STABILIZATION OF A FAMILY OF DIFFERENTIAL EQUATIONS

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Abstract. Differential equations have a central place in the invariant harmonic analysis of Harish-Chandra on real groups. Related differential equations also play a role in the noninvariant harmonic analysis that arises from the study of automorphic forms. We shall establish some interesting identities among these latter equations. The identities we obtain are likely to be useful for the comparison of automorphic forms on different groups.

1. The differential equations. Suppose that G is a real, reductive algebraic group, and that T is a maximal torus in G which is defined over $\mathbb{R}$. We write $T_{\mathrm{reg}}(\mathbb{R})$ for the open dense subset of elements in $T(\mathbb{R})$ that are strongly regular, in the sense that their centralizer in G equals T. Harish-Chandra reduced many fundamental questions on the harmonic analysis of $G(\mathbb{R})$ to the study of a family of functions on $T_{\mathrm{reg}}(\mathbb{R})$. We are referring to the invariant orbital integrals

$$I_G(\gamma, f) \;=\; J_G(\gamma, f) \;=\; f_G(\gamma)\,, \qquad\qquad \gamma \in T_{\mathrm{reg}}(\mathbb{R}),\ f \in \mathcal{C}\big(G(\mathbb{R})\big),$$

which can be defined for any function f in Harish-Chandra's Schwartz space $\mathcal{C}\big(G(\mathbb{R})\big)$ by

$$I_G(\gamma, f) \;=\; |D(\gamma)|^{\frac{1}{2}} \int\limits_{T(\mathbb{R})\backslash G(\mathbb{R})} f(x^{-1}\gamma x)dx\,, \qquad\qquad \gamma \in T_{\mathrm{reg}}(\mathbb{R}). \tag{1.1}$$

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(See [HC6].) As usual, dx stands for a $G(\mathbb{R})$-invariant measure on $T(\mathbb{R})\backslash G(\mathbb{R})$,

$$D(\gamma) \ = \ \det\bigl(1 - \mathrm{Ad}(\gamma)\bigr)_{\mathfrak{g}/\mathfrak{t}}$$

is the Weyl discriminant, and $\mathfrak{g}$ and $\mathfrak{t}$ denote the respective Lie algebras of G and T.

Differential equations play a central role in Harish-Chandra's theory of orbital integrals. We first recall the Harish-Chandra homomorphism, which is a map

$$h_T \ = \ h_{T,G} : \ \mathcal{Z} \longrightarrow S\bigl(\mathfrak{t}(\mathbb{C})\bigr),$$

from the center $\mathcal{Z} = \mathcal{Z}(G)$ of the universal enveloping algebra of $\mathfrak{g}(\mathbb{C})$ to the symmetric algebra on $\mathfrak{t}(\mathbb{C})$. The homomorphism is injective, and its image is the subalgebra $S\bigl(\mathfrak{t}(\mathbb{C})\bigr)^{\Omega}$ of elements in $S\bigl(\mathfrak{t}(\mathbb{C})\bigr)$ that are invariant under the Weyl group $\Omega = \Omega(G,T)$ of (G,T). Any element $h \in S\bigl(\mathfrak{t}(\mathbb{C})\bigr)$ can be identified with a translation invariant differential operator $\partial(h)$ on $T(\mathbb{R})$. Harish-Chandra's differential equations take the form

$$I_G(\gamma, zf) \ = \ \partial\bigl(h_T(z)\bigr) I_G(\gamma, f), \qquad z \in \mathcal{Z}. \tag{1.2}$$

It is of course implicit in the equations that $I_G(\gamma, f)$ is a smooth function of $\gamma \in T_{\mathrm{reg}}(\mathbb{R})$. This is a surprisingly deep fact. Harish-Chandra actually used the analogous equations for functions in $C_c^\infty\bigl(G(\mathbb{R})\bigr)$ [HC3, Theorem 3] to establish the absolute convergence of the orbital integral (1.1), the smoothness of the resulting function of γ, and the differential equations (1.2) for Schwartz functions. (See [HC4, Theorem 3], [HC5, Theorem 4], [HC6, Theorem 5], and [HC7, §17].)

Invariant orbital integrals are a special case of a more general family of tempered distributions. These distributions are the weighted orbital integrals, which first arose in the study of automorphic forms. They depend on a Levi component M of some parabolic subgroup of G over $\mathbb{R}$ which contains T, as well as a choice of maximal compact subgroup $K_{\mathbb{R}}$ of $G(\mathbb{R})$. Let $\mathfrak{a}_M$ be the Lie algebra of the $\mathbb{R}$-split component A_M of the center of

M. We assume that $\mathfrak{a}_M$ is orthogonal to the Lie algebra of $K_{\mathbb{R}}$ with respect to the Killing form.

The weighted orbital integral of a Schwartz function $f \in \mathcal{C}\big(G(\mathbb{R})\big)$ is defined by a noninvariant integral

$$(1.3) \qquad J_M(\gamma, f) \;=\; |D(\gamma)|^{\frac{1}{2}} \int_{T(\mathbb{R})\backslash G(\mathbb{R})} f(x^{-1}\gamma x) v_M(x) dx \,, \qquad \gamma \in T_{\mathrm{reg}}(\mathbb{R}),$$

over the conjugacy class of γ. The weight factor

$$v_M(x) \;=\; \lim_{\zeta \to 0} \sum_{P \in \mathcal{P}(M)} v_P(\zeta, x)\theta_P(\zeta)^{-1}$$

is obtained from the (G, M)-family of functions

$$v_P(\zeta, x) \;=\; \mathrm{e}^{-\zeta(H_P(x))}, \qquad P \in \mathcal{P}(M),$$

of $\zeta \in i\mathfrak{a}_M^*$, according to the limit in [A2, Lemma 6.2]. We are following standard notation and terminology, as for example in [A2]. Thus, $\mathcal{P}(M)$ denotes the set of parabolic subgroups $P = MN_P$ of G over F with Levi component M, while $H_P\colon G(\mathbb{R}) \to \mathfrak{a}_M$ is the function of Harish-Chandra defined by the decomposition

$$G(\mathbb{R}) \;=\; N_P(\mathbb{R})M(\mathbb{R})K_{\mathbb{R}}\,.$$

The function $\theta_P(\zeta)$ is a product of linear forms $\zeta(\alpha^\vee)$, taken over the simple roots α of (P, A_M), and scaled by a factor that depends on a choice of metric on $\mathfrak{a}_M$. The integral (1.3) converges absolutely, and defines a smooth function of γ in $T_{\mathrm{reg}}(\mathbb{R})$ [A1, §8]. The proof of these facts exploits the techniques Harish-Chandra applied to (1.1). In particular, it relies on a family of differential equations parametrized by the operators in $\mathcal{Z}$.

The differential equations satisfied by $J_M(\gamma, f)$ are more complicated than the earlier ones. Instead of having just one term, the right hand side consists of a sum of terms, taken over the finite set $\mathcal{L}(M) = \mathcal{L}^G(M)$ of Levi subgroups of G that contain M. If L belongs to $\mathcal{L}(M)$, $S\big(\mathfrak{t}(\mathbb{C})\big)^{\Omega}$ is a subalgebra of $S\big(\mathfrak{t}(\mathbb{C})\big)^{\Omega(L,T)}$. There is consequently an injective homomorphism

$$h_{T,L}^{-1} \circ h_T : \ z \longrightarrow z_L \,, \qquad\qquad z \in \mathcal{Z},$$

from $\mathcal{Z}$ into $\mathcal{Z}(L)$. The differential equations take the form

$$J_M(\gamma, zf) \ = \sum_{L \in \mathcal{L}(M)} \partial_M^L(\gamma, z_L) J_L(\gamma, f)\,, \tag{1.4}$$

where $\partial_M^L(\gamma, z_L)$ is a (real) analytic differential operator on $T_{\mathrm{reg}}(\mathbb{R})$ that depends only on L. (See [A3, Proposition 11.1] or [A1, Lemma 8.5].) For the term with $L = M$, we have

$$\partial_M^M(\gamma, z_M) \ = \ \partial\big(h_{T,M}(z_M)\big) \ = \ \partial\big(h_T(z)\big) \tag{1.5}$$

[A3, Lemma 12.4]. This is just the differential operator that occurs on the right hand side of (1.2).

Weighted orbital integrals have two drawbacks. They depend on $K_{\mathbb{R}}$, and they are not invariant under conjugation of f by $G(\mathbb{R})$. However, there is a natural way to construct a parallel family of distributions with better properties. Following [A5, §3], for example, we define invariant tempered distributions

$$I_M(\gamma, f) \ = \ I_M^G(\gamma, f)\,, \qquad\qquad \gamma \in T_{\mathrm{reg}}(\mathbb{R}),$$

on $G(\mathbb{R})$ inductively by a formula

$$I_M(\gamma, f) \ = \ J_M(\gamma, f) - \sum_{\substack{L \in \mathcal{L}(M) \\ L \neq G}} \widehat{I}_M^L\big(\gamma, \phi_L(f)\big)\,. \tag{1.6}$$

Here

$$\phi_L : \ \mathcal{C}\big(G(\mathbb{R})\big) \longrightarrow \mathcal{I}\big(L(\mathbb{R})\big)$$

is the continuous linear map from $\mathcal{C}\big(G(\mathbb{R})\big)$ to the invariant Schwartz space $\mathcal{I}\big(L(\mathbb{R})\big)$ of $L(\mathbb{R})$, defined by the canonically normalized weighted characters of [A5]. The space

$\mathcal{I}\big(L(\mathbb{R})\big)$ can be identified with the family of functions on $\Pi_{\text{temp}}\big(L(\mathbb{R})\big)$ (the set of irreducible tempered representations of $L(\mathbb{R})$) of the form

$$g_L:\ \pi \longrightarrow g_L(\pi) \ = \ \operatorname{tr}\big(\pi(g)\big)\,, \qquad\qquad g \in \mathcal{C}\big(L(\mathbb{R})\big),$$

while $\widehat{I}^L_M(\gamma)$ is the continuous linear form on $\mathcal{I}\big(L(\mathbb{R})\big)$ defined by

$$\widehat{I}^L_M(\gamma, g_L) \ = \ I^L_M(\gamma, g)\,.$$

The tempered distributions $I_M(\gamma)$ on $G(\mathbb{R})$ are both invariant and independent of the choice of $K_{\mathbb{R}}$. They are the more natural generalizations of invariant orbital integrals.

The invariant distributions satisfy the same differential equations as the noninvariant ones. It follows from the definition of ϕ_L that

$$\phi_L(zf) \ = \ \widehat{z}_L\phi_L(f)\,, \qquad\qquad z \in \mathcal{Z},\ f \in \mathcal{C}\big(G(\mathbb{R})\big),$$

where $\widehat{z}_L$ is the function on $\Pi_{\text{temp}}\big(L(\mathbb{R})\big)$ defined by the infinitesimal character. An easy induction argument on (1.4) and (1.6) then yields differential equations

$$I_M(\gamma, zf) \ = \sum_{L \in \mathcal{L}(M)} \partial^L_M(\gamma, z_L) I_L(\gamma, f)\,, \tag{1.7}$$

for elements $z \in \mathcal{Z}$. Recall that $\partial^M_M(\gamma, z_M)$ is the differential operator that occurs on the right hand side of (1.2). In particular, (1.7) represents a natural generalization of (1.2). We can regard it as a nonhomogeneous family of linear differential equations in an unknown function $\gamma \to I_M(\gamma, f)$. One is often in a position to assume inductively that the functions $\gamma \to I_L(\gamma, f)$ with $L \neq M$ are, in one sense or another, known. The corresponding summands in (1.7) can therefore be regarded as the nonhomogeneous terms.

We are going to investigate the behaviour of the differential equations (1.7) under endoscopic transfer. To allow for induction arguments, it is convenient to work with a slight generalization of the objects above. Suppose that Z is a central torus in G over $\mathbb{R}$, and that ζ is a character on $Z(\mathbb{R})$. We assume that Z is induced, by which we mean that $Z(\mathbb{R})$ is a product of a number of a number of copies of $\mathbb{C}^*$ and of $\mathbb{R}^*$. There is a natural Schwartz space $\mathcal{C}\big(G(\mathbb{R}),\zeta\big)$ of ζ^{-1}-equivariant functions on $G(\mathbb{R})$, and the distributions $I_G(\gamma,f)$, $J_M(\gamma,f)$ and $I_M(\gamma,f)$ can all be defined for functions f in $\mathcal{C}\big(G(\mathbb{R}),\zeta\big)$. The resulting functions of γ then lie in the space $C^\infty\big(T_{\mathrm{reg}}(\mathbb{R}),\zeta\big)$ of smooth, ζ^{-1}-equivariant functions on $T_{\mathrm{reg}}(\mathbb{R})$. We can of course identify $\mathcal{C}\big(G(\mathbb{R}),\zeta\big)$ with a space of sections of a line bundle on $G(\mathbb{R})/Z(\mathbb{R})$. Let $\mathcal{Z}(\zeta)=\mathcal{Z}(G,\zeta)$ be the algebra of $G(\mathbb{R})$-biinvariant differential operators on this line bundle. The differential equations (1.2), (1.4) and (1.7) then hold for functions $f\in\mathcal{C}\big(G(\mathbb{R}),\zeta\big)$ and elements $z\in\mathcal{Z}(\zeta)$.

2. Differential operators and transfer. In this section, we shall consider some elementary points on the transfer of differential operators. We recall that if ϕ: $X\to Y$ is a diffeomorphism between differential manifolds, and ϕ^*: $C^\infty(Y)\to C^\infty(X)$ is the corresponding pullback map of functions,

$$\phi:\ D\longrightarrow \phi D\ =\ (\phi^*)^{-1}\circ D\circ\phi^*$$

is an isomorphism from the space of (smooth) differential operators on X to the corresponding space for Y. A similar remark applies to differential operators on vector bundles. For example, $C^\infty\big(T_{\mathrm{reg}}(\mathbb{R}),\zeta\big)$ is the space of smooth sections of a line bundle on $T_{\mathrm{reg}}(\mathbb{R})/Z(\mathbb{R})$, so we can consider differential operators ∂ on this space. Let $\Omega_{\mathbb{R}}=\Omega_{\mathbb{R}}(G,T)$ be the subgroup of elements in the Weyl group Ω that map $T(\mathbb{R})$ to itself. Then $\Omega_{\mathbb{R}}$ is a group of diffeomorphisms of $T_{\mathrm{reg}}(\mathbb{R})$ that contains the subgroup $\Omega_{\mathbb{R}}^0=\Omega\big(G(\mathbb{R}),T(\mathbb{R})\big)$ of elements in Ω induced from $G(\mathbb{R})$. We shall say that ∂ is $\Omega_{\mathbb{R}}$-invariant if $\omega\partial=\partial$ for every ω in $\Omega_{\mathbb{R}}$.

We would like to examine the behaviour of differential operators on $C^\infty\big(T_{\mathrm{reg}}(\mathbb{R}),\zeta\big)$ under the Langlands-Shelstad transfer factors. Let ψ: $G\to G^*$ be a quasisplit inner twist for G, fixed for once and for all, and let G' be an endoscopic group for G. (See [L-S, (1.2)].)

We assume that G' has a maximal torus T' over $\mathbb{R}$ that is an image of T [L-S, (1.3)]. This means that there is an isomorphism from T to T' over $\mathbb{R}$ of the form

$$\phi \;=\; i^{-1} \circ \mathrm{Int}(h) \circ \psi\,,$$

where i is an admissible embedding of T' into G^* (in the sense of [L-S, (1.3)]), and h is an element in $G^*(\mathbb{C})$ such that $h\psi(T)h^{-1}$ equals $i(T')$. We identify ϕ with the associated diffeomorphism from $T_{\mathrm{reg}}(\mathbb{R})$ onto the set $T'_{G\text{-reg}}(\mathbb{R})$ of strongly G-regular elements in $T'(\mathbb{R})$. The map ϕ is not uniquely determined by T' and T. However, any other such map is of the form $\phi \circ \omega$, for some element $\omega \in \Omega_{\mathbb{R}}$. In particular, the restriction of ϕ to the subgroup Z of T is independent of the choice of ϕ, and allows us to identify Z with a subgroup of T'. If ∂ is a differential operator on $C^\infty\big(T_{\mathrm{reg}}(\mathbb{R}), \zeta\big)$, $\phi\partial$ is then a differential operator on $C^\infty\big(T'_{\mathrm{reg}}(\mathbb{R}), \zeta\big)$. If ∂ is $\Omega_{\mathbb{R}}$-invariant, we shall write $\overline{\partial}' = \phi\partial$, since this differential operator is independent of the choice of ϕ. We shall reserve the symbol ∂' for a differential operator that is more directly related to the transfer factor for G'.

Recall that G' really stands for a full endoscopic datum $(G', \mathcal{G}', s', \xi')$ for G [L-S, (1.2)]. The transfer factor depends on such a datum, as well as some supplementary objects. We fix a central extension $\widetilde{G}'$ of G' by an induced torus $\widetilde{C}'$ over $\mathbb{R}$, and an L-embedding $\widetilde{\xi}'\colon \mathcal{G}' \to {}^L\widetilde{G}'$. (See [L-S, (4.4)] and [A4, §2].) If $T' \subset G'$ is an image of T as above, let $\widetilde{T}'$ be the preimage of T' in $\widetilde{G}'$. The transfer factor can then be defined as a function $\Delta_G(\sigma', \gamma)$ on $\widetilde{T}'_{G\text{-reg}}(\mathbb{R}) \times T_{\mathrm{reg}}(\mathbb{R})$ [L-S, §3 and (4.4)], [K-S, §4 and (5.1)]. The preimage $\widetilde{Z}'$ of Z in $\widetilde{G}'$ is a central induced torus in $\widetilde{G}'$ over $\mathbb{R}$ that contains $\widetilde{C}'$. According to [L-S, Lemma 4.4A], there is a character $\widetilde{\eta}'$ on $\widetilde{Z}'(\mathbb{R})$ such that

$$\Delta_G(\sigma'\varepsilon, \gamma\varepsilon_G) \;=\; \widetilde{\eta}'(\varepsilon)^{-1}\Delta_G(\sigma', \gamma), \tag{2.1}$$

for any element $\varepsilon \in \widetilde{Z}'(\mathbb{R})$ with image ε_G in $Z(\mathbb{R})$. We write $\widetilde{\zeta}'$ for the character on $\widetilde{Z}'(\mathbb{R})$ that is the product of $\widetilde{\eta}'$ with pullback of ζ. (In [A4] and [A6], it was $\widetilde{C}'$ and $\widetilde{\eta}'$ that were denoted by $\widetilde{Z}'$ and $\widetilde{\zeta}'$.)

The role of the transfer factor is of course to map functions on $T_{\mathrm{reg}}(G)$ to functions on $\widetilde{T}'_{G\text{-reg}}(\mathbb{R})$. If a_G belongs to $C^\infty\big(T_{\mathrm{reg}}(\mathbb{R}),\zeta\big)$, the function

$$a'(\sigma') \;=\; \sum_{\gamma} \Delta_G(\sigma',\gamma)a_G(\gamma)\,, \qquad \sigma' \in \widetilde{T}'_{G\text{-reg}}(\mathbb{R}), \tag{2.2}$$

defined by a sum over γ in the set of $\Omega^0_{\mathbb{R}}$-orbits in $T_{\mathrm{reg}}(\mathbb{R})$, belongs to $C^\infty\big(\widetilde{T}'_{G\text{-reg}}(\mathbb{R}),\widetilde{\zeta}'\big)$. The sum can be taken over a finite set, of order $|\Omega^0_{\mathbb{R}}\backslash\Omega_{\mathbb{R}}|$, that represents the set of conjugacy classes in the stable conjugacy class of σ' in $G(\mathbb{R})$. We shall extend the differential operator $\overline{\partial}'$ on $C^\infty\big(T'_{G\text{-reg}}(\mathbb{R})$, defined above, to a differential operator ∂' on $C^\infty\big(\widetilde{T}'_{G\text{-reg}}(\mathbb{R}),\widetilde{\zeta}'\big)$.

Following Shelstad [S3], we first construct a homomorphism from $\mathbb{C}^*$ into $Z(\widehat{\widetilde{G}'})^0$, where as usual, $Z(\widehat{\widetilde{G}'})$ denotes the center of the dual group $\widehat{\widetilde{G}'}$ of $\widetilde{G}'$. Recall that G' represents an endoscopic datum $(G',\mathcal{G}',s',\xi')$, in which $\mathcal{G}'$ is a split extension of the Weil group $W_{\mathbb{R}}$ of $\mathbb{R}$ by $\widehat{G}'$, and ξ' is an L-embedding of $\mathcal{G}'$ into ${}^LG$. For any $u\in\mathbb{C}^*$, we choose a point $\delta'(u)\in\mathcal{G}'$ such that the element $\xi'\big(\delta'(u)\big)$ in ${}^LG$ is of the form $\varepsilon(u)\times u$, for some $\varepsilon(u)\in Z(\widehat{G})$. The element $\widetilde{\xi}'\big(\delta'(u)\big)$ in ${}^L\widetilde{G}'$ is then of the form $\widetilde{\varepsilon}'(u)\times u$, for some $\widetilde{\varepsilon}'(u)\in z(\widehat{\widetilde{G}'})$. Now $Z(\widehat{G})^0$ is a subgroup of $Z(\widehat{\widetilde{G}})^0$. The quotient $\varepsilon(u)\widetilde{\varepsilon}'(u)^{-1}$ is independent of the choice of $\delta'(u)$, and we obtain a homomorphism

$$u \longrightarrow z(u) \;=\; \varepsilon(u)\widetilde{\varepsilon}'(u)^{-1}\,, \qquad u\in\mathbb{C}^*,$$

from $\mathbb{C}^*$ into $Z(\widehat{\widetilde{G}'})^0$.

Let $X = X_*\big(Z(\widehat{\widetilde{G}'})^0\big)$ be the dual of the character module of $Z(\widehat{\widetilde{G}'})^0$. We can identify $X\otimes\mathbb{C}$ with the Lie algebra of $Z(\widehat{\widetilde{G}'})^0$, which in turn is an extension of the Lie algebra of $\widehat{\widetilde{Z}'}$. We write

$$z(u) \;=\; u^{\mu'}\overline{u}^{\nu'}\,, \qquad u\in\mathbb{C}^*,$$

for elements $\mu',\nu'\in X\otimes\mathbb{C}$ such that $\mu'-\nu'$ lies in X.

Lemma 2.1. *The projection of μ' onto the Lie algebra of $\widehat{\widetilde{Z}'}$ equals the differential $d\widetilde{\eta}'$ of the character $\widetilde{\eta}'$ on $\widetilde{Z}'(\mathbb{R})$.*

Proof. The proof is an exercise in a few of the basic definitions from [L-S]. We can take $\widetilde{G}'$ to be an endoscopic datum for $\widetilde{G}$, where $\widetilde{G}$ is a z-extension of G [K, §1]. Given our construction of μ', and the definition of the transfer factor for (G, G') in terms of $(\widetilde{G}', \widetilde{G})$ [L-S, (4.4)], we see that the lemma holds for G if it can be established for $\widetilde{G}$. We may therefore assume that $\widetilde{G}' = G'$, $\mathcal{G}' = {}^L G'$, and $\widetilde{\xi}' = 1$. Consequently, $\widetilde{Z}'$ equals Z, and

$$\xi'(u) \;=\; z(u) \times u \;=\; u^{\mu'}\overline{u}^{\nu'} \times u\,, \qquad u \in \mathbb{C}^*.$$

The character $\widetilde{\eta}'$ is constructed in the proof of [L-S, Lemma 4.4A]. It equals the restriction to $Z(\mathbb{R})$ of a character on $T(\mathbb{R})$ that is defined by the Langlands correspondence for tori. That is, $\widetilde{\eta}'$ is dual to a 1-cocycle of $W_{\mathbb{R}}$ in $\widehat{Z}$ which is the composition of a 1-cocycle a of $W_{\mathbb{R}}$ in $\widehat{T}$ with the projection $\widehat{T} \to \widehat{Z}$. The latter is defined in [L-S, (3.5)]. It is given by

$$a(w) \;=\; \xi'\big(\xi'_T(w)\big)\xi_T(w)^{-1}\,, \qquad w \in W_{\mathbb{R}},$$

for certain L-embeddings ξ'_T: ${}^L T \to {}^L G'$ and ξ_T: ${}^L T \to {}^L G$. The L-embeddings are such that they have the same image under the projections of ${}^L G'$ and ${}^L G$ onto ${}^L Z$. Thus, $a(w)$ and $\xi'(w)$ have the same image in $\widehat{Z}$, and $\widetilde{\eta}'$ is dual to the 1-cocycle of $W_{\mathbb{R}}$ in $\widehat{Z}$ obtained from ξ'. The lemma then follows easily from the Langlands correspondence for Z. □

A differential operator ∂ on $T_{\mathrm{reg}}(\mathbb{R})$ can of course be identified with its symbol. This is a function

$$\partial(\gamma):\; \lambda \longrightarrow \partial(\gamma, \lambda)\,, \qquad \gamma \in T_{\mathrm{reg}}(\mathbb{R}),\; \lambda \in \mathfrak{t}^*(\mathbb{C}),$$

on $T_{\mathrm{reg}}(\mathbb{R})$, with values in the algebra of polynomials on the dual space $\mathfrak{t}^*(\mathbb{C})$ of $\mathfrak{t}(\mathbb{C})$. More generally, let $\mathfrak{t}^*(\mathbb{C}, -d\zeta)$ be the affine subspace of elements in $\mathfrak{t}^*(\mathbb{C})$ whose projections onto the Lie algebra of $\widehat{Z}$ equal $-d\zeta$. The symbol of a differential operator ∂ on $C^\infty\big(T_{\mathrm{reg}}(\mathbb{R}), \zeta\big)$ is then a function $\partial(\gamma, \xi)$ on $T_{\mathrm{reg}}(\mathbb{R})/Z(\mathbb{R})$ with values in the algebra of polynomial functions on the affine space $\mathfrak{t}^*(\mathbb{C}, -d\zeta)$. Now the dual $(\mathfrak{t}')^*$ of the Lie algebra of T' is contained in the dual $(\widetilde{\mathfrak{t}}')^*$ of the Lie algebra of $\widetilde{T}'$. We can certainly identify the vector μ' above with an element in $(\widetilde{\mathfrak{t}}')^*(\mathbb{C})$. It is an easy consequence of Lemma 2.1 that

$$\lambda' \longrightarrow \overline{\lambda}' \;=\; \lambda' + \mu'$$

is an isomorphism from the affine space $(\widetilde{\mathfrak{t}}')^*(\mathbb{C}, -d\widetilde{\zeta}')$ onto the affine space $(\mathfrak{t}')^*(\mathbb{C}, -d\zeta)$. It follows that there is a canonical map $\overline{\partial}' \to \partial'$, from the space of differential operators on $C^\infty\big(T'_{\mathrm{reg}}(\mathbb{R}), \zeta\big)$ to the space of differential operators on $C^\infty\big(\widetilde{T}'_{G\text{-reg}}(\mathbb{R}), \widetilde{\zeta}'\big)$, such that

$$\partial'(\sigma', \lambda') \;=\; \overline{\partial}'(\overline{\sigma}', \overline{\lambda}'), \qquad \lambda' \in (\widetilde{\mathfrak{t}}')^*(\mathbb{C}, -d\widetilde{\zeta}'). \tag{2.3}$$

Here, σ' denotes a point in $\widetilde{T}'_{G\text{-reg}}(\mathbb{R})$ with image $\overline{\sigma}'$ in $T'_{G\text{-reg}}(\mathbb{R})$. In particular, the composition

$$\partial \longrightarrow \overline{\partial}' \longrightarrow \partial'$$

provides a map from the space of $\Omega_{\mathbb{R}}$-invariant differential operators on $C^\infty\big(T_{\mathrm{reg}}(\mathbb{R}), \zeta\big)$ to the space of differential operators on $C^\infty\big(\widetilde{T}'_{G\text{-reg}}(\mathbb{R}), \widetilde{\zeta}'\big)$.

The following lemma, which is more or less implicit in the work of Shelstad, justifies the construction.

Lemma 2.2. *If ∂ is a $\Omega_{\mathbb{R}}$-invariant differential operator on $C^\infty\big(T_{\mathrm{reg}}(\mathbb{R}), \zeta\big)$, we have*

$$(\partial a_G)' \;=\; \partial' a', \qquad a_G \in C^\infty\big(T_{\mathrm{reg}}(\mathbb{R}), \zeta\big).$$

Proof. As in Lemma 2.1, it is easy to reduce the proof to a special case. It follows from the definition of ∂', the formula (2.1), and the reduction in [L-S, (4.4)], that the required identity,

$$\sum_\gamma \Delta_G(\sigma', \gamma)(\partial a_G)(\gamma) \;=\; \partial'\Big(\sum_\gamma \Delta_G(\sigma', \gamma) a_G(\gamma)\Big), \tag{2.4}$$

holds for G if it can be established for a z-extension $\widetilde{G}$ of G. We may therefore assume that $\widetilde{G}' = G'$, $\mathcal{G}' = {}^L G'$, and $\widetilde{\xi}' = 1$. In particular, we can use the basic construction of the transfer factor in [L-S, §3].

Let C be a small open set in $T_{\mathrm{reg}}(\mathbb{R})$, and let C be a fixed image [L-S, (1.3)] of C in $T'_{G\text{-reg}}(\mathbb{R})$. Then for any $\sigma' \in C'$, there is a unique point $\gamma' \in C$ such that $\Delta_G(\sigma', \gamma') \neq 0$. The sums in (2.4) are over orbits γ of $\Omega^0_{\mathbb{R}}$ in $T_{\mathrm{reg}}(\mathbb{R})$. They can each be replaced by a product of $|\Omega^0_{\mathbb{R}}|^{-1}$ with the corresponding sum over the points γ in $\{\omega\gamma' : \ \omega \in \Omega_{\mathbb{R}}\}$. It is not hard to see from the definitions in [L-S, (3.4)] that the functions

$$\sigma' \longrightarrow \Delta_G(\sigma', \omega\gamma')\Delta_G(\sigma', \gamma')^{-1}, \qquad \sigma' \in C', \ \omega \in \Omega_{\mathbb{R}},$$

are constant. (See for example [K-S, Theorem 5.1.D].) Since ∂ is $\Omega_{\mathbb{R}}$-invariant, we see that (2.4) would follow if we could prove the identity

$$\Delta_G(\sigma', \gamma') \circ \partial \ = \ \partial' \circ \Delta_G(\sigma', \gamma'), \qquad \sigma' \in C'.$$

Now $\Delta_G(\sigma', \gamma')$ is a product of four terms, which depend individually on a fixed admissible embedding i: $T' \to T^*$ of T' into G^* (as well as auxiliary a- and χ-data). (The fifth term $\Delta_{\mathrm{IV}}(\sigma', \gamma')$ from [L-S, (3.6)] does not occur, since we have included the factor $|D(\gamma)|^{\frac{1}{2}}$ in the definition of orbital integral.) The terms $\Delta_{\mathrm{II}}(\sigma', \gamma')$ and $\Delta_2(\sigma', \gamma')$ in the product are defined explicitly as functions of the image σ^* of σ' in $T^*_{\mathrm{reg}}(\mathbb{R})$ [L-S, (3.3), (3.5)]. The other two terms are independent of (σ', γ'). Identifying ∂ and ∂' with differential operators on $T^*_{\mathrm{reg}}(\mathbb{R})$, we conclude that (2.4) would be valid if we could establish the identity

$$\big(\Delta_{\mathrm{II}}(\sigma^*)\Delta_2(\sigma^*)\big) \circ \partial \ = \ \partial' \circ \big(\Delta_{\mathrm{II}}(\sigma^*)\Delta_2(\sigma^*)\big). \tag{2.5}$$

The proof of (2.5) is not difficult, but would entail a recapitulation of a number of other notions from [L-S]. Rather than take the paper too far from its original focus, we shall leave the details to the reader. It is instructive to compare the definitions in [L-S, (3.3), (3.5)] with the earlier constructions of Shelstad [S.4, (3.1)-(3.3)] for real groups.

Shelstad is actually planning a paper that would relate the results of [S.1]–[S.4] with the general definitions of [L-S]. $\square$

We have two examples in mind. The first is the standard case of a $\Omega_{\mathbb{R}}$-invariant differential operator ∂ on $C^\infty\big(T_{\mathrm{reg}}(\mathbb{R}),\zeta\big)$ of constant coefficients. The symbol of ∂ is then an $\Omega_{\mathbb{R}}$-invariant polynomial on $\mathfrak{t}^*(\mathbb{C},-d\zeta)$. In other words, ∂ equals $\partial\big(h_T(z)\big)$, for a differential operator z in the algebra $\mathcal{Z}(G,\zeta)$ defined at the end of §1. The symbol of $\overline{\partial}\big(h_T(z)\big)'$ is a polynomial on $(\mathfrak{t}')^*(\mathbb{C},-d\zeta)$ that is $\Omega(G',T')$-invariant. The symbol of $\partial\big(h_T(z)\big)'$ is a polynomial on $(\widetilde{\mathfrak{t}}')^*(\mathbb{C},-d\widetilde{\zeta}')$ that is easily seen to be invariant under the Weyl group $\Omega(\widetilde{G}',\widetilde{T}')$. We obtain injective homomorphisms $z\to\overline{z}'$ and $z\to z'$ from $Z(G,\zeta)$ to $Z(G',\zeta)$ and $Z(\widetilde{G}',\widetilde{\zeta}')$ respectively, such that

$$\overline{\partial}\big(h_T(z)\big)' \;=\; \partial\big(h_{T'}(\overline{z}')\big)$$

and

$$\partial\big(h_T(z)\big)' \;=\; \partial\big(h_{\widetilde{T}'}(z')\big)\,. \tag{2.6}$$

The other example comes from the differential equations (1.7). As we remarked at the end of §1, these equations hold if f is a function in $\mathcal{C}\big(G(\mathbb{R}),\zeta\big)$ and z is an operator in $\mathcal{Z}(G,\zeta)$. Recall that M is a Levi subgroup of G that contains T. Then

$$\partial_M^G(z) \;=\; \partial_M^G(\gamma,z)\,, \qquad \gamma\in T_{\mathrm{reg}}(\mathbb{R}),$$

is a differential operator on $C^\infty\big(T_{\mathrm{reg}}(\mathbb{R}),\zeta\big)$. (As in §1, we shall usually include γ in the notation to keep track of the variable of differentiation.)

Lemma 2.3. *For any $z\in\mathcal{Z}(G,\zeta)$, the differential operator $\partial_M^G(z)$ is $\Omega_{\mathbb{R}}(M,T)$-invariant.*

Proof. The lemma is a consequence of the construction in [A3, §12], the point of which was to compute the operators $\partial_M^L(z_L)=\partial_M^L(\gamma,z_L)$ from the nonstandard terms in Harish-Chandra's radial decomposition [HC1]. Suppose that a given element $\omega\in\Omega_{\mathbb{R}}(M,T)$

is induced from the adjoint action of an element $m \in M(\mathbb{C})$. Then $\mathrm{Ad}(m)$ maps z to itself. Therefore $\mathrm{Ad}(m)$ also maps the right hand side of the equation [A3, (12.1)] to itself. Combining this property with the various definitions from [A3, §12], one sees without difficulty that $\partial_M^G(z)$ is invariant under ω. (See the remarks on p. 286 of [A3].) $\square$

We can therefore apply the earlier discussion (with G replaced by M) to the operator $\partial_M^G(z)$. It follows from Lemma 2.3 that $\overline{\partial}_M^G(z)'$ and $\partial_M^G(z)'$ are well defined differential operators on $C^\infty\big(T'_{G\text{-reg}}(\mathbb{R}), \zeta\big)$ and $C^\infty\big(\widetilde{T}'_{G\text{-reg}}(\mathbb{R}), \widetilde{\zeta}'\big)$ respectively. To match the notation of §1, we shall generally write

$$\overline{\partial}_M^G(z)' \;=\; \partial_M^G(\overline{\sigma}', z)\,, \qquad \overline{\sigma}' \in T'_{G\text{-reg}}(\mathbb{R}),$$

and

$$\partial_M^G(z)' \;=\; \partial_M^G(\sigma', z)\,, \qquad \sigma' \in \widetilde{T}'_{G\text{-reg}}(\mathbb{R}). \tag{2.7}$$

3. The stabilization. We shall now stabilize the differential operators that occur in the equation (1.7). Lemma 2.2, or rather its analogue for M, provides a stabilization of sorts for these differential operators. However, we are looking for something more. We would like to stabilize the operators $\partial_M^G(\gamma, z)$ as functions of z, as well as of γ. We shall apply a construction that is suggested by the conjectural stabilization of weighted orbital integrals in [A5] and [A6].

We begin by allowing M to play the role of G in the general discussion of §2. In particular, we let M' denote a fixed endoscopic datum $(M', \mathcal{M}', s'_M, \xi'_M)$ for M. We assume that M' is elliptic [A4, §2], and that $T' \subset M'$ is a fixed maximal torus over $\mathbb{R}$ that is an image of T (relative to M). We also fix a central extension $\widetilde{M}'$ of M by an induced torus $\widetilde{C}'$, and an L-embedding $\widetilde{\xi}'_M\colon\ \mathcal{M}' \to {}^L\widetilde{M}'$. We then obtain objects $\widetilde{Z}'$, $\widetilde{\eta}'$ and $\widetilde{\zeta}'$, relative to M. There is no loss of generality in assuming that $\mathcal{M}'$ is actually an L-subgroup of ${}^LM$ and that ξ'_M is the identity embedding of $\mathcal{M}'$ into ${}^LM$. This allows us to introduce the family $\mathcal{E}_{M'}(G)$ of endoscopic data $(G', \mathcal{G}', s', \xi')$ for G that was defined in [A6]. The objects in $\mathcal{E}_{M'}(G)$ are taken only modulo translation of s' by $Z(\widehat{G})^\Gamma$, the group of invariants in $Z(\widehat{G})$ under $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. They are parametrized by $Z(\widehat{M})^\Gamma / Z(\widehat{G})^\Gamma$, and are constructed in a simple way from M'. The choices of $\widetilde{M}'$ and $\widetilde{\xi}'_M$ for M determine a central extension $\widetilde{G}'$ of any G' in $\mathcal{E}_{M'}(G)$ by $\widetilde{C}'$, and an L-embedding $\mathcal{G}' \to {}^L\widetilde{G}'$. (See [A6, §3].) The set $\mathcal{E}_{M'}(G)$ is infinite if $M \neq G$. However, there are only finitely many elements G' in $\mathcal{E}_{M'}(G)$ that are elliptic, or equivalently, such that the coefficient

$$\iota_{M'}(G, G') \; = \; |Z(\widehat{M}')^\Gamma / Z(\widehat{M})^\Gamma| \, |Z(\widehat{G}')^\Gamma / Z(\widehat{G})^\Gamma|^{-1}$$

is nonzero.

The stabilization of $\partial_M^G(\gamma, z)$ is an example of a general construction that includes both a definition and an identity that has to be proved. We shall state it as a theorem. Since the construction is inductive, we shall formulate it so that the objects (G, T, M, ζ) and (M', T') (together with $(\widetilde{M}', \widetilde{\xi}'_M)$) above are allowed to vary.

Theorem 3.1. *For any (G, T, M, ζ), with G quasisplit, there are differential operators*

$$\delta_M^G(\sigma, z)\,, \qquad\qquad \sigma \in T_{\mathrm{reg}}(\mathbb{R}),\ z \in \mathcal{Z}(G, \zeta),$$

on $C^\infty\big(T_{\mathrm{reg}}(\mathbb{R}), \zeta\big)$ with the following property. For any (G, T, M, ζ) at all, and any (M', T'), the identity

$$\text{(3.1)} \qquad \partial_M^G(\sigma', z) \; = \sum_{G' \in \mathcal{E}_{M'}(G)} \iota_{M'}(G, G') \delta_{\widetilde{M}'}^{\widetilde{G}'}(\sigma', z') \qquad\qquad \sigma' \in \widetilde{T}'_{G\text{-reg}}(\mathbb{R}),$$

is valid for every operator $z \in \mathcal{Z}(G, \zeta)$.

The proof of the theorem will take up the rest of the section. There will be two ingredients. One is the construction of the differential operators $\partial_M^G(\gamma, z)$ [A3, §12] that was based on Harish-Chandra's radial decomposition [HC1]. The other, which will come later, is a stable descent formula like those of [A6, §7].

We shall first observe that $\partial_M^G(\gamma, z)$ comes from a (G, M)-family. Following notation from [A3, §12], we set

$$\partial_P(\Lambda, \gamma, z) \;=\; \sum_{i=1}^{r} \big\langle \mu_P(X_i), \Lambda \big\rangle \partial_i(\gamma, z)\,, \tag{3.2}$$

for any $P \in \mathcal{P}(M)$ and $\Lambda \in i\mathfrak{a}_M^*$. For each i, $\partial_i(\gamma, z)$ can be taken to be a differential operator on $C^\infty\big(T_{\mathrm{reg}}(\mathbb{R}), \zeta\big)$, and $\mu_P(X_i)$ is an element in the symmetric algebra $S\big(\mathfrak{a}_M(\mathbb{C})\big)$. It follows from the definition of $\mu_P(X_i)$ on p. 283 of [A3] that

$$\big\langle \mu_P(X_i), \Lambda \big\rangle\,, \qquad P \in \mathcal{P}(M),$$

is a (G, M)-family. Therefore

$$\partial_P(\Lambda, \gamma, z)\,, \qquad P \in \mathcal{P}(M),$$

is a (G, M)-family of functions of Λ, with values in the space of differential operators on $C^\infty\big(T_{\mathrm{reg}}(\mathbb{R}), \zeta\big)$. It is then an easy consequence of Lemma 12.1 of [A3] that

$$\partial_M^G(\gamma, z) \;=\; \lim_{\Lambda \to 0} \sum_{P \in \mathcal{P}(M)} \partial_P(\Lambda, \gamma, z)\theta_P(\Lambda)^{-1}\,.$$

The next step will be to reduce Theorem 3.1 to a statement about the complex group $G(\mathbb{C})$. For this, we shall assume that ζ equals 1 (the trivial character on the trivial torus $Z = 1$). The terms $\partial_i(\gamma, z)$ in (3.2) are real analytic differential operators on $T_{\mathrm{reg}}(\mathbb{R})$. One of the points of the construction is that these operators do not really depend on the real form $\mathfrak{g}(\mathbb{R})$ of $\mathfrak{g}(\mathbb{C})$. They are the restrictions of complex analytic differential operators obtained by applying the construction to the complex group $G(\mathbb{C})$. We have been working strictly with real groups, so we shall write $H_1 = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(H_{\mathbb{C}})$, for any group H over $\mathbb{R}$. Then M_1 is a Levi subgroup of G_1, and $\mathfrak{a}_M$ comes with a linear isometric embedding into the real vector space $\mathfrak{a}_{M_1}$ (relative to suitable metrics on the two spaces). In fact, the triplet $(\mathbb{R}, G_1, M_1)$ satisfies the conditions in [A6, §4] for being a satellite of $(\mathbb{R}, G, M)$.

Let $z_1 \in \mathcal{Z}(G_1)$ be the biinvariant complex analytic differential operator on $G_1(\mathbb{R}) = G(\mathbb{C})$ corresponding to a given $z \in \mathcal{Z}(G)$. We could certainly apply the definition (3.2) to (G_1, M_1, T_1, z_1). We can equally well apply it to (G_1, T_1, T_1, z_1), since T_1 is also a Levi subgroup of G_1. Taking the latter course, we obtain a (G_1, T_1)-family of functions

$$\partial_{P_1}(\Lambda_1, \gamma_1, z_1)\,, \qquad\qquad P_1 \in \mathcal{P}(T_1),$$

of $\Lambda_1 \in i\mathfrak{a}^*_{T_1}$, with values in the space of complex analytic differential operators on $T_{1,\mathrm{reg}}(\mathbb{R}) = T_{\mathrm{reg}}(\mathbb{C})$. Now in each of these functions, we can restrict Λ_1 to the linear subspace $i\mathfrak{a}^*_M$ of $i\mathfrak{a}^*_{T_1}$, and we also can restrict γ_1 to the real submanifold $T_{\mathrm{reg}}(\mathbb{R})$ of $T_{\mathrm{reg}}(\mathbb{C})$. The first operation is an example of a general procedure in [A6, §4]. It provides a (G, M)-family of functions

$$\partial_P(\Lambda, \gamma_1, z_1) \;=\; \partial_{P_1}(\Lambda, \gamma_1, z_1)\,, \qquad\qquad P \in \mathcal{P}(M),$$

of $\Lambda \in i\mathfrak{a}^*_M$, where $P_1 \in \mathcal{P}(M_1)$ is any group whose closed chamber in $\mathfrak{a}_{T_1}$ contains the chamber of P in $\mathfrak{a}_M$. The second operation, the restriction of γ_1, then gives a (G, M)-family of functions

$$\partial_P(\Lambda, \gamma, z_1)\,, \qquad\qquad P \in \mathcal{P}(M),$$

with values in the space of real analytic differential operators on $T_{\mathrm{reg}}(\mathbb{R})$. It follows easily from the construction of [A3] that this last (G, M)-family is the same as the first family (3.2). We conclude that the original differential operator $\partial^G_M(\gamma, z)$ equals the restriction to $T_{\mathrm{reg}}(\mathbb{R})$ of the complex analytic differential operator

$$\partial^G_M(\gamma_1, z_1) \;=\; \lim_{\Lambda \to 0} \sum_{P \in \mathcal{P}(M)} \partial_P(\Lambda, \gamma_1, z_1)\theta_P(\Lambda)^{-1}$$

on $T_{1,\mathrm{reg}}(\mathbb{R})$.

Suppose that (M', T') is an in Theorem 3.1. Then M'_1 represents an endoscopic datum for M_1, and T'_1 is a maximal torus in M'_1 over $\mathbb{R}$. Since M'_1 amounts to an endoscopic datum for a complex group, we can take M_1' itself for the extension of §2. We will have no need of the group $\widetilde{M}_1'$ obtained from $\widetilde{M}'$. By the conventions of §2, we can form the complex analytic differential operator

$$\partial_M^G(\sigma_1', z_1) \;=\; \lim_{\Lambda\to 0} \sum_{P\in\mathcal{P}(M)} \partial_P(\Lambda, \sigma_1', z_1)\theta_P(\Lambda)^{-1}$$

on $T'_{1,G\text{-reg}}(\mathbb{R})$. It is clear that $\partial_M^G(\sigma_1', z) = \partial_M^G(\overline{\sigma}_1', z_1)$, in the notation at the end of §2, and that the restriction of this complex analytic differential operator to $T'_{G\text{-reg}}(\mathbb{R})$ equals

$$\partial_M^G(\overline{\sigma}', z) \;=\; \lim_{\Lambda\to 0} \sum_{P\in\mathcal{P}(M)} \partial_P(\Lambda, \overline{\sigma}', z)\theta_P(\Lambda)^{-1}\,.$$

Let us summarize the discussion so far. Given objects (G, T, M), (M', T') and $z \in \mathcal{Z}(G)$ as in the theorem, we obtain a real analytic differential operator $\partial_M^G(\overline{\sigma}', z)$ on $T'_{G\text{-reg}}(\mathbb{R})$ that comes from a (G, M)-family. The original objects also determine a second set of objects (G_1, T_1, M_1), (M_1', T_1') and $z_1 \in \mathcal{Z}(G_1)$. From these, we construct a complex analytic differential operator $\partial_M^G(\sigma_1', z_1)$ on $T'_{1,G\text{-reg}}(\mathbb{R})$, that comes from the (G, M)-image of a (G_1, T_1)-family. The restriction of this second differential operator to the real submanifold $T'_{G\text{-reg}}(\mathbb{R})$ of $T'_{1,G\text{-reg}}(\mathbb{R})$ is then equal to the original one.

We can now state a lemma that is essentially a reformulation of the theorem in terms of G_1.

Lemma 3.2. *For any (G, T, M), with G quasisplit, there are complex analytic differential operators*

$$\delta_M^G(\sigma_1, z_1)\,, \qquad\qquad \sigma_1 \in T_{1,\mathrm{reg}}(\mathbb{R}),\ z \in \mathcal{Z}(G),$$

on $T_{1,\mathrm{reg}}(\mathbb{R})$ such that for any (G, T, M) at all, and any (M', T'), the identity

$$\textit{(3.3)}\qquad \partial_M^G(\sigma_1', z_1) \;=\; \sum_{G'\in\mathcal{E}_{M'}(G)} \iota_{M'}(G, G')\delta_{M'}^{G'}(\sigma_1', z_1')\,, \qquad \sigma_1' \in T'_{1,G\text{-reg}}(\mathbb{R}),$$

is valid for every $z \in \mathcal{Z}(G)$.

Proof. The operators $\delta_M^G(\sigma_1, z_1)$ are uniquely determined by the special case of the identity (3.3) in which G is quasisplit, and $(M', T') = (M, T)$. In this case, we define

$$\delta_M^G(\sigma_1, z_1) \;=\; \partial_M^G(\sigma_1, z_1) - \sum_{G' \in \mathcal{E}_M^0(G)} \iota_M(G, G')\delta_M^{G'}(\sigma_1, z_1')\,, \tag{3.4}$$

where $\mathcal{E}_M^0(G)$ denotes the set of elements $G' \in \mathcal{E}_M(G)$ with $G' \neq G$. Since the coefficient $\iota_M(G, G')$ vanishes unless G' is elliptic, the sum can be taken over a finite set. Once having defined the operators $\delta_M^G(\sigma_1, z_1)$, we must then establish the identity (3.3) in general. We have to show that for any M' and T', the endoscopic differential operator

$$\partial_M^{G,\mathcal{E}}(\sigma_1', z_1) \;=\; \sum_{G' \in \mathcal{E}_{M'}(G)} \iota_{M'}(G, G')\delta_{M'}^{G'}(\sigma_1', z_1')\,, \qquad \sigma_1' \in T'_{1,G\text{-reg}}(\mathbb{R}),$$

equals $\partial_M^G(\sigma_1', z_1)$. We reiterate that if (M', T') is isomorphic to (M, T), there is nothing to prove. The identity in this case is just the definition of $\delta_M^G(\sigma_1, z_1)$ above.

This is the stage that we apply the descent formulas. The operator $\partial_M^G(\sigma_1', z_1)$ is defined in terms of the (G, M)-family $\{\partial_P(\Lambda, \sigma_1', z_1)\}$. However, this (G, M)-family is obtained by the restriction process of [A6, §4] from a (G_1, T_1)-family $\{\partial_{P_1}(\Lambda_1, \sigma_1', z_1)\}$. This was of course our reason for going to the complex group. One sees easily from [3, §12] that the construction of the operators $\partial_{P_1}(\Lambda_1, \sigma_1', z_1)$ for z_1 is compatible with the construction of the corresponding operators for $z_{L_1} = z_{1,L_1}$. It follows from [A6, Lemma 4.1] that

$$\partial_M^G(\sigma_1', z_1) \;=\; \sum_{L_1 \in \mathcal{L}(T_1)} d_{T_1}^G(M, L_1)\partial_{T_1}^{L_1}(\sigma_1', z_{L_1})\,. \tag{3.5}$$

Here $d_{T_1}^G(M, L_1)$ is the familiar Jacobian of a linear map

$$\mathfrak{a}_{T_1}^M \oplus \mathfrak{a}_{T_1}^{L_1} \longrightarrow \mathfrak{a}_{T_1}^G\,,$$

described for example in the preamble to [A6, Lemma 4.1]. Notice that for any L_1, $\partial_{T_1}^{L_1}(\sigma_1', z_{L_1})$ equals $\partial_{T_1}^{L_1,\mathcal{E}}(\sigma_1', z_{L_1})$ by definition, since the torus T_1' is isomorphic to T_1. To complete the proof of Lemma 3.2, then, we have only to show that $\partial_M^{G,\mathcal{E}}(\sigma_1', z_1)$ satisfies a descent formula that is parallel to (3.5).

Lemma 3.3. *(a) Suppose that (G, T, M) and (M', T') are as in (3.3). Then*

$$\partial_M^{G,\mathcal{E}}(\sigma_1', z_1) \;=\; \sum_{L_1 \in \mathcal{L}(T_1)} d_{T_1}^G(M, L_1)\partial_{T_1}^{L_1,\mathcal{E}}(\sigma_1', z_{L_1})\,. \tag{3.6}$$

(b) Suppose that (G, M, T) is given, with G quasisplit. Then

$$\delta_M^G(\sigma_1, z_1) \;=\; \sum_{L_1 \in \mathcal{L}(T_1)} \mathrm{e}_{T_1}^G(M, L_1)\delta_{T_1}^{L_1}(\sigma_1, z_{L_1})\,, \tag{3.7}$$

where

$$\mathrm{e}_{T_1}^G(M, L_1) \;=\; d_{T_1}^G(M, L_1)\big|Z(\widehat{M})^\Gamma \cap Z(\widehat{L}_1)^\Gamma / Z(\widehat{G})^\Gamma\big|^{-1}\,.$$

(According to the conventions of [A6], $Z(\widehat{M})$ and $Z(\widehat{L}_1)$ are both embedded subgroups of a fixed dual torus $\widehat{T}_1$ of T_1 in $\widehat{G}_1$.)

Proof. The lemma is an example of a general descent formula of the kind proved in [A6, §7]. However, it is not, strictly speaking, a special case of [A6, Theorem 7.1]. To spare the reader the task of relating the abstract framework of [A6] to the present situation, we shall give a direct proof.

We assume inductively that the formula (3.7) holds if G is replaced by any element G' in $\mathcal{E}_{M'}^0(G)$. Given the data of part (a), we set $\varepsilon(M, M')$ equal to 1 or 0, according to whether M' is isomorphic to M or not. In particular, if $\varepsilon(M, M')$ equals 1, G is quasisplit. In this case we shall assume that $M' = M$, $T' = T$, and $\sigma_1' = \sigma_1$, in order to match the notation of (b). It follows from the definition (3.4) that the difference

$$\partial_M^{G,\mathcal{E}}(\sigma_1', z_1) - \varepsilon(M, M')\delta_M^G(\sigma_1, z_1) \tag{3.8}$$

equals

$$\sum_{G' \in \mathcal{E}_{M'}^0(G)} \iota_{M'}(G, G')\delta_{M'}^{G'}(\sigma_1', z_1')\,.$$

According to our induction assumption, we have

$$\delta_{M'}^{G'}(\sigma_1', z_1') \;=\; \sum_{L_1' \in \mathcal{L}^{G_1'}(T_1')} \mathrm{e}_{T_1'}^{G'}(M', L_1')\delta_{T_1'}^{L_1'}(\sigma_1', z_1^{L_1'})\,,$$

for any $G' \in \mathcal{E}^0_{M'}(G)$. Since

$$\mathcal{E}^0_{M'}(G) \;=\; \begin{cases} \mathcal{E}_{M'}(G), & \text{if } \varepsilon(M,M')=0, \\ \mathcal{E}_{M'}(G)-\{G\}, & \text{if } \varepsilon(M,M')=1, \end{cases}$$

we see that (3.8) equals the difference between

$$\sum_{G'\in\mathcal{E}_{M'}(G)} \iota_{M'}(G,G') \sum_{L_1'\in\mathcal{L}^{G_1'}(T_1')} \mathrm{e}^{G'}_{T_1'}(M',L_1')\delta^{L_1'}_{T_1'}(\sigma_1', z_1^{L_1'}) \tag{3.9}$$

and

$$\varepsilon(M,M') \sum_{L_1\in\mathcal{L}(T_1)} \mathrm{e}^{G}_{T_1}(M,L_1)\delta^{L_1}_{T_1}(\sigma_1, z_{L_1})\,. \tag{3.10}$$

Consider a term in (3.9) corresponding to G' and L_1'. We can assume that the dual group $\widehat{T}_1'$ has been identified with the torus $\widehat{T}_1$ in $\widehat{G}_1$. Then both $Z(\widehat{G}')$ and $Z(\widehat{L}_1')$ are subgroups of $\widehat{T}_1$. The group L_1' therefore determines a Levi subgroup $L_1 \in \mathcal{L}(T_1)$ of G_1, with $\big(Z(\widehat{L}_1)^\Gamma\big)^0 = \big(Z(\widehat{L}_1')^\Gamma\big)^0$. The product of coefficients

$$\iota_{M'}(G,G')\mathrm{e}^{G'}_{T_1'}(M',L_1')$$

from (3.9) is equal to

$$\big|Z(\widehat{M}')^\Gamma/Z(\widehat{M})^\Gamma\big|\,\big|Z(\widehat{G}')^\Gamma/Z(\widehat{G})^\Gamma\big|^{-1}\big|Z(\widehat{M}')^\Gamma\cap Z(\widehat{L}_1')^\Gamma/Z(\widehat{G}')^\Gamma\big|^{-1} d^{G'}_{T_1'}(M',L_1')\,.$$

This in turn can be written as

$$\big|Z(\widehat{M}')^\Gamma/Z(\widehat{M})^\Gamma\big|\,\big|Z(\widehat{M}')^\Gamma\cap Z(\widehat{L}_1')^\Gamma/Z(\widehat{G})^\Gamma\big|^{-1} d^{G}_{T_1}(M,L_1)\,,$$

since the ellipticity of M' in M implies that $d^G_{T_1'}(M',L_1') = d^G_{T_1}(M,L_1)$. In particular, the term in (3.9) is independent of G'. On the other hand, if we are given L_1 instead of L_1', we can recover L_1' from L_1 and G' as the element in $\mathcal{E}_{T_1'}(L_1)$ such that s'_{L_1} is the projection of the point s' in $s'_M Z(\widehat{M})^\Gamma/Z(\widehat{G})^\Gamma$ onto $\big(\widehat{T}_1^\Gamma/Z(\widehat{L}_1)\big)^\Gamma$. (The reader may want to recall again the definition of the sets $\mathcal{E}_{M'}(G)$ and $\mathcal{E}_{T_1'}(L_1)$ from [A6, §3].)

We need only consider terms in (3.9) that contain nonvanishing coefficients. Given the formula for the product above, we may therefore assume that $\big(Z(\widehat{T}_1)^\Gamma\big)^0$ equals $\big(Z(\widehat{L}_1)^\Gamma\big)^0\big(Z(\widehat{M})^\Gamma\big)^0$, and that the group $Z(\widehat{M})^\Gamma \cap Z(\widehat{L}_1)^\Gamma/Z(\widehat{G})^\Gamma$ is finite. It follows that the map $G' \to L_1'$ from $\mathcal{E}_{M'}(G)$ to $\mathcal{E}_{T_1'}(L_1)$ is surjective, and has finite fibres that are in bijection with $Z(\widehat{M})^\Gamma \cap Z(\widehat{L}_1)^\Gamma/Z(\widehat{G})^\Gamma$. Since its terms are independent of G_1', we can rewrite the expression (3.9) as the sum over $L_1 \in \mathcal{L}(T_1)$ and $L_1' \in \mathcal{E}_{T_1'}(L_1)$ of the product of the differential operator

$$d_{T_1}^G(M, L_1)\delta_{T_1'}^{L_1'}(\sigma_1', z_1^{L_1'})$$

with the coefficient

$$\big|Z(\widehat{M}')^\Gamma/Z(\widehat{M})^\Gamma\big|\,\big|Z(\widehat{M}')^\Gamma \cap Z(\widehat{L}_1')^\Gamma/Z(\widehat{G})^\Gamma\big|^{-1}\big|Z(\widehat{M})^\Gamma \cap Z(\widehat{L}_1)^\Gamma/Z(\widehat{G})^\Gamma\big|\,.$$

The coefficient is easily seen to reduce to

$$\begin{aligned}&\big|Z(\widehat{M}')^\Gamma/Z(\widehat{M})^\Gamma\big|\,\big|Z(\widehat{M}')^\Gamma \cap Z(\widehat{L}_1')^\Gamma/Z(\widehat{M})^\Gamma \cap Z(\widehat{L})^\Gamma\big|^{-1}\\ =\ &\big|Z(\widehat{L}_1')^\Gamma/Z(\widehat{L}_1)^\Gamma\big|^{-1}\\ =\ &\iota_{T_1'}(L_1, L_1')\,.\end{aligned}$$

The expression (3.9) becomes

$$\sum_{L_1 \in \mathcal{L}(T_1)} d_{T_1}^G(M, L_1) \sum_{L_1' \in \mathcal{E}_{T_1'}(L_1)} \iota_{T_1'}(L_1, L_1')\delta_{T_1'}^{L_1'}(\sigma_1', z_1^{L_1'})\,.$$

Recalling the definition of the endoscopic differential operators, we conclude that (3.9) is equal to

$$\sum_{L_1 \in \mathcal{L}(T_1)} d_{T_1}^G(M, L_1)\partial_{T_1}^{L_1,\mathcal{E}}(\sigma_1', z_{L_1})\,. \tag{3.11}$$

We can now complete the proof. If $\varepsilon(M, M') = 0$, (3.8) reduces to $\partial_M^{G,\mathcal{E}}(\sigma_1', z_1)$, while (3.10) vanishes. The identity (3.6) then follows from the fact that (3.8) equals (3.11). If $\varepsilon(M, M') = 1$, $\partial_M^{G,\mathcal{E}}(\sigma_1', z_1)$ equals $\partial_M^G(\sigma_1', z_1)$ by definition. Since $\partial_{T_1}^{L_1,\mathcal{E}}(\sigma_1', z_{L_1})$ is always equal to $\partial_{T_1}^{L_1}(\sigma_1', z_{L_1})$, the required identity (3.6) in this case reduces to (3.5). In particular, the term $\partial_M^{G,\mathcal{E}}(\sigma_1', z_1)$ in (3.8) equals (3.11). The other term $\delta_M^G(\sigma_1, z_1)$ in (3.8) therefore equals (3.10). This gives us the required identity (3.7) of part (b). $\square$

With the parallel descent formulas (3.5) and (3.6) established, we have completed the proof of Lemma 3.2. To prove the theorem, we have only to restrict the complex analytic differential operators of Lemma 3.2 to the real submanifold $T_{\mathrm{reg}}(\mathbb{R})$. We obtain differential operators

$$\delta_M^G(\overline{\sigma}, z)\,, \qquad \overline{\sigma} \in T_{\mathrm{reg}}(\mathbb{R}),\ z \in \mathcal{Z}(G),$$

in the case that G is quasisplit, such that

$$\partial_M^G(\overline{\sigma}', z) \ = \sum_{G' \in \mathcal{E}_{M'}(G)} \iota_{M'}(G, G')\delta_{M'}^{G'}(\overline{\sigma}', \overline{z}')\,, \qquad \overline{\sigma}' \in T'_{G\text{-reg}}(\mathbb{R}),\ z \in \mathcal{Z}(G), \tag{3.12}$$

in general. This follows inductively from Lemma 3.2 and the discussion preceding the statement of the lemma. To treat the more general case that Z and ζ are nontrivial, we note that the differential operators $\partial_M^G(\overline{\sigma}', z)$ are invariant under translation of $T'_{G\text{-reg}}(\mathbb{R})$ by the central subgroup $Z(\mathbb{R})$. This follows easily from either the definition (1.4) or the construction in [A3, §12]. The same property for the operators $\delta_M^G(\overline{\sigma}, z)$ then follows inductively from the case that $M' = M$ of (3.12). The differential operators $\partial_M^G(\overline{\sigma}', z)$ and $\delta_{M'}^{G'}(\overline{\sigma}', \overline{z}')$ on $T'_{\mathrm{reg}}(\mathbb{R})$ can therefore be projected to differential operators on $C^\infty\big(T'_{\mathrm{reg}}(\mathbb{R}), \zeta\big)$, that depend only on the image of z in $\mathcal{Z}(G, \zeta)$, and for which the identity (3.12) continues to hold. The last step is to apply the construction of §2. From the differential operators $\partial_M^G(\overline{\sigma}', z)$ and $\delta_{M'}^{G'}(\overline{\sigma}', \overline{z}')$ on $C^\infty\big(T'_{\mathrm{reg}}(\mathbb{R}), \zeta\big)$ that occur in (3.12), we obtain the required differential operators $\partial_M^G(\sigma', z)$ and $\delta_{\widetilde{M}'}^{\widetilde{G}'}(\sigma', z')$ on $C^\infty\big(\widetilde{T}'_{\mathrm{reg}}(\mathbb{R}), \widetilde{\zeta}'\big)$. The process obviously converts (3.12) to the required identity (3.1) of the theorem. With this discussion, our proof of Theorem 3.1 is complete. $\square$

4. Application to weighted orbital integrals.

The transfer factors were introduced to stabilize invariant orbital integrals. In general terms, the point is to solve certain problems for orbital integrals that arise from the comparison of trace formulas. For the Archimedean case under consideration, Shelstad solved these problems (and more) in [S1]-[S4].

If G, T, G' and T' are as in §2, the map

$$f \longrightarrow f'(\sigma') \;=\; \sum_{\gamma} \Delta_G(\sigma',\gamma) f_G(\gamma)\,, \qquad \sigma' \in \widetilde{T}'_{G\text{-reg}}(\mathbb{R}), \tag{4.1}$$

sends functions $f \in \mathcal{C}\big(G(\mathbb{R}),\zeta\big)$ to functions $f' = f^{G'}$ in $C^\infty\big(\widetilde{T}'_{G\text{-reg}}(\mathbb{R}),\widetilde{\zeta}'\big)$. The value $f'(\sigma')$ depends only on the stable conjugacy class of σ' in $\widetilde{G}'(\mathbb{R})$. If T and T' are allowed to vary, f' can be identified with a function on the strongly G-regular stable conjugacy classes in $\widetilde{G}'(\mathbb{R})$. One of Shelstad's main results is that f' can also be regarded as a family of stable orbital integrals from $\widetilde{G}'(\mathbb{R})$. More precisely, there is a function h in $\mathcal{C}\big(\widetilde{G}'(\mathbb{R}),\widetilde{\zeta}'\big)$ whose stable orbital integral $h'(\sigma') = h^{\widetilde{G}'}(\sigma')$ equals $f'(\sigma')$, for every G-regular, stable conjugacy class σ' in $\widetilde{G}'(\mathbb{R})$. We recall that a tempered, $\widetilde{\zeta}'$-equivariant distribution S' on $\widetilde{G}'(\mathbb{R})$ is said to be *stable* if its value at any function $h \in \mathcal{C}\big(\widetilde{G}'(\mathbb{R}),\widetilde{\zeta}'\big)$ depends only on h'. In this case, we can write

$$S'(h) \;=\; \widehat{S}'(h')\,,$$

for a unique continuous linear form $\widehat{S}'$ on the stably invariant Schwartz space

$$S\mathcal{I}\big(\widetilde{G}'(\mathbb{R}),\widetilde{\zeta}'\big) \;=\; \big\{h' : h \in \mathcal{C}\big(\widetilde{G}'(\mathbb{R}),\widetilde{\zeta}'\big)\big\}\,.$$

The point of Shelstad's theorem is that $f \to \widehat{S}'(f')$ is then a well defined linear form on $\mathcal{C}\big(G(\mathbb{R}),\zeta\big)$.

One would also like to stabilize weighted orbital integrals, or rather, the associated invariant distributions $I_M(\gamma, f)$. At first glance, it might not be clear even how to formulate such a problem. We can certainly set

$$I_M(\sigma', f) \;=\; \sum_{\gamma} \Delta_M(\sigma',\gamma) I_M(\gamma, f)\,, \qquad \sigma' \in \widetilde{T}'_{G\text{-reg}}(\mathbb{R}), \tag{4.2}$$

for (M', T') as in the last section, and γ summed over the $\Omega\big(M(\mathbb{R}), T(\mathbb{R})\big)$-orbits in $T_{\mathrm{reg}}(\mathbb{R})$. In the special case that $M = G$, this matches the definition (4.1). The problem in general is to relate $I_M(\sigma', f)$ with stable distributions on the groups $\widetilde{G}'(\mathbb{R})$. Considerations from the trace formula suggest a conjectural solution [A5, §4], [A6, §3]. Stated in a form that is parallel to Theorem 3.1, the conjecture is as follows. For every (G, T, M, ζ), with G quasisplit, there are stable, tempered, ζ-equivariant distributions

$$S_M^G(\sigma, f)\,, \qquad\qquad \sigma \in T_{\mathrm{reg}}(\mathbb{R}),\ f \in \mathcal{C}\big(G(\mathbb{R}), \zeta\big),$$

such that for any (G, T, M, ζ) at all, and any (M', T'), the identity

$$(4.3) \qquad I_M(\sigma', f) \;=\; \sum_{G' \in \mathcal{E}_{M'}(G)} \iota_{M'}(G, G')\widehat{S}_{\widetilde{M}'}^{\widetilde{G}'}(\sigma', f')\,, \qquad \sigma' \in \widetilde{T}'_{G\text{-reg}}(\mathbb{R}),$$

holds.

The conjecture we have just stated is essentially the archimedean case of [A5, Conjecture 4.1]. As in the case of the slightly more general conjecture in [A6], it is convenient to separate what amounts to an inductive definition from what has to be proved. We assume inductively that for any $G' \in \mathcal{E}^0_{M'}(G)$, the distributions $S_{\widetilde{M}'}^{\widetilde{G}'}(\sigma')$ are defined and stable. If

$$\varepsilon(G) \;=\; \begin{cases} 1, & \text{if } G \text{ is quasisplit,} \\ 0, & \text{otherwise,} \end{cases}$$

we then define distributions $I_M^{\mathcal{E}}(\sigma', f)$, and also $S_M^G(M', \sigma', f)$ in case G is quasisplit, by setting

$$(4.4) \qquad I_M^{\mathcal{E}}(\sigma', f) \;=\; \sum_{G' \in \mathcal{E}^0_{M'}(G)} \iota_{M'}(G, G')\widehat{S}_{\widetilde{M}'}^{\widetilde{G}'}(\sigma', f') + \varepsilon(G) S_M^G(M', \sigma', f)$$

in general, and

$$I_M^{\mathcal{E}}(\sigma', f) \;=\; I_M(\sigma', f)$$

for G quasisplit. Suppose first that G is quasisplit. In the special case that $M' = M$, the conjecture asserts that the distributions

$$S_M^G(\sigma, f) \;=\; S_M^G(M, \sigma, f)\,, \qquad\qquad \sigma \in T_{\mathrm{reg}}(\mathbb{R}),$$

are stable. In case $M' \neq M$, the conjecture asserts that the distributions $S^G_M(M', \sigma', f)$ all vanish. If G is not quasisplit, the conjecture is just the assertion that $I^{\mathcal{E}}_M(\sigma', f)$ equals $I_M(\sigma', f)$.

Theorem 3.1 can be regarded as the first step towards a proof of the conjecture. Roughly speaking, it asserts that the conjecture is compatible with the differential equations (1.7). To state this more precisely, we fix (G, T, M, ζ) and (M', T'). According to Lemmas 2.2 and 2.3, the equations (1.7) can be written in the form

$$I_M(\sigma', zf) = \sum_{L \in \mathcal{L}(M)} \partial^L_M(\sigma', z_L) I_L(\sigma', f), \qquad \sigma' \in \widetilde{T}'_{G\text{-reg}}(\mathbb{R}), \tag{4.5}$$

for any $z \in \mathcal{Z}(G, \zeta)$. We assume that the distributions $S^{\widetilde{G}'}_{\widetilde{L}'}(\sigma')$ are defined and stable for any $G' \in \mathcal{E}^0_{M'}(G)$ and $L' \in \mathcal{L}^{G'}(M')$. In the case that G is quasisplit and $M' \neq M$, we also carry what can be regarded as a second induction assumption, that the distributions $S^{G'}_M(M', \sigma')$ vanish for any $G' \in \mathcal{E}^0_M(G)$. Theorem 3.1 then has the following corollary, that applies to operators $z \in \mathcal{Z}(G, \zeta)$ and functions $f \in \mathcal{C}\big(G(\mathbb{R}), \zeta\big)$.

Corollary 4.1. *(a) Suppose that G is arbitrary. Then*

$$I^{\mathcal{E}}_M(\sigma', zf) = \sum_{L \in \mathcal{L}(M)} \partial^L_M(\sigma', z_L) I^{\mathcal{E}}_L(\sigma', f), \qquad \sigma' \in \widetilde{T}'_{G\text{-reg}}(\mathbb{R}). \tag{4.6}$$

(b) Suppose that G is quasisplit. Then

$$S^G_M(\sigma, zf) = \sum_{L \in \mathcal{L}(M)} \delta^L_M(\sigma, z_L) S^G_L(\sigma, f), \qquad \sigma \in T_{\mathrm{reg}}(\mathbb{R}). \tag{4.7}$$

(b') Suppose that G is quasisplit, and that $M' \neq M$. Then

$$S^G_M(M', \sigma', zf) = \partial\big(h_T(z)\big)' S^G_M(M', \sigma', f), \qquad \sigma' \in \widetilde{T}'_{G\text{-reg}}(\mathbb{R}). \tag{4.8}$$

Proof. The proof is a variant of the argument used to establish Lemma 3.3. Assume inductively that the obvious analogue of (4.7) is valid for $(\widetilde{G}', \widetilde{M}')$, if G' is any element in $\mathcal{E}^0_{M'}(G)$. The difference

$$I^{\mathcal{E}}_M(\sigma', zf) - \varepsilon(G) S^G_M(M', \sigma', zf) \tag{4.9}$$

can then be written as

$$\begin{aligned}
&\sum_{G'\in\mathcal{E}^0_{M'}(G)} \iota_{M'}(G,G')\widehat{S}^{\widetilde{G}'}_{\widetilde{M}'}\big(\sigma',(zf)'\big) \\
&= \sum_{G'} \iota_{M'}(G,G')\widehat{S}^{\widetilde{G}'}_{\widetilde{M}'}(\sigma',z'f') \\
&= \sum_{G'} \sum_{\widetilde{L}'\in\mathcal{L}(\widetilde{M}')} \iota_{M'}(G,G')\delta^{\widetilde{L}'}_{\widetilde{M}'}(\sigma',z'_{\widetilde{L}'})\widehat{S}^{\widetilde{G}'}_{\widetilde{L}'}(\sigma',f')\,,
\end{aligned}$$

by (4.4), the definition of z', and our induction assumption.

Consider an element $\widetilde{L}'\in\mathcal{L}(\widetilde{M}')$. Then $\widetilde{L}'$ comes from a unique Levi subgroup $L'\in\mathcal{L}(M')$ of G', and this in turn determines a Levi subgroup $L\in\mathcal{L}(M)$ with $\big(Z(\widehat{L})^\Gamma\big)^0 = \big(Z(\widehat{L}')^\Gamma\big)^0$. It follows easily that L' belongs to $\mathcal{E}_{M'}(L)$ and that G' belongs to $\mathcal{E}^0_{L'}(G)$. It is also clear that

$$\iota_{M'}(G,G') \;=\; \iota_{M'}(L,L')\iota_{L'}(G,G')$$

and

$$z'_{\widetilde{L}'} \;=\; (z^{G'})_{\widetilde{L}'} \;=\; (z_L)^{L'} \;=\; z'_L\,.$$

Replacing the last double sum over G' and $\widetilde{L}'$ by a triple sum over $L\in\mathcal{L}(M)$, $L'\in\mathcal{E}_{M'}(L)$ and $G'\in\mathcal{E}^0_{L'}(G)$, we see that (4.9) equals

$$\sum_{L}\sum_{L'} \iota_{M'}(L,L')\delta^{\widetilde{L}'}_{\widetilde{M}'}(\sigma',z'_L) \sum_{G'\in\mathcal{E}^0_{L'}(G)} \iota_{L'}(G,G')\widehat{S}^{\widetilde{G}'}_{\widetilde{L}'}(\sigma',f')\,.$$

It then follows from the definition (4.4) that (4.9) is equal to the difference between

$$\sum_{L\in\mathcal{L}(M)}\ \sum_{L'\in\mathcal{E}_{M'}(L)} \iota_{M'}(L,L')\delta^{\widetilde{L}'}_{\widetilde{M}'}(\sigma',z'_L)I^{\mathcal{E}}_L(\sigma',f) \tag{4.10}$$

and

$$\varepsilon(G)\sum_{L\in\mathcal{L}(M)}\ \sum_{L'\in\mathcal{E}_{M'}(L)} \iota_{M'}(L,L')\delta^{\widetilde{L}'}_{\widetilde{M}'}(\sigma',z'_L)S^G_L(L',\sigma',f)\,. \tag{4.11}$$

Consider the expression (4.10). It follows from Theorem 3.1 that

$$\sum_{L' \in \mathcal{E}_{M'}(L)} \iota_{M'}(L, L') \delta_{\widetilde{M}'}^{\widetilde{L}'}(\sigma', z'_L) \;=\; \partial_M^L(\sigma', z_L)\,.$$

Consequently, (4.10) is just equal to the right hand side of (4.6). To deal with (4.11), we assume that G is quasisplit. Suppose that $L \neq M$ and that $L' \neq L$. The stable descent formulas of [A6, §7] then imply that $S_L^G(L', \sigma', f) = 0$. (This is essentially the vanishing assertion of [A6, Theorem 7.1(b′)]. Rather than trying to compare the conditions here with the more formal Assumption 5.1 of [A6], we simply note that the proof in [A6, §7] provides an implicit expansion of $S_L^G(L', \sigma', f)$ in terms of distributions $\widehat{S}_M^{L_1}(M', \sigma', f^{L_1})$, for Levi subgroups $L_1 \in \mathcal{L}(M)$ with $L_1 \neq G$. Since any such L_1 belongs to $\mathcal{E}_M^0(G)$, $\widehat{S}_M^{L_1}(M', \sigma', f^{L_1})$ vanishes by assumption.) If $L = M$, on the other hand, $L' = M'$ is the only element in $\mathcal{E}_{M'}(L)$. It follows from this discussion that (4.11) reduces to the right hand side of (4.7) in the case that $M' = M$ and $\sigma' = \sigma$. In case $M' \neq M$, the set $\mathcal{E}_{M'}(L)$ does not contain L. The summands in (4.11) then all vanish, except when $L = M$ and $L' = M'$. Since

$$\delta_{\widetilde{M}'}^{\widetilde{M}'}(\sigma', z'_M) \;=\; \partial_{\widetilde{M}'}^{\widetilde{M}'}(\sigma', z'_M) \;=\; \partial\big(h_{\widetilde{T}'}(z')\big) \;=\; \partial\big(h_T(z)\big)',$$

(4.11) reduces to the right hand side of (4.8) in this case.

Suppose that G is not quasisplit. Copying the last part of the proof of Lemma 3.3, we note that (4.9) reduces to $I_M^{\mathcal{E}}(\sigma', zf)$, while (4.11) vanishes. The identity (4.6) then follows from the fact that (4.9) equals (4.10). If G is quasisplit, $I_M^{\mathcal{E}}(\sigma', zf)$ equals $I_M(\sigma', zf)$ by definition. The required identity (4.6) in this case reduces to the original differential equation (4.5). In particular, the term $I_M^{\mathcal{E}}(\sigma', zf)$ in (4.9) equals (4.10). The other term $S_M^G(M', \sigma', zf)$ in (4.9) therefore equals (4.11). The required identities (4.7) and (4.8) then follow separately, according to whether M' equals M or not. □

It is obvious that every aspect of this paper is fundamentally dependent on the work of Harish-Chandra. The debt is implicit as well as explicit. Suppose for example that G

is not quasisplit. The conjecture in this case is that $I^{\mathcal{E}}_M(\sigma', f)$ equals $I_M(\sigma', f)$. Corollary 4.1 establishes that the two distributions satisfy the same differential equations. The next step would be to show that they also satisfy the same jump conditions, for σ' near a G-semiregular element in $\widetilde{T}'(\mathbb{R})$. Weighted orbital integrals actually combine the two kinds of jump conditions discovered by Harish-Chandra. We recall that these are the conditions satisfied by invariant orbital integrals about a noncompact imaginary root [HC7, Theorem 9.1], and the conditions satisfied by invariant eigendistributions about a real root [HC4]. One would like to establish a stabilization of the second kind of jump conditions that could be combined with Shelstad's stabilization [S1], [S2] of the first. It would then be possible to use Harish-Chandra's powerful analytic techniques [HC3] to study the singularities of the difference

$$I^{\mathcal{E}}_M(\sigma', f) - I_M(\sigma', f)\,, \qquad \sigma' \in \widetilde{T}'_{G\text{-reg}}(\mathbb{R}).$$

The goal would be to show inductively that the difference lies in $S\mathcal{I}\big(\widetilde{M}'(F), \widetilde{\zeta}'\big)$, or in other words, is given by the stable orbital integrals of a function in $\mathcal{C}\big(\widetilde{M}'(\mathbb{R}), \widetilde{\zeta}'\big)$. For inner forms of $GL(n)$, the process was carried out in [A-C, §2.14]. Global methods then eventually lead to a proof of the conjecture in this special case.

References

[A1] J. Arthur, *The characters of discrete series as orbital integrals,* Invent. Math. **32** (1976), 205–261.

[A2] ——————, *The trace formula in invariant form,* Ann. of Math. (2) **114** (1981), 1–74.

[A3] ——————, *The local behaviour of weighted orbital integrals,* Duke Math. J. **56** (1988), 223–293.

[A4] ——————, *On local character relations,* Selecta Math. **2** (1996), 501–579.

[A5] ——————, *Canonical normalization of weighted characters and a transfer conjecture,* C.R. Math. Acad. Sci. Canada **20** (2) (1998), 33–52.

[A6] ——————, *On the transfer of distributions: weighted orbital integrals,* to appear in Duke Math. J.

[A-C] J. Arthur and L. Clozel, *Simple Algebras, Base Change and the Advanced Theory of the Trace Formula,* Annals of Math. Studies, Vol. 120, Princeton Univ. Press, 1989.

[HC1] Harish-Chandra, *The characters of semisimple Lie groups,* Trans. Amer. Math. Soc. **83** (1956), 98–163.

[HC2] ———, *Fourier transforms on a semisimple Lie algebra. I,* Amer. J. Math. **79** (1957), 193–257.

[HC3] ———, *A formula for semisimple Lie groups,* Amer. J. Math. **79** (1957), 733–760.

[HC4] ———, *Invariant eigendistributions on a semisimple Lie group,* Trans. Amer. Math. Soc. **119** (1965), 457–508.

[HC5] ———, *Two theorems on semisimple Lie groups,* Ann. of Math. **83** (1966), 74–128.

[HC6] ———, *Discrete series for semisimple Lie groups. II. Explicit determination of the characters,* Acta Math. **116** (1966), 1–111.

[HC7] ———, *Harmonic analysis on real reductive groups. I. The theory of the constant term,* J. Funct. Anal. **19** (1975), 104–204.

[K] R. Kottwitz, *Rational conjugacy classes in reductive groups,* Duke Math. J. **49** (1982), 785–806.

[K-S] R. Kottwitz and D. Shelstad, *Twisted endoscopy I: Definitions, norm mappings and transfer factors,* preprint.

[L-S] R.P. Langlands and D. Shelstad, *On the definition of transfer factors,* Math. Ann. **278** (1987), 219–271.

[S1] D. Shelstad, *Characters and inner forms of a quasisplit group over* $\mathbb{R}$*,* Compositio Math. **39** (1979), 11–45.

[S2] ———, *Orbital integrals and a family of groups attached to a real reductive group,* Ann. Sci. École Norm. Sup. **12** (1979), 1–31.

[S3] ———, *Embeddings of L-groups,* Canad. J. Math. **33** (1981), 513–558.

[S4] ———, *L-indistinguishability for real groups,* Math. Ann. **259** (1982), 385–430.

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