$ of functions with admissible support in the ζ_S^{-1}-equivariant Hecke algebra on G_S [I, §1] onto the corresponding space $\mathcal{H}_{\mathrm{adm}}(\widetilde{G}, S, \widetilde{\zeta})$ of functions on $\widetilde{G}_S$. We can assume that the bases $\Gamma(\widetilde{G}_S, \widetilde{\zeta}_S)$, $\Delta^{\mathcal{E}}(\widetilde{G}_S, \widetilde{\zeta}_S)$, etc., for $\widetilde{G}$ are the images of the corresponding bases $\Gamma(G_S, \zeta_S)$, $\Delta^{\mathcal{E}}(\Gamma_S, \zeta_S)$, etc., for G, under the canonical maps $\dot{\gamma}_S \to \widetilde{\dot{\gamma}}_S$, $\dot{\delta}_S \to \widetilde{\dot{\delta}}_S$, etc., of distributions. We shall show that the proposition holds for (G, ζ) if it holds for $(\widetilde{G}, \widetilde{\zeta})$.

We first check that the original coefficients satisfy

$$a_{\mathrm{ell}}^{G}(\dot{\gamma}_S) = a_{\mathrm{ell}}^{\widetilde{G}}(\widetilde{\dot{\gamma}}_S), \tag{2.1}$$

for any admissible element $\dot{\gamma}_S$ in $\Gamma(G_S, \zeta_S)$. This identity is plausible enough. However, it has to be verified indirectly, since the construction of the coefficients goes back to the indirect definitions in [A3]. We shall apply an induction argument to the expansion

$$J(\dot{f}_S) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\dot{\gamma}_S \in \Gamma(M,S,\zeta)} a_{\mathrm{ell}}^M(\dot{\gamma}_S) J_M(\dot{\gamma}_S, \dot{f}_S)$$

that was derived for any $\dot{f}_S \in \mathcal{H}_{\mathrm{adm}}(G, S, \zeta)$ in the course of proving Proposition 2.2 of [I]. (The expansion is the special case of [I, (2.10)] in which $V = S$.) The linear form $J(\dot{f}_S) = J^G(\dot{f}_S)$ was constructed from the distribution [I, (2.1)] on $G(\mathbb{A})^1$. It follows easily from the construction [A1, §8], [A2, §2] of this original distribution in terms of a truncated kernel, together with the simple definitions at the beginning of §2 of [I], that

$$J^G(\dot{f}_S) = J^{\widetilde{G}}(\widetilde{\dot{f}}_S).$$

Consider the terms in the expansions of these two linear forms. The terms that depend on $\dot{f}_S$ are weighted orbital integrals. They are constructed in such a way that

$$J_M^G(\dot{\gamma}_S, \dot{f}_S) = J_{\widetilde{M}}^{\widetilde{G}}(\widetilde{\dot{\gamma}}_S, \widetilde{\dot{f}}_S), \qquad \dot{\gamma}_S \in \Gamma(M_S, \zeta_S).$$

As for the coefficients, we assume inductively that (2.1) holds if G is replaced by any Levi subgroup $M \neq G$. This implies that the terms with $M \neq G$ in the expansions for $J^G(\dot{f}_S)$

and $J^{\widetilde{G}}(\widetilde{\dot{f}}_S)$ match. The terms with $M = G$ then also match. By varying $\dot{f}_S$, we conclude that the identity (2.1) is valid.

If G' belongs to $\mathcal{E}_{\mathrm{ell}}(G, S)$, there is a canonical central extension $\widetilde{G}'$ of G' by $\widetilde{C}$ that is determined by $\widetilde{G}$. Since $\widetilde{G}_{\mathrm{der}}$ is simply connected, there is also an L-embedding ${}^L\widetilde{G}' \to {}^L\widetilde{G}$ [L1], which we can assume is unramified outside of S. (See the remark following Lemma 7.1 of [I].) Recall that G' represents an endoscopic datum $(G', \mathcal{G}', s', \xi')$ for G. The composition of L-embeddings

$$\mathcal{G}' \xrightarrow{\xi'} {}^L G \longrightarrow {}^L\widetilde{G}$$

maps $\mathcal{G}'$ into the image of ${}^L\widetilde{G}'$ in ${}^L\widetilde{G}$, thereby providing an L-embedding $\widetilde{\xi}'\colon \mathcal{G}' \to {}^L\widetilde{G}'$. The pair $(\widetilde{G}', \widetilde{\xi}')$ serves as the auxiliary datum for G'. On the other hand, $\widetilde{G}'$ can be identified with an endoscopic datum for $\widetilde{G}$, and it is easy to see that the correspondence $G' \to \widetilde{G}'$ is a bijection from $\mathcal{E}_{\mathrm{ell}}(G, S)$ to $\mathcal{E}_{\mathrm{ell}}(\widetilde{G}, S)$. Moreover, there is an identity

$$\dot{f}'_S = (\widetilde{\dot{f}}_S)', \qquad\qquad \dot{f}_S \in \mathcal{H}_{\mathrm{adm}}(G, S, \zeta),$$

between the two transfer maps. (See [LS1, §4.2].) For the convenience of the reader, we check that G' and $\widetilde{G}'$ also satisfy the identity

$$\iota(G, G') = \iota(\widetilde{G}, \widetilde{G}'). \tag{2.2}$$

Recall [K2, Theorem 8.3.1 and (5.1.1)] that

$$\iota(G, G') = \big|\mathrm{Out}_G(G')\big|^{-1} \big|Z(\widehat{G}')^\Gamma / Z(\widehat{G})^\Gamma\big|^{-1} \big|\ker^1\big(F, Z(\widehat{G}')\big)\big| \big|\ker^1\big(F, Z(\widehat{G})\big)\big|^{-1},$$

where $\ker^1(F, \cdot)$ denotes the subset of locally trivial elements in $H^1(F, \cdot)$. (We are using the identity

$$\big|Z(\widehat{G}')^\Gamma / Z(\widehat{G})^\Gamma\big| = \big|\pi_0\big(Z(\widehat{G}')^\Gamma\big)\big| \big|\pi_0\big(Z(\widehat{G})^\Gamma\big)\big|^{-1},$$

which follows from the fact that G' is elliptic.) The centers $Z(\widehat{G})$, $Z(\widehat{\widetilde{G}})$, $Z(\widehat{G}')$, and $Z(\widehat{\widetilde{G}'})$ are related by the two short exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & Z(\widehat{G}) & \longrightarrow & Z(\widehat{\widetilde{G}}) & \longrightarrow & \widehat{\widetilde{C}} & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow & & \| & & \\
1 & \longrightarrow & Z(\widehat{G}') & \longrightarrow & Z(\widehat{\widetilde{G}'}) & \longrightarrow & \widehat{\widetilde{C}} & \longrightarrow & 1.
\end{array}$$

Corollary 2.3 of [K2] then provides two long exact sequences, which include the two exact sequences

$$\begin{array}{ccccccc}
X_*(\widehat{\widetilde{C}}^{\,\Gamma}) & \longrightarrow & \pi_0\big(Z(\widehat{G})^\Gamma\big) & \longrightarrow & \pi_0\big(Z(\widehat{\widetilde{G}})^\Gamma\big) & \longrightarrow & \pi_0(\widehat{\widetilde{C}}^{\,\Gamma}) = 1 \\
\| & & \big\downarrow & & \big\downarrow & & \| \\
X_*(\widehat{\widetilde{C}}^{\,\Gamma}) & \longrightarrow & \pi_0\big(Z(\widehat{G}')^\Gamma\big) & \longrightarrow & \pi_0\big(Z(\widehat{\widetilde{G}'})^\Gamma\big) & \longrightarrow & \pi_0(\widehat{\widetilde{C}}^{\,\Gamma}) = 1.
\end{array}$$

It follows that the map

$$Z(\widehat{G}')^\Gamma/Z(\widehat{G})^\Gamma = \pi_0\big(Z(\widehat{G}')^\Gamma\big)/\pi_0\big(Z(\widehat{G})^\Gamma\big) \longrightarrow \pi_0\big(Z(\widehat{\widetilde{G}'})^\Gamma\big)/\pi_0\big(Z(\widehat{\widetilde{G}})^\Gamma\big) = Z(\widehat{\widetilde{G}'})^\Gamma/Z(\widehat{\widetilde{G}})^\Gamma$$

is an isomorphism. Moreover, the H^1 terms of the long exact sequences of [K2, Corollary 2.3], applied both locally and globally as in the proof of [K2, Lemma 4.3.2 (a)], tell us that the maps

$$\ker^1\big(F, Z(\widehat{G})\big) \longrightarrow \ker^1\big(F, Z(\widehat{\widetilde{G}})\big)$$

and

$$\ker^1\big(F, Z(\widehat{G}')\big) \longrightarrow \ker\big(F, Z(\widehat{\widetilde{G}'})\big)$$

are isomorphisms. Finally, it is easy to check that the group

$$\mathrm{Out}_G(G') = \mathrm{Aut}_G(G')/\widehat{G}$$

maps isomorphically onto the corresponding group $\mathrm{Out}_{\widetilde{G}}(\widetilde{G}')$ for $\widetilde{G}$ and $\widetilde{G}'$. The formula (2.2) follows.

We can now extend the identity (2.1) to the associated endoscopic and stable coefficients. It is perhaps simplest to make use of the linear form

$$I_{\mathrm{ell}}(\dot{f}_S) = \sum_{\dot{\gamma}_S} a^G_{\mathrm{ell}}(\dot{\gamma}_S)\dot{f}_{S,G}(\dot{\gamma}_S), \qquad \dot{f}_S \in \mathcal{H}_{\mathrm{adm}}(G, S, \zeta),$$

together with associated linear forms $I^{\mathcal{E}}_{\mathrm{ell}}(\dot{f}_S)$ and $S^G_{\mathrm{ell}}(\dot{f}_S)$ defined in [I, §7]. It follows from the definition [I, (7.5)] and the remarks above that

$$\begin{aligned}
&I_{\mathrm{ell}}^{\mathcal{E}}(\dot{f}_S) - \varepsilon(G) S_{\mathrm{ell}}^{G}(\dot{f}_S) \\
= &\sum_{G' \in \mathcal{E}_{\mathrm{ell}}^{0}(G,S)} \iota(G,G') \widehat{S}_{\mathrm{ell}}^{\widetilde{G}'}(\dot{f}'_S) \\
= &\sum_{\widetilde{G}' \in \mathcal{E}_{\mathrm{ell}}^{0}(\widetilde{G},S)} \iota(\widetilde{G},\widetilde{G}') \widehat{S}_{\mathrm{ell}}^{\widetilde{G}'}(\widetilde{\dot{f}}'_S) \\
= &I_{\mathrm{ell}}^{\mathcal{E}}(\widetilde{\dot{f}}_S) - \varepsilon(\widetilde{G}) S_{\mathrm{ell}}^{\widetilde{G}}(\widetilde{\dot{f}}_S).
\end{aligned}$$

The original identity (2.1) implies that $I_{\mathrm{ell}}(\dot{f}_S) = I_{\mathrm{ell}}(\widetilde{\dot{f}}_S)$. This in turn implies that

$$I_{\mathrm{ell}}^{\mathcal{E}}(\dot{f}_S) = I_{\mathrm{ell}}(\dot{f}_S) = I_{\mathrm{ell}}(\widetilde{\dot{f}}_S) = I_{\mathrm{ell}}^{\mathcal{E}}(\widetilde{\dot{f}}_S),$$

in the case that $\varepsilon(G) = 1$. We conclude that $I_{\mathrm{ell}}^{\mathcal{E}}(\dot{f}_S) = I_{\mathrm{ell}}^{\mathcal{E}}(\widetilde{\dot{f}}_S)$ in general, and that $S_{\mathrm{ell}}^{G}(\dot{f}_S) = S_{\mathrm{ell}}^{\widetilde{G}}(\widetilde{\dot{f}}_S)$ in case $\varepsilon(G) = 1$. The general induction hypothesis we took on at the end of §1 allows us to apply the expansions for the distributions $I_{\mathrm{ell}}^{\mathcal{E}}$ and S_{ell}^{G} in [I, Lemma 7.2]. It follows easily from these expansions that

$$a_{\mathrm{ell}}^{G,\mathcal{E}}(\dot{\gamma}_S) = a_{\mathrm{ell}}^{\widetilde{G},\mathcal{E}}(\widetilde{\dot{\gamma}}_S), \tag{2.3}$$

and in the case $\varepsilon(G) = 1$, that

$$b_{\mathrm{ell}}^{G}(\dot{\delta}_S) = b_{\mathrm{ell}}^{\widetilde{G}}(\widetilde{\dot{\delta}}_S), \tag{2.4}$$

for admissible elements $\dot{\gamma}_S \in \Gamma_{\mathrm{ell}}^{\mathcal{E}}(G,S,\zeta)$ and $\dot{\delta}_S \in \Delta_{\mathrm{ell}}^{\mathcal{E}}(G,S,\zeta)$. In particular, if the two assertions of the proposition are valid for the elements $\widetilde{\dot{\gamma}}_S$ and $\widetilde{\dot{\delta}}_S$, they are also valid for $\dot{\gamma}_S$ and $\dot{\delta}_S$. The proposition thus holds for (G,ζ) if it is valid for $(\widetilde{G},\widetilde{\zeta})$.

We have reduced the proposition to the case that $G = \widetilde{G}$. It remains to show that if it holds for a given G, with (Z,ζ) trivial, then it holds for arbitrary (Z,ζ). This will again be a straightforward consequence of formal constructions from [I].

As in [I, §2], we have a natural projection $\dot{f}_S^1 \to \dot{f}_S^\zeta$ from $\mathcal{H}_{\mathrm{adm}}(G,S)$ onto $\mathcal{H}_{\mathrm{adm}}(G,S,\zeta)$. The linear form I_{ell} on $\mathcal{H}_{\mathrm{adm}}(G,S,\zeta)$ is related to the corresponding linear form on $\mathcal{H}_{\mathrm{adm}}(G,S)$ (in which (Z,ζ) is taken to be trivial) by the formula

$$I_{\mathrm{ell}}(\dot{f}_S) = \int_{Z_{S,\mathfrak{o}}\backslash Z_S^1} I_{\mathrm{ell}}(\dot{f}_{S,z}^1)\zeta(z)dz, \qquad \dot{f}_S \in \mathcal{H}_{\mathrm{adm}}(G,S,\zeta), \tag{2.5}$$

where $\dot{f}_S^1$ is any function in $\mathcal{H}_{\mathrm{adm}}(G,S)$ such that $\dot{f}_S^\zeta$ equals $\dot{f}_S$, and $\dot{f}_{S,z}^1$ is the translate of $\dot{f}_S^1$ by z. In particular, the integrand is invariant under translation of z by elements in the image of the discrete group

$$Z_{S,\mathfrak{o}} = Z(F) \cap Z_S Z(\mathfrak{o}^S)$$

in Z_S^1. We shall show that the linear forms $I_{\mathrm{ell}}^{\mathcal{E}}$ and S_{ell}^G satisfy similar formulas.

We have first to note that the projection $\dot{f}_S^1 \to \dot{f}_S^\zeta$ commutes with endoscopic transfer. Suppose that $G' \in \mathcal{E}(G)$ is an endoscopic datum. If $\dot{f}_S^1$ belongs to $\mathcal{H}(G,S)$, the transfer $(\dot{f}_S^1)'$ lies in $S\mathcal{I}(\widetilde{G}', S, \widetilde{\eta}')$, where $\widetilde{\eta}'$ is the automorphic character on $\widetilde{C}'$ attached to $\widetilde{G}'$ [I, §4]. A variant of the projection above then maps $(\dot{f}_S^1)'$ to a function in $S\mathcal{I}(\widetilde{G}', S, \widetilde{\zeta}')$, for the automorphic character $\widetilde{\zeta}' = \widetilde{\eta}'\widetilde{\zeta}$ on the image $\widetilde{Z}'$ of $\widetilde{Z}$ in $\widetilde{G}'$. We claim that the image of $(\dot{f}_S^1)'$ in $S\mathcal{I}(\widetilde{G}', S, \widetilde{\zeta}')$ coincides with the transfer $(\dot{f}_S^\zeta)'$ of $\dot{f}_S^\zeta$. To check this, it suffices to compare the values of the two functions in $S\mathcal{I}(\widetilde{G}', S, \widetilde{\zeta}')$ at any point $\widetilde{\sigma}_S'$ in $\Delta_{G\text{-reg}}(\widetilde{G}_S', \widetilde{\zeta}_S')$. It follows from the original definition of transfer [I, (4.9)], together with the formula [LS1, Lemma 4.4A] that provides the extension of the automorphic character $\widetilde{\eta}'$ from $\widetilde{C}'$ to $\widetilde{Z}'$, that

$$(f_{S,z}^1)' = (f_S')_z'\widetilde{\eta}'(z), \qquad z \in Z_S^1.$$

We need only integrate the product of each of these functions with $\zeta(z)$. The claim follows.

Let $\dot{f}_S$ be any function in $\mathcal{H}_{\mathrm{adm}}(G,S,\zeta)$, and let $\dot{f}_S^1$ be some function in $\mathcal{H}_{\mathrm{adm}}(G,S)$ such that $\dot{f}_S^\zeta$ equals $\dot{f}_S^1$. If z belongs to Z_S^1, the expression

$$I_{\mathrm{ell}}^{\mathcal{E}}(\dot{f}_{S,z}^1) - \varepsilon(G)S_{\mathrm{ell}}^G(\dot{f}_{S,z}^1)$$

equals

$$\sum_{G' \in \mathcal{E}_{\mathrm{ell}}^0(G,S)} \iota(G,G')\widehat{S}_{\mathrm{ell}}^{\widetilde{G}'}\big((\dot{f}_S^1)_z'\big)\widetilde{\eta}(z),$$

since $(\dot{f}^1_{S,z})'$ equals $(\dot{f}^1_S)'_z\widetilde{\eta}'(z)$. We assume inductively that for each $G' \in \mathcal{E}^0_{\text{ell}}(G,S)$, the integrand in

$$\int_{Z_{S,\mathfrak{o}}\backslash Z^1_S} \widehat{S}^{\widetilde{G}'}_{\text{ell}}\big((\dot{f}^1_S)'_z\big)\widetilde{\zeta}'(z)dz$$

is invariant under translation of z by $Z_{S,\mathfrak{o}}$, and that the integral itself equals the value taken by the linear form $\widehat{S}^{\widetilde{G}'}_{\text{ell}}$ on $S\mathcal{I}(\widetilde{G}',S,\widetilde{\zeta}')$ at the image of $(\dot{f}^1_S)'$. We have just observed that the image of $(\dot{f}^1_S)'$ in $S\mathcal{I}(\widetilde{G}',S,\widetilde{\zeta}')$ is equal to the function $(\dot{f}^\zeta_S)' = \dot{f}'_S$. Our induction hypothesis therefore asserts that the last integral equals $\widehat{S}^{\widetilde{G}'}_{\text{ell}}(\dot{f}'_S)$. It follows that the integral

$$\begin{aligned}&\int_{Z_{S,\mathfrak{o}}\backslash Z^1_S} \big(I^{\mathcal{E}}_{\text{ell}}(\dot{f}^1_{S,z}) - \varepsilon(G)S^G_{\text{ell}}(\dot{f}^1_{S,z})\big)\zeta(z)dz\\ = &\int_{Z_{S,\mathfrak{o}}\backslash Z^1_S} \Big(\sum_{G'\in\mathcal{E}^0_{\text{ell}}(G,S)} \iota(G,G')\widehat{S}^{\widetilde{G}'}_{\text{ell}}\big((\dot{f}^1_S)'_z\big)\Big)\widetilde{\zeta}'(z)dz\end{aligned}$$

is well defined, and equal to

$$\sum_{G'\in\mathcal{E}^0_{\text{ell}}(G,S)} \iota(G,G')\widehat{S}^{\widetilde{G}'}_{\text{ell}}(\dot{f}'_S),$$

an expression that in turn equals

$$I^{\mathcal{E}}_{\text{ell}}(\dot{f}_S) - \varepsilon(G)S^G_{\text{ell}}(\dot{f}_S).$$

We combine this formula with the original formula (2.5) for $I_{\text{ell}}(\dot{f}_S)$. We deduce in the usual way that

$$I^{\mathcal{E}}_{\text{ell}}(\dot{f}_S) = \int_{Z_{S,\mathfrak{o}}\backslash Z^1_S} I^{\mathcal{E}}_{\text{ell}}(\dot{f}^1_{S,z})\zeta(z)dz, \tag{2.6}$$

and in the case that $\varepsilon(G)=1$, that

$$S^G_{\text{ell}}(\dot{f}_S) = \int_{Z_{S,\mathfrak{o}}\backslash Z^1_S} S^G_{\text{ell}}(\dot{f}^1_{S,z})\zeta(z)dz. \tag{2.7}$$

We shall now prove the assertions of the proposition by establishing the corresponding assertions for the linear forms $I_{\text{ell}}(\dot{f}_S)$, $I^{\mathcal{E}}_{\text{ell}}(\dot{f}_S)$ and $S^G_{\text{ell}}(\dot{f}_S)$. Let $\dot{f}_S$ be a function in

$\mathcal{H}_{\mathrm{adm}}(G,S,\zeta)$ such that $\dot{f}_{S,G}(\dot{\gamma}_S)$ vanishes for any $\dot{\gamma}_S$ in $\Gamma(G_S,\zeta_S)$ whose semisimple part is not central. If G is quasisplit, we assume also that $\dot{f}_S^{G^*}=0$. We can choose the associated function $\dot{f}_S^1$ so that the functions $\dot{f}_{S,z}^1$ on the right hand sides of the identities (2.5), (2.6) and (2.7) have similar properties. We are assuming that the proposition holds for G, with (Z,ζ) trivial. This implies that for each $z\in Z_S^1$, $I_{\mathrm{ell}}^{G,\mathcal{E}}(\dot{f}_{S,z}^1)=I_{\mathrm{ell}}^G(\dot{f}_{S,z}^1)$ and $S_{\mathrm{ell}}^G(\dot{f}_S)=0$. We conclude from the expansions in [I, Lemma 7.2] that the two assertions of the proposition are valid for admissible elements $\dot{\gamma}_S\in\Gamma_{\mathrm{ell}}^{\mathcal{E}}(G,S,\zeta)$ and $\dot{\delta}_S\in\Delta_{\mathrm{ell}}^{\mathcal{E}}(G,S,\zeta)$ whose semisimple parts are not central. In other words, the proposition holds for arbitrary G and ζ. □

The proof of the proposition can be used to extend the descent formulas of Theorem 1.1 to the general case.

Corollary 2.2. *Assume that Theorem 1.1 has been proved for some z-extension $\widetilde{G}$ of G.*

(a) Let $\dot{\gamma}_S$ be an admissible element in $\Gamma_{\mathrm{ell}}^{\mathcal{E}}(G,S,\zeta)$ with Jordan decomposition $\dot{\gamma}_S=c_S\dot{\alpha}_S$. Then

$$a_{\mathrm{ell}}^{G,\mathcal{E}}(\dot{\gamma}_S)=\sum_c\sum_{\dot{\alpha}} i^{\overline{G}}(S,c)\big|\overline{G}_{c,+}(F)/\overline{G}_c(F)\big|^{-1}a_{\mathrm{ell}}^{G_c,\mathcal{E}}(\dot{\alpha}), \tag{2.8}$$

for c and $\dot{\alpha}$ summed as in (1.4).

(b) Suppose that G is quasisplit, and that $\dot{\delta}_S$ is an admissible element in $\Delta_{\mathrm{ell}}(G,S,\zeta)$ with Jordan decomposition $\dot{\delta}_S=d_S\dot{\beta}_S$. Then

$$b_{\mathrm{ell}}^{G}(\dot{\delta}_S)=\sum_d\sum_{\dot{\beta}} j^{\overline{G}^*}(S,d)\big|(\overline{G}_{d,+}^*/\overline{G}_d^*)(F)\big|^{-1}b_{\mathrm{ell}}^{G_d^*}(\dot{\beta}), \tag{2.9}$$

where d is summed over the elements in $\Delta_{\mathrm{ss}}(\overline{G}^*)$ whose image in $\Delta_{\mathrm{ss}}(\overline{G}_S^*)$ equals d_S (a set of order 0 or 1), and $\dot{\beta}$ is summed over the orbit of $(\overline{G}_{d,+}^*/\overline{G}_d^*)(F)$ in $\Delta_{\mathrm{unip}}(G_{d,S},\zeta_S)$ determined by $\dot{\beta}_S$.

Proof. The proofs of (a) and (b) are similar. The formulas follow from the special cases given by Theorem 1.1, together with the identities established in the proof of the proposition. We shall prove only (b), since we will have no need of (a), and since (2.8) will in any case be a consequence of Global Theorem 1 (together with (1.4)). We assume therefore that G is quasisplit.

Let $\widetilde{G}$ be the given z-extension of G. The first step is to check that (2.9) holds for G and ζ, if it is valid for $\widetilde{G}$ and $\widetilde{\zeta}$. By (2.4), the left hand side of the identity (2.9) equals the left hand side of the corresponding identity for $(\widetilde{G}, \widetilde{\zeta})$. To compare the expansions of the two right hand sides, we note that $\overline{\widetilde{G}} = \overline{G}$. In particular, the outer sums in the two expansions can be taken over the same set. If d belongs to this set, $\widetilde{G}^*_d$ is a central extension of G^*_d by $\widetilde{Z}$, so that $\overline{\widetilde{G}}{}^*_d = \overline{G}{}^*_d$ and $\overline{\widetilde{G}}{}^*_{d,+} = \overline{G}{}^*_{d,+}$, by definition. Moreover, there is a canonical bijection $\dot{\beta} \to \widetilde{\dot{\beta}}$ between the sets that index the two inner sums, which satisfies

$$b^{G_d}_{\mathrm{ell}}(\dot{\beta}) = b^{\widetilde{G}_d}_{\mathrm{ell}}(\widetilde{\dot{\beta}}),$$

by (2.4). It follows that there is a term by term identification of (2.9) with the right hand side of the corresponding identity for $(\widetilde{G}, \widetilde{\zeta})$. This proves that (2.9) holds for G and ζ if it is valid for $\widetilde{G}$ and $\widetilde{\zeta}$.

We have reduced the proof of (b) to the case that $\widetilde{G} = G$. The second step is to show that if G_{der} is simply connected, and if (2.9) holds for (Z, ζ) trivial, then (2.9) also holds for arbitrary (Z, ζ). This will follow from a comparison of expansions of the two sides of (2.7).

Let $\dot{f}_S$ be a fixed function in $\mathcal{H}_{\mathrm{adm}}(G, S, \zeta)$ such that the associated function $\dot{f}^{\mathcal{E}}_{S,G}$ is supported on the subset $\Delta(G_S, \zeta_S)$ of $\Delta^{\mathcal{E}}(G_S, \zeta_S)$. The expansion of [I, Lemma 7.2] for the left hand side of (2.7) becomes

$$\sum_{\dot{\delta}_S \in \Delta_{\mathrm{ell}}(G,S,\zeta)} b^G_{\mathrm{ell}}(\dot{\delta}_S) \dot{f}^G_S(\dot{\delta}_S). \tag{2.10}$$

Similarly, we obtain an expansion

$$\int_{Z_{S,\mathfrak{o}} \backslash Z^1_S} \sum_{\dot{\delta}^1_S \in \Delta_{\mathrm{ell}}(G,S)} b^G_{\mathrm{ell}}(\dot{\delta}^1_S) \dot{f}^{1,G}_S(z\dot{\delta}^1_S)\zeta(z)dz$$

for the right hand side of (2.7). The coefficients $b^G_{\mathrm{ell}}(\dot\delta^1_S)$ in this second expansion pertain to the case that (Z,ζ) is trivial. We are assuming that they satisfy the formula (2.9), which in this case reduces to (1.9). The second expansion becomes

$$(2.11)\qquad \int_{Z_{S,\mathfrak{o}}\backslash Z^1_S}\sum_{d^1}\sum_{\dot\beta^1} j^{G^*}(S,d^1)b^{G^*_{d^1}}_{\mathrm{ell}}(\dot\beta^1)\dot f^{1,G^*}_S(zd^1_S\dot\beta^1)\zeta(z)dz,$$

where d^1 is summed over the classes in $\Delta_{\mathrm{ss}}(G^*)$ that are bounded at each $v\notin S$, d^1_S is the image of d^1 in $\Delta_{\mathrm{ss}}(G^*_S)$, and $\dot\beta^1$ is summed over $\Delta_{\mathrm{unip}}(G^*_{d^1,S})$. The functions $\dot f_S$ and $\dot f^1_S$ are related by

$$\int_{Z^1_S}\dot f^1_S(zx)\zeta(z)dz=\dot f_S(x),\qquad x\in G^1_S.$$

Our task is to compare the coefficients of $\dot f^G_S$ and $\dot f^{1,G^*}_S$ in the two expansions.

There is a surjective map $d^1\to d$, from the classes in $\Delta_{\mathrm{ss}}(G^*)$ that are bounded away from S, onto those classes in $\Delta_{\mathrm{ss}}(\overline{G}^*)$ that are bounded away from S. The group $Z_{S,\mathfrak{o}}$ acts transitively on the fibres of this map, and the stabilizer of d^1 in $Z_{S,\mathfrak{o}}$ is isomorphic to the group $(\overline{G}^*_{d,+}/\overline{G}^*_d)(F)$ under the map

$$g\longrightarrow z=g^{-1}d^1g(d^1)^{-1},\qquad g\in(\overline{G}^*_{d,+}/\overline{G}^*_d)(F).$$

We claim that

$$j^{G^*}(S,d^1)=j^{\overline{G}^*}(S,d).$$

We can assume that d^1 is F-elliptic in G^*, since $j^{G^*}(S,d^1)$ would otherwise vanish. This implies that $X^*(G^*)^\Gamma=X^*(G^*_{d^1})^\Gamma$. Now the identity [K2, (5.2.3)] for Tamagawa numbers is actually valid for any central extension of a group by Z. Applying it to the pairs $(G^*,\overline{G}^*)$ and $(G^*_{d^1},\overline{G}^*_d)$, we see that

$$\tau(G^*)\tau(G^*_{d^1})^{-1}=\tau(\overline{G}^*)\tau(\overline{G}^*_d)^{-1}.$$

The claim follows from the definition (1.7). As for the elements $\dot\beta^1$ in (2.11), we recall from the remarks at the end of §1 that $b^{G^*_{d^1}}_{\mathrm{ell}}(\dot\beta^1)$ is supported on the subset $\Delta_{\mathrm{unip,der}}(G^*_{d^1,S})$ of

$\Delta_{\text{unip}}(G^*_{d^1,S})$. We can therefore apply the canonical bijection $\dot\beta^1 \to \dot\beta$ from $\Delta_{\text{unip,der}}(G^*_{d^1,S})$ to the corresponding subset $\Delta_{\text{unip,der}}(G^*_{d,S},\zeta_S)$ of $\Delta_{\text{unip}}(G^*_{d,S},\zeta_S)$. We assume inductively that if d^1 is not central in G, then

$$b_{\text{ell}}^{G^*_{d^1}}(\dot\beta^1) = b_{\text{ell}}^{G^*_d}(\dot\beta). \tag{2.12}$$

In fact, by (2.4) it is enough to assume that the analogue of (2.12) holds for some z-extension of the group $G^*_{d^1} = G^*_d$. This takes care of the second coefficient in (2.11), and leaves us in a position to change the sum over $(d^1,\dot\beta^1)$ to a sum over $(d,\dot\beta)$.

Suppose for a moment that $\dot f_S$ vanishes on an invariant neighbourhood of the center of G_S. The function $\dot f_S^{1,G^*}(zd^1_S\dot\beta^1)$ in (2.11) then vanishes if d^1 is central in G. It follows from the discussion above that we can write (2.11) in the form

$$\sum_d \sum_{\dot\beta} j^{\overline{G}^*}(S,d)\big|(\overline{G}^*_{d,+}/\overline{G}^*_d)(F)\big|^{-1} b_{\text{ell}}^{G^*_d}(\dot\beta)\dot f_S^{G^*}(ds\dot\beta). \tag{2.13}$$

We have established that the expressions (2.10) and (2.13) are equal. Since we can vary $\dot f_S$, subject of course to the given constraints, we deduce that the coefficients of $\dot f_S^G$ and $\dot f_S^{G^*}$ in the two expansions are equal. Comparing these coefficients, we conclude that the descent formula (2.9) is valid for any $\dot\delta_S$ whose semisimple part is not central.

We now remove the condition that $\dot f_S$ vanish on an invariant neighbourhood of the center. We still have expansions for the two sides of (2.7). Given what we have just proved, we see that the terms with noncentral semisimple parts cancel from the two expansions. Comparing the coefficients in the remaining terms, we conclude that (2.9) holds if the semisimple part of $\dot\delta_S$ is central, and hence in general. We also deduce that

$$b_{\text{ell}}^{G^*}(\dot\beta^1) = b_{\text{ell}}^{G^*}(\dot\beta), \tag{2.14}$$

for any $\dot\beta$ in the subset $\Delta_{\text{unip,der}}(G^*_S,\zeta_S)$ of $\Delta_{\text{unip}}(G^*_S,\zeta_S)$ on which $b_{\text{ell}}^{G^*}$ is supported. This completes the induction argument, and our reduction to the case of trivial (Z,ζ). $\square$

As we recalled during the proof of the last corollary, there is a canonical bijection between the sets $\Delta_{\mathrm{unip,der}}(G^*_S)$ and $\Delta_{\mathrm{unip,der}}(G^*_S, \zeta_S)$. In the special case that ζ is the trivial automorphic character on Z, we can identify $\Delta_{\mathrm{unip,der}}(G^*_S, \zeta_S)$ with the set $\Delta_{\mathrm{unip,der}}(\overline{G}^*_S)$ attached to the group $\overline{G}^* = G^*/Z$. We therefore actually have a pair of bijections

$$\Delta_{\mathrm{unip,der}}(G^*_S, \zeta_S) \stackrel{\sim}{\longleftarrow} \Delta_{\mathrm{unip,der}}(G^*_S) \stackrel{\sim}{\longrightarrow} \Delta_{\mathrm{unip,der}}(\overline{G}^*_S).$$

In particular, we can identify the two sets $\Delta_{\mathrm{unip,der}}(G^*_S, \zeta_S)$ and $\Delta_{\mathrm{unip,der}}(\overline{G}^*_S)$.

Corollary 2.3. *Assume that G is quasisplit, and that Theorem 1.1 has been established for some z-extension of G. Then*

$$b^{G^*}_{\mathrm{ell}}(\dot{\beta}) = b^{G^*}_{\mathrm{ell}}(\dot{\beta}^1) = b^{\overline{G}^*}_{\mathrm{ell}}(\dot{\beta}), \qquad \dot{\beta} \in \Delta_{\mathrm{unip,der}}(G^*_S, \zeta_S),$$

where $\dot{\beta}^1$ is the preimage of $\dot{\beta}$ in $\Delta_{\mathrm{unip,der}}(G^*_S)$.

Proof. The formula (2.4) reduces the problem immediately to the case that G^*_{der} is simply connected. We can therefore apply (2.14). This gives the first half of the required identity. The second half follows from (2.4) and the special case of (2.14) in which ζ is trivial. □

There are similar identities for the values of the coefficients a^G_{ell} and $a^{G,\mathcal{E}}_{\mathrm{ell}}$ at unipotent elements. They can be established from (2.5) and (2.6), in the same way that (2.14) was defined from (2.7) in the proof of Corollary 2.2.

§3. The mapping $\mathcal{X}(G) \to \mathcal{Y}(G)$

We have reduced the general problem of global descent to the special case treated in Theorem 1.1. The proof of Theorem 1.1 will take up the rest of the paper. We assume from now on that G_{der} is simply connected, and that (Z, ζ) is trivial.

In this section, we shall investigate a descent mapping for endoscopic data. The mapping is a global analogue of the local mapping in [LS2, §1.4], which was a starting point for the Langlands-Shelstad descent theorem for transfer factors. As in the local case, the mapping depends only on the quasisplit inner form G^* of G attached to the underlying inner twist ψ: $G \to G^*$.

The domain of the mapping will be the set $\mathcal{X}(G)$ of equivalent classes of pairs (G', d'), where G' is an elliptic endoscopic datum for G over F, and d' is a semisimple, elliptic element in $G'(F)$. Two such pairs (G', d') and $(\overline{G}', \overline{d}')$ are defined to be equivalent if there is an isomorphism $\overline{G}' \to G'$ of endoscopic data for G [LS1, (1.2)] that carries $\overline{d}'$ to an element in $G'(F)$ that is stably conjugate to d'. (The notation $\overline{G}'$ here is unrelated to the earlier notation $\overline{G} = G/Z$. Since Z is trivial in this section, there should be no danger of confusion.) The codomain of the mapping will be the set $\mathcal{Y}(G)$ of equivalence classes of pairs (d, G'_d), where d is a semisimple elliptic element in $G^*(F)$, and G'_d is an elliptic endoscopic datum over F for the centralizer G^*_d. Two such pairs (d, G'_d) and $(\overline{d}, G'_{\overline{d}})$ will be called equivalent if there is an inner automorphism of G^* that maps $\overline{d}$ to d, and maps $G'_{\overline{d}}$ to an endoscopic datum for G^*_d that is isomorphic [LS1, (1.2)] to G'_d. Observe that if α is such an inner automorphism, $\alpha\tau(\alpha)^{-1}$ is an inner automorphism of G^*_d for each τ in the Galois group $\Gamma = \mathrm{Gal}(\overline{F}/F)$, since $G^*_d = G^*_{d,+}$. Therefore α: $G^*_{\overline{d}} \to G^*_d$ is an inner twist, which serves to identify the isomorphism classes of endoscopic data for $G^*_{\overline{d}}$ and G^*_d.

Before we describe the mapping, we shall define some simple invariants attached to points in $\mathcal{X}(G)$ and $\mathcal{Y}(G)$. Consider an element $y \in \mathcal{Y}(G)$. If (d, G'_d) and $(\overline{d}, G'_{\overline{d}})$ both represent y, and $\widehat{G}^*_d$ and $\widehat{G}^*_{\overline{d}}$ are dual groups for G^*_d and $G^*_{\overline{d}}$, there is a canonical Γ-isomorphism from $Z(\widehat{G}^*_d)$ onto $Z(\widehat{G}^*_{\overline{d}})$. We may as well introduce an abstract group $\widehat{Z}_y$, equipped with a Γ-action, together with a canonical isomorphism $\widehat{Z}_y \to Z(\widehat{G}^*_d)$ for each (d, G'_d) in the class y. Recall [LS1, (1.2)] that the global endoscopic datum G'_d for G^*_d comes with a locally trivial 1-cocycle a'_d from Γ to $Z(\widehat{G}^*_d)$. The preimage of a'_d in $\widehat{Z}_y$ maps to a canonical element a_y in the group $\ker^1(F, \widehat{Z}_y)$ of locally trivial classes in $H^1(F, \widehat{Z}_y)$. Now there is a canonical Γ-embedding of $Z(\widehat{G})$ into $\widehat{Z}_y$. The image of the map $\mathcal{X}(G) \to \mathcal{Y}(G)$ will turn out to be the set of y such that a_y belongs to the image of $\ker^1\big(F, Z(\widehat{G})\big)$ in $\ker^1(F, \widehat{Z}_y)$. Let $\mathcal{K}_y$ be the subgroup of elements in $\big(\widehat{Z}_y / Z(\widehat{G})\big)^\Gamma$ whose image in $H^1\big(\Gamma, Z(\widehat{G})\big)$ is locally trivial. Once we have defined the map, we will construct a transitive action of $\mathcal{K}_y$ on the fibre of any y in the range. We note in passing that the order

$$\mathfrak{o}_y = \big|\mathrm{Out}_{G^*_d}(G'_d)\big|$$

of the group of outer automorphisms of the endoscopic datum G'_d depends only on the class y. We also note that if (G', d') represents a point x in $\mathcal{X}(G)$, the number

$$c_x = \big|(G'_{d',+}/G'_{d'})(F)\big|$$

of rational components in $G'_{d',+}$ depends only on x, as does the order

$$\mathfrak{o}_x = \big|\mathrm{Out}_G(G', d')\big|$$

of the stabilizer in $\mathrm{Out}_G(G')$ of the stable conjugacy class of d' in G'.

To define the mapping from $\mathcal{X}(G)$ to $\mathcal{Y}(G)$, we have to attach some noncanonical auxiliary data to a given point x in $\mathcal{X}(G)$. First, we fix a representative (G', d') of the class x with the property that the connected centralizer $G'_{d'}$ is quasisplit. This is possible [K1, Theorem 3.3], since we can always replace d' by a stable conjugate in $G'(F)$. We also assume that the full endoscopic datum $(G', \mathcal{G}', s', \xi')$ represented by G' is such that $\mathcal{G}'$ is an L-subgroup of ${}^LG$, and ξ' is the identity embedding of $\mathcal{G}'$ into ${}^LG$. Next, we choose an admissible embedding $\rho'\colon\ T' \to T$ [LS1, (1.3)], of a maximal torus T' in G' over F that contains d' into a maximal torus T in G^* over F, with the property that if d is the image of d' in $G^*(F)$, the centralizer G^*_d is also quasisplit. Again this is possible, since we can always replace T by a stable conjugate in G^*. Note that T can then be identified with a maximal torus in G^*_d. Finally, we choose an admissible L-embedding η_d: ${}^LT \to {}^LG^*_d$. By this, we mean an L-homomorphism from the L-group ${}^LT = \widehat{T} \rtimes W_F$ of T into the L-group of G^*_d, such that the restriction of η_d to $\widehat{T}$ belongs to the canonical $\widehat{G}^*_d$-conjugacy class of embeddings of $\widehat{T}$ into $\widehat{G}^*_d$ [LS1, (2.6)]. The existence of η_d follows from [L1, Lemma 4].

Having fixed auxiliary data (G', d'), ρ' and η_d, we construct a pair (d, G'_d) by following [LS2, (1.4)]. The element d is just the image of d', as above. The symbol G'_d represents an elliptic endoscopic datum $(G'_d, \mathcal{G}'_d, s'_d, \xi'_d)$ for G^*_d whose components we have to describe. We define the first component to be a fixed quasisplit group G'_d, equipped with an isomorphism $G'_d \to G'_{d'}$ over F. Having fixed G'_d, we write T'_d for the preimage of T' in G'_d. The third component is a semisimple element in $\widehat{G}^*_d$. It is defined by

$$s'_d = \eta_d(s'_T) = \eta_d\big(\widehat{\rho'}(s')\big),$$

where $s'_T \in \widehat{T}$ is the image of the point $s' \in Z(\widehat{G}')$ under the map $\widehat{\rho'}$: $\widehat{T}' \to \widehat{T}$ that is dual to ρ'. (See [LS1, (3.1)]. We have identified $Z(\widehat{G}')$ with the canonical subgroup of $\widehat{T}'$ determined by any admissible embedding of $\widehat{T}'$ into $\widehat{G}'$.) For the second component, we set

$$\mathcal{G}'_d = \widehat{G}'_d \cdot \eta_d({}^LT),$$

where $\widehat{G}'_d$ is the connected centralizer of s'_d in $\widehat{G}^*_d$. The last component ξ'_d of the endoscopic datum we define simply to be the identity embedding of $\mathcal{G}'_d$ into ${}^LG^*_d$. To see that $(G'_d, \mathcal{G}'_d, s'_d, \xi'_d)$ is an endoscopic datum for G^*_d, we observe that the map $\eta_d \circ \widehat{\rho'}$ provides a Γ-isomorphism from the dual $\widehat{T}'_d = \widehat{T}'$ of the maximal torus T'_d of G'_d to the maximal torus $\eta_d(\widehat{T})$ of $\widehat{G}^*_d$, which maps the coroots of (G'_d, T'_d) onto the roots of $\big(\widehat{G}'_d, \eta_d(\widehat{T})\big)$. This isomorphism identifies $\widehat{G}'_d$ with a dual group of G'_d. By assumption [LS, (1.2)],

$$\mathrm{Int}(s') \circ \xi' = a' \otimes \xi',$$

where a' is a locally trivial 1-cocycle from Γ to $Z(\widehat{G})$. It follows that

$$\mathrm{Int}(s'_d) \circ \xi'_d = a'_d \otimes \xi'_d,$$

where a'_d is the locally trivial 1-cocyle from Γ to $Z(\widehat{G}^*_d)$ that is the composition of a' with the canonical embedding of $Z(\widehat{G})$ in $Z(\widehat{G}^*_d)$. Therefore $(G'_d, \mathcal{G}'_d, s'_d, \xi'_d)$ is indeed an endoscopic datum for G^*_d.

Observe that the group $Z(\widehat{G}'_d)^\Gamma / Z(\widehat{G}^*_d)^\Gamma$ is a quotient of $Z(\widehat{G}'_{d'})^\Gamma / Z(\widehat{G})^\Gamma$. Since G' is elliptic for G, and d' is an elliptic element in $G'(F)$, this group is finite. Therefore G'_d is an elliptic endoscopic datum for G^*_d. In particular, (d, G'_d) represents a point $y \in \mathcal{Y}(G)$. We shall write ρ'_d: $T'_d \to T$ for the composition of the underlying isomorphism $T'_d \to T'$ with ρ'. The construction is such that ρ'_d is an admissible embedding of T'_d into G^*_d.

Suppose that κ belongs to the group $\mathcal{K}_y$, where y is the image of (d, G'_d) in $\mathcal{Y}(G)$ as above. We shall define a second pair (G^κ, d^κ) in terms of the data (G', d'), ρ' and η_d. To do so, we have also to fix an admissible L-embedding η: ${}^LT \to {}^LG$ such that $s' = \eta(s'_T)$. The existence of η follows from the definition of s'_T and [L1, Lemma 4]. The symbol G^κ represents an endoscopic datum $(G^\kappa, \mathcal{G}^\kappa, s^\kappa, \xi^\kappa)$ for G, which we shall describe. Let κ_T be the element in $\widehat{T}$ given by

$$\kappa_T = \eta_d^{-1}(\kappa_d),$$

where κ_d is a fixed preimage of κ in $Z(\widehat{G}^*_d)$. (Since the restriction of η_d to $\widehat{T}$ is canonically defined up to $\widehat{G}^*_d$-conjugacy, κ_T is actually independent of η_d.) We define

$$s^\kappa = s'\eta(\kappa_T) = \eta(s'_T\kappa_T),$$

and

$$\mathcal{G}^\kappa = \widehat{G}^\kappa \eta({}^LT),$$

where $\widehat{G}^\kappa$ is the connected centralizer of s^κ in $\widehat{G}$. We take ξ^κ to be the identity embedding of $\mathcal{G}^\kappa$ into ${}^LG$, and we take G^κ to be any quasisplit group over F for which $\widehat{G}^\kappa$ is a dual group. This gives an endoscopic datum for G. We still have to construct a semisimple, elliptic element $d^\kappa \in G^\kappa(F)$. Since $\eta({}^L T)$ is contained in $\mathcal{G}^\kappa$, one sees easily from [LS1, Lemma 1.3.A] that the maximal torus $T \subset G^*$ transfers to G^κ. In other words, there is a maximal torus $T^\kappa \subset G^\kappa$ over F, and an admissible embedding ρ^κ of T^κ into G^* that takes T^κ to T. We define d^κ to be the preimage of d in $T^\kappa(F)$. We can assume that ρ^κ has been chosen so that the centralizer $G^\kappa_{d^\kappa}$ is quasisplit. We can also arrange that the point s^κ_T in $\widehat{T}$ associated to ρ^κ and s^κ equals $s'_T\kappa_T$, or equivalently, that η has the property that $\eta(s^\kappa_T) = s^\kappa$.

The pair (G^κ, d^κ) is already equipped with the requisite auxiliary data ρ^κ and η_d. What is its image in $\mathcal{Y}(G)$? Since $d = \rho^\kappa(d^\kappa)$, the construction above assigns a pair (d, G^κ_d) to (G^κ, d^κ), where

$$s^\kappa_d = \eta_d(s^\kappa_T) = \eta_d(s'_T\kappa_T) = s'_d\kappa_d.$$

Therefore, the connected centralizers $\widehat{G}^\kappa_d$ and $\widehat{G}'_d$ of s^κ_d and s'_d in $\widehat{G}^*_d$ coincide. It follows that $\mathcal{G}^\kappa_d = \mathcal{G}'_d$ and $\xi^\kappa_d = \xi'_d$, so that G^κ_d represents the endoscopic datum $(G^\kappa_d, \mathcal{G}'_d, s'_d\kappa_d, \xi'_d)$. It follows that G^κ_d and G'_d are isomorphic as endoscopic data for G^*_d. We have shown that (G', d') and (G^κ, d^κ) map to the same point y in $\mathcal{Y}(G)$.

Proposition 3.1. *(i) The correspondence*

$$(G', d') \longrightarrow (d, G'_d)$$

provides a well defined mapping from $\mathcal{X}(G)$ onto the set of y in $\mathcal{Y}(G)$ such that a_y lies in the image of $\ker^1\big(F, Z(\widehat{G})\big)$ in $\ker^1(F, \widehat{Z}_y)$.

(ii) The correspondence

$$(G', d') \longrightarrow (G^\kappa, d^\kappa), \qquad \kappa \in \mathcal{K}_y,$$

provides a well defined transitive action $x \to x^\kappa$ of $\mathcal{K}_y$ on the fibre of y in $\mathcal{X}(G)$.

(iii) If $y \in \mathcal{Y}(G)$ is the image of $x \in \mathcal{X}(G)$, let $\mathcal{K}_{y,x}$ be the stabilizer of x in $\mathcal{K}_y$. Then

$$|\mathcal{K}_{y,x}| = \mathfrak{o}_y \mathfrak{o}_x^{-1} c_x^{-1}.$$

Proof. We shall prove each of the assertions in turn. For the assertion (i), the main point is to show that the map is well defined. This is perhaps already clear to the reader, but for the benefit of the author, at least, we shall check the conditions in detail. We have then to show that the equivalence class y of (d, G'_d) in $\mathcal{Y}(G)$ is independent of the auxiliary data (G', d'), ρ' and η_d attached to x.

Suppose that $(\overline{G}', \overline{d}')$, $\overline{\rho}'\colon \overline{T}' \to \overline{T}$, and $\overline{\eta}_{\overline{d}}\colon {}^L\overline{T} \to {}^L G^*_{\overline{d}}$ is a second family of auxiliary data for x, with $\overline{d} = \overline{\rho}'(\overline{d}')$. The construction then yields a second pair $(\overline{d}, \overline{G}'_{\overline{d}})$. We have to show that $(\overline{d}, \overline{G}'_{\overline{d}})$ is equivalent to (d, G'_d). By assumption, there is an isomorphism of endoscopic data from $\overline{G}'$ to G' that maps $\overline{d}'$ to a stable conjugate of d'. We are free to replace $\overline{d}'$, $\overline{T}'$ and $\overline{\rho}'$ by stably conjugate data for $\overline{G}'$, without affecting the image $(\overline{d}', \overline{G}'_{\overline{d}})$. We can in fact do so in such a way that the isomorphism from $\overline{G}'$ to G' actually maps $\overline{d}'$ to d'. (This last point requires a standard application of Steinberg's theorem, which shows that the maximal torus $\overline{T}'$ in $\overline{G}'_{\overline{d}}$ transfers to the quasisplit group $G'_{d'}$.) We may therefore assume that $(\overline{G}', \overline{d}') = (G', d')$. Next, observe that $\overline{d}$ and d are $G^*(\overline{F})$-conjugate, since they both belong to the image of the $G^*(\overline{F})$-conjugacy class of d' under the map $\mathcal{A}_{G'/G^*}$ defined in [LS1, p. 225]. It follows from the fact that G^*_{der} is simply connected that $\overline{d}$ is stably conjugate to d. Replacing $\overline{d}$, $\overline{T}$, and $\overline{\rho}'$ by stably conjugate data for G^*, we see (again using Steinberg's theorem) that we may also assume that $\overline{d} = d$. It remains to check that the endoscopic data G'_d and $\overline{G}'_d$ for G^*_d, determined by the different auxiliary data, are isomorphic.

It is consequence of the definitions in [LS1, (1.3)] that the admissible embedding $\overline{\rho}'\colon \overline{T}' \to \overline{T}$ is a composition of three isomorphisms

$$\overline{T}' \xrightarrow{\alpha} T' \xrightarrow{\rho'} T \xrightarrow{\beta} \overline{T},$$

where α is any fixed element in $\mathrm{Int}\big(G'(\overline{F})\big)$ that maps $\overline{T}'$ to T', and β is some element in $\mathrm{Int}\big(G^*(\overline{F})\big)$ (depending on the choice of α) that maps T to $\overline{T}$. Since $\overline{T}'$ and T' are both contained in $G'_{d'}$, we can arrange that α belongs to $\mathrm{Int}(G'_{d'})$. Then β maps d to itself. This means that β lies in $\mathrm{Int}(G_d^*)$, since G_{der}^* is simply connected. Write $\widehat{\alpha}$, $\widehat{\rho}'$ and $\widehat{\beta}$ for the corresponding dual isomorphisms. Then $\overline{\eta}_d\widehat{\beta}$ is an embedding of $\widehat{T}$ into $\widehat{G}_d^*$ that belongs to the same $\widehat{G}_d^*$-conjugacy class of embeddings as the restriction of η_d to $\widehat{T}$. In other words, there is an element $g_d \in \widehat{G}_d^*$, unique up to right translations by $\eta_d(\widehat{T})$, such that $\overline{\eta}_d\widehat{\beta}$ equals the restriction of $\mathrm{Int}(g_d)\eta_d$ to $\widehat{T}$. We claim that $\mathrm{Int}(g_d)$ define an isomorphism between the endoscopic data G_d' and $\overline{G}_d'$.

Observe that

$$\overline{s}_d' = \overline{\eta}_d(s_{\overline{T}}') = (\overline{\eta}_d\widehat{\overline{\rho}}')(s') = (\overline{\eta}_d\widehat{\beta}\widehat{\rho}'\widehat{\alpha})(s') = (\overline{\eta}_d\widehat{\beta}\widehat{\rho}')(s'),$$

since s' lies in $Z(\widehat{G}')$. It follows that

$$\overline{s}_d' = (\overline{\eta}_d\widehat{\beta})(s_T') = (\overline{\eta}_d\widehat{\beta}\eta_d^{-1})(s_d') = \mathrm{Int}(g_d)(s_d').$$

The second point to check is that $\mathrm{Int}(g_d)$ intertwines the L-actions of W_F on $\widehat{G}_d'$ and $\widehat{\overline{G}}{}_d'$. This entails showing that $g_d^{-1}\overline{\eta}_d(\tau_{\overline{T}})g_d$ belongs to $\widehat{G}_d'\eta_d(\tau_T)$, for any element $\tau \in W_F$ with images τ_T and $\tau_{\overline{T}}$ in ${}^LT$ and ${}^L\overline{T}$. To this end, we fix a general point $t \in \widehat{T}$, and set $t_d = \eta_d(t)$. Then

$$\begin{aligned}
\mathrm{Int}\big(g_d^{-1}\overline{\eta}_d(\tau_{\overline{T}})g_d\big)(t_d) \;&=\; \mathrm{Int}\big(g_d^{-1}\overline{\eta}_d(\tau_{\overline{T}})\big)\big((\mathrm{Int}(g_d)\eta_d)(t)\big)\\
= \big(\mathrm{Int}(g_d^{-1})\mathrm{Int}\big(\overline{\eta}_d(\tau_{\overline{T}})\big)\big)\big(\overline{\eta}_d\big(\widehat{\beta}(t)\big)\big) \;&=\; \mathrm{Int}(g_d)^{-1}\big(\overline{\eta}_d\big(\tau\big(\widehat{\beta}(t)\big)\big)\big)\\
= \big(\eta_d\widehat{\beta}^{-1}\tau(\widehat{\beta})\big)\big(\tau(t)\big) \;&=\; \big(\eta_d\widehat{\beta}^{-1}\tau(\widehat{\beta})\eta_d^{-1}\big)\big(\eta_d\big(\tau(t)\big)\big)\\
= \big(\eta_d\widehat{\beta}^{-1}\tau(\widehat{\beta})\eta_d^{-1}\big)\big(\mathrm{Int}\big(\eta_d(\tau_T)\big)(t_d)\big).&
\end{aligned}$$

Since both ρ' and $\overline{\rho}' = \beta\rho'\alpha$ are defined over F, the dual maps $\widehat{\rho}'$ and $\widehat{\beta}\widehat{\rho}'\widehat{\alpha}$ commute with the relevant actions of Γ. Therefore

$$\widehat{\beta}^{-1}\tau(\widehat{\beta}) = \widehat{\rho}'\widehat{\alpha}\tau(\widehat{\alpha})^{-1}(\widehat{\rho}')^{-1}.$$

The isomorphism $\widehat{\alpha}\tau(\widehat{\alpha})^{-1}$ of $\widehat{T}'$ is dual to an element in the Weyl group of $(G_{d'}', T')$. The isomorphism

$$\eta_d\widehat{\beta}^{-1}\tau(\widehat{\beta})\eta_d^{-1} = \eta_d\widehat{\rho}'\widehat{\alpha}\tau(\widehat{\alpha})^{-1}(\widehat{\rho}')^{-1}\eta_d^{-1}$$

of the torus $\eta_d(\widehat{T}) = (\eta_d\widehat{\rho})(\widehat{T}')$ is the corresponding dual element in the Weyl group of $\widehat{G}'_{d'} = \widehat{G}'_d$. Combining this with the first identity, we see that

$$\mathrm{Int}\big(g_d^{-1}\overline{\eta}_d(\tau_{\overline{T}})g_d\big)(t_d) = \mathrm{Int}\big(g(\tau)\eta_d(\tau_T)\big)(t_d),$$

for some element $g(\tau)$ in $\widehat{G}'_d$. Since t_d represents a general point in $\eta(\widehat{T})$, we conclude that $g_d^{-1}\overline{\eta}_d(\tau_{\overline{T}})g_d$ lies in $\widehat{G}'_d\eta_d(\tau_T)$, as required. We have shown that $\mathrm{Int}(g_d)$ defines an isomorphism between G'_d and $\overline{G}'_d$. The pairs (d, G'_d) and $(d, \overline{G}'_d)$ thus represent the same element in $\mathcal{Y}(G)$, and the map $\mathcal{X}(G) \to \mathcal{Y}(G)$ is well defined.

It is clear from the original definition that the map sends $\mathcal{X}(G)$ into the set of y in $\mathcal{Y}(G)$ such that a_y belongs to the image of $\ker^1\big(F, Z(\widehat{G})\big)$ in $\ker^1(F, \widehat{Z}_y)$. Conversely, suppose that y is an element in this set. Choose a representative (d, G'_d) of y such that G^*_d is quasisplit, and such that if G'_d represents the endoscopic datum $(G'_d, \mathcal{G}'_d, s'_d, \xi'_d)$, then $\mathcal{G}'_d$ is an L-subgroup of ${}^LG'_d$, and ξ'_d is the identity embedding of $\mathcal{G}'_d$ into ${}^LG_d$. We also assume that

$$\mathrm{Int}(s'_d) \circ \xi'_d = a' \circ \xi'_d,$$

where a' is a locally trivial 1-cocycle of Γ that takes values in the subgroup $Z(\widehat{G})$ of $Z(\widehat{G}_d)$. We shall construct a pair (G', d') that maps to (d, G'_d). Let ρ'_d: $T'_d \to T$ be an admissible embedding of a maximal torus T'_d of G'_d over F to a maximal torus T in G^*_d. Then T is also a maximal torus in G, for which we fix an admissible L-embedding η: ${}^LT \to {}^LG$. To obtain the pair (G', d'), we need only reverse the construction in the original definition. Set

$$s' = \eta(s'_{d,T}) = \eta\big(\widehat{\rho}'_d(s'_d)\big),$$

and if $\widehat{G}'$ is the connected centralizer of s' in $\widehat{G}$, let ξ' be the identity embedding of $\mathcal{G}' = \widehat{G}'\eta({}^LT)$ into ${}^LG$. Then

$$\mathrm{Int}(s') \circ \xi' = a' \otimes \xi'.$$

Let G' be any fixed quasisplit group over F for which $\widehat{G}'$ is a dual group. Since $\eta({}^LT)$ is contained in $\mathcal{G}'$, the maximal torus $T \subset G^*$ transfers to G'. We can find a maximal torus T' of G' over F, an element $d' \in T'(F)$ such that $G'_{d'}$ is quasisplit, and an admissible embedding ρ': $T' \to T$ that takes d' to d. Then $(G', \mathcal{G}', s', \xi')$ is an elliptic endoscopic datum for G, and the corresponding pair (G', d') represents a point $x \in \mathcal{X}(G)$ that maps to y. We have characterized the image of the map, and completed the proof of (i).

We turn next to (ii). We have to show that the correspondence $(G', d') \to (G^\kappa, d^\kappa)$ gives a well defined action of $\mathcal{K}_y$ on the fibre of y. Recalling the definition of the correspondence, we observe immediately that the image x^κ of (G^κ, d^κ) in $\mathcal{X}(G)$ is independent of the actual choice of the quasisplit group G^κ and the admissible embedding ρ^κ. It remains to check that it is also independent of the auxiliary data (G', d'), ρ' and η attached to the fixed point x in the fibre of y. Let $(\overline{G}', \overline{d}')$, $\overline{\rho}'$: $\overline{T}' \to \overline{T}$, and $\overline{\eta}$: ${}^L\overline{T} \to {}^LG$ be a second set of such data for x. Arguing as at the beginning of the proof of (i), we see that it is enough to treat the case in which the pair $(\overline{G}', \overline{d}')$ equals (G', d'), and the point $\overline{d} = \overline{\rho}'(\overline{d}')$ equals $d = \rho'(d')$. Following the proof of (i) further, we deal with this special case by writing $\overline{\rho}' = \beta\rho'\alpha$, for automorphisms $\alpha \in \mathrm{Int}(G'_{d'})$ and $\beta \in \mathrm{Int}(G^*_d)$. There is then an element $g \in \widehat{G}$, unique up to right translation by $\eta(\widehat{T})$, such that $\overline{\eta}\widehat{\beta}$ equals the restriction of $\mathrm{Int}(g)\eta$ to $\widehat{T}$. The embeddings η and $\overline{\eta}$ are assumed to be such that $\overline{\eta}(s'_{\overline{T}})$ and $\eta(s'_T)$ both equal the point s'. Therefore

$$
\begin{aligned}
s' = \overline{\eta}(s'_{\overline{T}}) &= (\overline{\eta}\,\widehat{\overline{\rho}}')(s') = (\overline{\eta}\widehat{\beta}\widehat{\rho}'\widehat{\alpha})(s') = (\overline{\eta}\widehat{\beta}\widehat{\rho}')(s') \\
&= (\overline{\eta}\widehat{\beta})(s'_T) = (\overline{\eta}\widehat{\beta}\eta^{-1})(s') = \mathrm{Int}(g)(s'),
\end{aligned}
$$

so that $\mathrm{Int}(g)$ centralizes s'. The two sets of auxiliary data assign two elements s^κ and $\overline{s}^\kappa$ in $\widehat{G}'$ to any x in $\mathcal{K}_y$. They are related by

$$
\begin{aligned}
\overline{s}^\kappa = s'\overline{\eta}(\kappa_{\overline{T}}) &= s'(\overline{\eta}\widehat{\beta})(\kappa_T) = s'\big(\mathrm{Int}(g)\eta\big)(\kappa_T) \\
&= \mathrm{Int}(g)\big(s'\eta(\kappa_T)\big) = \mathrm{Int}(g)(s^\kappa),
\end{aligned}
$$

since $\kappa_{\overline{T}} = \widehat{\beta}(\kappa_T)$. The second point is to compare the L-actions of W_F on the two groups $\widehat{G}^\kappa$ and $\widehat{\overline{G}}^\kappa$. Arguing again as in the proof of (i), we deduce that for any $\tau \in W_F$, $g^{-1}\overline{\eta}(\tau_{\overline{T}})g$ lies in $\widehat{G}^\kappa\eta(\tau_T)$. Therefore, $\mathrm{Int}(g)$ represents an isomorphism between the two endoscopic data G^κ and $\overline{G}^\kappa$ for G. It follow easily from the definitions that this isomorphism takes d^κ to a stable conjugate of $\overline{d}^\kappa$ in $\overline{G}^\kappa$. The pairs (G^κ, d^κ) and $(\overline{G}^\kappa, \overline{d}^\kappa)$ therefore represent the same element x^κ in $\mathcal{X}(G)$.

We have constructed a well defined action of $\mathcal{K}_y$ on the fibre of y in $\mathcal{X}(G)$. It remains to show that the action is transitive. We equip our fixed point x in the fibre with a complete set of auxiliary data (G', d'), ρ', η and η_d. Let $\overline{x}$ be any other point in the fibre, equipped with a corresponding set of auxiliary data $(\overline{G}', \overline{d}')$, $\overline{\rho}'$: $\overline{T}' \to \overline{T}$, $\overline{\eta}$: ${}^L\overline{T} \to {}^LG$, and $\overline{\eta}_{\overline{d}}$: ${}^L\overline{T} \to {}^L\widehat{G}_{\overline{d}}$. We must show that $\overline{x} = x^\kappa$, for some κ in $\mathcal{K}_y$.

By assumption, the pairs (d, G'_d) and $(\overline{d}, \overline{G}'_{\overline{d}})$ attached to the two sets of data are equivalent. In particular, $\overline{d}$ is stably conjugate to d. Replacing $\overline{d}$, $\overline{T}$, and $\overline{\rho}'$ by stably conjugate data for G^*, we may assume that $\overline{d} = d$. There is also an isomorphism between the endoscopic data $\overline{G}'_d$ and G'_d for G^*_d. Redefining $\overline{\rho}'$, if necessary, we can assume that the admissible embeddings $\overline{\rho}'_d$ and ρ'_d are compatible with respect to this isomorphism, and in particular, that $\overline{T} = T$. The isomorphism includes an element g_d in $\widehat{G}^*_d$ such that

$$g_d^{-1} \overline{s}'_d g_d = s'_d \kappa_d,$$

for some element κ_d in $Z(\widehat{G}^*_d)$, and such that

$$g_d^{-1} \widehat{\overline{G}}'_d \overline{\eta}_d({}^LT) g_d = \widehat{G}'_d \eta_d({}^LT).$$

Replacing $\overline{\eta}_d$ by its $\mathrm{Int}(g_d^{-1})$ conjugate, we can assume that $g_d = 1$, and that $\overline{\eta}_d = \eta_d$. We are also free to take $\overline{\eta} = \eta$. The points $\overline{s}'_d$ and s'_d are obtained from the semisimple elements $\overline{s}'$ and s' attached to $\overline{G}'$ and G'. It follows without difficulty that κ_d maps to an element κ in $\mathcal{K}_y$. From this, we deduce that

$$\begin{aligned} \overline{s}' &= \overline{\eta}(\overline{s}'_T) = \overline{\eta}\big(\overline{\eta}_d^{-1}(\overline{s}'_d)\big) = \eta\big(\eta_d^{-1}(s'_d \kappa_d)\big) \\ &= \eta(s'_T \kappa_T) = s' \eta(\kappa_T) = s^\kappa. \end{aligned}$$

We are assuming that the groups $\overline{\mathcal{G}}'$ and $\mathcal{G}'$ attached to $\overline{G}'$ and G' are L-subgroups of ${}^LG$, and that $\overline{\xi}'$ and ξ' are the trivial embeddings. Consequently,

$$\overline{\mathcal{G}}' = \widehat{\overline{G}}' \overline{\eta}({}^LT) = \widehat{G}^\kappa \eta({}^LT) = \mathcal{G}^\kappa.$$

We have established that $(\overline{G}', \overline{d}')$ equals (G^κ, d^κ), for an element $\kappa \in \mathcal{K}_y$. Therefore $\overline{x} = x^\kappa$, and so the actions of $\mathcal{K}_y$ on the fibre is transitive. This completes the proof of (ii).

The final assertion (iii) is an identity among four constants attached to a point $x \in \mathcal{X}(G)$, and its image $y \in \mathcal{Y}(G)$. Fix auxiliary data (G', d'), ρ': $T' \to T$, η: ${}^L T \to {}^L G$, and η_d: ${}^L T \to {}^L G_d$ for x, as above, and form the corresponding representative (d, G'_d) of y. We may as well assume that the maximal torus T' in the quasisplit group $G'_{d'} \cong G'_d$ is maximally split. Let W be the Weyl group of (G^*, T). The embedding η allows us to identity W with the Weyl group of $\big(\widehat{G}, \eta(\widehat{T})\big)$, and the action of Γ on either T or $\widehat{T}$ provides an action of Γ on W. We shall express each of the four constants in terms of various groups of Γ-invariant cosets in W.

Let W_d, W' and W'_d be the Weyl groups of (G^*_d, T), (G', T'), and (G'_d, T'_d), respectively. The embeddings η_d, $\eta\widehat{\rho}'$ and $\eta_d\widehat{\rho}'_d$ allow us to identify these three groups with the respective Weyl groups of $\big(\widehat{G}^*_d, \eta_d(\widehat{T})\big)$, $\big(\widehat{G}', \eta(\widehat{T})\big)$ and $\big(\widehat{G}'_d, \eta_d(\widehat{T})\big)$. It follows that each of the three groups is identified, in one way or another, with a subgroup of W. We have also to introduce two other subgroups

$$\widetilde{W}' = \big\{w \in W : \ ws'w^{-1} \in s'Z(\widehat{G})\big\}$$

and

$$\widetilde{\widetilde{W}}' = \big\{w \in W : \ ws'w^{-1} \in s'(\eta\eta_d^{-1})\big(Z(\widehat{G}_d)\big)\big\}$$

of W. We shall be interested in the chain

$$W'_d \subset (W' \cap W_d) \subset (\widetilde{W}' \cap W_d) \subset (\widetilde{\widetilde{W}}' \cap W_d)$$

of Γ-stable subgroups of W_d. We note that the largest of these subgroups can also be written as

$$\widetilde{\widetilde{W}}' \cap W_d = \big\{w \in W_d : \ ws'_d w^{-1} \in s'_d Z(\widehat{G}_d)\big\}.$$

The constant c_x is the number of F-rational points in the quotient $G'_{d',+}/G'_{d'}$. Any coset in this quotient has a representative that normalizes T'. We identify this representative with an element in the intersection $W' \cap W_d$ that is uniquely determined modulo the

Weyl group W'_d of $G'_{d'} \cong G'_d$. Conversely, any coset in $W' \cap W_d/W'_d$ determines a coset in $G'_{d',+}/G'_{d'}$. We thus obtain an isomorphism from $W' \cap W_d/W'_d$ to $G'_{d',+}/G'_{d'}$, which is clearly compatible with the two actions of Γ. Therefore

$$c_x = \left|(W' \cap W_d/W'_d)^\Gamma\right|.$$

The constant $\mathfrak{o}_y$ equals the number of outer automorphisms of the endoscopic datum G'_d of G^*_d. Any such automorphism can be represented by a coset g_d in $\widehat{G}^*_d/\widehat{G}'_d$ such that $g_d s'_d g_d^{-1}$ belongs to $s'_d Z(\widehat{G}^*_d)$, and such that $g_d \mathcal{G}'_d g_d^{-1} = \mathcal{G}'_d$. By choosing a representative of g_d in $\widehat{G}^*_d$ that normalizes $\eta_d(\widehat{T})$, we obtain an element w in the Weyl group W_d that is uniquely determined modulo W'_d. The first condition of g_d asserts that the image of w in W_d/W'_d is contained in $\widetilde{\widetilde{W}}' \cap W_d/W'_d$. The second condition asserts that it belongs to the subset $(\widetilde{\widetilde{W}}' \cap W_d/W'_d)^\Gamma$ of Γ-invariant elements. On the other hand, any element in this subset can be identified with a class g_d that satisfies the two conditions. We obtain a homomorphism

$$(\widetilde{\widetilde{W}}' \cap W_d/W'_d)^\Gamma \longrightarrow \mathrm{Out}_{G^*_d}(G'_d),$$

which from the discussion above is surjective. Since any outer automorphism is completely determined by its action on a maximal torus, the homomorphism is also injective. It follows that

$$\mathfrak{o}_y = \left|\left(\widetilde{\widetilde{W}}' \cap W_d/W'_d\right)^\Gamma\right|.$$

The constant $\mathfrak{o}_x$ equals the order of the group $\mathrm{Out}_{G^*}(G', d')$ of outer automorphisms of the endoscopic datum G' of G that map the stable conjugacy class of d' to itself. Consider an element in this group. We can represent such an element by an F-rational automorphism ϕ: $G' \to G'$ that maps d' to a stable conjugate. Having chosen ϕ, we select an element α in $\mathrm{Int}\big(G'(\overline{F})\big)$ such that $\alpha\phi$ maps d' to itself, and such that for any $\tau \in \Gamma$, $\alpha\tau(\alpha)^{-1}$ belongs to $\mathrm{Int}\big(G'_{d'}(\overline{F})\big)$. The torus $(\alpha\phi)(T')$ centralizes d', and hence is contained in $G'_{d'}$. Replacing α by the left translate by an element in $\mathrm{Int}\big(G'_{d'}(\overline{F})\big)$ if necessary, we can in fact assume that $(\alpha\phi)(T') = T'$. The composition $w = \rho'\alpha\phi(\rho')^{-1}$ then stabilizes the pair (T, d), and therefore belongs to W_d. According to our conventions, W also acts on $\eta(\widehat{T})$. Since it represents an automorphism of the endoscopic datum G', the element w maps s' to a point in $s'Z(\widehat{G})$. Therefore, w belongs to the subgroup $\widetilde{W}' \cap W_d$ of W_d. Moreover, if τ belongs to the Galois group Γ, we see that

$$\begin{aligned} w\tau(w)^{-1} &= \rho'\alpha\phi(\rho')^{-1}\tau(\rho')\tau(\phi)^{-1}\tau(\alpha)^{-1}\tau(\rho')^{-1} \\ &= \rho'\alpha\tau(\alpha)^{-1}(\rho')^{-1}, \end{aligned}$$

since ρ and ϕ are F-rational maps. Therefore $w\tau(w)^{-1}$ belongs to W'_d, the subgroup of elements in W_d induced by $G'_{d'}$. In other words, the projection of w onto $\widetilde{W}' \cap W_d/W'_d$ belongs to the subgroup $(\widetilde{W}' \cap W_d/W'_d)^\Gamma$ of Γ-invariant cosets. We have shown how to associate a coset in $(\widetilde{W}' \cap W_d/W'_d)^\Gamma$ to any element in $\mathrm{Out}_{G^*}(G', d')$. On the other hand, any coset in $(\widetilde{W}' \cap W_d/W'_d)^\Gamma$ clearly determines a unique element in $\mathrm{Out}_{G^*}(G', d')$. We obtain a homomorphism

$$(\widetilde{W}' \cap W_d/W'_d)^\Gamma \longrightarrow \mathrm{Out}_{G^*}(G', d'),$$

which from the discussion above is surjective. The kernel of this homomorphism is the subgroup of cosets induced from the Weyl group W' of $\widehat{G}'$, which is just $(W' \cap W_d/W'_d)^\Gamma$. We conclude that

$$\mathfrak{o}_x = \big|(\widetilde{W}' \cap W_d/W'_d)^\Gamma\big|\big|(W' \cap W_d/W'_d)^\Gamma\big|^{-1}.$$

We evaluate the last constant $|\mathcal{K}_{y,x}|$ by almost identical means. Suppose that κ belongs to the stabilizer $\mathcal{K}_{y,x}$. The pairs (G^κ, d^κ) and (G', d') are then equivalent. We can find an F-rational isomorphism ψ: $G^\kappa \to G'$, which represents an isomorphism of endoscopic data, such that $\psi(d^\kappa)$ is stably conjugate to d'. We choose an element β in $\mathrm{Int}\big(G'(\overline{F})\big)$ such that $\beta\psi$ maps d^κ to d', and such that $\beta\tau(\beta)^{-1}$ belongs to $\mathrm{Int}\big(G'_{d'}(\overline{F})\big)$, for each $\tau \in \Gamma$. The torus $(\beta\psi)(T^\kappa)$ centralizes d', and is therefore contained in $G'_{d'}$. Translating β by an element in $\mathrm{Int}\big(G'_{d'}(\overline{F})\big)$ if necessary, we can assume that $(\beta\psi)(T^\kappa) = T'$. The composition

$w = \rho' \beta \psi(\rho^\kappa)^{-1}$ then stabilizes the pair (T, d), and therefore belongs to W_d. The action of w on $\eta(\widehat{T})$ maps s' to an element in $s^\kappa Z(\widehat{G}) = s'\eta(\kappa_T)Z(\widehat{G})$. Therefore, w belongs to the subgroup $\widetilde{\widetilde{W}}' \cap W_d$ of W_d. We also obtain

$$w\tau(w)^{-1} = \rho'\beta\tau(\beta)^{-1}(\rho')^{-1}, \qquad \tau \in T,$$

as above, so that $w\tau(w)^{-1}$ lies in W'_d. Therefore, the projection of w onto $\widetilde{\widetilde{W}}' \cap W_d/W'_d$ lies in $(\widetilde{\widetilde{W}}' \cap W_d/W'_d)^\Gamma$. We have shown how to associate a coset in $(\widetilde{\widetilde{W}}' \cap W_d/W'_d)^\Gamma$ to the element κ in $\mathcal{K}_{y,x}$. On the other hand, suppose that w is any coset $(\widetilde{\widetilde{W}}' \cap W_d/W'_d)^\Gamma$. Then $ws'w^{-1}$ belongs to $s'\eta(\kappa_T)Z(\widehat{G})$, for a unique element κ in $\mathcal{K}_y$. Moreover, w determines an isomorphism from G^κ to G' that takes d^κ to a stable conjugate of d'. It follows that κ belongs to the subgroup $\mathcal{K}_{y,x}$ of $\mathcal{K}_y$. The correspondence $w \to x$ gives us a homomorphism

$$\big(\widetilde{\widetilde{W}}' \cap W_d/W'_d\big)^\Gamma \longrightarrow \mathcal{K}_{y,x},$$

which from the discussion above is surjective. The kernel of this homomorphism is the subgroup of cosets w such that $ws'w^{-1}$ belongs to $s'Z(\widehat{G})$. The kernel is therefore equal to $(\widetilde{W}' \cap W_d/W'_d)^\Gamma$. We conclude that

$$|\mathcal{K}_{y,x}| = \big|\big(\widetilde{\widetilde{W}}' \cap W_d/W'_d\big)^\Gamma\big|\big|\big(\widetilde{W}' \cap W_d/W'_d\big)^\Gamma\big|^{-1}.$$

We have now only to compare the four formulas we have obtained for the four constants. We see immediately that

$$|\mathcal{K}_{y,x}| = \mathfrak{o}_y \mathfrak{o}_x^{-1} c_x^{-1}.$$

This gives the final assertion (iii) of Proposition 3.1. □

§4. Descent and global transfer factors

The mapping $\mathcal{X}(G) \to \mathcal{Y}(G)$ is useful for describing the behaviour of transfer factors under descent. In this section we shall discuss some implications of the main theorem of [LS2] for the global transfer factors in (1.1). Since we will be applying the results to the proof of Theorem 1.1, we need only consider elements in $\mathcal{Y}(G)$ that map to a fixed semisimple stable conjugacy class in $G(F)$. In fact for the next two sections, we can fix a point $y \in \mathcal{Y}(G)$, together with a representative (d, G'_d) such that G^*_d is quasisplit. For the time being, we assume that y lies in the image of the map $\mathcal{X}(G) \to \mathcal{Y}(G)$. In particular, we can fix a point x in $\mathcal{X}(G)$ that maps to y, together with a representative (G', d') such that $G'_{d'}$ is quasisplit.

We are assuming that G_{der} is simply connected, and that $Z = \{1\}$. To study the transfer factors for G', we recall from §2 that we can set $\widetilde{G}' = G'$, with $\widetilde{\xi}'$ then being a fixed L-isomorphism of $\mathcal{G}'$ with the L-group ${}^LG'$. The derived group of G^*_d need not be simply connected. To study the transfer factors for G'_d, we have to fix a suitable central extension $\widetilde{G}'_d \to G'_d$ over F, together with an L-embedding $\widetilde{\xi}'_d$: $\mathcal{G}'_d \to {}^L\widetilde{G}'_d$. We may as well assume that $\widetilde{G}'_d$ is obtained from a fixed z-extension $\widetilde{G}^*_d$ of G^*_d, as in [LS1, (4.4)].

Let S be a finite set of valuations outside of which G and G' are unramified, and such that d is S-admissible [I, §1] and bounded away from S. It follows from [K3, Lemma 7.1] that the centralizers G^*_d and $G'_{d'}$ are also unramified outside of S. Since the endoscopic group G'_d for G^*_d is isomorphic to $G'_{d'}$, it too is unramified outside of S. We can assume that the same is true of the extension $\widetilde{G}'_d$ and the L-embedding $\widetilde{\xi}'_d$ [1, Lemma 7.1].

Assume that

$$c_S = \prod_{v \in S} c_v$$

is a semisimple element in $G(F_S)$ such that each of the local components d_v of d is an image of c_v, relative to the fixed inner twist ψ: $G \to G^*$. In particular, we can choose elements $g_v^* \in G_{\mathrm{sc}}^*(\overline{F}_v)$ such that $\mathrm{Int}(g_v^*)\psi(c_v) = d_v$. For each v, the isomorphism

$$\psi_{c_v} = \mathrm{Int}(g_v^*)\psi : \ G_{c_v} \to G_{d_v}^*$$

is then an inner twist, whose inner class is independent of g_v^*. It allows us to identify G'_{d_v} with a local endoscopic datum for G_{c_v}. We assume that (d, G'_d) is such that G'_{d_v} is relevant to G_{c_v}. The main result of [LS2] relates the local relative transfer factors attached to the pairs (G_v, G'_v) and (G_{c_v}, G'_{d_v}).

Before we recall the descent formula from [LS2], we shall thicken c_S to an adelic element $c_{\mathbb{A}} \in G(\mathbb{A})$. We have fixed an open maximal compact subgroup $K^S = \prod_{v \notin S} K_v$ of $G^S(\mathbb{A}^S)$ such that each $K_v \subset G_v$ is hyperspecial. For every $v \notin S$, we can choose a semisimple element $c_v \in K_v$ for which d_v is an image. This follows from [T, (3.2)], the S-admissibility of d, and the fact that any two hyperspecial maximal compact subgroups of $G_v(F_v)$ are conjugate under $G_{v,\mathrm{ad}}(F_v)$. By the last statement of Proposition 7.1 of [K3], the conjugacy class of c_v in $G_v(F_v)$ is unique. We set

$$c_{\mathbb{A}} = c_S \cdot \prod_{v \notin S} c_v.$$

We have introduced $c_{\mathbb{A}}$ in order to make use of a general construction in [K3, §6] (which extends the special case of [L2, Chapter 7]). Applied to $c_{\mathbb{A}}$, the construction yields a character $\mathrm{obs}(c_{\mathbb{A}})$ on the group $\mathcal{K}(G_d^*, G^*) \cong \mathcal{K}_y$ of element in $\big(Z(\widehat{G}_d^*)/Z(\widehat{G})\big)^\Gamma$ whose image in $H^1\big(\Gamma, Z(\widehat{G})\big)$ is locally trivial. According to [K3, Theorem 6.6], $\mathrm{obs}(c_{\mathbb{A}})$ is trivial if and only if $c_{\mathbb{A}}$ is $G(\mathbb{A})$-conjugate to the diagonal image of an element in $G(F)$, or equivalently, if and only if c_S has a representative in $G(F)$ whose conjugacy class in $G^S(\mathbb{A}^S)$ meets K^S.

We now describe the descent formula from [LS2]. For each $v \in S$, choose maximal tori T'_v and $\overline{T}'_v$ in $G'_{d'_v}$ that are images of F_v-rational maximal tori T_{c_v} and $\overline{T}_{c_v}$ in G_{c_v}. We can identify T'_v and $\overline{T}'_v$ with maximal tori in G'_{d_v}, and we write $\widetilde{T}'_v$ and $\widetilde{\overline{T}}'_v$ for their preimages in $\widetilde{G}'_{d_v}$. As usual, we write T'_S, T_{c_S}, $\widetilde{T}'_S$, etc., for the relevant products over $v \in S$. Choose strongly G-regular elements $\sigma'_S \in T'_S$ and $\overline{\sigma}'_S \in \overline{T}'_S$ that are images of points $\rho_S \in T_{c_S}$ and

$\overline{\rho}_S \in \overline{T}_{c_S}$, together with preimages $\widetilde{\sigma}'_S \in \widetilde{T}'_S$ and $\widetilde{\overline{\sigma}}'_S \in \widetilde{\overline{T}}'_S$. We can then form the relative (local) transfer factors

$$\Delta(\sigma'_S, \rho_S; \overline{\sigma}'_S, \overline{\rho}_S) = \prod_{v \in S} \Delta(\sigma'_v, \rho_v; \overline{\sigma}'_v, \overline{\rho}_v)$$

for (G_S, G'_S), and corresponding objects

$$\Delta_{c_S}(\widetilde{\sigma}'_S, \rho_S; \widetilde{\overline{\sigma}}'_S, \overline{\rho}_S) = \prod_{v \in S} \Delta_{c_v}(\widetilde{\sigma}'_v, \rho_v; \widetilde{\overline{\sigma}}'_v, \overline{\rho}_v)$$

for (G_{c_S}, G'_{c_S}). The results of [LS2] apply to the quotient

$$\Theta(\widetilde{\sigma}'_S, \rho_S; \widetilde{\overline{\sigma}}'_S, \overline{\rho}_S) = \Delta(\sigma'_S, \rho_S; \overline{\sigma}'_S, \overline{\rho}_S) \Delta_{c_S}(\widetilde{\sigma}'_S, \rho_S; \widetilde{\overline{\sigma}}'_S, \overline{\rho}_S)^{-1}.$$

Let $\widetilde{d}'$ be a fixed preimage of d' in $\widetilde{G}'(F)$ that is bounded at each place $v \notin S$. One of the easier results of [LS2] is that $\Theta(\widetilde{\sigma}'_S, \rho_S; \widetilde{\overline{\sigma}}'_S, \overline{\rho}_S)$ extends to a smooth function of $(\widetilde{\sigma}', \rho_S)$ and $(\widetilde{\overline{\sigma}}'_S, \overline{\rho}_S)$ in a neighbourhood of $(\widetilde{d}_S, c_S)$. The main result of [LS2] asserts that the limit of $\Theta(\widetilde{\sigma}'_S, \rho_S; \widetilde{\overline{\sigma}}'_S, \overline{\rho}_S)$, as $(\widetilde{\sigma}'_S, \rho_S)$ and $(\widetilde{\overline{\sigma}}'_S, \overline{\rho}_S)$ both approach $(\widetilde{d}'_S, c_S)$, is independent of the various tori.

We can also form the absolute (global) transfer factor

$$\Delta(\sigma'_S, \rho_S) = \Delta(\sigma'_S, \rho_S; \overline{\sigma}'_S, \overline{\rho}_S) \Delta(\overline{\sigma}'_S, \overline{\rho}_S)$$

for (G_S, G'_S), defined as in [I, §4]. Here $\overline{\sigma}'_S$ is assumed to come from a rational element $\overline{\sigma}' \in \overline{T}'(F)$, in which $\overline{T}' \subset G'$ is a maximal torus over F, while $\overline{\rho}_S$ comes from an element $\overline{\rho} \in G(\mathbb{A})$ of which $\overline{\sigma}'$ is an adelic image. The factor on the right is the canonical preassigned value

$$\Delta(\overline{\sigma}'_S, \overline{\rho}_S) = d(\overline{\sigma}', \overline{\rho})^{-1} \prod_{v \notin S} \Delta_{K_v}(\overline{\sigma}'_v, \overline{\rho}_v)^{-1},$$

described in [I, §4]. Since each $G'_{d'_v}$ is relevant to G_{c_v}, we can arrange that $\overline{T}'$ is contained in $G'_{d'}$ and that each point $\overline{\rho}'_v$ lies in G_{c_v}, as before. (See the discussion preceding [I, Lemma

4.3].) Now the group G_{c_S}, unlike G_S, does not generally have an F-rational structure. In this case, we just set

$$\Delta_{c_S}(\widetilde{\sigma}'_S, \rho_S) = \Delta_{c_S}(\widetilde{\sigma}'_S, \rho_S; \widetilde{\overline{\sigma}}'_S, \overline{\rho}_S)\Delta_{c_S}(\widetilde{\overline{\sigma}}'_S, \rho_S),$$

where $\Delta_{c_S}(\widetilde{\overline{\sigma}}'_S, \overline{\rho}_S)$ is some arbitrary preassigned value taken at the fixed base point $(\widetilde{\overline{\sigma}}'_S, \overline{\rho}_S)$. We then form an absolute quotient

$$\Theta(\widetilde{\sigma}'_S, \rho_S) = \Delta(\sigma'_S, \rho_S)\Delta_{c_S}(\widetilde{\sigma}'_S, \rho_S)^{-1}.$$

The earlier relative quotient has the transitivity property

$$\Theta(\widetilde{\sigma}'_S, \rho_S; \widetilde{\overline{\sigma}}'_S, \overline{\rho}_S) = \Theta(\widetilde{\sigma}'_S, \rho_S; \widetilde{\overline{\overline{\sigma}}}'_S, \overline{\overline{\rho}}_S)\Theta(\widetilde{\overline{\overline{\sigma}}}'_S, \overline{\overline{\rho}}_S; \widetilde{\overline{\sigma}}'_S, \overline{\rho}_S).$$

The point $(\widetilde{\overline{\sigma}}_S, \overline{\rho}_S)$ has now been fixed, but we are free to let $(\widetilde{\overline{\overline{\sigma}}}'_S, \overline{\overline{\rho}}_S)$ approach $(\widetilde{d}'_S, c_S)$. We see that $\Theta(\widetilde{\overline{\sigma}}'_S, \rho_S)$ extends to a smooth function of $(\widetilde{\sigma}'_S, \rho_S)$ in a neighbourhood of $(\widetilde{d}'_S, c_S)$ in $(\widetilde{T}'_S, T_S)$, whose value

$$\Theta(x, c_{\mathbb{A}}) = \lim_{(\widetilde{\sigma}'_S, \rho_S)\to(\widetilde{d}'_S, c_S)} \Theta(\widetilde{\sigma}'_S, \rho_S) \tag{4.1}$$

at $(\widetilde{d}'_S, c_S)$ is independent of T'_S and T_S. (The language is slightly careless, since the domain of $\Theta(\widetilde{\sigma}'_S, \rho_S)$ is a proper subset of $\widetilde{T}'_S \times T_{c_S}$. To be more precise, we have first to take $\widetilde{\sigma}'_S$ to be in a neighbourhood of $\widetilde{d}'_S$ in $\widetilde{T}'_S$, and then let ρ_S range over a corresponding neighbourhood of c_S in T_{c_S} in such a way that $\{\sigma'_S\}$ is a smoothly varying family of images of $\{\rho_S\}$. The assertion is that the resulting function of $\widetilde{\sigma}'_S$ is smooth.)

Consider the special case that $c_{\mathbb{A}}$ is the diagonal image of an element c in $G(F)$. In this case, we choose an inner twist

$$\psi_c = \mathrm{Int}(g^*)\psi : \ G_c \longrightarrow G^*_d, \qquad g^* \in G^*_{\mathrm{sc}}(\overline{F}),$$

that maps c to d, and we set $\psi_{c_v} = \psi_c$, for each $v \in S$. If $v \notin S$, G_{c_v} is unramified, and $K_{c_v} = K_v \cap G_{c_v}(F_v)$ is a hyperspecial maximal compact subgroup of $G_{c_v}(F_v)$ [K3, Proposition 7.1]. We can therefore take the canonical preassigned value

$$\Delta_c(\widetilde{\overline{\sigma}}'_S, \overline{\rho}_S) = \Delta_{c_S}(\widetilde{\overline{\sigma}}'_S, \overline{\rho}_S) = d_{c_S}(\overline{\sigma}', \overline{\rho})^{-1} \prod_{v \notin S} \Delta_{K_{c_v}}(\widetilde{\overline{\sigma}}'_v, \overline{\rho}_v)^{-1}$$

in the absolute transfer factor

$$\Delta_c(\widetilde{\sigma}'_S, \rho_S) = \Delta_{c_S}(\widetilde{\sigma}'_S, \rho_S).$$

With this preassigned value, we set

$$\Theta(x, c) = \Theta(x, c_{\mathbb{A}}).$$

Lemma 4.1. *(i) The action of $\mathcal{K}_y$ on the fibre of y satisfies*

$$\Theta(x^\kappa, c_{\mathbb{A}}) = \langle \mathrm{obs}(c_{\mathbb{A}}), \kappa \rangle^{-1} \Theta(x, c_{\mathbb{A}}), \qquad \kappa \in \mathcal{K}_y.$$

(ii) Suppose that $\mathrm{obs}(c_{\mathbb{A}}) = 1$. *Then* $\Theta(x, c) = 1$, *for any element* $c \in G(F)$ *that is* $G(\mathbb{A})$*-conjugate to* $c_{\mathbb{A}}$.

Proof. We can certainly arrange that the representative (G', d') of x maps directly to the representative (d', G'_d) of y, as in the preamble to Proposition 3.1. In particular, we fix an admissible embedding ρ': $T' \to T$, of a maximal torus $T' \subset G'$ over F to a maximal torus $T \subset G^*$ over F, as well as dual admissible L-embeddings η'_d: ${}^L T \to {}^L G^*_d$ and η: ${}^L T \to {}^L G$, which satisfy the conditions of §3. We can then represent the action $\mathcal{K}_y$ on the fibre of y by the correspondence $(G', d') \to (G^\kappa, d^\kappa)$.

We can assume that the rational tori T' and T transfer locally to each of the groups G_v. For each v, T is then a local image at v of a fixed maximal torus $T_{c_v} \subset G_{c_v}$, which is defined over F_v. The limit (4.1) is independent of the original maximal tori. In particular, we can evaluate it by taking σ'_S and ρ_S to be in the tori $T'_S = T'(F_S)$ and $T_{c_S} = \prod\limits_v T_{c_v}(F_v)$ we have just fixed. The point is that it is easier to study the quotient

$$\Theta(\widetilde{\sigma}'_S, \rho_S) = \Delta(\sigma'_S, \rho_S) \Delta_{c_S}(\widetilde{\sigma}'_S, \rho_S)^{-1}$$

when σ'_S lies in a torus that is defined over F.

The notation suggests that the limit (4.1) is independent of S. To verify this, we observe that if S is augmented by an unramified place v, the quotient $\Theta(\widetilde{\sigma}'_S, \rho_S)$ is changed by a factor

$$\Theta_{K_v}(\widetilde{\sigma}'_v, \rho_v) = \Delta_{K_v}(\sigma'_v, \rho_v)\Delta_{K_{c_v}}(\widetilde{\sigma}'_v, \rho_v)^{-1}.$$

Since

$$\lim_{(\widetilde{\sigma}_v, \rho_v)\to(\widetilde{d}_v, c_v)} \Theta_{K_v}(\widetilde{\sigma}'_v, \rho_v) = 1,$$

by [H, Lemma 8.1], the limit (4.1) is indeed independent of S. We are therefore free to enlarge the finite set S in our computation of the limit.

With the preliminary remarks out of the way, let us consider the numerator $\Delta(\sigma'_S, \rho_S)$ of the quotient $\Theta(\widetilde{\sigma}'_S, \rho_S)$. We shall examine the factors in the product

$$\Delta(\sigma'_S, \rho_S) = \Delta(\sigma'_S, \rho_S; \overline{\sigma}'_S, \overline{\rho}_S) \cdot d(\overline{\sigma}', \overline{\rho})^{-1} \cdot \prod_{v \notin S} \Delta_{K_v}(\overline{\sigma}'_v, \overline{\rho}_v)^{-1}.$$

The factors in the infinite product on the right are almost all equal to 1 [LS1, Corollary 6.4.B]. Enlarging S, if necessary, we can assume that the infinite product itself is equal to 1. The relative factor on the left is a product

$$\Delta(\sigma'_S, \rho_S; \overline{\sigma}'_S, \overline{\rho}_S) = \prod_{\iota} \Delta_\iota(\sigma'_S, \rho_S; \overline{\sigma}'_S, \rho_S)$$

of four terms, in which ι ranges over indices I, II, 1 and 2, as in [LS1, (3.7)]. If ι equals one of the three indices I, II or 2, the corresponding term is by definition a quotient

$$\Delta_\iota(\sigma'_S, \rho_S; \overline{\sigma}'_S, \overline{\rho}_S) = \Delta_\iota(\sigma'_S, \rho_S)\Delta_\iota(\overline{\sigma}'_S, \overline{\rho}_S)^{-1}$$

of absolute factors, whose denominator $\Delta_\iota(\overline{\sigma}'_S, \overline{\rho}_S)$ depends only on the rational element $\overline{\sigma}' \in \overline{T}'(F)$. Enlarging S if necessary, we deduce from Lemma 6.4.A of [LS1] that

$$\Delta_\iota(\sigma'_S, \rho_S; \overline{\sigma}'_S, \overline{\rho}_S) = \Delta_\iota(\sigma'_S, \rho_S), \qquad \iota = \mathrm{I}, \mathrm{II}, 2.$$

The fourth term $\Delta_1(\sigma'_S, \rho_S; \overline{\sigma}'_S, \overline{\rho}_S)$ is a product over $v \in S$ of the local relative factors defined in [LS1, (3.4)]. However, the term itself is really a global object. It is subject to the argument on pp. 267–268 of [LS1]. The argument was presented for the case that σ'_S comes from an element in $T'(F)$, but it holds more generally, so long as the torus T' is defined over F. We obtain

$$\Delta_1(\sigma'_S, \rho_S; \overline{\sigma}'_S, \overline{\rho}_S) = d(\sigma'_S, \rho_S)^{-1} d(\overline{\sigma}'_S, \overline{\rho}_S),$$

where $d(\sigma'_S, \rho_S)$ is defined as a global Tate-Nakayama pairing on the torus T_{sc} by an obvious extension of the construction on p. 267 of [LS1]. The factor $d(\overline{\sigma}'_S, \overline{\rho}_S)$ is defined the same way. However, the arguments $\overline{\sigma}'_S$ and $\overline{\rho}_S$ in this case come from the adelic points $\overline{\sigma}'$ and $\overline{\rho}$. If S is sufficiently large, $d(\overline{\sigma}'_S, \overline{\rho}_S)$ equals $d(\overline{\sigma}', \overline{\rho})$, and therefore cancels the middle factor from the original product. We conclude that

$$\Delta(\sigma'_S, \rho_S) = \Delta_{\mathrm{I}}(\sigma'_S, \rho_S)\Delta_{\mathrm{II}}(\sigma'_S, \rho_S)\Delta_2(\sigma'_S, \rho_S)d(\sigma'_S, \rho_S)^{-1}. \tag{4.2}$$

A similar discussion applies to the denominator $\Delta_{c_S}(\widetilde{\sigma}'_S, \rho_S)$ of $\Theta(\widetilde{\sigma}'_S, \rho_S)$, even though G_{c_S} does not have to come from a group that is defined over F. This is because the points $\widetilde{\sigma}'_S$ and $\widetilde{\overline{\sigma}}'_S$ lie in maximal tori in $\widetilde{G}'_d$ that are each defined over F. We deduce that $\Delta_{c_S}(\widetilde{\sigma}'_S, \rho_S)$ has a product decomposition that is completely parallel to (4.2), except for a multiplicative constant C_S that depends on the preassigned value $\Delta_{c_S}(\overline{\sigma}'_S, \overline{\rho}_S)$. It follows that

$$\Theta(\widetilde{\sigma}'_S, \rho_S) = C_S \Theta_{\mathrm{I}}(\widetilde{\sigma}'_S, \rho_S)\Theta_{\mathrm{II}}(\widetilde{\sigma}'_S, \rho_S)\Theta_2(\widetilde{\sigma}'_S, \rho_S)d(\sigma'_S, \rho_S)^{-1} d_{c_S}(\widetilde{\sigma}'_S, \rho_S),$$

where

$$\Theta_\iota(\widetilde{\sigma}'_S, \rho_S) = \Delta_\iota(\sigma'_S, \rho_S)\Delta_{c_S,\iota}(\widetilde{\sigma}'_S, \rho_S)^{-1}, \qquad \iota = \mathrm{I}, \mathrm{II}, 2.$$

If c_S has a rational representative c, and $\psi_{c_S} = \psi_c$ as above, the constant C_S equals 1.

We shall show that for $\iota = $ I, II or 2, we can assume that the formula

$$\lim_{(\widetilde{\sigma}'_S, \rho_S) \to (\widetilde{d}'_S, c_S)} \Theta_\iota(\widetilde{\sigma}'_S, \rho_S) = 1 \tag{4.3}$$

holds. The three functions to be considered in (4.3) are each independent of the point ρ_S. They do depend on choices of a-data $\{a_{\alpha,S}\}$ and χ-data $\{\chi_{\alpha,S}\}$, but we can assume that these objects have been chosen relative to the global field F, as in [LS1, (2.6.5)]. The first function depends only on the admissible embedding $\rho'\colon T' \to T$ of rational tori, rather than the point $\widetilde{\sigma}'_S$. The relevant part of the proof of [LS1, Theorem 6.4.A] tells us that

$$\Delta_I(\sigma'_S, \rho_S) = \Delta_{c_S,I}(\widetilde{\sigma}'_S, \rho_S) = 1,$$

so long as S is sufficiently large. Therefore (4.3) holds if $\iota = I$. For the second function, we observe as on p. 504 of [LS2] that

$$\Theta_{\mathrm{II}}(\widetilde{\sigma}'_S, \rho_S) = \prod_\alpha \chi_{\alpha,S}\Big(\frac{\alpha(\sigma^*_S) - 1}{a_{\alpha,S}}\Big),$$

where $\sigma^*_S = \rho'(\sigma'_S)$ is the image of $\widetilde{\sigma}'_S$ in T_S, and α runs over the Γ-orbits of roots of (G^*, T) that are not roots of G^*_d and are not coroots of $\widehat{G}'$ (relative to the embedding $\eta\colon {}^LT \to {}^LG$). For each α, χ_α is a character on $\mathbb{A}^*_{F_\alpha}/F^*_\alpha$, for a finite Galois extension $F_\alpha \supset F$ such that the quotient $\big(\alpha(d) - 1\big)a_\alpha^{-1}$ belongs to F^*_α. Enlarging S if necessary, we obtain

$$\lim_{\widetilde{\sigma}'_S \to \widetilde{d}'_S} \chi_\alpha\Big(\frac{\alpha(\sigma^*_S) - 1}{a_{\alpha,S}}\Big) = \chi_\alpha\Big(\frac{\alpha(d_S) - 1}{a_{\alpha,S}}\Big) = \chi_\alpha\Big(\frac{\alpha(d) - 1}{a_\alpha}\Big) = 1.$$

Therefore (4.3) also holds if $\iota = \mathrm{II}$. The third function equals

$$\Theta_2(\widetilde{\sigma}'_S, \rho_S) = \langle a_\chi, \sigma^*_S\rangle \langle a_{d,\chi}, \widetilde{\sigma}^*_S\rangle^{-1},$$

where $a_\chi \in H^1(W_F, \widehat{T})$ is the element constructed from the χ-data and the embedding ${}^LG' \xrightarrow{\sim} \mathcal{G}' \hookrightarrow {}^LG$, as in [LS1, (3.5)], and $a_{d,\chi}$ is the corresponding element for G^*_d. (We have written $\widetilde{\sigma}^*_S$ for the image of $\widetilde{\sigma}'_S$ in the z-extension $\widetilde{G}^*_d$ of G^*_d used to construct G'_d.) Since $\langle a_\chi, \cdot\rangle$ stands for a character on $T(\mathbb{A})/T(F)$, we obtain

$$\lim_{\widetilde{\sigma}'_S \to \widetilde{d}'_S} \langle a_\chi, \sigma^*_S\rangle = \langle a_\chi, d_S\rangle = \langle a_\chi, d\rangle = 1,$$

so long as S is sufficiently large. A similar formula holds for the limit of $\langle a_{d,\chi}, \widetilde{\sigma}_S^*\rangle$. We conclude therefore that (4.3) is valid for $\iota = 2$.

We have shown that if S is sufficiently large, (4.3) is valid for each of the three indices $\iota = \mathrm{I}, \mathrm{II}, 2$. It follows that the limit (4.1) equals

$$\Theta(x, c_{\mathbb{A}}) = C_S \lim_{(\widetilde{\sigma}'_S, \rho_S)\to(\widetilde{d}'_S, c_S)} d(\sigma'_S, \rho_S)^{-1} d_{c_S}(\widetilde{\sigma}'_S, \rho_S). \tag{4.4}$$

We shall use this formula to establish the two assertions of the lemma.

For the first assertion, we fix an element $\kappa \in \mathcal{K}_y$. Then x^κ is the point in the fibre of y that is represented by the pair (G^κ, d^κ) constructed in §3. Recall that G^κ comes with a maximal torus T^κ that contains d^κ, together with an admissible embedding ρ^κ of T^κ into G_d^* that takes (T^κ, d^κ) to (T, d). Suppose that $\sigma'_S \in T'_S$ is a given point, with image σ_S^* in T_S, as above. We then take σ_S^κ to be the preimage of σ_S^* in T_S^κ. Since $d_{c_S}(\widetilde{\sigma}_S^\kappa, \rho_S)$ depends on κ only through the endoscopic datum G_d^κ for G_d^*, and since (d, G'_d) and (d, G_d^κ) represent the same point y in $\mathcal{Y}(G)$, we see that

$$d_{c_S}(\widetilde{\sigma}_S^\kappa, \rho_S) = d_{c_S}(\widetilde{\sigma}'_S, \rho_S).$$

To compare the factors $d(\sigma_S^\kappa, \rho_S)$ and $d(\sigma'_S, \rho_S)$, we recall that the factor $d(\sigma'_S, \rho_S)$ is defined by a global Tate-Nakayama pairing on T_{sc}. It equals $\langle \mu_T, \overline{s}'_T\rangle$, where as in [LS1, (6.3)], $\mu_T = \mu_T(\sigma_S^*, \rho_S)$ is a class in $H^1\big(\Gamma, T_{\mathrm{sc}}(\overline{\mathbb{A}})/T_{\mathrm{sc}}(\overline{F})\big)$ attached to $\sigma_S^* = \rho'(\sigma'_S)$ and ρ_S, and $\overline{s}'_T$ is the image of s'_T in the group $\pi_0(\widehat{T}_{\mathrm{ad}}^\Gamma) = \pi_0\big((\widehat{T}/Z(\widehat{G}))^\Gamma\big)$. We have written $\overline{\mathbb{A}}$ here for the ring of adeles of $\overline{F}$. Since σ_S^* also equals $\rho^\kappa(\sigma_S^\kappa)$, $d(\sigma_S^\kappa, \rho_S)$ equals $\langle \mu_T, \overline{s}_T^\kappa\rangle$. But according to the definitions in §3, $\overline{s}_T^\kappa$ equals $\overline{s}'_T\kappa$. It follows from (4.4) that

$$\Theta(x^\kappa, c_{\mathbb{A}})\Theta(x, c_{\mathbb{A}})^{-1} = \lim_{(\widetilde{\sigma}'_S, \rho_S)\to(\widetilde{d}'_S, c_S)} \langle \mu_T(\sigma_S^*, \rho_S), \kappa\rangle^{-1}. \tag{4.5}$$

The precise definition of $\mu_T(\sigma_S^*, \rho_S)$ is a minor extension of that on p. 267 of [LS1]. To describe it, we write $\overline{\mathbb{A}}_S$ for the subring of adeles in $\overline{\mathbb{A}}$ that are supported at the places of $\overline{F}$ over S. Let h_S be a point in $G_{\mathrm{sc}}^*(\overline{\mathbb{A}}_S)$ such that

$$h_S \psi(\rho_S) h_S^{-1} = \sigma_S^*.$$

Then $\mu_T(\sigma_S^*, \rho_S)$ is defined to be the image in $H^1\big(\Gamma, T_{\mathrm{sc}}(\overline{\mathbb{A}})/T_{\mathrm{sc}}(\overline{F})\big)$ of the 1-chain

$$v_S(\tau) = h_S u(\tau)\tau(h_S)^{-1}, \qquad \tau \in \Gamma,$$

from Γ to $T_{\mathrm{sc}}(\overline{\mathbb{A}}_S)$, for any points in $u(\tau)$ in $G^*_{\mathrm{sc}}(\overline{F})$ such that $\psi\tau(\psi)^{-1}$ equals $\mathrm{Int}\big(u(\tau)\big)$. This definition is quite similar to the construction of the element $\mathrm{obs}(c_{\mathbb{A}})$ in [K3, §6]. In the case of $\mathrm{obs}(c_{\mathbb{A}})$, the role of the torus

$$T_{\mathrm{sc}} = G^*_{\mathrm{sc}} \cap T = G^*_{\mathrm{der}} \cap T$$

is played by the preimage

$$G^*_{\mathrm{sc},d} = G^*_{\mathrm{sc}} \cap G^*_d = G^*_{\mathrm{der}} \cap G^*_d$$

of G^*_d in G^*_{sc}, while the role of (σ^*_S, ρ_S) is played by $(d, c_{\mathbb{A}})$. The construction in [K3] then exhibits $\mathrm{obs}(c_{\mathbb{A}})$ as a class in $H^1\big(\Gamma, G^*_{\mathrm{sc},d}(\overline{\mathbb{A}})/Z^*_{\mathrm{sc},d}(\overline{F})\big)$, where $Z^*_{\mathrm{sc},d}$ denotes the center of $G^*_{\mathrm{sc},d}$. It is the pairing of this cohomology set with $\pi_0\Big(\big(Z(\widehat{G}^*_d)/Z(\widehat{G})\big)^{\Gamma}\Big)$, provided by [K3, Theorem 2.2], that defines $\mathrm{obs}(c_{\mathbb{A}})$ as a character on $\mathcal{K}_y$. Now the 1-chain $v_S(\tau)$ above is a cocycle modulo translation by the center of G^*_{sc}. It projects to a class in $H^1\big(\Gamma, T_{\mathrm{sc}}(\overline{\mathbb{A}})/Z^*_{\mathrm{sc},d}(\overline{F})\big)$, which can be mapped in turn to a class in $H^1\big(\Gamma, G^*_{\mathrm{sc},d}(\overline{\mathbb{A}})/Z^*_{\mathrm{sc},d}(\overline{F})\big)$. The pairing of the latter class with κ is equal to the pairing that occurs on the right hand side of (4.5), which is between the original class μ_T and the image of κ in $\pi_0(\widehat{T}^{\Gamma}_{\mathrm{ad}})$. (This follows from the functoriality assertion in [K3, Theorem 2.2], and the remark at the top of p. 370 of [K3].) A review of the definition in [K3, §6] of $\mathrm{obs}(c_{\mathbb{A}})$ then leads directly to the conclusion that

$$\lim_{(\widetilde{\sigma}'_S, \rho_S) \to (\widetilde{d}'_S, c_S)} \langle \mu_T(\sigma^*_S, \rho_S), \kappa\rangle^{-1} = \langle \mathrm{obs}(c_{\mathbb{A}}), \kappa\rangle^{-1},$$

so long as S is sufficiently large. The assertion (i) of the lemma follows from (4.5).

For the assertion (ii), we assume that $\mathrm{obs}(c_{\mathbb{A}}) = 1$. Then there is an element $c \in G(F)$ that is $G(F)$-conjugate to $c_{\mathbb{A}}$. We fix an inner twist $\psi_c = \mathrm{Int}(g^*)\psi$ from G_c to G^*_d, as in the preamble to the lemma. The constant C_S is then equal to 1. It follows from (4.4) that

$$\begin{aligned}\Theta(c,x) &= \lim\big(d(\sigma'_S,\rho_S)^{-1}d_c(\widetilde{\sigma}'_S,\rho_S)\big)\\ &= \lim\big(\langle\mu_T,\overline{s}'_T\rangle^{-1}\langle\mu_{c,T},\overline{s}'_{d,T}\rangle\big).\end{aligned}$$

The class $\mu_T = \mu_T(\sigma^*_S,\rho_S)$ was defined in terms of the elements h_S and $u(\tau)$ above. The class $\mu_{c,T} = \mu_{c,T}(\widetilde{\sigma}^*_S,\rho_S)$ is defined in exactly the same way, except that the role of G and ψ has to be played by G_c and ψ_c. In particular, $\mu_{c,T}$ takes values in the preimage of T in $G^*_{d,\mathrm{sc}}$, rather than its preimage T_{sc} in G^*_{sc}. Now, according to the definitions in §3, $s'_{d,T}$ and s'_T represent the same point in $\widehat{T}$. Moreover, $\overline{s}'_{d,T}$ equals the image of $\overline{s}'_T$ under the map from $\widehat{T}/Z(\widehat{G})$ to $\widehat{T}/Z(\widehat{G}^*_d)$. Therefore

$$\langle\mu_{c,T},\overline{s}'_{d,T}\rangle = \langle\overline{\mu}_{c,T},\overline{s}'_T\rangle,$$

where $\overline{\mu}_{c,T}$ is the image of $\mu_{c,T}$ in $H^1\big(\Gamma, T_{\mathrm{sc}}(\mathbb{A})/T_{\mathrm{sc}}(\overline{F})\big)$ under the map from $G^*_{d,\mathrm{sc}}$ to G^*_{sc}. Set $h_{S,c} = h_S(g^*)^{-1}$, where h_S is the element used above to construct μ_T. It is an immediate consequence of the definitions that

$$h_{S,c}\psi_c(\rho_S)h^{-1}_{S,c} = \sigma^*_S,$$

and that

$$\psi_c\tau(\psi_c)^{-1} = \mathrm{Int}\big(g^*u(\tau)\tau(g^*)^{-1}\big), \qquad \tau\in\Gamma.$$

It follows that $\overline{\mu}_{c,T} = \mu_T$, from which we conclude that $\Theta(c,x) = 1$. This completes the proof of the remaining assertion of the lemma. □

The descent theorem of Langlands and Shelstad will be a major part of our proof of Theorem 1.1. We shall actually use a minor variant of the theorem, which applies to the extended transfer factors $\Delta_G(\dot{\delta}'_S,\dot{\gamma}_S)$ in the definition (1.1). Assume (d,G'_d), (G',d') and c_S are as at the beginning of the section. We fix an element $\dot{\gamma}_S$ in $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G,S)$, with Jordan decomposition $\dot{\gamma}_S = c_S\dot{\alpha}_S$, as in assertion (a) of Theorem 1.1. We also fix an element $\dot{\delta}'_S$ in $\Delta_{\mathrm{ell}}(G',S)$, with Jordan decomposition $\dot{\delta}'_S = d'_S\dot{\beta}'_S$, where d'_S is the image of d' in $G'(F_S)$. Since $\widetilde{G}' = G'$, we can form the extended transfer factor $\Delta_G(\dot{\delta}'_S, \dot{\gamma}_S)$. The element $\dot{\beta}'_S$ belongs a priori to the basis $\Delta_{\text{unip}}(G'_{d'_S})$. Suppose that it actually lies in the subset $\Delta_{\text{unip,der}}(G'_{d'_S})$ of this basis. As we noted before stating Corollary 2.3, there is a canonical bijection from $\Delta_{\text{unip,der}}(G'_{d'_S})$ onto the corresponding subset $\Delta_{\text{unip,der}}(\widetilde{G}'_{d'_S}, \widetilde{\eta}'_S)$ of $\Delta_{\text{unip}}(\widetilde{G}'_{d'_S}, \widetilde{\eta}'_S)$. Identifying $\dot{\beta}'_S$ with its image in $\Delta_{\text{unip,der}}(\widetilde{G}'_{d'_S}, \widetilde{\eta}'_S)$, we can also form the extended transfer factor $\Delta_{G,c_S}(\dot{\beta}'_S, \dot{\alpha}_S)$ for G_{c_S}.

Lemma 4.2. *Assume that $\dot{\beta}'_S$ lies in the subset $\Delta_{\text{unip,der}}(G'_{d'_S})$ of $\Delta_{\text{unip}}(G'_{d'_S})$. Then*

$$\Delta_G(\dot{\delta}'_S, \dot{\gamma}_S) = \Theta(x, c_{\mathbb{A}})\Delta_{G,c_S}(\dot{\beta}'_S, \dot{\alpha}_S) \tag{4.6}$$

Proof. We can write

$$\Theta(x, c_{\mathbb{A}}) = \prod_{v \in S} \Theta(x, c_v),$$

where

$$\Theta(x, c_v) = \lim_{(\widetilde{\sigma}'_v, \rho_v) \to (\widetilde{d}'_v, c_v)} \Theta(\widetilde{\sigma}'_v, \rho_v)$$

is a limit of the v-component

$$\Theta(\widetilde{\sigma}'_v, \rho_v) = \Delta(\sigma'_v, \rho_v)\Delta_{c_v}(\widetilde{\sigma}'_v, \rho_v)^{-1}$$

of $\Theta(\widetilde{\sigma}'_S, \rho_S)$. This is of course because the descent theorem of [LS2] applies to each v. The theorem asserts that the quotient $\Theta(\widetilde{\sigma}'_v, \rho_v)$ extends to a smooth function of $(\widetilde{\sigma}'_v, \rho_v)$ in a neighbourhood of $(\widetilde{d}'_v, c_v)$, and that the limit $\Theta(x, c_v)$ is independent of the choice of the tori $\widetilde{T}'_v$ and T_v that are the domains of $\widetilde{\sigma}'_v$ and ρ_v.

The local components $\dot{\gamma}_v = c_v\dot{\alpha}_v$ and $\widetilde{\dot{\delta}}'_v = \widetilde{d}'_v\dot{\beta}'_v$ are of course not assumed to be semisimple. However, we can approximate them by the strongly G-regular elements ρ_v and σ'_v. If v is a p-adic place, the function $\Theta(\widetilde{\sigma}'_v, \rho_v)$ is actually constant for $(\widetilde{\sigma}'_v, \rho_v)$ in a neighbourhood of $(\widetilde{d}'_v, c_v)$. It is then not hard to establish from Harish-Chandra's theory of descent for smooth functions [LS2, (1.5)] that

$$\Delta_G(\dot{\delta}'_v, \dot{\gamma}_v) = \Theta(x, c_v)\Delta_{G,c_v}(\widetilde{\dot{\delta}}'_v, \dot{\gamma}_v). \tag{4.7}$$

If v is archimedean, $\Theta(\widetilde{\sigma}'_v, \rho_v)$ need not be constant near $(\widetilde{d}'_v, c_v)$. However, the dependence on $(\widetilde{\sigma}'_v, \rho_v)$ is quite mild. Of the four terms in [LS1, (3.2)-(3.5)] that make up the product $\Delta(\sigma'_v, \rho_v)$, it is only the factors $\Delta_{\mathrm{II}}(\sigma'_v, \rho_v)$ and $\Delta_2(\sigma'_v, \rho_v)$ that are not locally constant. An examination of the definitions in [LS1, (3.3), (3.5)] reveals that the product

$$\Delta_{\mathrm{II}}(\sigma'_v, \rho_v)\Delta_2(\sigma'_v, \rho_v)$$

depends only on the image of σ'_v in $G'_v/G'_{\mathrm{der},v}$. Since the corresponding terms in $\Delta_{c_v}(\widetilde{\sigma}'_v, \rho_v)$ have a similar property, the quotient $\Theta(\widetilde{\sigma}'_v, \rho_v)$ then depends only on the image of $\widetilde{\sigma}'_v$ in $\widetilde{G}'_{d'_v}/\widetilde{G}'_{d'_v,\mathrm{der}}$. We are assuming that $\dot{\beta}'_v$ lies in the image of $\Delta_{\mathrm{unip}}(\widetilde{G}'_{d'_v,\mathrm{der}})$ in $\Delta_{\mathrm{unip}}(\widetilde{G}'_{d'_v}, \widetilde{\eta}'_v)$. This condition can therefore be used in conjunction with Harish-Chandra descent to show that (4.7) holds in the archimedean case as well. (See the forthcoming paper [A5].)

Taking the product of the terms in (4.7) over $v \in S$, we obtain

$$\Delta_G(\dot{\delta}'_S, \dot{\gamma}_S) = \Theta(x, c_{\mathbb{A}})\Delta_{G,c_S}(\widetilde{d}'_S\dot{\beta}'_S, c_S\dot{\alpha}_S).$$

To remove the arguments $\widetilde{d}'_S$ and c_S from the factor on the right, we appeal to Lemma 3.5.A of [LS2]. This result implies that

$$\Delta_{G,c_S}(\widetilde{d}'_S\dot{\beta}'_S, c_S\dot{\alpha}_S) = \widetilde{\eta}'_S(\widetilde{d}'_S)\Delta_{G,c_S}(\dot{\beta}', \dot{\alpha}_S),$$

where $\widetilde{\eta}'_S = \prod_{v\in S} \widetilde{\eta}'_v$ is a character on the center of $\widetilde{G}^*_{d_S}$ that restricts to the earlier character on $\widetilde{C}'_S$. We recall here that $\widetilde{G}^*_d$ is the fixed z-extension of G^*_d used to construct the extension $\widetilde{G}'_d$ of G'_d. We also recall that for any v, $\widetilde{\eta}'_v$ is the restriction to the center of $\widetilde{G}^*_{d_v}$ of the character

$$\lambda_v(\widetilde{\sigma}'_v) = \Delta_{c_v,2}(\widetilde{\sigma}'_v, \rho_v), \qquad \widetilde{\sigma}'_v \in \widetilde{T}'_v,$$

where $\widetilde{T}'_v$ is a maximal torus in $\widetilde{G}'_{d_v}$ over F_v, and $\Delta_{c_v,2}$ is the component in [LS1, (3.5)] of the transfer factor for (G'_{d_v}, G_{c_v}). The point of [LS2, Lemma 3.5.A] was to show that $\widetilde{\eta}'_v$ is independent of the choice of $\widetilde{T}'_v$. It remains for us to check that $\widetilde{\eta}'_S(\widetilde{d}'_S)$ equals 1.

The factor $\Delta_{c_v,2}$ is defined in terms of the local embedding $\widetilde{\xi}'_{d_v}\colon {}^L\widetilde{G}'_{d_v} \to {}^L\widetilde{G}^*_{d_v}$, and the admissible embeddings ${}^L\widetilde{T}'_v \to {}^L\widetilde{G}'_{d_v}$ and ${}^L\widetilde{T}'_v \to {}^L\widetilde{G}^*_{d_v}$ determined by local χ-data [LS1, (2.5)]. Since $\widetilde{\eta}'_v$ is independent of $\widetilde{T}'_v$, we can take $\widetilde{T}'_v$ to be the localization of a maximal torus $\widetilde{T}'$ in $\widetilde{G}'_d$ over F. We can also arrange that the admissible embeddings are localizations of global embeddings ${}^L\widetilde{T}' \to {}^L\widetilde{G}'_d$ and ${}^L\widetilde{T}' \to {}^L\widetilde{G}_d$, as in [LS1, (2.6.5)]. Since $\widetilde{\xi}'_{d_v}$ is the localization of the global embedding $\widetilde{\xi}'_d$, λ_v is the local component of a character λ on $\widetilde{T}'(F)\backslash\widetilde{T}'(\mathbb{A})$. It follows that

$$\widetilde{\eta}'_S(\widetilde{d}'_S) = \lambda_S(\widetilde{d}'_S) = \lambda(\widetilde{d}')\Big(\prod_{v\notin S}\lambda_v(\widetilde{d}'_v)^{-1}\Big) = \prod_{v\notin S}\widetilde{\eta}'_v(\widetilde{d}'_v)^{-1}.$$

We have assumed that $\widetilde{G}^*_d$, $\widetilde{G}'_d$ and $\widetilde{\xi}'_d$ are unramified at any v not in S. For any such v, we now take $\widetilde{T}'_v$ to be an unramified torus, equipped with unramified local χ-data. The corresponding character λ_v on $\widetilde{T}'_v$ is then unramified. We also assumed earlier in this section that $\widetilde{d}'$ was bounded at any $v \notin S$. In other words, $\widetilde{d}'_v$ lies in a compact subgroup of the unramified torus $\widetilde{T}'_v$. It follows that

$$\widetilde{\eta}'_v(\widetilde{d}'_v) = \lambda_v(\widetilde{d}'_v) = 1, \qquad v \notin S.$$

Therefore $\widetilde{\eta}'_S(\widetilde{d}'_S) = 1$, as required. $\square$

5. Transfer factors for G_c

In this section, we shall establish another property of global transfer factors. This result pertains strictly to the transfer factors for groups G_c obtained from G by global descent. Our concern will be the global endoscopic data G'_d for G_c that do not come from endoscopic data for G. We shall establish a key formula for any such G'_d, which will be used in §6 in the form of a vanishing property.

We fix $y \in \mathcal{Y}(G)$ and (d, G'_d) as in §4, but we relax the condition that y lie in the image of the map $\mathcal{X}(G) \to \mathcal{Y}(G)$. We continue to assume that d transfers locally to G. Rather than deal with a finite set S of valuations, however, we simply fix a semisimple element $c_{\mathbb{A}}$ in $G(\mathbb{A})$ of which d is an adelic image. We assume that $\mathrm{obs}(c_{\mathbb{A}}) = 1$. Then we can find a rational element $c \in G(F)$ that is $G(\mathbb{A})$-conjugate to $c_{\mathbb{A}}$. Given c, we define an inner twist $\psi_c\colon\ G_c \to G^*_d$, as in the last section. We assume that G'_d belongs to $\mathcal{E}_{\mathrm{ell}}(G_c)$. Then G'_d represents an elliptic endoscopic datum for G^*_d that is locally relevant to each of the groups G_{c_v}. We shall consider the canonically normalized adelic transfer factors $\Delta_c(\widetilde{\sigma}', \rho_c)$, for strongly G-regular elements $\widetilde{\sigma}' \in \widetilde{G}'_d(\mathbb{A})$ and $\rho_c \in G_c(\mathbb{A})$ such that the projection σ' of $\widetilde{\sigma}'$ onto $G'_d(\mathbb{A})$ is an adelic image of ρ_c.

The conjugacy class of c in $G(F)$ is not uniquely determined by $c_{\mathbb{A}}$. Are the transfer factors $\Delta_c(\cdot, \cdot)$ independent of c? The question will be of considerable interest to us when we turn to the proof of Theorem 1.1 in the next section. For we shall encounter some extraneous terms that will be impossible to suppress unless the transfer factors do actually vary with c. But the isomorphism class of the group $G_c(\mathbb{A})$ is independent of c. What mechanism could cause the corresponding transfer factors to vary?

To examine the question, we first recall how to classify the set of rational conjugacy classes $c \in \Gamma_{\mathrm{ss}}(G)$ that map to the $G(\mathbb{A})$-conjugacy class of $c_{\mathbb{A}}$. Fix one such class c_1, and set $G_1 = G_{c_1}$. If c is another such class, we can find an element $g \in G(\overline{F})$ such that

$$c_1 = \mathrm{Int}(g)c.$$

Since $G_{\rm der}$ is simply connected, the cocycle

$$\tau \longrightarrow g\tau(g)^{-1}, \qquad \tau \in \Gamma,$$

takes values in G_1. Since c and c_1 both lie in the $G(\mathbb{A})$-conjugacy class of $c_{\mathbb{A}}$, the image λ_c of this cocycle in $H^1(F, G_1)$ lies in the subset $\ker^1(F, G_1)$ of locally trivial classes. The map $c \to \lambda_c$ is then a bijection from the set of rational conjugacy classes $c \in \Gamma_{\rm ss}(G)$ in the $G(\mathbb{A})$-conjugacy class of $c_{\mathbb{A}}$, onto the set

$$\ker\bigl(\ker^1(F, G_1) \longrightarrow \ker^1(F, G)\bigr). \tag{5.1}$$

On the other hand, there is a canonical bijection $\lambda \to \langle \lambda, \cdot \rangle$ from the set (5.1) onto the dual of the finite abelian group

$$\operatorname{coker}\bigl(\ker^1(F, Z(\widehat{G})) \longrightarrow \ker^1\bigl(F, Z(\widehat{G}_1)\bigr)\bigr). \tag{5.2}$$

This follows from the commutative diagram

$$\begin{array}{ccc} \ker^1(F, G_1) & \longrightarrow & \ker^1(F, G) \\ \downarrow & & \downarrow \\ \ker^1\bigl(F, Z(\widehat{G}_1)\bigr)^* & \longrightarrow & \ker^1\bigl(F, Z(\widehat{G})\bigr)^*, \end{array}$$

as in [K3, p. 394], where the vertical arrows are the bijections defined in [K2, §4].

For any c, there is an isomorphism $\rho_1 \to \rho_c$ from $G_1(\mathbb{A})$ onto $G_c(\mathbb{A})$, which is uniquely determined up to conjugacy in $G_1(\mathbb{A})$. Moreover, the inner twist ψ_c allows us to identity the dual group $\widehat{G}_c$ with $\widehat{G}_d$. In particular, we can identity the groups $\ker^1\bigl(F, Z(\widehat{G}_1)\bigr)$ and $\ker^1\bigl(F, Z(\widehat{G}_d)\bigr)$. Recall that the endoscopic datum G'_d for G^*_d determines a class a'_d in $\ker^1\bigl(F, Z(\widehat{G}_d)\bigr)$. We shall sometimes identity this class with its image in the quotient (5.2). According to Proposition 3.1, the point y represented by (d, G'_d) lies in the image of the mapping $\mathcal{X}(G) \to \mathcal{Y}(G)$ if and only if the image of a'_d in (5.2) is trivial.

Lemma 5.1. *The absolute adelic transfer factors for (G_c, G'_d) and (G_1, G'_d) are related by*

$$\Delta_c(\widetilde{\sigma}'_c, \rho_c) = \langle \lambda_c, a'_d \rangle \Delta_{c_1}(\widetilde{\sigma}', \rho_1). \tag{5.3}$$

Proof. We have assumed that the endoscopic datum G'_d is locally relevant to G_c. It is therefore locally relevant to G_1 (since G_c and G_1 are isomorphic over any of the completions F_v). As we noted in [I, §4], this implies that there is a maximal torus T' in G'_d over F that transfers locally to G_1. We fix such a T', as well as an admissible embedding of T' into a maximal torus T in G^*_d over F. We also fix an adelic torus $T_{1,\mathbb{A}}(\mathbb{A})$ in $G_1(\mathbb{A})$ which transfers to $T'(\mathbb{A})$. By this we mean the centralizer of some strongly regular element in $G_1(\mathbb{A})$ that has an adelic image in $T'(\mathbb{A})$. Then $T_{1,\mathbb{A}}(\mathbb{A})$ is a restricted direct product of groups $T_{1,v}(F_v)$, where $T_{1,v}$ is a maximal torus in G_1 over F_v that transfers to T' over F_v.

We can regard $T_{1,\mathbb{A}}$ as a group scheme over the ring $\mathbb{A}$, which is embedded as a subgroup scheme in the product $G_{1,\mathbb{A}} = G_1 \times_F \mathbb{A}$. As in the last section, we shall work with the ring of adeles $\overline{\mathbb{A}}$ of the algebraic closure $\overline{F}$ of F. The Galois group $\Gamma = \mathrm{Gal}(\overline{F}/F)$ acts on $\overline{\mathbb{A}}$, as well as on the group $T_{1,\mathbb{A}}(\overline{\mathbb{A}})$ of points in $T_{1,\mathbb{A}}$ with values in $\overline{\mathbb{A}}$. We note that $T_{1,\mathbb{A}}(\overline{\mathbb{A}})$ is a restricted direct product of groups

$$T_{1,v}(\overline{\mathbb{A}}_v) = \prod_{w|v} T_{1,v}(\overline{F}_w),$$

where w runs over the valuations of $\overline{F}$ that lie above the valuation v of F. The action of Γ on $T_{1,\mathbb{A}}(\overline{\mathbb{A}})$ is then given by the corresponding product of actions of Γ on each of the groups $T_{1,v}(\overline{\mathbb{A}})$. (The arguments that follow will actually depend only on the ring $\mathbb{A}_E$ of adeles of some large finite Galois extension E of F. The reader is free to replace $\overline{\mathbb{A}}$ by $\mathbb{A}_E$, and Γ by the quotient $\Gamma_E = \mathrm{Gal}(E/F)$.)

The lemma pertains to a fixed rational conjugacy class $c \in \Gamma_{\mathrm{ss}}(G)$ that is conjugate to c_1 over $G(\mathbb{A})$. The corresponding class λ_c is the image in (5.1) of the cocycle $g\tau(g)^{-1}$, where g is a fixed element in $G(\overline{F})$ such that $c_1 = \mathrm{Int}(g)c$. From a general result [S, Corollary 5.4] for adjoint groups, it is known that $\ker^1(F, G_{1,\mathrm{ad}})$ is trivial. The image of λ_c in $\ker^1(F, G_{1,\mathrm{ad}})$ therefore splits. Replacing g by a left translate by some element in $G_1(\overline{F})$, if necessary, we can then assume that the cocycle takes values in the center of $G_1(\overline{F})$. The group $G_1(\overline{F})$ is of course contained in $G_1(\overline{\mathbb{A}})$, and the center of $G_1(\overline{F})$ is contained in the subgroup $T_{1,\mathbb{A}}(\overline{\mathbb{A}})$ of $G_1(\overline{\mathbb{A}})$. The cocycle therefore takes values in $T_{1,\mathbb{A}}(\overline{\mathbb{A}})$, and maps to a class in $H^1\big(\Gamma, T_{1,\mathbb{A}}(\overline{\mathbb{A}})\big)$. We claim that this class lies in the subset

$$\ker\Big(H^1\big(\Gamma, T_{1,\mathbb{A}}(\overline{\mathbb{A}})\big) \to H^1\big(\Gamma, G_1(\overline{\mathbb{A}})\big)\Big) \tag{5.4}$$

of $H^1\big(\Gamma, T_{1,\mathbb{A}}(\overline{\mathbb{A}})\big)$. To see this, we write

$$H^1\big(\Gamma, T_{1,\mathbb{A}}(\overline{\mathbb{A}})\big) = \bigoplus_v H^1\big(\Gamma, T_{1,v}(\overline{\mathbb{A}}_v)\big) \cong \bigoplus_v H^1(\Gamma_v, T_{1,v})$$

and

$$H^1\big(\Gamma, G_1(\overline{\mathbb{A}})\big) = \bigoplus_v H^1\big(\Gamma, G_1(\overline{\mathbb{A}}_v)\big) \cong \bigoplus_v H^1(\Gamma_v, G_1),$$

by Shapiro's lemma. (The summands on the right of course depend on a choice of embedding of $\overline{F}$ into $\overline{F}_v$, for each v.) Since the original class λ_c is locally trivial, the class in $H^1\big(\Gamma, T_{1,\mathbb{A}}(\overline{\mathbb{A}})\big)$ projects to the subset

$$\ker\big(H^1(F_v, T_{1,v}) \to H^1(F_v, G_1)\big)$$

of $H^1(F, T_{1,v})$, for each v. The claim follows.

Let $T_{1,\mathbb{A},\mathrm{sc}}$ be the preimage of $T_{1,\mathbb{A}}$ in the scheme $G_{1,\mathrm{sc}} \times_F \mathbb{A}$. The set (5.4) is then the image of a surjective map, whose domain is the set

$$\ker\Big(H^1\big(\Gamma, T_{1,\mathbb{A},\mathrm{sc}}(\overline{\mathbb{A}})\big) \longrightarrow H^1\big(\Gamma, G_{1,\mathrm{sc}}(\overline{\mathbb{A}})\big)\Big).$$

This follows from the fact that $G_{1,\mathrm{sc}}(\overline{\mathbb{A}})$ maps surjectively to $G_1(\overline{\mathbb{A}})/T_{1,\mathbb{A}}(\overline{\mathbb{A}})$. (See the remark near the bottom of p. 381 of [K3].) We can therefore find an element k_{sc} in $G_{1,\mathrm{sc}}(\overline{\mathbb{A}})$ such that the cocycle

$$k_{\mathrm{sc}}^{-1}\tau(k_{\mathrm{sc}}), \qquad \tau \in \Gamma,$$

takes values in $T_{1,\mathbb{A},\mathrm{sc}}(\overline{\mathbb{A}})$, and maps to the same class in (5.4) as the cocycle $g\tau(g)^{-1}$. Let k be the image of k_{sc} in $G_1(\overline{\mathbb{A}})$. Then there is an element t in $T_{1,\mathbb{A}}(\overline{\mathbb{A}})$ such that

$$k^{-1}\tau(k) = g\tau(g)^{-1}t\tau(t)^{-1} = (tg)\tau(tg)^{-1},$$

for any τ. It follows that the point

$$x = g^{-1}t^{-1}k^{-1}$$

in $G(\overline{\mathbb{A}})$ is Γ-invariant. In other words, x belongs to $G(\mathbb{A})$. Since the point $gx = t^{-1}k^{-1}$ belongs to $G_1(\overline{\mathbb{A}})$, $\mathrm{Int}(x)$ maps c_1 to c. The map $\mathrm{Int}(x)$ is a representative of the canonical class of isomorphisms from $G_1(\mathbb{A})$ to $G_c(\mathbb{A})$.

We have to compare transfer factors. Recall that $\widetilde{\sigma}'$ is a G-regular point in $\widetilde{G}'_d(\mathbb{A})$ whose projection σ' onto $G'_d(\mathbb{A})$ is an adelic image of a point ρ_1 in $G_1(\mathbb{A})$. Then σ' is also the image of the point $\rho_c = \mathrm{Int}(x)\rho_1$. To compare the canonical transfer factors $\Delta_c(\widetilde{\sigma}', \rho_c)$ and $\Delta_{c_1}(\widetilde{\sigma}', \rho_1)$, we need to introduce base points. Let $\overline{\sigma}'$ be a fixed G-regular element in $T'(F)$, with preimage $\widetilde{\overline{\sigma}}'$ in $\widetilde{G}'_d(F)$, and with image $\overline{\sigma}^*$ in $T(F)$ under the admissible embedding $T' \to T$. We have assumed that $\overline{\sigma}'$ is an adelic image of some point in $T_{1,\mathbb{A}}(\mathbb{A})$. Rather than fix such a point, however, we shall introduce another adelic torus in $G_1(\mathbb{A})$. The group

$$\overline{T}_{1,\mathbb{A}} = \mathrm{Int}(k_{\mathrm{sc}})T_{1,\mathbb{A}} = \mathrm{Int}(k)T_{1,\mathbb{A}}$$

is a subscheme of $G_{1,\mathbb{A}}$. Since $k^{-1}\tau(k)$ belongs to $T_{1,\mathbb{A}}(\overline{\mathbb{A}})$, for every $\tau \in \Gamma$, $\overline{T}_{1,\mathbb{A}}$ is actually defined over $\mathbb{A}$. We fix a point $\overline{\rho}_1$ in $\overline{T}_{1,\mathbb{A}}$ such that $\overline{\sigma}'$ is an adelic image of $\overline{\rho}_1$. Then $\overline{\sigma}'$ is also an adelic image of the point $\overline{\rho}_c = \mathrm{Int}(x)\overline{\rho}_1$ in $G_c(\mathbb{A})$. The pairs $(\widetilde{\overline{\sigma}}', \overline{\rho}_1)$ and $(\widetilde{\overline{\sigma}}', \overline{\rho}_c)$ will serve as base points for the two transfer factors.

The relative transfer factors are symmetric under the isomorphism $\mathrm{Int}(x)$ from $G_1(\mathbb{A})$ to $G_c(\mathbb{A})$. It follows from the definitions in [I, §4] that

$$\begin{aligned}
\Delta_c(\widetilde{\sigma}', \rho_c) &= \Delta_c(\widetilde{\sigma}', \rho_c; \widetilde{\overline{\sigma}}', \overline{\rho}_c) d(\overline{\sigma}', \overline{\rho}_c)^{-1} \\
&= \Delta_{c_1}(\widetilde{\sigma}, \rho_1; \widetilde{\overline{\sigma}}', \overline{\rho}_1) d(\overline{\sigma}', \overline{\rho}_c)^{-1} \\
&= \Delta_{c_1}(\widetilde{\sigma}, \rho_1) d(\overline{\sigma}', \overline{\rho}_1) d(\overline{\sigma}', \overline{\rho}_c)^{-1}.
\end{aligned}$$

Furthermore,

$$d(\overline{\sigma}', \overline{\rho}_1) d(\overline{\sigma}', \overline{\rho}_c)^{-1} = \langle \mu_T(\overline{\sigma}^*, \overline{\rho}_1) \mu_T(\overline{\sigma}^*, \overline{\rho}_c)^{-1}, \overline{s}'_{d,T} \rangle.$$

As in §4, $s'_d \in \widehat{G}^*_d$ is the semisimple element attached to the endoscopic datum G'_d, and $\overline{s}'_{d,T}$ is the image of $s'_{d,T}$ in the group $\pi_0(\widehat{T}^{\Gamma}_{\mathrm{ad}}) = \pi_0\big((\widehat{T}/Z(\widehat{G}^*_d))^{\Gamma}\big)$. The underlying K-group G plays only a peripheral role in this section, so we shall write T_{sc} for the preimage of T in $G^*_{d,\mathrm{sc}}$ (rather than in G^*_{sc} as before). We have reduced the problem to a comparison of two classes $\mu_T(\overline{\sigma}^*, \overline{\rho}_1)$ and $\mu_T(\overline{\sigma}^*, \overline{\rho}_c)$ in $H^1\big(\Gamma, T_{\mathrm{sc}}(\overline{\mathbb{A}})/T_{\mathrm{sc}}(\overline{F})\big)$.

The class $\mu_T(\overline{\sigma}^*, \overline{\rho}_1)$ is defined exactly as in [LS1, (6.3)]. Let h_1 be an element in $G^*_{d,\mathrm{sc}}(\overline{\mathbb{A}})$ such that

$$\overline{\sigma}^* = h_1 \psi_1(\overline{\rho}_1) h_1^{-1},$$

where $\psi_1 = \psi_{c_1}$ is the inner twist from G_1 to G^*_d. Then $\mu_T(\overline{\sigma}^*, \overline{\rho}_1)$ is the class in $H^1\big(\Gamma, T_{\mathrm{sc}}(\overline{\mathbb{A}})/T_{\mathrm{sc}}(\overline{F})\big)$ that is represented by the cocyle obtained composing the function

$$v_1(\tau) = h_1 u_1(\tau) \tau(h_1)^{-1}, \qquad \tau \in \Gamma,$$

with the projection of $T_{\mathrm{sc}}(\overline{\mathbb{A}})$ onto $T_{\mathrm{sc}}(\overline{\mathbb{A}})/T_{\mathrm{sc}}(\overline{F})$. Here $u_1(\tau) = u_{c_1}(\tau)$ is any element in $G^*_{d,\mathrm{sc}}(\overline{F})$ such that $\psi_1 \tau(\psi_1)^{-1}$ equals $\mathrm{Int}\big(u_1(\tau)\big)$. What is the corresponding function $v_c(T)$ associated to $\overline{\rho}_c$?

To deal with this question, we let the map $\psi_c = \psi_1 \mathrm{Int}(g)$ serve as our inner twist from G_c to G^*_d. Then

$$\begin{aligned} \psi_c \tau(\psi_c) &= \psi_1 \mathrm{Int}(g) \tau\big(\psi_1 \mathrm{Int}(g)\big)^{-1} \\ &= \psi_1 \mathrm{Int}\big(g\tau(g)^{-1}\big)\tau(\psi_1)^{-1} = \psi_1 \tau(\psi_1)^{-1}, \end{aligned}$$

since $g\tau(g)^{-1}$ lies in the center of $G_1(\overline{F})$. We can therefore take $u_c(\tau) = u_1(\tau)$. We also note that

$$\begin{aligned} \psi_c(\overline{\rho}_c) &= \psi_1\big(\mathrm{Int}(g)\mathrm{Int}(x)\overline{\rho}_1\big) = \psi_1\big(\mathrm{Int}(gx)\overline{\rho}_1\big) \\ &= \psi_1\big(\mathrm{Int}(t^{-1}k^{-1})\overline{\rho}_1\big) = \big(\psi_1 \mathrm{Int}(k^{-1})\big)\big(\mathrm{Int}(kt^{-1}k^{-1})\overline{\rho}_1\big) \\ &= \big(\psi_1 \mathrm{Int}(k^{-1})\big)(\overline{\rho}_1) = \mathrm{Int}\big(\psi_{1,\mathrm{sc}}(k_{\mathrm{sc}}^{-1})\big)\psi_1(\overline{\rho}_1), \end{aligned}$$

since the point $\overline{t}^{-1} = kt^{-1}k^{-1}$ lies in $\overline{T}_{1,\mathbb{A}}(\mathbb{A})$, and since

$$\psi_1 \mathrm{Int}(k^{-1}) = \mathrm{Int}\big(\psi_{1,\mathrm{sc}}(k_{\mathrm{sc}}^{-1})\big)\psi_1.$$

We are writing $\psi_{1,\mathrm{sc}}\colon\ G_{1,\mathrm{sc}} \to G^*_{d,\mathrm{sc}}$, as usual, for the isomorphism of simply connected groups attached to ψ_1. It follows that

$$\overline{\sigma}^* = h_1\psi_{1,\mathrm{sc}}(k_{\mathrm{sc}}) \cdot \psi_c(\overline{\rho}_c) \cdot \big(h_1\psi_{1,\mathrm{sc}}(k_{\mathrm{sc}})\big)^{-1}.$$

Therefore

$$\begin{aligned} v_c(\tau) &= h_1\psi_{1,\mathrm{sc}}(k_{\mathrm{sc}})u_1(\sigma)\tau\big(h_1\psi_{1,\mathrm{sc}}(k_{\mathrm{sc}})\big)^{-1} \\ &= h_1\psi_{1,\mathrm{sc}}(k_{\mathrm{sc}})u_1(\sigma)\tau\big(\psi_{1,\mathrm{sc}}(k_{\mathrm{sc}}^{-1})\big)\tau(h_1)^{-1}. \end{aligned}$$

Observe that

$$\begin{aligned} u_1(\tau)\tau\big(\psi_{1,\mathrm{sc}}(k_{\mathrm{sc}}^{-1})\big) &= u_1(\sigma)(\tau\psi_{1,\mathrm{sc}})\big(\tau(k_{\mathrm{sc}}^{-1})\big) \\ &= u_1(\sigma)\Big(\big(\mathrm{Int}(u_1(\sigma))\big)^{-1}\psi_{1,\mathrm{sc}}\Big)\big(\tau(k_{\mathrm{sc}})^{-1}\big) \\ &= \psi_{1,\mathrm{sc}}\big(\tau(k_{\mathrm{sc}})^{-1}\big)u_1(\sigma). \end{aligned}$$

We may therefore write

$$v_c(\tau) = w_c(\tau)v_1(\tau),$$

where

$$w_c(\tau) = \big(\mathrm{Int}(h_1)\psi_{1,\mathrm{sc}}\big)\big(k_{\mathrm{sc}}\tau(k_{\mathrm{sc}})^{-1}\big).$$

The cocycle $k_{\mathrm{sc}}\tau(k_{\mathrm{sc}})^{-1}$ takes values in $\overline{T}_{1,\mathbb{A},\mathrm{sc}}(\overline{\mathbb{A}})$. Moreover, the restriction of $\mathrm{Int}(h_1)\psi_{1,\mathrm{sc}}$ to $\overline{T}_{1,\mathbb{A},\mathrm{sc}}(\overline{\mathbb{A}})$ is a Γ-isomorphism from $\overline{T}_{1,\mathbb{A},\mathrm{sc}}(\overline{\mathbb{A}})$ to $T_{\mathrm{sc}}(\overline{\mathbb{A}})$. The function $w_c(\tau)$ is therefore a 1-cocycle of Γ with values in $T_{\mathrm{sc}}(\overline{\mathbb{A}})$. It projects to a class ν_c in $H^1\big(\Gamma, T_{\mathrm{sc}}(\overline{\mathbb{A}})/T_{\mathrm{sc}}(\overline{F})\big)$.

We have shown that

$$\mu_T(\overline{\sigma}^*, \overline{\rho}_1)\mu_T(\overline{\sigma}^*, \overline{\rho}_c)^{-1} = \nu_c^{-1}.$$

It therefore follows that

$$\Delta_c(\widetilde{\sigma}', \rho_c) = \Delta_{c_1}(\widetilde{\sigma}', \rho_1)\langle \nu_c, \overline{s}'_{d,T}\rangle^{-1}.$$

To complete the proof of the lemma, we need only establish the identity

$$\langle \nu_c, \overline{s}'_{d,T}\rangle = \langle \lambda_c, a'_d\rangle^{-1}. \tag{5.5}$$

We claim that the image of ν_c under the canonical map from $H^1\big(\Gamma, T_{\mathrm{sc}}(\overline{\mathbb{A}})/T_{\mathrm{sc}}(\overline{F})\big)$ to $H^1\big(\Gamma, T(\overline{\mathbb{A}})/T(\overline{F})\big)$ is zero. The image of $w_c(\tau)$ under the map from $T_{\mathrm{sc}}(\overline{\mathbb{A}})$ to $T(\overline{\mathbb{A}})$ is

$$\big(\mathrm{Int}(h_1)\psi_1\big)\big(k\tau(k)^{-1}\big). \tag{5.6}$$

But

$$k\tau(k)^{-1} = \mathrm{Int}(k)\big(k^{-1}\tau(k)\big)^{-1},$$

and by construction, the cocycle $k^{-1}\tau(k)$ has the same image in $H^1\big(\Gamma, T_{1,\mathbb{A}}(\overline{\mathbb{A}})\big)$ as the cocycle $g\tau(g)^{-1}$. Since $\mathrm{Int}(k)$ is a Γ-isomorphism from $T_{1,\mathbb{A}}(\overline{\mathbb{A}})$ to $\overline{T}_{1,\mathbb{A}}(\overline{\mathbb{A}})$, and since $g\tau(g)^{-1}$ takes values in the center of $G_1(\overline{F})$, the cocycles $k\tau(k)^{-1}$ and $\big(g\tau(g)^{-1}\big)^{-1}$ have the same image in $H^1\big(\Gamma, \overline{T}_{1,\mathbb{A}}(\overline{\mathbb{A}})\big)$. Therefore the cocycle (5.6) has the same image in $H^1\big(\Gamma, T(\overline{\mathbb{A}})\big)$ as the cocycle

$$\big(\mathrm{Int}(h_1)\psi_1\big)\big(g\tau(g)^{-1}\big)^{-1} = \psi_1\big(g\tau(g)^{-1}\big)^{-1}.$$

We have used the fact here that $\psi_1\big(g\tau(g)^{-1}\big)^{-1}$ takes values in the center of $G_d^*(\overline{F})$. The center of $G_d^*(\overline{F})$ is of course contained in $T(\overline{F})$. It follows that the image of the cocycle (5.6) in $H^1\big(\Gamma, T(\overline{\mathbb{A}})/T(\overline{F})\big)$ is trivial. In other words, the image of ν_c is trivial, as claimed.

Recall that the dual group of T_{sc} is equal to the quotient

$$\widehat{T}_{\mathrm{ad}} = \widehat{T}/Z(\widehat{G}_d^*) = \widehat{T}/Z(\widehat{G}_1)$$

of the dual group of T. In particular, the pairing $\langle \nu_c, \overline{s}'_{d,T}\rangle$ in (5.5) is given by the Tate-Nakayama isomorphism from $H^1\big(\Gamma, T_{\mathrm{sc}}(\overline{\mathbb{A}})/T_{\mathrm{sc}}(\overline{F})\big)$ to the dual of the finite abelian group $\pi_0(\widehat{T}_{\mathrm{ad}}^{\Gamma})$. This isomorphism maps the subgroup

$$\ker\Big(H^1\big(\Gamma, T_{\mathrm{sc}}(\overline{\mathbb{A}})/T_{\mathrm{sc}}(\overline{F})\big)\longrightarrow H^1\big(\Gamma, T(\overline{\mathbb{A}})/T(\overline{F})\big)\Big)$$

onto the dual of the quotient group

$$\mathrm{coker}\big(\pi_0(\widehat{T}^{\Gamma})\longrightarrow \pi_0(\widehat{T}_{\mathrm{ad}}^{\Gamma})\big).$$

This quotient group is in turn isomorphic to the image of $\pi_0(\widehat{T}^{\Gamma}_{\mathrm{ad}})$ in $H^1\big(\Gamma, Z(\widehat{G}_1)\big)$, under the map that comes from the long exact sequence [K2, Corollary 2.3] of cohomology of diagonalizable groups over $\mathbb{C}$. Now the image of $\overline{s}'_{d,T}$ in $H^1\big(\Gamma, Z(\widehat{G})\big)$ is by definition equal to a'_d. It follows that the left hand side of (5.5) depends only on a'_d.

We can view the situation in terms of the commutative diagram

$$\begin{array}{ccc} H^1\big(\Gamma, T_{\mathrm{sc}}(\overline{\mathbb{A}})\big) & \xrightarrow{\alpha} & H^1\big(\Gamma, T_{\mathrm{sc}}(\overline{\mathbb{A}})/T_{\mathrm{sc}}(\overline{F})\big) \\ {\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle\delta} \\ H^1\big(\Gamma, T(\overline{\mathbb{A}})\big) & \xrightarrow{\gamma} & H^1\big(\Gamma, T(\overline{\mathbb{A}})/T(\overline{F})\big) \end{array}$$

and the corresponding dual diagram

$$\begin{array}{ccc} \pi_0(\widehat{T}^{\Gamma}_{\mathrm{ad},\mathbb{A}}) & \xleftarrow{\alpha^*} & \pi_0(\widehat{T}^{\Gamma}_{\mathrm{ad}}) \\ {\scriptstyle\beta^*}\big\uparrow & & \big\uparrow{\scriptstyle\delta^*} \\ \pi_0(\widehat{T}^{\Gamma}_{\mathbb{A}}) & \xleftarrow{\gamma^*} & \pi_0(\widehat{T}^{\Gamma}), \end{array}$$

both of which are suggested by the discussion on p. 638 of [K2]. We are writing $T_{\mathbb{A}}$ here for the projective limit

$$\varprojlim_{E,S}\Big(\prod_{v\in S}\prod_{w}\widehat{T}_w\Big)$$

of complex tori, in which E ranges over finite Galois extensions of F, S ranges over finite sets of valuations of F, and w is taken over the valuations of E above v. As an element in $H^1\big(\Gamma, Z(\widehat{G})\big)$, a'_d is not arbitrary. Since it lies in the image of $\pi_0(\widehat{T}^{\Gamma}_{\mathrm{ad}})$, it can identified with an element in $\mathrm{coker}(\delta^*)$. Moreover, it is locally trivial. It follows from the long exact sequence attached to

$$1 \longrightarrow Z(\widehat{G}_{1,\mathbb{A}}) \longrightarrow \widehat{T}_{\mathbb{A}} \longrightarrow \widehat{T}_{\mathrm{ad},\mathbb{A}} \longrightarrow 1$$

by [K2, Corollary 2.3] that the image of a'_d under α^* lies in the image of β^*. We may therefore identify a'_d with an element in the group

$$\ker\big(\mathrm{coker}(\delta^*) \longrightarrow \mathrm{coker}(\beta^*)\big). \tag{5.7}$$

Then

$$\langle \nu_c, \overline{s}'_{d,T} \rangle = \lambda^\vee_c(a'_d), \tag{5.8}$$

where $\lambda^\vee_c$ is the image of ν_c in the corresponding dual group

$$\mathrm{coker}\big(\ker(\beta) \longrightarrow \ker(\delta)\big).$$

The element ν_c is actually the image in $H^1\big(\Gamma, T_{\mathrm{sc}}(\overline{\mathbb{A}})/T_{\mathrm{sc}}(\overline{F})\big)$ of a class in $H^1\big(\Gamma, T_{\mathrm{sc}}(\overline{\mathbb{A}})\big)$, so $\lambda^\vee_c$ in fact belongs to the subgroup

$$\mathrm{coker}\big(\ker(\beta) \longrightarrow \ker(\delta) \cap \mathrm{im}(\alpha)\big). \tag{5.9}$$

The pairing $\lambda^\vee_c(a'_d)$ therefore depends only on the image of a'_d in the corresponding quotient of (5.7).

We have identified (5.7) with a subgroup of $\ker^1\big(F, Z(\widehat{G}_1)\big)$. We have also mapped $\ker^1(F, G_1)$ to the subgroup (5.9) of the dual group of (5.7). This map is defined by representing the class $\lambda_c \in \ker^1(F, G_1)$ by a 1-cocycle with values in $Z(G_1)$. It follows easily that the map is independent of the inner form G_1 of G^*_d. In fact the entire construction can be applied directly to an arbitrary connected reductive group H over F, which we take to be quasisplit. Given H, we take T to be any maximal torus in H over F, with preimage T_{sc} in H_{sc}. We can then identify the associated group (5.7) with a subgroup of $\ker^1\big(F, Z(\widehat{H})\big)$. We also obtain a canonical map $\lambda \to \lambda^\vee$ from $\ker^1(F, H)$ to the subgroup (5.9) of the dual group of (5.7). The problem is to show that

$$\lambda^\vee(a) = \langle \lambda, a \rangle^{-1}, \tag{5.10}$$

for any λ in $\ker^1(F, H)$, and any element a in (5.7).

We need to look more closely at the pairing between $\ker^1(F, H)$ and $\ker^1\big(F, Z(\widehat{H})\big)$ on the right hand side of (5.10). We follow the indirect definition of the pairing in [K2, §4], which has two steps. The first step is to construct the pairing in the case that H_{der}

is simply connected. The second is to reduce the definition for general H to this special case. (See [K2, (4.4)].)

Working in the reverse order, we assume that H is arbitrary. Let $\widetilde{H} \to H$ be a z-extension of H by $\widetilde{Z}$. Then $\widetilde{H}_{\mathrm{der}}$ is simply connected, and we can take $H_{\mathrm{sc}} = \widetilde{H}_{\mathrm{der}} = \widetilde{H}_{\mathrm{sc}}$. The bijection from $\ker^1(F,H)$ to the dual group $\ker^1\big(F, Z(\widehat{H})\big)^*$ is defined as a composition of bijections

$$\ker^1(F,H) \overset{\sim}{\longleftarrow} \ker^1(F,\widetilde{H}) \overset{\sim}{\longrightarrow} \ker^1\big(F, Z(\widehat{\widetilde{H}})\big)^* \overset{\sim}{\longrightarrow} \ker^1\big(F, Z(\widehat{H})\big)^*,$$

in which the middle bijection is assumed to have already been defined, and the outer two bijections are given by [K2, Lemma 4.3.2]. We take $\widetilde{T}_{\mathrm{sc}} = T_{\mathrm{sc}}$ and $\widehat{\widetilde{T}}_{\mathrm{ad}} = \widehat{T}_{\mathrm{ad}}$. From the exact sequence

$$1 \longrightarrow \widehat{T} \longrightarrow \widehat{\widetilde{T}} \longrightarrow \widehat{\widetilde{Z}} \longrightarrow 1,$$

and the fact that $\widehat{\widetilde{Z}}$ is an induced complex torus, we see that the maps $\pi_0(\widehat{T}^{\Gamma}) \to \pi_0(\widehat{\widetilde{T}}^{\Gamma})$ and $\pi_0(\widehat{T}^{\Gamma}_{\mathbb{A}}) \to \pi_0(\widehat{\widetilde{T}}^{\Gamma}_{\mathbb{A}})$ are both surjective. The groups (5.7) attached to H and $\widetilde{H}$, as subquotients of $\pi_0(\widehat{T}^{\Gamma}_{\mathrm{ad}})$, are then both the same. From the exact sequence

$$1 \longrightarrow \widetilde{Z} \longrightarrow \widetilde{T} \longrightarrow T \longrightarrow 1,$$

and the fact that $\widetilde{Z}$ is an induced torus over F, we see that the vertical arrows in the diagram

$$\begin{array}{ccccc} H^1\big(\Gamma, \widetilde{T}(\overline{F})\big) & \longrightarrow & H^1\big(\Gamma, \widetilde{T}(\overline{\mathbb{A}})\big) & \longrightarrow & H^1\big(\Gamma, \widetilde{T}(\overline{\mathbb{A}})/\widetilde{T}(\overline{F})\big) \\ \downarrow & & \downarrow & & \downarrow \\ H^1\big(\Gamma, T(\overline{F})\big) & \longrightarrow & H^1\big(\Gamma, T(\overline{\mathbb{A}})\big) & \longrightarrow & H^1\big(\Gamma, T(\overline{\mathbb{A}})/T(\overline{F})\big) \end{array}$$

are injections. In particular, the groups (5.9) attached to H and $\widetilde{H}$ are equal to the same subquotient of $H^1\big(\Gamma, T_{\mathrm{sc}}(\overline{\mathbb{A}})/T_{\mathrm{sc}}(\overline{F})\big)$. Since the bijection $\widetilde{\lambda} \to \lambda$ from $\ker^1(F,\widetilde{H})$ to $\ker^1(F,H)$ is compatible with the two maps into (5.9), we see that $\widetilde{\lambda}^\vee = \lambda^\vee$, and hence that $\widetilde{\lambda}^\vee(a) = \lambda^\vee(a)$. We have shown that (5.10) holds for H if it is valid for $\widetilde{H}$.

The last step is to establish (5.10) in the case that H_{der} is simply connected. Under this assumption on H, the group $H_{\mathrm{der}} = H_{\mathrm{sc}}$ is a subgroup of H, and the quotient $D = H/H_{\mathrm{sc}}$ is a torus that is defined over F. The bijection from $\ker^1(F,H)$ to the dual group $\ker^1\big(F, Z(\widehat{H})\big)^*$ is then defined as a composition of bijections

$$\ker^1(F,H) \xrightarrow{\ \sim\ } \ker^1(F,D) \xrightarrow{\ \sim\ } \ker^1(F,\widehat{D})^* \xleftarrow{\ \sim\ } \ker^1\big(F, Z(\widehat{H})\big)^*,$$

in which the map in the middle is defined by Tate-Nakayama duality for the torus D, and the outer two bijections are given by [K2, Lemma 4.3.1]. Now $Z(\widehat{H})$ is equal to the subgroup $\widehat{D}$ of $\widehat{H}$, and the bijection on the right is the identity map. The bijection on the left requires more discussion.

The short exact sequence of tori

$$1 \longrightarrow T_{\mathrm{sc}} \longrightarrow T \longrightarrow D \longrightarrow 1$$

provides columns for a commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & T_{\mathrm{sc}}(\overline{F}) & \longrightarrow & T_{\mathrm{sc}}(\overline{\mathbb{A}}) & \longrightarrow & T_{\mathrm{sc}}(\overline{\mathbb{A}})/T_{\mathrm{sc}}(\overline{F}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & T(\overline{F}) & \longrightarrow & T(\overline{\mathbb{A}}) & \longrightarrow & T(\overline{\mathbb{A}})/T(\overline{F}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & D(\overline{F}) & \longrightarrow & D(\overline{\mathbb{A}}) & \longrightarrow & D(\overline{\mathbb{A}})/D(\overline{F}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

whose rows and columns are exact. The corresponding exact sequences of cohomology yield a diagram that contains the original commutative square. More precisely, we obtain a diagram

$$
\begin{array}{ccccccccc}
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\to & D(\overline{F})^\Gamma & \to & D(\overline{\mathbb{A}})^\Gamma & \to & \big(D(\overline{\mathbb{A}})/D(\overline{F})\big)^\Gamma & \to & H^1\big(\Gamma, D(\overline{F})\big) \\
 & \downarrow & & \downarrow & & \downarrow & & \\
\to & H^1\big(\Gamma, T_{\mathrm{sc}}(\overline{F})\big) & \to & H^1\big(\Gamma, T_{\mathrm{sc}}(\overline{\mathbb{A}})\big) & \xrightarrow{\alpha} & H^1\big(\Gamma, T_{\mathrm{sc}}(\overline{\mathbb{A}})/T_{\mathrm{sc}}(\overline{F})\big) & \to & \\
 & \downarrow & & \beta\big\downarrow & & \big\downarrow\delta & & \\
\to & H^1\big(\Gamma, T(\overline{F})\big) & \to & H^1\big(\Gamma, T(\overline{\mathbb{A}})\big) & \xrightarrow{\gamma} & H^1\big(\Gamma, T(\overline{\mathbb{A}})/T(\overline{F})\big) & \to & \\
 & \downarrow & & \downarrow & & \downarrow & & \\
\to & H^1\big(\Gamma, D(\overline{F})\big) & \to & H^1\big(\Gamma, D(\overline{\mathbb{A}})\big) & \to & H^1\big(\Gamma, D(\overline{\mathbb{A}})/D(\overline{F})\big) & \to & \\
 & \downarrow & & \downarrow & & \downarrow & &
\end{array}
$$

whose rows and columns are exact, and whose constituent squares are either commutative or anticommutative [CE, Proposition III.4.1]. The bijection from $\ker^1(F,H)$ to $\ker^1(F,D)$ under study takes values in the group $H^1\big(\Gamma, D(\overline{F})\big)$ in the lower left hand corner of the diagram. Its value at a given $\lambda \in \ker^1(F,H)$ is just the image in $H^1\big(\Gamma, D(\overline{F})\big)$ of the class in $H^1\big(\Gamma, T(\overline{F})\big)$ obtained by representing λ by a 1-cocycle from Γ to $Z(H)$. The corresponding element $\lambda^\vee$ belongs to the subquotient (5.9) of $H^1\big(\Gamma, T_{\mathrm{sc}}(\overline{\mathbb{A}})/T_{\mathrm{sc}}(\overline{F})\big)$. Since it lies in the kernel of δ, $\lambda^\vee$ pulls back to an element in $\big(D(\overline{\mathbb{A}})/D(\overline{F})\big)^\Gamma$, which we can then map to the group $H^1\big(\Gamma, D(\overline{F})\big)$ in the upper right hand corner of the diagram. We can therefore identify the map $\lambda \to \lambda^\vee$ with a correspondence that goes from the group $H^1\big(\Gamma, D(\overline{F})\big)$ in the lower left hand corner of the diagram to the same group in the upper right hand corner. Our task is to show that the correspondence maps λ to λ^{-1}. It will be a simple exercise in diagram closing.

Suppose that λ belongs to $\ker^1\big(F, D(\overline{F})\big)$. As we have noted, λ is the image of a class in $H^1\big(\Gamma, T(\overline{F})\big)$, which we can represent by a 1-cocycle $\mu(\tau)$ from Γ to $T(\overline{F})$. We can of course also regard $\mu(\tau)$ as a 1-cocycle from Γ to the larger group $T(\overline{\mathbb{A}})$. Since λ is locally trivial, the projection of $\mu(\tau)$ onto the quotient $D(\overline{\mathbb{A}})$ of $T(\overline{\mathbb{A}})$ splits. We can therefore find an element $t \in T(\overline{\mathbb{A}})$ such that the 1-cocycle $\mu(\tau)\big(t\tau(t^{-1})\big)^{-1}$ takes values in the kernel $T_{\mathrm{sc}}(\overline{\mathbb{A}})$ of the projection $T(\overline{\mathbb{A}}) \to D(\overline{\mathbb{A}})$. Let $\bar{t}$, d and $\bar{d}$ be the images of t in $T(\overline{\mathbb{A}})/T(\overline{F})$,

$D(\overline{\mathbb{A}})$ and $D(\overline{\mathbb{A}})/D(\overline{F})$ respectively, provided by the commutative diagram

$$\begin{array}{ccc} T(\overline{\mathbb{A}}) & \longrightarrow & T(\overline{\mathbb{A}})/T(\overline{F}) \\ \downarrow & & \downarrow \\ D(\overline{\mathbb{A}}) & \longrightarrow & D(\overline{\mathbb{A}})/D(\overline{F}). \end{array}$$

The projection of $\mu(\tau)$ onto $D(\overline{\mathbb{A}})$ equals $d\tau(d)^{-1}$. It follows that $\overline{d}$ is Γ-invariant. The original class λ is just the image in $H^1\big(\Gamma, D(\overline{F})\big)$ of the element $\overline{d} \in \big(D(\overline{\mathbb{A}})/D(\overline{F})\big)^{\Gamma}$. On the other hand, $\lambda^\vee$ is constructed from $\alpha(\lambda_{\mathrm{sc}})$, where λ_{sc} is the class in $H^1\big(\Gamma, T_{\mathrm{sc}}(\overline{\mathbb{A}})\big)$ of the cocycle $\mu(\tau)\big(t\tau(t^{-1})\big)^{-1}$. By definition, $\lambda^\vee$ is the image in $H^1\big(\Gamma, D(\overline{F})\big)$ of any element in $\big(D(\overline{\mathbb{A}})/D(\overline{F})\big)^{\Gamma}$ in the preimage of $\alpha(\lambda_{\mathrm{sc}})$. We know that $\alpha(\lambda_{\mathrm{sc}})$ lies in the kernel of δ, so it does have a preimage in $\big(D(\overline{\mathbb{A}})/D(\overline{F})\big)^{\Gamma}$. Therefore the projection of $\mu(\tau)\big(t\sigma(t^{-1})\big)^{-1}$ onto $T(\overline{\mathbb{A}})/T(\overline{F})$ equals the cocycle

$$\big(\overline{t}\tau(\overline{t}^{-1})\big)^{-1} = \overline{t}^{-1}\tau(\overline{t}^{-1})^{-1}.$$

Since the projection of $\overline{t}$ onto $D(\overline{\mathbb{A}})/D(\overline{F})$ equals $\overline{d}$, the element $\overline{d}^{-1} \in \big(D(\overline{\mathbb{A}})/D(\overline{F})\big)^{\Gamma}$ lies in the preimage of $\alpha(\lambda_{\mathrm{sc}})$. It follows that $\lambda^\vee = \lambda^{-1}$.

We have established that

$$\lambda^\vee(a) = \langle \lambda^{-1}, a\rangle = \langle \lambda, a\rangle^{-1},$$

in the case that H_{der} is simply connected, and hence in general. The formula (5.10) therefore holds for any H. Setting $H = G_d^*$, $\lambda = \lambda_c$ and $a = a_d'$, we see from (5.8) that

$$\langle \nu_c, \overline{s}'_{d,T}\rangle = \lambda_c^\vee(a_d') = \langle \lambda_c, a_d'\rangle^{-1}.$$

This is the required formula (5.5). The proof of Lemma 5.1 is complete. □

We remark in passing that the last part of the proof of the lemma can be applied to any commutative diagram of Γ-modules whose rows and columns are short exact sequences. The corresponding cohomology groups then yield a planar diagram of long exact sequences. Given any group

$$\begin{array}{ccc} & \Big\downarrow{\scriptstyle d''_{i-1,j}} & \\ \xrightarrow{d'_{i,j-1}} & C_{ij} & \xrightarrow{d'_{ij}} \\ & \Big\downarrow{\scriptstyle d''_{i,j}} & \end{array}$$

in the latter diagram, we can set

$$Z_{ij} = \ker(d'_{ij}) \cap \ker(d''_{ij}),$$

$$B_{ij} = \mathrm{im}(d''_{i-1,j} d'_{i-1,j-1}) = \mathrm{im}(d'_{i,j-1} d''_{i-1,j-1}),$$

and

$$H_{ij} = Z_{ij}/B_{ij}.$$

For example, the group (5.9) is just the subquotient H_{ij} attached to the group $H^1\big(\Gamma, T_{\mathrm{sc}}(\overline{\mathbb{A}})/T_{\mathrm{sc}}(\overline{F})\big)$ in our earlier diagram. In general, one sees that there is a canonical isomorphism

$$\Lambda_{ij} : \ H_{ij} \longrightarrow H_{i-1,j+1},$$

for each (i, j). One also observes from the periodicity of the diagram that $H_{ij} = H_{i-3,j+3}$, so that $(\Lambda_{ij})^3$ can be regarded as an isomorphism of H_{ij} with itself. Finally, as in the proof of the last part of the lemma, one can show that that $(\Lambda_{ij})^3$ is in fact the automorphism $\lambda \to \lambda^{-1}$ (written in multiplicative notation) of H_{ij}. I would imagine that such things are known, but I do not have a reference.

§6. Proof of the theorem

We are now ready to prove Theorem 1.1. We are going to have to convert an expression obtained from the coefficient $a_{\mathrm{ell}}^{G,\mathcal{E}}(\dot{\gamma}_S)$ into a corresponding expression of descent. The lemmas of §4 and §5 will each be applied at some point to a transfer factor in the expression. We shall use the descent mapping of §3 to keep track of the combinatorics of the process.

We are assuming that G_{der} is simply connected, and that Z is trivial. In particular, the centralizer in G of any semisimple element is connected. Moreover, we can take $\widetilde{G}' = G'$, for any $G' \in \mathcal{E}(G)$. Suppose that $\dot{\gamma}_S$ is an admissible element in $\Gamma_{\mathrm{ell}}^{\mathcal{E}}(G,S)$ with Jordan decomposition $\dot{\gamma}_S = c_S\dot{\alpha}_S$, as in statement (a) of Theorem 1.1. According to the definition (1.1), the difference

$$a_{\mathrm{ell}}^{G,\mathcal{E}}(\dot{\gamma}_S) - \varepsilon(G) \sum_{\dot{\delta}_S \in \Delta_{\mathrm{ell}}^{\mathcal{E}}(G,S)} b_{\mathrm{ell}}^{G}(\dot{\delta}_S)\Delta_G(\dot{\delta}_S,\gamma_S) \tag{6.1}$$

equals

$$\sum_{G' \in \mathcal{E}_{\mathrm{ell}}^{0}(G,S)} \iota(G,G') \sum_{\dot{\delta}'_S \in \Delta_{\mathrm{ell}}(G',S)} b_{\mathrm{ell}}^{G'}(\dot{\delta}'_S)\Delta_G(\dot{\delta}'_S,\gamma_S).$$

We have to analyse this last expression.

The groups G' in $\mathcal{E}_{\mathrm{ell}}^{0}(G,S)$ need not have the property that G'_{der} is simply connected. However, we can assume inductively that any such G' has a z-extension for which Theorem 1.1 holds. Therefore Corollary 2.2 holds for G' itself. Suppose that $\dot{\delta}'_S$ is an element in $\Delta_{\mathrm{ell}}(G',S)$, with Jordan decomposition $\dot{\delta}'_S = d'_S\dot{\beta}'_S$, such that $\Delta_G(\dot{\delta}'_S,\dot{\gamma}_S)$ is nonzero. Then $\dot{\delta}'_S$ is admissible for G', and the coefficient $b_{\mathrm{ell}}^{G'}(\dot{\delta}'_S)$ is defined. Applying the expansion (2.9) of Corollary 2.2 to G', we obtain

$$b_{\mathrm{ell}}^{G'}(\dot{\delta}'_S) = \sum_{d'}\sum_{\dot{\beta}'} j^{G'}(S,d')\left|(G'_{d',+}/G'_{d'})(F)\right|^{-1} b_{\mathrm{ell}}^{G'_{d'}}(\dot{\beta}'),$$

where d' is summed over the elements in $\Delta_{\mathrm{ss}}(G')$ whose image in $\Delta_{\mathrm{ss}}(G'_S)$ equals d'_S, and $\dot{\beta}'$ is summed over the $(G'_{d',+}/G'_{d'})(F)$-orbit of $\dot{\beta}'_S$ in $\Delta_{\mathrm{unip}}(G'_{d'_S})$. According to our usual convention, d' represents both a stable semisimple conjugacy class, and a representative of the class with $G'_{d'}$ quasisplit. We substitute this formula into the expression above for (6.1). Since d'_S and $\dot\beta'_S$ are themselves to be summed, we can subsume these elements in a general sum over d' and $\dot\beta'$. We then write

$$\Delta_G(\dot\delta'_S, \dot\gamma_S) = \Delta_G(d'_S\dot\beta'_S, c_S\dot\alpha_S)$$

in the resulting expression, where d'_S and $\dot\beta'_S$ are the images of d' and $\dot\beta'$ in $\Delta_{\mathrm{ss}}(G'_S)$ and $\Delta_{\mathrm{unip}}(G'_{d',S})$ respectively. We also write

$$\begin{aligned}&\iota(G,G')j^{G'}(S,d')\\ &= \tau(G)\tau(G')^{-1}|\mathrm{Out}_G(G')|^{-1}i^{G'}(S,d')\tau(G')\tau(G'_{d'})^{-1}\\ &= i^{G'}(S,d')\tau(G)\tau(G'_{d'})^{-1}|\mathrm{Out}_G(G')|^{-1},\end{aligned}$$

by (1.7) and [K2, Theorem 8.3.1]. The Tamagawa number

$$\tau(G) = \tau(G_\alpha) = \tau(G^*), \qquad \alpha \in \pi_0(G),$$

here is well defined [K4]. Taking the definition of $i^{G'}(s,d')$ in §1 into account, we see that (6.1) equals

$$(6.2)\quad \sum_{G'}\sum_{d'}\sum_{\dot\beta'} |\mathrm{Out}_G(G')|^{-1}\tau(G)\tau(G'_{d'})^{-1}|(G'_{d',+}/G'_{d'})(F)|^{-1}b_{\mathrm{ell}}^{G'_{d'}}(\dot\beta')\Delta_G(d'_S\dot\beta'_S, c_S\dot\alpha_S),$$

where G' is summed over $\mathcal{E}^0_{\mathrm{ell}}(G,S)$, d' is summed over classes in $\Delta_{\mathrm{ss}}(G')$ that are elliptic and bounded at each $v \notin S$, and $\dot\beta'$ is summed over all classes in $\Delta_{\mathrm{unip}}(G'_{d',S})$.

In order to exploit the mapping of §3, it would be useful to frame our discussion in terms of a suitable rational stable class $d \in \Delta_{\mathrm{ss}}(G^*)$. We would ask that d be elliptic, that it be bounded at each $v \notin S$, and that it be a local image of each component c_v of c_S. Such a class of course might not exist. Suppose that it does not. If (G',d') indexes a nonzero summand in (6.2), d'_v will be an image of c_v, for each $v \in S$, since the local transfer factor $\Delta_G(d'_v\dot\beta'_v, c_v\dot\alpha_v)$ is nonzero. For any such d', we could take a maximal elliptic torus in G' over F that contains d', together with an admissible embedding of this torus into

G^*. The corresponding image d of d' would then represent a stable class of the required kind. Since we are assuming that d does not exist, there can be no pairs (G', d'). In other words, (6.2) vanishes, and so therefore does (6.1). The nonexistence of d also implies that c_S does not have a rational representative c which is elliptic, and is bounded at each $v \notin S$. In particular, the right hand side of the formula (1.8) also vanishes. Therefore, (1.8) is trivially valid. This also implies that the sum in (6.1) vanishes. It follows from the inversion formulas [I, (5.5)] for the transfer factors that $b^G_{\mathrm{ell}}(\dot{\delta}_S)$ equals zero for any $\dot{\delta}_S$ such that $\Delta_G(\dot{\delta}_S, \dot{\gamma}_S) \neq 0$. The nonexistence of d also implies that the corresponding right hand side of the formula (1.9) of part (b) of Theorem 1.1 vanishes. Therefore (1.9) is trivially valid for $\dot{\delta}_S$. We have shown that the assertions of Theorem 1.1 are trivial if d does not exist.

We can therefore assume that there is an elliptic class $d \in \Delta_{\mathrm{ss}}(G^*)$ that is bounded at each $v \notin S$, and that is a local image of c_v for each $v \in S$. Since G^*_{der} is simply connected, stable conjugacy in $G^*(F)$ is just conjugacy over $G^*(\overline{F})$. It follows that the class d is unique. We may as well also assume that d does not lie in the center of G^*, since the assertions of Theorem 1.1 would otherwise be trivial.

Now any class $d' \in \Delta_{\mathrm{ss}}(G')$ that indexes a nonzero summand in (6.2) is an image of d over F. Conversely, every $d' \in \Delta_{\mathrm{ss}}(G')$ that is an image of d over F will appear in the sum (6.2), since any such class will automatically be elliptic, and will be bounded at each $v \notin S$. We can therefore take the middle sum in (6.2) over all classes in $\Delta_{\mathrm{ss}}(G')$ that are images of d. We are going to apply the discussion of §3 to the resulting double sum over (G', d') in (6.2). Before doing so, however, let us first modify (6.2). We shall add to (6.2) an expression

$$\varepsilon(G) \sum_{\dot{\beta} \in \Delta_{\mathrm{unip}}(G^*_{d,S})} \tau(G^*)\tau(G^*_d)^{-1} b^{G^*_d}_{\mathrm{ell}}(\dot{\beta}) \Delta_G(d_S \dot{\beta}_S, c_S \dot{\alpha}_S), \tag{6.3}$$

which vanishes if $\varepsilon(G) = 0$, but which represents a supplementary summand in (6.2) with $G' = G^*$ and $d' = d$, in case $\varepsilon(G) = 1$. The effect of adding (6.3) to (6.2) is simply to change the sum over G' in $\mathcal{E}^0_{\mathrm{ell}}(G,S)$ into a sum over the full set $\mathcal{E}_{\mathrm{ell}}(G,S)$.

We can now express the augmented form of (6.2) in terms of the mapping $\mathcal{X}(G) \to \mathcal{Y}(G)$ of §3. Let us write $\mathcal{Y}_d(G,S)$ for the subset of elements in $\mathcal{Y}(G)$ that lie in the image of the map $\mathcal{X}(G) \to \mathcal{Y}(G)$, and that can be represented by a pair (d, G'_d) in which G'_d is unramified outside of S. Suppose that (G', d') is a pair that indexes a double summand in the augmented form of (6.2). Then (G', d') maps to an element x in $\mathcal{X}(G)$ whose image in $\mathcal{Y}(G)$ lies in $\mathcal{Y}_d(G,S)$. Conversely, suppose that x is an element in $\mathcal{X}(G)$ whose image y in $\mathcal{Y}(G)$ lies in the subset $\mathcal{Y}_d(G,S)$. Then x comes from a pair (G', d') that indexes a double summand. The group $\mathrm{Out}_G(G')$ acts transitively on the set of such (G', d'), and the stabilizer in $\mathrm{Out}_G(G')$ of (G', d') is equal to $\mathrm{Out}_G(G', d')$, by definition. We can therefore replace the double sum over (G', d') by a single sum over x, provided that we replace the coefficient $|\mathrm{Out}_G(G')|^{-1}$ by $|\mathrm{Out}_G(G', d')|^{-1}$. In the resulting expression, we are free to write

$$|\mathrm{Out}_G(G', d')|^{-1} |(G'_{d',+}/G'_{d'})(F)|^{-1} = \mathfrak{o}_x^{-1} c_x^{-1} = |\mathcal{K}_{y,x}||\mathfrak{o}_y|^{-1},$$

according to Proposition 3.1 (iii). The augmented form of (6.2) can therefore be written as

$$\sum_y \sum_x |\mathcal{K}_{y,x}||\mathfrak{o}_y|^{-1} \tau(G)\tau(G'_{d'})^{-1} \Big(\sum_{\dot\beta'} b^{G'_{d'}}_{\mathrm{ell}}(\dot\beta') \Delta_G(d'_S \dot\beta'_S, c_S \dot\alpha_S) \Big), \tag{6.4}$$

where y is summed over $\mathcal{Y}_d(G,S)$, x is summed over the fibre of y in $\mathcal{X}(G)$, (G', d') denotes a representative of x as in §3, and $\dot\beta'$ is summed over the classes in $\Delta_{\mathrm{unip}}(G'_{d',S})$.

We have thus far shown that the sum of (6.1) and (6.3) equals (6.4). The next step is to apply the Langlands-Shelstad descent theorem, or rather its variant Lemma 4.2, to the transfer factor $\Delta_G(d'\dot\beta', c_S\dot\alpha_S)$ in (6.4). We can assume that $\dot\beta'$ is such that the coefficient $b^{G'_{d'}}_{\mathrm{ell}}(\dot\beta')$ is nonzero. This implies that $\dot\beta'_S$ belongs to the subset $\Delta_{\mathrm{unip,der}}(G'_{d'_S})$ of $\Delta_{\mathrm{unip}}(G'_{d'_S})$, according to the discussion at the end of §1. It follows from Lemma 4.2 that

$$\Delta_G(d'\dot\beta', c_S\dot\alpha_S) = \Theta(x, c_{\mathbb{A}}) \Delta_{G,c_S}(\dot\beta'_S, \dot\alpha_S),$$

where $c_{\mathbb{A}} \in G(\mathbb{A})$ is the adelic element attached to c_S and K^S in §4. We shall substitute this into (6.4).

Most of the terms in (6.4) are independent of the point x in the fibre of y. We recall from §3 that $G'_{d'}$ comes with a canonical isomorphism with G'_d. The quasisplit group G'_d is of course part of the endoscopic datum taken from the pair (d, G'_d) that represents y. We can remove the dependence of $G'_{d'}$ on x simply by summing $\dot{\beta}'$ over the set $\Delta_{\mathrm{unip}}(G'_{d,S})$ instead of $\Delta_{\mathrm{unip}}(G'_{d',S})$. The expression (6.4) then takes the form

$$\sum_y |\mathfrak{o}_y|\tau(G)\tau(G'_d)^{-1}\Big(\sum_x |\mathcal{K}_{y,x}|\Theta(x, c_{\mathbb{A}})\Big)\Big(\sum_{\dot{\beta}'} b_{\mathrm{ell}}^{G'_d}(\dot{\beta}')\Delta_{G,c_S}(\dot{\beta}'_S, \dot{\alpha}_S)\Big).$$

Recall from Proposition 3.1 (ii) that the group $\mathcal{K}_y$ acts transitively on the fibre of y. We can therefore replace the sum over x with a sum over $\kappa \in \mathcal{K}_y$, provided that we divide by the order $|\mathcal{K}_{y,x}|$ of the stabilizer of any x in $\mathcal{K}_y$. Combining this with Lemma 4.1 (i), we see that

$$\sum_x |\mathcal{K}_{y,x}|\Theta(x, c_{\mathbb{A}}) = \sum_\kappa \Theta(x_y^\kappa, c_{\mathbb{A}}) = \Theta(x_y, c_{\mathbb{A}})\Big(\sum_\kappa \langle \mathrm{obs}(c_{\mathbb{A}}), \kappa\rangle^{-1}\Big),$$

where x_y is any fixed point in the fibre of y. We conclude that (6.4) equals

$$(6.5)\quad \sum_y |\mathfrak{o}_y|^{-1}\tau(G)\tau(G'_d)^{-1}\Theta(x_y, c_{\mathbb{A}})\Big(\sum_\kappa \langle \mathrm{obs}(c_{\mathbb{A}}), \kappa\rangle^{-1}\Big)\Big(\sum_{\dot{\beta}'} b_{\mathrm{ell}}^{G'_d}(\dot{\beta}')\Delta_{G,\dot{c}_S}(\dot{\beta}'_S, \dot{\alpha}_S)\Big),$$

where y, κ and $\dot{\beta}'$ are summed over $\mathcal{Y}_d(G, S)$, $\mathcal{K}_y$ and $\Delta_{\mathrm{unip}}(G'_{d,S})$ respectively. We have established that the sum of (6.1) and (6.3) equals the expression (6.5).

Suppose first that $\mathrm{obs}(c_{\mathbb{A}}) \neq 1$. Then the $G(\mathbb{A})$-conjugacy class of $c_{\mathbb{A}}$ does not have a rational representative. This of course implies that the right hand sides of both (1.4) and (1.8) vanish. It also implies that the sum over κ in (6.5) vanishes, so that (6.5) is itself equal to zero. In other words, the sum (6.1) and (6.3) equals zero. If $\varepsilon(G) = 0$, $a_{\mathrm{ell}}^{G,\mathcal{E}}(\dot{\gamma}_S)$ equals zero, since it is the only term in the sum of (6.1) and (6.3). Therefore the formula (1.8) of Theorem 1.1 is trivially valid. If $\varepsilon(G) = 1$, $a_{\mathrm{ell}}^{G,\mathcal{E}}(\dot{\gamma}_S)$ equals $a_{\mathrm{ell}}^{G}(\dot{\gamma}_S)$ by definition, which in turn equals zero by virtue of (1.4). Therefore (1.8) is trivially valid in this case as well. Thus, part (a) of Theorem 1.1 is trivial when $\mathrm{obs}(c_{\mathbb{A}}) \neq 1$. Moreover, the coefficients of $\varepsilon(G)$ in (6.1) and (6.3) are equal if $\varepsilon(G) = 1$, since the sum of the two expressions vanishes. We shall apply this later to the proof of part (b) of the theorem.

Assume now that $\mathrm{obs}(c_{\mathbb{A}}) = 1$. Then (6.5) reduces to

$$\sum_{y} |\mathfrak{o}_y|^{-1} \tau(G)\tau(G'_d)^{-1} \Theta(x_y, c_{\mathbb{A}}) |\mathcal{K}_y| \Big(\sum_{\dot{\beta}'} b_{\mathrm{ell}}^{G'_d}(\dot{\beta}') \Delta_{G,c_S}(\dot{\beta}'_S, \dot{\alpha}_S) \Big),$$

with y and $\dot{\beta}'$ summed as in (6.5). Since $\mathrm{obs}(c_{\mathbb{A}}) = 1$, there is a rational conjugacy class $c \in \Gamma_{\mathrm{ss}}(G)$ in the $G(\mathbb{A})$-conjugacy class of $c_{\mathbb{A}}$. The associated inner twist $\psi_c\colon\ G_c \rightarrow G_d^*$ allows us to identify the element $G'_d \in \mathcal{E}_{\mathrm{ell}}(G_d^*)$ with an endoscopic datum for G_c, and to define the canonical global transfer factor $\Delta_{G,c}(\dot{\beta}', \dot{\alpha})$ for (G_c, G'_d). The constant $\Theta(x_y, c_{\mathbb{A}})$ depends on a normalization for the transfer factor $\Delta_{G,c_S}(\dot{\beta}'_S, \dot{\alpha}_S)$. However, Lemma 4.1 (ii) tells us that

$$\Theta(x_y, c_{\mathbb{A}}) \Delta_{G,c_S}(\dot{\beta}'_S, \dot{\alpha}_S) = \Delta_{G,c}(\dot{\beta}', \dot{\alpha}),$$

where $\dot{\alpha}$ is the image of $\dot{\alpha}_S$ in $\Gamma_{\mathrm{unip}}(G_{c,S})$. In particular, the transfer factor $\Delta_{G,c}(\dot{\beta}', \dot{\alpha})$ is independent of the choice of c. (This property is also an easy consequence of Lemma 5.1.) The expression (6.5) therefore reduces to

$$\sum_{y} |\mathfrak{o}_y|^{-1} \tau(G)\tau(G'_d)^{-1} |\mathcal{K}_y| \Big(\sum_{\dot{\beta}'} b_{\mathrm{ell}}^{G'_d}(\dot{\beta}') \Delta_{G,c}(\dot{\beta}', \dot{\alpha}) \Big),$$

where c is any rational representative of the $G(\mathbb{A})$-conjugacy class $c_{\mathbb{A}}$. Now for any such c, we have the coefficient $\iota(G_c, G'_d)$ that occurs in the definition of $a_{\mathrm{ell}}^{G_c,\mathcal{E}}(\dot{\alpha})$. We shall substitute the formula

$$\iota(G_c, G'_d) = \tau(G_c)\tau(G'_d)^{-1} |\mathrm{Out}_{G_c}(G'_d)|^{-1} = \tau(G_c)\tau(G'_d)^{-1} |\mathfrak{o}_y|^{-1}$$

for this coefficient [K2, Theorem 8.3.1] into the expression above. Let $\mathcal{E}_{\mathrm{ell}}^{G}(G_d^*, S)$ be the set of isomorphism classes of endoscopic data G'_d in $\mathcal{E}_{\mathrm{ell}}(G_d^*, S)$ such that (d, G'_d) represents a point y in $\mathcal{Y}_d(G,S)$. In particular, $\mathcal{E}^G_{\text{ell}}(G_d,S)$ is bijective with $\mathcal{Y}_d(G,S)$. We can then write our expression for (6.5) in the form

$$\tau(G)\tau(G_c)^{-1}|\mathcal{K}(G_c,G)|\sum_{G'_d}\iota(G_c,G'_d)\sum_{\dot\beta'}b^{G'_d}_{\text{ell}}(\dot\beta')\Delta_{G,c}(\dot\beta',\dot\alpha), \tag{6.6}$$

where G'_d and $\dot\beta'$ are summed over $\mathcal{E}^G_{\text{ell}}(G^*_d,S)$ and $\Delta_{\text{unip}}(G'_{d,S})$ respectively, and where $\mathcal{K}(G_c,G)\cong\mathcal{K}_y$ stands for the subgroup of elements in $\big(Z(\widehat{G}_c)/Z(\widehat{G})\big)^\Gamma$ whose image in $H^1\big(F,Z(\widehat{G})\big)$ is locally trivial.

We claim that the product

$$\tau(G)\tau(G_c)^{-1}|\mathcal{K}(G_c,G)|$$

in (6.6) is equal to the number of rational classes $c\in\Delta_{\text{ss}}(G)$ in the $G(\mathbb{A})$-conjugacy class of $c_{\mathbb{A}}$. To see this, we need only follow the argument in [K3, §9]. In particular, we combine the exact sequence

$$1\longrightarrow\pi_0\big(Z(\widehat{G})^\Gamma\big)\longrightarrow\pi_0\big(Z(\widehat{G}_c)^\Gamma\big)\longrightarrow\mathcal{K}(G_c,G)\longrightarrow\ker^1\big(F,Z(\widehat{G})\big)\longrightarrow\ker^1\big(F,Z(\widehat{G}_c)\big)$$

with the formula

$$\tau(G)=\big|\pi_0\big(Z(\widehat{G})^\Gamma\big)\big|\big|\ker^1\big(F,\big(Z(\widehat{G})\big)\big)\big|^{-1}$$

for the Tamagawa number [K2, (5.1.1)]. As on p. 395 of [K3], we deduce that the given product equals the order of the finite abelian group

$$\operatorname{coker}\big(\ker^1\big(F,Z(\widehat{G})\big)\longrightarrow\ker^1\big(F,Z(\widehat{G}_c)\big)\big).$$

Recalling the discussion at the beginning of §5, we note that this group is bijective with the set

$$\ker\big(\ker^1(F,G_c)\to\ker^1(F,G)\big),$$

which can in turn be used to parametrize the set of classes $c\in\Gamma_{\text{ss}}(G)$ attached to $c_{\mathbb{A}}$. The claim follows. The terms $\iota(G_c,G'_d)$ and $\Delta_{G,c}(\dot\beta',\dot\alpha)$ in (6.6) are independent of the choice of c. It follows that (6.6) equals

$$\sum_{c}\sum_{G'_d}\iota(G_c,G'_d)\sum_{\dot\beta'} b_{\mathrm{ell}}^{G'_d}(\dot\beta')\Delta_{G,c}(\dot\beta',\dot\alpha), \tag{6.7}$$

where c is summed over the classes in $\Gamma_{\mathrm{ss}}(G)$ that map to the $G(\mathbb{A})$-conjugacy class of $c_{\mathbb{A}}$, G'_d and $\dot\beta'$ are summed over $\mathcal{E}^G_{\mathrm{ell}}(G^*_d,S)$ and $\Delta_{\mathrm{unip}}(G'_{d,S})$ respectively, and $\dot\alpha$ is the image of $\dot\alpha_S$ in $\Gamma_{\mathrm{unip}}(G_{c,S})$.

We have established that the sum of (6.1) and (6.3) equals the expression (6.7). Observe that the summand indexed by c in (6.7) comes close to matching the definition of $a^{G_c,\mathcal{E}}_{\mathrm{ell}}(\dot\alpha)$ in §1. The main discrepancy is that the inner sum in (6.7) is over the subset $\mathcal{E}^G_{\mathrm{ell}}(G^*_d,S)$ of $\mathcal{E}_{\mathrm{ell}}(G^*_d,S)$, while the definition of $a^{G_c,\mathcal{E}}_{\mathrm{ell}}(\dot\alpha)$ requires a sum over the full set $\mathcal{E}_{\mathrm{ell}}(G^*_d,S)$. Can the two be reconciled? The supplementary contribution to (6.7) that would come from an element G'_d in the complement of $\mathcal{E}^G_{\mathrm{ell}}(G_d,S)$, namely

$$\sum_{c}\iota(G_c,G'_d)\sum_{\dot\beta'\in\Delta_{\mathrm{unip}}(G'_{d,S})} b_{\mathrm{ell}}^{G'_d}(\dot\beta')\Delta_{G,c}(\dot\beta',\dot\alpha),$$

can be written as the sum over $\dot\beta'$ of the product of

$$\iota(G^*_d,G'_d)b_{\mathrm{ell}}^{G'_d}(\dot\beta')$$

with

$$\sum_{c}\Delta_{G,c}(\dot\beta',\dot\alpha). \tag{6.8}$$

We shall use Lemma 5.1 to deal with this last sum.

Let $c_1\in\Gamma_{\mathrm{ss}}(G)$ be a fixed rational class that maps to the adelic class of $c_{\mathbb{A}}$. Lemma 5.1 asserts that for any c, the canonical adelic transfer factor $\Delta_c=\Delta_{G,c}$ for (G_c,G'_d) satisfies

$$\Delta_c(\widetilde\sigma',\rho_c)=\langle\lambda_c,a'_d\rangle\Delta_{c_1}(\widetilde\sigma',\rho_1).$$

We are of course following the notation of §5. In particular, ρ_1 is a strongly regular point in $G_{c_1}(\mathbb{A})$, and ρ_c is the image of ρ_1 under the canonical inner class of automorphisms from $G_{c_1}(\mathbb{A})$ to $G_c(\mathbb{A})$. Since c is $G(F_v)$-conjugate to the component $c_v \in K_v$ of $c_{\mathbb{A}}$, for each $v \notin S$, we can define the maximal compact subgroup $K_c^S = \prod_{v \notin S} K_{c,v}$ of $G_c(\mathbb{A}^S)$ as in (1.5). It is the image of the corresponding group $K_{c_1}^S$ for $G_{c_1}(\mathbb{A}^S)$ under an automorphism from the canonical inner class. It follows that

$$\Delta_{K_{c,v}}(\widetilde{\sigma}'_v, \rho_{c,v}) = \Delta_{K_{c_1,v}}(\widetilde{\sigma}'_v, \rho_{1,v}), \qquad v \notin S.$$

Since

$$\Delta_c(\widetilde{\sigma}'_S, \rho_{c,S}) = \Delta_c(\widetilde{\sigma}', \rho_c) \prod_{v \notin S} \Delta_{K_c,v}(\widetilde{\sigma}'_v, \rho_v)^{-1},$$

the assertion of Lemma 5.1 becomes

$$\Delta_c(\widetilde{\sigma}'_S, \rho_{c,S}) = \langle \lambda_c, a'_d \rangle \Delta_{c_1}(\widetilde{\sigma}'_S, \rho_{1,S}).$$

Now the transfer factors in (6.8) are taken at elements that are unipotent (in the general set of §1 and [I, §1]). However, unipotent elements can be approximated by strongly regular conjugacy classes. Moreover, the unipotent element $\dot{\alpha}$ in (6.8) attached to c is the image of the corresponding element $\dot{\alpha}_1$ for c_1 under the canonical class of isomorphisms from $G_{c_1}(\mathbb{A})$ to $G_c(\mathbb{A})$. It follows from the construction [I, §5] of the extended transfer factors that

$$\Delta_{G,c}(\dot{\beta}', \dot{\alpha}) = \langle \lambda_c, a'_d \rangle \Delta_{G,c_1}(\dot{\beta}', \dot{\alpha}_1).$$

We are assuming that G'_d belongs to the complement of $\mathcal{E}^G_{\mathrm{ell}}(G_d, S)$. In other words, the pair (d, G'_d) represents a point $y \in \mathcal{Y}(G)$ that does not lie in the image of the map $\mathcal{X}(G) \to \mathcal{Y}(G)$. It follows from Proposition 3.1 (i) that the element $a'_d \in \ker^1\big(F, Z(\widehat{G}_1)\big)$ maps to a nontrivial element in the group (5.2). Since the map $c \to \lambda_c$ is a bijection from the set of c to the dual of the group (5.2), we see that

$$\sum_c \Delta_{G,c}(\dot{\beta}', \dot{\alpha}) = \Big(\sum_c \langle \lambda_c, a'_d \rangle\Big) \Delta_{G,c_1}(\dot{\beta}', \dot{\alpha}_1) = 0.$$

Lemma 5.1 has thus served us in the form of a vanishing property for the sum (6.8).

Since (6.8) equals zero, the supplementary contributions to (6.7) vanish. The value of (6.7) therefore remains unchanged if G'_d is summed over the entire set $\mathcal{E}_{\mathrm{ell}}(G^*_d, S)$. To be able to match this expanded sum with the definition of $a^{G_c,\mathcal{E}}_{\mathrm{ell}}(\dot{\alpha})$, we require only a couple of elementary observations.

Recall that $\widetilde{G}'_d$ is a central extension of G'_d. The summand of $\dot{\beta}'$ in (6.7) is supported on $\Delta_{\mathrm{unip,der}}(G'_{d,S})$, a subset of $\Delta_{\mathrm{unip}}(G'_{d,S})$ we have agreed to identify with the corresponding subset $\Delta_{\mathrm{unip,der}}(\widetilde{G}'_{d,S}, \widetilde{\eta}'_{d,S})$ of $\Delta_{\mathrm{unip}}(\widetilde{G}'_{d,S}, \widetilde{\eta}'_{d,S})$. Indeed, it is only with this understanding that the transfer factor in (6.7) is defined. Corollary 2.3 (b) asserts that $b^{G'_d}_{\mathrm{ell}}(\dot{\beta}')$ equals $b^{\widetilde{G}'_d}_{\mathrm{ell}}(\dot{\beta}')$. Moreover, the function

$$b^{\widetilde{G}'_d}_{\mathrm{ell}}(\dot{\delta}'_{d,S})\Delta_{G,c}(\dot{\delta}'_{d,S}, \dot{\alpha}), \qquad \dot{\delta}'_{d,S} \in \Delta_{\mathrm{ell}}(\widetilde{G}'_d, S, \widetilde{\eta}'_d),$$

is supported on the subset $\Delta_{\mathrm{unip}}(\widetilde{G}'_{d,S}, \widetilde{\eta}'_{d,S}) \cap \Delta_{\mathrm{ell}}(\widetilde{G}'_d, S, \widetilde{\eta}'_d)$ of $\Delta_{\mathrm{ell}}(\widetilde{G}'_d, S, \widetilde{\eta}'_d)$. We can therefore write (6.7) in the form

$$\sum_c \Big(\sum \iota(G_c, G'_d) \sum b^{\widetilde{G}'_d}_{\mathrm{ell}}(\dot{\delta}'_{d,S})\Delta_{G,c}(\dot{\delta}'_{d,S}, \dot{\alpha}) \Big),$$

where two summations in the brackets are taken over $G'_d \in \mathcal{E}_{\mathrm{ell}}(G^*_d, S)$ and $\dot{\delta}'_{d,S} \in \Delta_{\mathrm{ell}}(\widetilde{G}'_d, S, \widetilde{\eta}'_d)$. Observe that the entire expression inside the brackets equals the right hand side of the streamlined form (1.1*) of the definition of $a^{G_c,\mathcal{E}}_{\mathrm{ell}}(\dot{\alpha})$. This form of the definition is applicable here because our induction hypothesis, coupled with the fact that d is not central, implies that Global Theorem 1(b) is valid for G^*_d. It follows that (6.7) equals the sum

$$\sum_c a^{G_c,\mathcal{E}}_{\mathrm{ell}}(\dot{\alpha}).$$

Now any c that occurs in this sum is F-elliptic in G, and is $G^S(\mathbb{A}^S)$-conjugate to an element in K^S. In particular, the coefficient $i^G(S,c)$ that occurs in the putative formula (1.8) is equal to 1. We can therefore write (6.7) as

$$\sum_c i^G(S,c) a^{G_c,\mathcal{E}}_{\mathrm{ell}}(\dot{\alpha}), \tag{6.9}$$

where c is now summed as in (1.8). This is just the right hand side of the required formula (1.8) of Theorem 1.1. We have shown that it is equal to the sum of (6.1) and (6.3).

We shall now complete the proof of the theorem. Suppose that $\varepsilon(G) = 0$. Then the sum of (6.1) and (6.3) reduces to the left hand side $a_{\mathrm{ell}}^{G,\mathcal{E}}(\dot{\gamma}_S)$ of (1.8). Since this also equals the right hand (6.9) of (1.8), the formula (1.8) follows. Suppose that $\varepsilon(G) = 1$. Then $a_{\mathrm{ell}}^{G,\mathcal{E}}(\dot{\gamma}_S)$ equals $a_{\mathrm{ell}}^{G}(\dot{\gamma}_S)$, by the definition (1.2). If c occurs in the sum (6.9), the dimension of $G_{c,\mathrm{der}}$ is smaller than that of G_{der}, since c does not lie in the center of G. It follows from our induction hypothesis that the term $a_{\mathrm{ell}}^{G_c,\mathcal{E}}(\dot{\alpha})$ on the right hand side (6.9) of (1.8) equals the corresponding term $a_{\mathrm{ell}}^{G_c}(\dot{\alpha})$ on the right hand side of (1.4). The formula (1.8) in this case then follows from (1.4). We have established part (a) of Theorem 1.1, in the case $\mathrm{obs}(c_{\mathbb{A}}) = 1$ we have been considering, and hence in general.

To establish part (b), we assume that $\varepsilon(G) = 1$. It is clear how to combine the identity (1.8) we have just proved with the fact that the sum of (6.1) and (6.3) equals (6.9). We find that the coefficients of $\varepsilon(G)$ in (6.1) and (6.3) are equal when $\mathrm{obs}(c_{\mathbb{A}}) = 1$, and hence in general. Taking into account the properties of d, we see that we can replace the factor $\tau(G^*)\tau(G^*_d)^{-1}$ in (6.3) by the left hand side $j^{G^*}(S,d)$ of (1.7). We conclude that the expression

$$\sum_{\dot{\delta}_S \in \Delta_{\mathrm{ell}}^{\mathcal{E}}(G,S)} b_{\mathrm{ell}}^{G}(\dot{\delta}_S)\Delta_G(\dot{\delta}_S, \dot{\gamma}_S)$$

equals

$$\sum_{\dot{\beta} \in \Delta_{\mathrm{unip}}(G^*_{d,S})} j^{G^*}(S,d) b_{\mathrm{ell}}^{G^*_d}(\dot{\beta})\Delta_G(d\dot{\beta}, \dot{\gamma}_S).$$

We have only to invert these expressions. Let us index the two variables of summation by $\dot{\delta}_{1,S}$ and $\dot{\beta}_1$, instead of $\dot{\delta}_S$ and $\dot{\beta}$. The symbol $\dot{\delta}_S$ is then free to denote a fixed element in $\Delta_{\mathrm{ell}}^{\mathcal{E}}(G,S)$ with semisimple part equal to the image ds of d. We multiply each of the two expressions by the adjoint transfer factor $\Delta_G(\dot{\gamma}_S, \dot{\delta}_S)$, and then sum over $\dot{\gamma}_S$ in $\Gamma_{\mathrm{ell}}^{\mathcal{E}}(G,S,\zeta)$. The adjoint relation [I, (5.5)] provides the desired inversion. We conclude that $b^G_{\mathrm{ell}}(\dot{\delta}_S)$ vanishes unless $\dot{\delta}_S$ has a Jordan decomposition of the form

$$\dot{\delta}_S = d_S\dot{\beta}_S, \qquad \dot{\beta}_S \in \Delta_{\mathrm{unip}}(G^*_{d_S}),$$

in which case

$$b^G_{\mathrm{ell}}(\dot{\delta}_S) = j^G(S,d)b^{G^*_d}_{\mathrm{ell}}(\dot{\beta}),$$

where $\dot{\beta}$ is the image of $\dot{\beta}_S$ in $\Delta_{\mathrm{unip}}(G^*_{d,S})$. This is assertion (b) of the theorem. We have completed the proof of Theorem 1.1. □

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