Canonical normalization of weighted characters and a transfer conjecture

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§1. Introduction

Suppose that G is a connected reductive algebraic group over a local field F of characteristic 0. If $\pi \in \Pi_{\text{unit}}(G)$ is any irreducible unitary representation of $G(F)$, the character

$$f \longrightarrow f_G(\pi) = \operatorname{tr}\bigl(\pi(f)\bigr), \qquad f \in \mathcal{H}(G),$$

is an invariant linear form on the Hecke algebra $\mathcal{H}(G)$ of $G(F)$. Among the irreducible characters, there is a special place for the induced characters

$$f_M(\pi) = f_G(\pi^G) = \operatorname{tr}\bigl(\mathcal{I}_P(\pi, f)\bigr), \qquad \pi \in \Pi_{\text{unit}}(M).$$

Here M is a Levi subgroup of G, that will remain fixed throughout the paper, $P \in \mathcal{P}(M)$ is a parabolic subgroup of G over F with Levi component M, and $\pi^G = \mathcal{I}_P(\pi)$ is the corresponding induced representation of $G(F)$. As the notation suggests, $f_M(\pi)$ is independent of P. Another elementary property is that $f_M(\pi_\lambda)$ is an analytic function of a real variable λ in the vector space

$$i\mathfrak{a}_M^* = i \operatorname{Hom}\bigl(X(M)_F, \mathbb{R}\bigr)$$

that parametrizes the unramified unitary twists of π. (We are following standard notation here and below; the reader can consult [5, §1-2] for more explanation.)

Induced characters are really too easy. They are hardly worthy companions of the more fundamental discrete series. There is a related family of objects, however, that are just as interesting (and difficult) as noninduced characters. They are the weighted characters

$$J_M^P(\pi, f) = \operatorname{tr}\bigl(\mathcal{J}_M(\pi, P)\mathcal{I}_P(\pi, f)\bigr),$$

in which $\mathcal{J}_M(\pi, P)$ is a nonscalar operator on the space of $\mathcal{I}_P(\pi)$, constructed from the basic intertwining operators

$$J_{Q|P}(\pi) : \mathcal{I}_P(\pi) \longrightarrow \mathcal{I}_Q(\pi), \qquad Q \in \mathcal{P}(M).$$

The use of unnormalized intertwining operators in the construction has the effect of leaving $J_M^P(\pi, f)$ dependent on the choice of P. Moreover, $J_M^P(\pi_\lambda, f)$ can have poles at some of the point $\lambda \in i\mathfrak{a}_M^*$. One can sidestep these problems by using normalized intertwining operators. Such a construction was used in [1], [2], [3], [5], and other

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papers on the trace formula. Given our present knowledge, however, there is no canonical way to normalize the intertwining operators in general. The corresponding weighted characters will therefore depend on a choice of normalizing factors. This is likely to complicate the problem of relating automorphic representations on different groups.

One purpose of this paper is to normalize the weighted characters in a different way. Roughly speaking, the role of normalizing factors for intertwining operators will be played by Plancherel densities. In §2, we shall define a weighted character

$$J_M(\pi, f) \;=\; J^{\mu}_M(\pi, f) \;=\; \mathrm{tr}\big(\mathcal{M}_M(\pi, P)\mathcal{I}_P(\pi, f)\big)\,,$$

in which $\mathcal{M}_M(\pi, P)$ is an operator constructed in a certain way from the unnormalized intertwining operators $J_{Q|P}(\pi)$ and Harish-Chandra's μ-functions. On the one hand, $J_M(\pi, f)$ is independent of any arbitrary choice of normalizing factors. On the other hand, we shall show that it is independent of P (Corollary 2.2) and that $J_M(\pi_\lambda, f)$ is an analytic function of $\lambda \in i\mathfrak{a}^*_M$ (Proposition 2.3). The distribution $J_M(\pi, f)$ is thus closer to being a canonical object. It is a natural generalization of the induced character $f_M(\pi)$.

The geometric analogues of irreducible characters are the invariant orbital integrals

$$f_G(\gamma) \;=\; |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x)dx\,,$$

that are defined for conjugacy classes γ in $G(F)$. If γ is a G-regular conjugacy class in $M(F)$, we can also form the weighted orbital integral

$$J_M(\gamma, f) \;=\; |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x)v_M(x)dx\,,$$

with a weight factor $v_M(x)$ that plays the role of the operator $\mathcal{M}_M(\pi, P)$. Neither of the distributions $J_M(\pi)$ or $J_M(\gamma)$ is invariant under conjugation by $G(F)$. However, one can use the weighted characters as correction terms to build an invariant distribution out of $J_M(\gamma)$. This process was carried out originally in [1, §10], and gave invariant distributions that were implicitly dependent on a choice of normalizing factors. In §3 we shall apply the same construction to the new weighted characters we have defined. We shall obtain invariant distributions

$$I_M(\gamma, f) \;=\; I^{\mu}_M(\gamma, f)$$

that are independent of any choice of normalizing factors. We shall also check that the distributions are independent of any choice of maximal compact subgroup, and behave well under automorphisms.

As canonical objects on $G(F)$, the invariant distributions $I_M(\gamma)$ should be related to their counterparts on the endoscopic groups of G. We recall that the theory of endoscopy, still largely conjectural, is a general framework of Langlands [8] for comparing trace formulas and automorphic representations on different groups. A second purpose of this paper is to state a conjecture that attempts to summarize the role of the distributions in this theory. The conjecture includes an identity that describes an interplay between the distributions on various groups, and the Langlands-Shelstad transfer mappings among these groups.

The conjecture appears to be quite deep. It depends intrinsically on the invariance of the distributions $I_M(\gamma)$, whereas the distributions themselves can be described explicitly only in terms of their noninvariant components $J_M(\gamma)$ and $J_M(\pi)$. A proof will probably have to await the construction and deeper analysis of a stable trace formula. The conjectural identity is in fact likely to be an essential part of such an analysis.

§2. Weighted characters

We have fixed the Levi subgroup M of G. Suppose that π belongs to the set $\Pi(M)$ of equivalence classes of irreducible representations of $M(F)$, and that λ lies in the complex vector space

$$\mathfrak{a}^*_{M,\mathbb{C}} \;=\; \mathfrak{a}^*_M \otimes_{\mathbb{R}} \mathbb{C} \;=\; \mathrm{Hom}\big(X(M)_F, \mathbb{C}\big)\,.$$

Then we have the unnormalized operators

$$J_{Q|P}(\pi_\lambda):\ \mathcal{H}_P(\pi) \longrightarrow \mathcal{H}_Q(\pi)\,, \qquad P, Q \in \mathcal{P}(M),$$

which intertwine the actions of the induced representations $\mathcal{I}_P(\pi_\lambda)$ and $\mathcal{I}_Q(\pi_\lambda)$. Recall that $J_{Q|P}(\pi_\lambda)$ is defined by an absolutely convergent integral over $N_Q(F) \cap N_P(F) \backslash N_Q(F)$ when the real part of λ lies in a certain chamber, and can be analytically continued to a meromorphic function of $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$.

Suppose that π is in general position, in the sense that the operators $J_{Q|P}(\pi_\lambda)$ are all analytic at $\lambda = 0$. For fixed P, we set

$$\mathcal{J}_Q(\zeta, \pi, P) \;=\; J_{Q|P}(\pi)^{-1} J_{Q|P}(\pi_\zeta)\,, \qquad Q \in \mathcal{P}(M),$$

for $\zeta \in i\mathfrak{a}^*_M$ near 0. We claim that $\{\mathcal{J}_Q(\zeta, \pi, P)\}$ is a (G, M)-family of operator valued functions, in the sense of [1, §6]. According to the definition [1, p. 36], we must show that if Q and Q' are adjacent groups in $\mathcal{P}(M)$, and if ζ belongs to the hyperplane spanned by the common wall of the chambers of Q and Q', then $\mathcal{J}_{Q'}(\zeta, \pi, P)$ equals $\mathcal{J}_Q(\zeta, \pi, P)$. Since they have not been normalized, the operators $J_{Q|P}(\pi)$ are not multiplicative in Q and P. However, it is easily seen that

$$J_{Q'|P}(\pi_\zeta) \;=\; \mu_\alpha(\pi_\zeta)^{\varepsilon} J_{Q'|Q}(\pi_\zeta) J_{Q|P}(\pi_\zeta)\,, \tag{2.1}$$

where ε equals 0 or 1 according to whether the number of singular hyperplanes which separate the chambers of Q' and P is greater than or less than the corresponding number for Q and P, $\alpha \in \Delta_Q$ is the simple root of Q that defines the hyperplane in question, and $\mu_\alpha(\pi_\zeta)$ is a function that depends only on $\zeta(\alpha^\vee)$. We are assuming that $\zeta(\alpha^\vee) = 0$. This implies that $\mu_\alpha(\pi_\zeta) = \mu_\alpha(\pi)$, and also that $J_{Q'|Q}(\pi_\zeta) = J_{Q'|Q}(\pi)$. It follows that $\mathcal{J}_{Q'}(\zeta, \pi, P)$ equals $\mathcal{J}_Q(\zeta, \pi, P)$. The claim is therefore valid. According to [1, Lemma 6.2], we can take the limit

$$\mathcal{J}_M(\pi, P) \;=\; \lim_{\zeta \to 0} \sum_{Q \in \mathcal{P}(M)} \mathcal{J}_Q(\zeta, \pi, P)\theta_Q(\zeta)^{-1}\,,$$

for

$$\theta_Q(\zeta) \;=\; \mathrm{vol}\big(\mathfrak{a}^G_M/\mathbb{Z}(\Delta^\vee_Q)\big)^{-1} \prod_{\alpha \in \Delta_Q} \zeta(\alpha^\vee)\,,$$

in the notation of [1].

The operator $\mathcal{J}_M(\pi, P)$ on $\mathcal{H}_P(\pi)$ plays the role of a weight factor. We can use it to define an (unnormalized) weighted character

$$J^P_M(\pi, f) \;=\; \mathrm{tr}\big(\mathcal{J}_M(\pi, P)\mathcal{I}_P(\pi, f)\big)\,, \tag{2.2}$$

for any $f \in \mathcal{H}(G)$. However this object has the disadvantage of being dependent on the group $P \in \mathcal{P}(M)$, as one oberves easily from the formula (2.1). More seriously, it has singularities at any points π where the intertwining operators have poles. We would like an object that is defined at least for all π is the subset $\Pi_{\mathrm{temp}}(M)$ of tempered representations in $\Pi(M)$. It was for these reasons that weighted characters were defined in terms of normalized intertwining operators in [1, §8] (and in other papers on the trace formula).

Recall that the normalized intertwining operators

$$R_{Q|P}(\pi_\lambda) \ = \ r_{Q|P}(\pi_\lambda)^{-1} J_{Q|P}(\pi_\lambda)$$

are constructed from meromorphic scalar valued functions $r_{Q|P}(\pi_\lambda)$ of λ. Langlands [7, Appendix II] conjectured that the normalizing factors $r_{Q|P}(\pi_\lambda)$ could be defined canonically in terms of local L-functions and ε-factors. However, the existence of these objects depends on the local Langlands classification, which in the case of p-adic groups is a long way away. As a substitute, one can simply prove the existence of a general family

$$r \ = \ \{r_{Q|P}(\pi_\lambda)\}$$

of normalizing factors that satisfy a list of natural conditions [5, (r.1)–(r.8)], and for which the corresponding operators $R_{Q|P}(\pi_\lambda)$ also satisfy certain conditions [5, (R.2)–(R.7)]. Among the latter is the property that if π is unitary, $R_{Q|P}(\pi_\lambda)$ is analytic for all $\lambda \in i\mathfrak{a}_M^*$. If π is any representation for which the normalized operators are all analytic at $\lambda = 0$, we form the (G, M)-family

$$\mathcal{R}_Q(\zeta, \pi, P) \ = \ R_{Q|P}(\pi)^{-1} R_{Q|P}(\pi_\zeta)\,, \qquad Q \in \mathcal{P}(M),$$

of operator valued functions of $\zeta \in i\mathfrak{a}_M^*$ (near 0), and the corresponding limit

$$\mathcal{R}_M(\pi, P) \ = \ \lim_{\zeta \to 0} \sum_{Q \in \mathcal{P}(M)} \mathcal{R}_Q(\zeta, \pi, P)\theta_Q(\zeta)^{-1}\,.$$

We then set

$$J_M^r(\pi, f) \ = \ \mathrm{tr}\big(\mathcal{R}_M(\pi, P)\mathcal{I}_P(\pi, f)\big)\,, \tag{2.3}$$

for any $f \in \mathcal{H}(G)$. This is the weighted character that was used in earlier papers, where it was denoted simply by $J_M(\pi, f)$. It is well defined whenever π is unitary, and in particular, if π is any tempered representation. Moreover, one sees easily that $J_M^r(\pi, f)$ is independent of P. (See [1, p. 44].)

Thus, $J_M^r(\pi, f)$ does not have the two disadvantages of the unnormalized weighted characters $J_M^P(\pi, f)$. However, it does depend on the choice of the family r. One would eventually like to compare weighted characters, or rather invariant distributions constructed from weighted characters, on different groups. Since it is not clear how to compare abstract normalizing factors on different groups, the dependence of $J_M^r(\pi, f)$ on r is a problem we would like to avoid. We shall do so by defining a normalized weighted character that is independent of r.

Instead of the normalizing factors, we shall use Harish-Chandra's canonical family

$$\mu \ = \ \{\mu_{Q|P}(\pi_\lambda)\}$$

of μ-functions. Recall that

$$\mu_{Q|P}(\pi_\lambda) \ = \ \big(J_{Q|P}(\pi_\lambda)J_{P|Q}(\pi_\lambda)\big)^{-1} \ = \ \big(r_{Q|P}(\pi_\lambda)r_{P|Q}(\pi_\lambda)\big)^{-1}\,.$$

Also,

$$\mu_{Q|P}(\pi_\lambda) \ = \ \prod_{\alpha \in \Sigma_Q^r \cap \Sigma_{\overline{P}}^r} \mu_\alpha(\pi_\lambda)\,,$$

where Σ_Q^r stands for the set of reduced roots of (Q, A_M), and $\mu_\alpha(\pi_\lambda)$ is a μ-function of rank 1 that depends only on $\lambda(\alpha^\vee)$ (as in (2.1)). Suppose that π is in general enough position that the μ-functions $\mu_{Q|P}(\pi_\lambda)$ and the unnormalized operators $J_{Q|P}(\pi_\lambda)$ are all analytic at $\lambda = 0$. If P is fixed, we define a family of scalar valued functions

$$\mu_Q(\zeta, \pi, P) \;=\; \mu_{Q|P}(\pi)^{-1}\mu_{Q|P}(\pi_{\frac{1}{2}\zeta})\,, \qquad Q \in \mathcal{P}(M),$$

of $\zeta \in i\mathfrak{a}_M^*$ (near 0). Arguing as we did for the operator valued functions $\mathcal{J}_Q(\zeta, \pi, P)$, we see directly that this is a (G, M)-family. (Notice, however, that $\frac{1}{2}\zeta$ appears here instead of ζ.) Therefore the product

$$\mathcal{M}_Q(\zeta, \pi, P) \;=\; \mu_Q(\zeta, \pi, P)\mathcal{J}_Q(\zeta, \pi, P)$$

is also a (G, M)-family, and we can form the limit

$$\mathcal{M}_M(\pi, P) \;=\; \lim_{\zeta \to 0} \sum_{Q \in \mathcal{P}(M)} \mathcal{M}_Q(\zeta, \pi, P)\theta_Q(\zeta)^{-1}\,.$$

We then define

$$J_M^\mu(\pi, f) \;=\; \mathrm{tr}\big(\mathcal{M}_M(\pi, P)\mathcal{I}_P(\pi, f)\big)\,, \tag{2.4}$$

for any $f \in \mathcal{H}(G)$.

The distributions $J_M^\mu(\pi, f)$ are the normalized weighted characters of the title. They seem to be natural objects, having been defined without recourse to the family r. We must still show that they have the good properties of the earlier weighted characters $J_M^r(\pi, f)$. To do so, we shall establish a simple relationship between the two families of distributions.

Like the μ-functions, the abstract normalizing factors satisfy a product formula

$$r_{Q|P}(\pi_\lambda) \;=\; \prod_{\alpha \in \Sigma_Q^r \cap \Sigma_{\overline{P}}^r} r_\alpha(\pi_\lambda) \;=\; r_{\overline{P}|\overline{Q}}(\pi_\lambda)\,,$$

where $r_\alpha(\pi_\lambda)$ is a meromorphic function that depends only on $\lambda(\alpha^\vee)$ [5, (r.2)]. It follows that the functions

$$r_Q(\zeta, \pi) \;=\; r_{Q|\overline{Q}}(\pi)^{-1} r_{Q|\overline{Q}}(\pi_{\frac{1}{2}\zeta})\,, \qquad Q \in \mathcal{P}(M),$$

of $\zeta \in i\mathfrak{a}_M^*$ form a (G, M)-family (assuming of course that π is in general position). If L belongs to $\mathcal{L}(M)$, the set of Levi subgroups of G which contain M, and Q_L belongs to $\mathcal{P}(L)$, the limit

$$r_M^L(\pi) \;=\; \lim_{\zeta \to 0} \sum_{\substack{Q \in \mathcal{P}(M) \\ Q \subset Q_L}} r_Q(\zeta, \pi)\theta_{Q \cap L}(\zeta)^{-1} \tag{2.5}$$

exists, and is independent of Q_L. The first assertion here is a general property of (G, M)-families, while the second is a consequence of the product formula above.

Lemma 2.1. *We have*

$$J_M^\mu(\pi, f) \;=\; \sum_{L\in\mathcal{L}(M)} r_M^L(\pi) J_L^r(\pi^L, f)\,, \tag{2.6}$$

for $\pi \in \Pi(M)$ in general position.

Proof. The right hand side of (2.6) is the kind of expression that comes from a product of (G, M)-families. According to the product decomposition [1, Corollary 6.5], the expression equals

$$\mathrm{tr}\big(\mathcal{N}_M(\pi, P)\mathcal{I}_P(\pi, f)\,,$$

where

$$\mathcal{N}_M(\pi, P) \;=\; \lim_{\zeta\to 0} \sum_{Q\in\mathcal{P}(M)} r_Q(\zeta, \pi)\mathcal{R}_Q(\zeta, \pi, P)\theta_Q(\zeta)^{-1}\,.$$

Looking back at the definition (2.4), we see that it will be enough to prove that the operators $\mathcal{M}_M(\pi, P)$ and $\mathcal{N}_M(\pi, P)$ are equal.

We can also write

$$\mathcal{N}_M(\pi, P) \;=\; \lim_{\zeta\to 0} \sum_{Q\in\mathcal{P}(M)} \nu_Q(\zeta, \pi, P)\mathcal{J}_Q(\zeta, \pi, P)\theta_Q(\zeta)^{-1}\,,$$

for a scalar valued (G, M)-family

$$\nu_Q(\zeta, \pi, P) \;=\; r_Q(\zeta, \pi)\big(r_{Q|P}(\pi)^{-1} r_{Q|P}(\pi_\zeta)\big)^{-1}\,, \qquad Q \in \mathcal{P}(M),$$

of functions of $\zeta \in i\mathfrak{a}_M^*$. We apply the product decomposition [1, Corollary 6.5] to this expression, and also to the parallel expression

$$\mathcal{M}_M(\pi, P) \;=\; \lim_{\zeta\to 0} \sum_{Q\in\mathcal{P}(M)} \mu_Q(\zeta, \pi, P)\mathcal{J}_Q(\zeta, \pi, P)\theta_Q(\zeta)^{-1}\,.$$

The comparison of the operators $\mathcal{M}_M(\pi, P)$ and $\mathcal{N}_M(\pi, P)$ reduces to a comparison of the (G, M)-families $\{\mu_Q\}$ and $\{\gamma_Q\}$. We see that it will suffice to prove that the numbers $\mu_M^L(\pi, P)$ and $\nu_M^L(\pi, P)$, defined for any $L \in \mathcal{L}(M)$ by the analogues of (2.5), are equal.

Let us write

$$\nu_Q(\zeta, \pi, P) \;=\; c_Q(\zeta, \pi, P)\overline{\nu}_Q(\zeta, \pi, P)\,,$$

where

$$c_Q(\zeta, \pi, P) \;=\; \big(r_{Q|P}(\pi)^{-1} r_{Q|P}(\pi_\zeta)\big)^{-1}\big(r_{Q|P}(\pi)^{-1} r_{Q|P}(\pi_{\frac{1}{2}\zeta})\big)^2\,.$$

Once again we have a product of (G, M)-families. The product decomposition again gives us a formula

$$\nu_M^L(\pi, P) \;=\; \sum_{\substack{L_1\in\mathcal{L}(M)\\ L_1\subset L}} c_L^{L_1}(\pi, P)\overline{\nu}_{L_1}^L(\pi, P)\,.$$

Now

$$c_Q(\zeta, \pi, P) \;=\; \prod_{\alpha\in\Sigma_Q^r\cap\Sigma_{\overline{P}}^r} c_\alpha\big(\zeta(\alpha^\vee)\big)\,,$$

where

$$c_\alpha\big(\zeta(\alpha^\vee)\big) \;=\; \big(r_\alpha(\pi)^{-1} r_\alpha(\pi_\zeta)\big)^{-1}\big(r_\alpha(\pi)^{-1} r_\alpha(\pi_{\frac{1}{2}\zeta})\big)^2\,.$$

As a product of functions of one variable, $c_Q(\zeta,\pi,P)$ gives a (G,M)-family of the special sort considered in [2, §7]. According to [2, Lemma 7.1], each limit $c_M^{L_1}(\pi,P)$ can be expressed as a linear combination of products of derivatives $c'_\alpha(0)$. But

$$c_\alpha(t) \;=\; \big(d_\alpha(0)^{-1}d_\alpha(t)\big)^{-1}\big(d_\alpha(0)^{-1}d_\alpha(\tfrac{1}{2}t)\big)^2\,, \qquad t\in\mathbb{R},$$

for the function $d_\alpha\big(\zeta(\alpha^\vee)\big) = r_\alpha(\pi_\zeta)$. It follows that $c'_\alpha(0) = 0$. Therefore the terms with $L_1 \neq M$ in the decomposition above vanish. We are left simply with the identity

$$\nu_M^L(\pi,P) \;=\; \overline{\nu}_M^L(\pi,P)\,.$$

Now we have

$$\begin{aligned}\overline{\nu}_Q(\zeta,\pi,P) \;&=\; r_Q(\zeta,\pi)\big(r_{Q|P}(\pi)^{-1}r_{Q|P}(\pi_{\frac{1}{2}\zeta})\big)^{-2}\\ &=\; r_{Q|\overline{Q}}(\pi)^{-1}r_{Q|\overline{Q}}(\pi_{\frac{1}{2}\zeta})r_{Q|P}(\pi)^2 r_{Q|P}(\pi_{\frac{1}{2}\zeta})^{-2}\,.\end{aligned}$$

Since

$$r_{Q|\overline{Q}}(\pi) \;=\; r_{Q|P}(\pi)r_{P|\overline{Q}}(\pi) \;=\; r_{Q|P}(\pi)r_{Q|\overline{P}}(\pi)$$

by [5, (r.1)], we can write $\overline{\nu}_Q(\zeta,\pi,P)$ as

$$r_{Q|\overline{P}}(\pi)^{-1}r_{Q|\overline{P}}(\pi_{\frac{1}{2}\zeta})r_{Q|P}(\pi)r_{Q|P}(\pi_{\frac{1}{2}\zeta})^{-1}\,.$$

Applying the same formula to $r_{P|\overline{P}}$, we then write $\overline{\nu}_Q(\zeta,\pi,P)$ as the product of

$$r_{P|\overline{P}}(\pi)^{-1}r_{P|\overline{P}}(\pi_{\frac{1}{2}\zeta})$$

and

$$r_{P|Q}(\pi)r_{P|Q}(\pi_{\frac{1}{2}\zeta})^{-1}r_{Q|P}(\pi)r_{Q|P}(\pi_{\frac{1}{2}\zeta})^{-1}\,.$$

The first of these functions is independent of Q and is equal to 1 at $\zeta = 0$. The second function equals

$$\mu_{Q|P}(\pi)^{-1}\mu_{Q|P}(\pi_{\frac{1}{2}\zeta}) \;=\; \mu_Q(\zeta,\pi,P)\,.$$

It follows that

$$\begin{aligned}\overline{\nu}_M^L(\pi,P) \;&=\; \lim_{\zeta\to 0}\sum_{Q\subset Q_L}\overline{\nu}_Q(\zeta,\pi,P)\theta_{Q\cap L}(\zeta)^{-1}\\ &=\; \lim_{\zeta\to 0}\big(r_{P|\overline{P}}(\pi)^{-1}r_{P|\overline{P}}(\pi_{\frac{1}{2}\zeta}\big)\sum_Q \mu_Q(\zeta,\pi,P)\theta_{Q\cap L}(\zeta)^{-1}\\ &=\; \mu_M^L(\pi,P)\,.\end{aligned}$$

We have now established that

$$\mu_M^L(\pi,P) \;=\; \overline{\nu}_M^L(\pi,P) \;=\; \nu_M^L(\pi,P)\,,$$

for any $L\in\mathcal{L}(M)$. Therefore the operators $\mathcal{M}_M(\pi,P)$ and $\mathcal{N}_M(\pi,P)$ are equal, and the identity (2.6) holds. □

Corollary 2.2. *The distribution $J_M^\mu(\pi,f)$ is independent of the fixed group $P\in\mathcal{P}(M)$.*

Proof. We have already noted that $J_L^r(\pi^L,f)$ is independent of any fixed parabolic subgroup. Since the definition of $r_M^L(\pi)$ is independent of P, the corollary follows from the lemma. □

We can now establish the basic regularity property.

Proposition 2.3. *Suppose that π belongs to the subset $\Pi_{\mathrm{unit}}(M)$ of unitary representations in $\Pi(M)$. Then if $f \in \mathcal{H}(G)$, $J_M^\mu(\pi_\lambda, f)$ is an analytic function of $\lambda \in i\mathfrak{a}_M^*$.*

Proof. It is a consequence of the definition that $J_M^\mu(\pi_\lambda, f)$ is a meromorphic function of $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$. We have to show that the function has no poles on $i\mathfrak{a}_M^*$. Since π_λ remains unitary for $\lambda \in i\mathfrak{a}_M^*$, it is enough to show that $J_M^\mu(\pi_\lambda, f)$ has no pole at $\lambda = 0$. We shall combine Lemma 2.1 with an argument that was used in the local trace formula. (See [4, Lemma 12.1].)

The functions $J_L^r(\pi_\lambda^L, f)$ which occur in the expansion (2.6) for $J_M^\mu(\pi_\lambda, f)$ are analytic at $\lambda = 0$. This follows from the fact that the normalized intertwining operators are analytic at any π that is unitary. It remains only to deal with the functions $r_M^L(\pi_\lambda)$ in the expansion.

If $P \in \mathcal{P}(M)$ is fixed, we can write

$$\begin{aligned} r_{Q|\overline{Q}}(\pi_\lambda) &= \prod_{\alpha \in \Sigma_Q^r} r_\alpha(\pi_\lambda) \\ &= \prod_{\alpha \in \Sigma_P^r} r_\alpha(\pi_\lambda) \prod_{\alpha \in \Sigma_Q^r \cap \Sigma_{\overline{P}}^r} r_\alpha(\pi_\lambda) \prod_{\alpha \in \Sigma_{\overline{Q}}^r \cap \Sigma_P^r} r_\alpha(\pi_\lambda)^{-1} \\ &= r_{P|\overline{P}}(\pi_\lambda) \prod_{\alpha \in \Sigma_Q^r \cap \Sigma_{\overline{P}}^r} \big(r_\alpha(\pi_\lambda) r_{-\alpha}(\pi_\lambda)^{-1}\big)\,. \end{aligned}$$

Now for any root α, we have

$$r_{-\alpha}(\pi_\lambda) = \overline{r_\alpha(\pi_{-\overline{\lambda}})}\,,$$

by [5, (r.4)]. It follows that quotient

$$r_\alpha(\pi_\lambda) r_{-\alpha}(\pi_\lambda)^{-1}$$

is analytic at $\lambda = 0$. To exploit this, we write the function

$$r_Q(\zeta, \pi_\lambda) = r_{Q|\overline{Q}}(\pi_\lambda)^{-1} r_{Q|\overline{Q}}(\pi_{\lambda + \frac{1}{2}\zeta})$$

as the product of

$$r_{P|\overline{P}}(\pi_\lambda)^{-1} r_{P|\overline{P}}(\pi_{\lambda + \frac{1}{2}\zeta})$$

and

$$\prod_{\alpha \in \Sigma_Q^r \cap \Sigma_{\overline{P}}^r} \big(r_\alpha(\pi_\lambda) r_{-\alpha}(\pi_\lambda)^{-1}\big)^{-1} \big(r_\alpha(\pi_{\lambda + \frac{1}{2}\zeta}) r_{-\alpha}(\pi_{\lambda + \frac{1}{2}\zeta})^{-1}\big)\,.$$

The first of these functions is independent of Q, and equals 1 at $\zeta = 0$. The second, which we will denote by $d_Q(\zeta, \pi_\lambda, P)$, is an analytic function of the two variables ζ and λ in a neighbourhood of 0 in $i\mathfrak{a}_M^*$. It follows that

$$\begin{aligned} r_M^L(\pi_\lambda) &= \lim_{\zeta \to 0} \sum_{Q \subset Q_L} r_Q(\zeta, \pi_\lambda) \theta_{Q \cap L}(\zeta)^{-1} \\ &= \lim_{\zeta \to 0} \sum_{Q \subset Q_L} d_Q(\zeta, \pi_\lambda, P) \theta_{Q \cap L}(\zeta)^{-1}\,. \end{aligned}$$

This last sum is a smooth function of λ and ζ in $i\mathfrak{a}_M^*$ [1, Lemma 6.2]. Therefore $r_M^L(\pi_\lambda)$ is analytic at $\lambda = 0$. It follows from Lemma 2.1 that $J_M^\mu(\pi_\lambda, f)$ is analytic at $\lambda = 0$. □

Corollary 2.4. *Suppose that $\pi \in \Pi_{\mathrm{unit}}(M)$ and that L lies in $\mathcal{L}(M)$. Then $r_M^L(\pi_\lambda)$ is an analytic function of $\lambda \in i\mathfrak{a}_M^*$.*

Proof. We established that $r_M^L(\pi_\lambda)$ is analytic at $\lambda = 0$ during the proof of the proposition. If we replace π by a fixed $i\mathfrak{a}_M^*$-translate, we obtain the general assertion. □

The construction of $J_M^\mu(\pi, f)$ was for any π in general position. However, if π is unitary, we can define $J_M^\mu(\pi, f)$ to be the value at $\lambda = 0$ of $J_M^\mu(\pi_\lambda, f)$. Similarly, we define $r_M^L(\pi)$ to be the value at $\lambda = 0$ if $r_M^L(\pi_\lambda)$. The formula of Lemma 2.1 is then valid for π.

§3. Canonical invariant distributions

One of the purposes of weighted characters is to build invariant distributions out of weighted orbital integrals. When the weighted characters $J_M^r(\pi, f)$ are used, as in earlier papers, the resulting invariant distributions depend on the family r of normalizing factors. We shall recall the construction, with a view to replacing $J_M^r(\pi, f)$ by $J_M^\mu(\pi, f)$. This will lead to invariant distributions that are independent of r.

The construction relies on the interpretation of a weighted character as a transform, which sends functions on $G(F)$ to functions on $\Pi(M)$. One actually restricts attention to the subset $\Pi_{\mathrm{temp}}(M)$ of tempered representations in $\Pi(M)$. Tempered representations are unitary, so $J_M^r(\pi, f)$ and $J_M^\mu(\pi, f)$ are both defined for $\pi \in \Pi_{\mathrm{temp}}(M)$. As long as we are considering only tempered representations, it is natural to take f to be in the larger space $\mathcal{C}(G) = \mathcal{C}\big(G(F)\big)$ of Schwartz functions on $G(F)$, rather than the Hecke algebra $\mathcal{H}(G)$. If π lies in $\Pi_{\mathrm{temp}}(M)$, $J_M^r(\pi, f)$ is defined by (2.3) for any f in $\mathcal{C}(G)$. (See [5, p. 175].) The normalized weighted $J_M^\mu(\pi, f)$ can also be defined, by applying (2.4) to $\mathcal{C}(G)$. In fact, all the definitions and constructions of §2 hold for any $f \in \mathcal{C}(G)$, as long as π belongs to $\Pi_{\mathrm{temp}}(M)$.

Given $f \in \mathcal{C}(G)$, we define $\phi_M^r(f)$ to be the function on $\Pi_{\mathrm{temp}}(M)$ whose value at π equals $J_M^r(\pi, f)$. Then $\phi_M^r(f)$ lies in the Schwartz space $\mathcal{I}(M) = \mathcal{I}\big(M(F)\big)$ of functions on $\Pi_{\mathrm{temp}}(M)$, defined (somewhat more generally) in [1, §5]. According to [1, Corollary 9.2], ϕ_M^r is a continuous linear map from $\mathcal{C}(G)$ to $\mathcal{I}(M)$. (The proof of this fact relies on an estimate [1, (7.6)], which was later proved in [5, Lemma 2.1].) However, ϕ_M^r depends on r. To obtain a map which is independent of r, we define $\phi_M^\mu(f)$ to be the function on $\Pi_{\mathrm{temp}}(M)$ whose value at π equals $J_M^\mu(\pi, f)$. Then

$$\phi_M^\mu(f, \pi) = \sum_{L \in \mathcal{L}(M)} r_M^L(\pi) \phi_L^r(f, \pi^L), \qquad \pi \in \Pi_{\mathrm{temp}}(M), \tag{3.1}$$

by Lemma 2.1. (The induced representation π^L here could be reducible, but the function $\phi_L^r(f, \pi^L)$ is to be interpreted obviously as the sum of the values of $\phi_L^r(f)$ at the irreducible constituents of π^L.)

Lemma 3.1. *The transform ϕ_M^μ is a continuous linear map from $\mathcal{C}(G)$ to $\mathcal{I}(M)$.*

Proof. If $L \in \mathcal{L}(M)$, the map which sends any $h \in \mathcal{I}(L)$ to the function

$$\pi \longrightarrow h(\pi^L), \qquad \pi \in \Pi_{\mathrm{temp}}(M),$$

sends $\mathcal{I}(L)$ continuously to $\mathcal{I}(M)$. Since ϕ_L^r maps $\mathcal{C}(G)$ continuously to $\mathcal{I}(L)$, the transform

$$(f, \pi) \longrightarrow \phi_L^r(f, \pi^L),$$

maps $\mathcal{C}(G)$ continuously to $\mathcal{I}(M)$. On the other hand, the function $r^L_M(\pi)$ is a smooth function on $\Pi_{\text{temp}}(M)$. More precisely, suppose that M_1 is a Levi subgroup of M, and that $\pi_1 \in \Pi_{\text{temp}}(M_1)$. Then $r^L_M(\pi^M_{1,\Lambda})$ is a smooth function of $\Lambda \in i\mathfrak{a}^*_{M_1}$. This follows from Corollary 2.4, if we apply the descent formula [3, Corollary 7.2] to $r^L_M(\pi^M_{1,\Lambda})$. The lemma will follow from (3.1) if we can show that any derivative in Λ of $r^L_M(\pi^M_{1,\Lambda})$ is a slowly increasing function of the infinitesimal character of $\pi^M_{1,\Lambda}$. In the p-adic case, the definition of the Schwartz space makes this a vacuous condition, and the lemma follows immediately. In the archimedean case, we use [2, Corollary 7.4] to express $r^L_M(\pi^M_{1,\Lambda})$ as a linear combination of products of logarithmic derivatives

$$r_\alpha(\pi^M_{1,\Lambda})^{-1} r'_\alpha(\pi^M_{1,\Lambda})\,,$$

taken with respect to the real variable $\Lambda(\alpha^\vee)$. The fact that derivatives of $r^L_M(\pi^M_{1,\Lambda})$ are slowly increasing can then be inferred from the inequality [5, (r.8)]. □

Remark. The presence of reducible induced tempered representations complicates the description of $\mathcal{I}(M)$. It is actually better to identify $\mathcal{I}(M)$ with a space of functions on a certain basis $T_{\text{temp}}(M)$ of virtual tempered characters, as in [5, §2-3]. However, we have no need of this refinement here, since we shall only be dealing with formal properties of the maps ϕ^r_M and ϕ^μ_M.

We now consider weighted orbital integrals. Recall that the weighted orbital integral

$$J_M(\gamma) \;=\; J^G_M(\gamma)$$

is a tempered distribution on $G(F)$ that depends on M and on a conjugacy class γ in $M(F)$. We shall take γ to lie in the set $\Gamma_G(M) = \Gamma_{G\text{-reg}}\big(M(F)\big)$ of strongly G-regular conjugacy classes in $M(F)$. Then

$$J_M(\gamma, f) \;=\; |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x) v_M(x)dx\,, \qquad f \in \mathcal{C}(G),$$

where

$$D(\gamma) \;=\; \det\big(1 - \mathrm{Ad}(\gamma)\big)_{\mathfrak{g}/\mathfrak{g}_\gamma}$$

is the Weyl discriminant, G_γ is the centralizer of γ in G, and

$$v_M(x) \;=\; \lim_{\zeta\to 0} \sum_{P\in\mathcal{P}(M)} v_P(\zeta, x)\theta_P(\zeta)^{-1}$$

is a weight factor on $M(F)\backslash G(F)$ obtained from the (G, M)-family

$$v_P(\zeta, x) \;=\; e^{-\zeta(H_P(x))}\,, \qquad P \in \mathcal{P}(M),$$

of functions of $\zeta \in i\mathfrak{a}^*_M$. (See [1, §8] and [5, §1].) As a distribution in $G(F)$, $J_M(\gamma)$ is not invariant. More precisely, if

$$f^y(x) \;=\; f(yxy^{-1})\,, \qquad x, y \in G(F),$$

then

$$J_M(\gamma, f^y) \;=\; \sum_{Q\in\mathcal{F}(M)} J^{M_Q}_M(\gamma, f_{Q,y})\,, \tag{3.2}$$

where $\mathcal{F}(M)$ denotes the set of parabolic subgroups of G over F that contain M, and

$$f_{Q,y}:\ m \longrightarrow \delta_Q(m)^{\frac{1}{2}} \int_K \int_{N_Q(F)} f(k^{-1}mnk)u'_Q(k,y)dn\,dk$$

is the function in $\mathcal{C}(M_Q)$ defined in [1, (3.3)]. (See [1, Lemma 8.2].)

The maps ϕ^r_M are used to build an invariant distribution out of $J_M(\gamma)$. What makes this possible is the fact that the distribution $J^r_M(\pi)$ behaves in the same way under conjugation as $J_M(\gamma)$. That is,

$$J^r_M(\pi, f^y) \ = \sum_{Q\in\mathcal{F}(M)} J^{r,M_Q}_M(\pi, f_{Q,y})$$

[1, Lemma 8.3]. Therefore

$$\phi^r_M(f^y) \ = \sum_{Q\in\mathcal{F}(M)} \phi^{r,M_Q}_M(f_{Q,y})\,. \tag{3.3}$$

This allows us to define an invariant tempered distribution

$$I^r_M(\gamma) \ = \ I^{r,G}_M(\gamma)$$

on $G(F)$ inductively by

$$I^r_M(\gamma, f) \ = \ J_M(\gamma, f) - \sum_{L\in\mathcal{L}^0(M)} \widehat{I}^{r,L}_M\big(\gamma, \phi^r_L(f)\big)\,, \tag{3.4}$$

where $\mathcal{L}^0(M) = \mathcal{L}(M) - \{G\}$ denotes the set of Levi subgroups of G distinct from G. The invariance of $I^r_M(\gamma)$ follows by induction and the two formulas (3.2) and (3.3). To complete the inductive definition, one has still to show that $I^r_M(\gamma)$ vanishes on the kernel of the surjective trace map $f \to f_G$ from $\mathcal{C}(G)$ to $\mathcal{I}(G)$, or in the terminology of [5], that $I^r_M(\gamma)$ is supported on $\mathcal{I}(G)$. This permits one to write

$$I^r_M(\gamma, f) \ = \ \widehat{I}^r_M(\gamma, f_G)\,,$$

for a unique continuous linear form $\widehat{I}^r_M(\gamma)$ on $\mathcal{I}(G)$, and justifies the inductive use of the symbol $\widehat{I}^{r,L}_M(\gamma)$ in (3.4). The required property was established in [5, Corollary 5.2].

To construct an invariant distribution that is independent of r, we carry out the same process with ϕ^r_M replaced by ϕ^μ_M. If we combine (3.3) with (3.1), we obtain the analogous formula

$$\phi^\mu_M(f^y) \ = \sum_{Q\in\mathcal{F}(M)} \phi^{\mu,M_Q}_M(f_{Q,y}) \tag{3.5}$$

for ϕ^μ_M. This allows us to define an invariant tempered distribution

$$I^\mu_M(\gamma) \ = \ I^{\mu,G}_M(\gamma)$$

on $G(F)$ inductively by

$$I^\mu_M(\gamma, f) \ = \ J_M(\gamma, f) - \sum_{L\in\mathcal{L}^0(M)} \widehat{I}^{\mu,L}_M\big(\gamma, \phi^\mu_L(f)\big)\,. \tag{3.6}$$

The invariance of $I^\mu_M(\gamma)$ again follows from the two relevant covariance formulas, in this case (3.2) and (3.5). To show that $I^\mu_M(\gamma)$ is supported on $\mathcal{I}(G)$, we introduce another transform from $\mathcal{C}(G)$ to $\mathcal{I}(M)$. If $f \in \mathcal{C}(G)$, let $\rho_M(f)$ be the function whose value at $\pi \in \Pi_{\mathrm{temp}}(M)$ equals

$$r^G_M(\pi) f_G(\pi^G) \ = \ r^G_M(\pi) f_M(\pi)\,.$$

Arguing as in the proof of Lemma 3.1, we see that ρ_M is a continuous map from $\mathcal{C}(G)$ to $\mathcal{I}(M)$.

Lemma 3.2. *We have*

$$I_M^\mu(\gamma, f) \;=\; I_M^r(\gamma, f) - \sum_{L\in\mathcal{L}^0(M)} \widehat{I}_M^{\mu,L}\big(\gamma, \rho_L(f)\big)\,.$$

Therefore, $I_M^\mu(\gamma)$ is supported on $\mathcal{I}(G)$, and

$$I_M^r(\gamma, f) \;=\; \sum_{L\in\mathcal{L}(M)} \widehat{I}_M^{\mu,L}\big(\gamma, \rho_L(f)\big)\,. \tag{3.7}$$

Proof. Comparing the definitions (3.4) and (3.6), and noting the cancellation of $J_M(\gamma, f)$ from each formula, we write

$$I_M^\mu(\gamma, f) - I_M^r(\gamma, f)$$

as the sum of

$$-\sum_{L\in\mathcal{L}^0(M)} \widehat{I}_M^{\mu,L}\big(\gamma, \phi_L^\mu(f)\big)$$

and

$$\sum_{L_1\in\mathcal{L}^0(M)} \widehat{I}_M^{r,L_1}\big(\gamma, \phi_{L_1}^r(f)\big)\,.$$

We assume inductively that (3.7) holds if G is replaced by any $L_1 \in \mathcal{L}^0(M)$. The second expression in the sum can then be written as

$$\sum_{L\in\mathcal{L}(M)} \widehat{I}_M^{\mu,L}\Big(\gamma, \sum_{L_1\in\mathcal{L}^0(L)} \widehat{\rho}_L^{L_1}\big(\phi_{L_1}^r(f)\big)\Big)\,.$$

By definition,

$$\sum_{L_1\in\mathcal{L}^0(L)} \widehat{\rho}_L^{L_1}\big(\phi_{L_1}^r(f)\big)$$

is the function in $\mathcal{I}(L)$ whose value at $\pi \in \Pi_{\mathrm{temp}}(L)$ equals

$$\sum_{L_1\in\mathcal{L}^0(L)} r_L^{L_1}(\pi)\phi_{L_1}^r(f, \pi^{L_1})\,,$$

an expression which in turn can be written

$$\sum_{L_1\in\mathcal{L}(L)} r_L^{L_1}(\pi)\phi_{L_1}^r(f, \pi^{L_1}) - r_L^G(\pi)\phi_G^r(f, \pi^G) \;=\; \phi_L^\mu(f, \pi) - r_L^G(\pi) f_G(\pi^G)\,,$$

by (3.1). The function is therefore equal to $\phi_L^\mu(f) - \rho_L(f)$. Since this vanishes if $L = G$, we may as well assume that $L \in \mathcal{L}^0(M)$. The second expression in the original sum becomes

$$\sum_{L\in\mathcal{L}^0(M)} \widehat{I}_M^{\mu,L}\big(\gamma, \phi_L^\mu(f)\big) - \sum_{L\in\mathcal{L}^0(M)} \widehat{I}_M^{\mu,L}\big(\gamma, \rho_L(f)\big)\,.$$

The first part of this expression cancels the first expression in the original sum. What remains leads immediately to the identity

$$I_M^\mu(\gamma, f) - I_M^r(\gamma, f) \;=\; -\sum_{L\in\mathcal{L}^0(M)} \widehat{I}_M^{\mu,L}\big(\gamma, \rho_L(f)\big)$$

asserted in the lemma. The second assertion of the lemma follows by induction and the fact that $I_M^r(\gamma)$ is supported on $\mathcal{I}(G)$. □

We have now constructed invariant distributions $I^{\mu}_M(\gamma)$ which do not depend on a choice of normalizing factors. In recognition of their intrinsic nature, we shall allow ourselves to suppress the superscript μ. From now on, we shall write $J_M(\gamma, f) = J^{\mu}_M(\gamma, f)$, $\phi_M(f) = \phi^{\mu}_M(f)$, and $J_M(\pi, f) = J^{\mu}_M(\pi, f)$. This notation differs from that of [1], [3], and [5]. In the earlier papers, it was only the objects $I^r_M(\gamma, f)$, $\phi^r_M(f)$ and $J^r_M(\gamma, f)$ that were being considered, and these were denoted without the superscript r. We hope that the change in notation will not cause confusion.

The distributions behave in a simple way under isomorphism. Suppose that

$$\theta : \, x \longrightarrow \theta x \, , \qquad x \in G,$$

is an isomorphism from G onto another group G_1 which is defined over F. We obtain a bijection $L \to \theta L$ between the corresponding sets of Levi subgroups. If f is a Schwartz function on $G(F)$, the function

$$(\theta f)(x_1) \; = \; f(\theta^{-1} x_1) \, , \qquad x_1 \in G_1(F),$$

belongs to the Schwartz space on $G_1(F)$.

Lemma 3.3. $I_{\theta M}(\theta\gamma, \theta f) = I_M(\gamma, f)$.

The lemma will be easy to establish, but we should first clear up another point. The noninvariant distributions $J_M(\gamma, f)$ and $J_M(\pi, f)$ that make up $I_M(\gamma, f)$ depend on the choice of a suitable maximal compact subgroup K of $G(F)$. (The condition on K is that it be admissible relative to M, in the sense of [1, p. 9].) As further evidence of the instrinsic nature of $I_M(\gamma)$, we have

Lemma 3.4. *The invariant distribution $I_M(\gamma)$ is independent of K.*

Proof. If F is archimedean, all maximal compact subgroups are conjugate under $G(F)$, and the lemma is an easy consequencxe of the invariance of $I_M(\gamma)$. We cannot use quite the same argument in general. We shall instead simply copy the original proofs [1, Lemmas 8.2 and 8.3] of the identities (3.2) and (3.3) which imply the invariance of $I_M(\gamma)$.

Let K_1 be another maximal compact subgroup of $G(F)$ which is admissible relative to M. We shall use the subscript K_1 to denote objects taken with respect to K_1 instead of K. Then

$$J_{M,K_1}(\gamma, f) \; = \; |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x) v_{M,K_1}(x) dx \, ,$$

where

$$v_{M,K_1}(x) \; = \; \lim_{\zeta \to 0} \sum_{P \in \mathcal{P}(M)} v_{P,K_1}(\zeta, x) \theta_P(\zeta)^{-1} \, .$$

Following the proof of [1, Lemma 8.2], we write

$$\begin{aligned} v_{P,K_1}(\zeta, x) \; &= \; e^{-\zeta(H_{P,K_1}(x))} \\ &= \; e^{-\zeta(H_P(x))} e^{-\zeta(H_{P,K_1}(K_P(x)))} \\ &= \; v_P(\zeta, x) u_P(\zeta, x, K_1) \, , \end{aligned}$$

where $K_P(x)$ is the component of x in K relative to the decomposition $G(F) = P(F)K$, and

$$u_P(\zeta, x, K_1) \; = \; e^{-\zeta(H_{P,K_1}(K_P(x)))} \, , \qquad P \in \mathcal{P}(M),$$

is a (G, M)-family of functions of $\zeta \in i\mathfrak{a}_M^*$. We can then write

$$v_{M,K_1}(x) \;=\; \sum_{Q\in\mathcal{F}(M)} v_M^Q(x)u'_Q(x, K_1)\,,$$

in the notation of the formula [1, Lemma 6.3]. If we substitute this into the original integral, we obtain a decomposition

$$J_{M,K_1}(\gamma, f) \;=\; \sum_{Q\in\mathcal{F}(M)} J_M^{M_Q}(\gamma, f_{Q,K_1})\,,$$

in which f_{Q,K_1} denotes the Schwartz function

$$m \longrightarrow \delta_Q(m)^{\frac{1}{2}} \int_K \int_{N_Q(F)} f(k^{-1}mnk)u'_Q(k, K_1)dn\,dk$$

on $M_Q(F)$. Similar modifications of the proof of [1, Lemma 8.3], which we leave to the reader, lead to a parallel decomposition

$$J_{M,K_1}(\pi, f) \;=\; \sum_{Q\in\mathcal{F}(M)} J_M^{M_Q}(\pi, f_{Q,K_1})$$

for weighted characters. This in turn implies that

$$\phi_{L,K_1}(f) \;=\; \sum_{Q\in\mathcal{F}(L)} \phi_L^{M_Q}(f_{Q,K_1})$$

for any $L \in \mathcal{L}(M)$. It follows inductively from the definitions of the invariant distributions that

$$I_{M,K_1}(\gamma, f) \;=\; I_M(\gamma, f)\,. \qquad \square$$

Proof of Lemma 3.3. Let K be a fixed maximal compact subgroup of $G(F)$ which is admissible relative to M. Then $K_1 = \theta K$ is a maximal compact subgroup of $G_1(F)$ which is admissible relative to θM. Having established Lemma 3.4, we are free to use K_1 as the "base point" for constructing the constituents of $I_{\theta M}(\theta\gamma, \theta f)$. The lemma is then a consequence of the various definitions.

Consider, for example, the weighted orbital integral $J_M(\gamma, f)$. To describe the effect of θ on the weight factor $v_M(x)$, we use the fact that for any $P \in \mathcal{P}(M)$, θ is compatible with the decompositions $G(F) = P(F)K$ and $G_1(F) = (\theta P)(F)K_1$. It follows that there is a linear isomorphism θ: $\mathfrak{a}_M \to \mathfrak{a}_{\theta M}$ such that $\theta H_P(x) = H_{\theta P}(\theta x)$, which leads to the identity $v_M(x) = v_{\theta M}(\theta x)$. We may therefore write

$$J_M(\gamma, f) \;=\; |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F)\backslash G(F)} (\theta f)\big((\theta x)^{-1}\theta\gamma(\theta x)\big)v_{\theta M}(\theta x)dx\,.$$

Since $D(\gamma) = D(\theta\gamma)$, and since $x \to \theta x$ is a measure preserving diffeomorphism from $G_\gamma(F)\backslash G(F)$ onto $G_{1,\theta\gamma}(F)\backslash G_1(F)$, we can conclude that

$$J_{\theta M}(\theta\gamma, \theta f) \;=\; J_M(\gamma, f)\,.$$

A similar argument for the weighted characters leads to the identity

$$J_{\theta M}(\theta\pi, \theta f) \;=\; J_M(\pi, f)\,, \qquad \pi \in \Pi_{\mathrm{temp}}(M).$$

Therefore, $\theta\big(\phi_L(f)\big) = \phi_{\theta L}(\theta f)$ for any $L \in \mathcal{L}(M)$. The lemma then follows from the inductive definitions of $I_{\theta M}(\theta\gamma, \theta f)$ and $I_M(\gamma, f)$. □

§4. A conjectural transfer identity

It is important to understand how the distributions $I_M(\gamma, f)$ behave under endoscopic transfer. We shall state a conjecture which seems to lie at the heart of the problem of comparing general trace formulas on different groups.

Let $\mathcal{E}_{\text{ell}}(G)$ be the finite set of equivalence classes of elliptic endoscopic data for G over F ([8], [9]). Following the usual convention, we denote an element in $\mathcal{E}_{\text{ell}}(G)$ by a symbol G', even though G' is only the first component of a representative $(G', \mathcal{G}', s', \xi')$ of an isomorphism class. (See [6, §2] for a description of the objects $\mathcal{G}'$, s' and ξ'.) Then G' is a quasisplit group over F, which shares some of its maximal tori with G. We assume for simplicity that we can fix an L-isomorphism from $\mathcal{G}'$ onto the L-group ${}^LG'$. Then ξ' can be identified with an L-embedding ${}^LG' \to {}^LG$ of L-groups. Langlands and Shelstad [9] define a transfer map

$$f \longrightarrow f'(\delta') \ = \sum_{\gamma \in \Gamma_G(G)} \Delta_G(\delta', \gamma) f_G(\gamma)$$

from functions f on $G(F)$ to functions $f' = f^{G'}$ on the set $\Sigma_G(G') = \Sigma_{G\text{-reg}}\big(G'(F)\big)$ of G-(strongly) regular stable conjugacy classes in $G'(F)$. The transfer factor $\Delta_G(\delta', \gamma)$ is an explicit, complex-valued function on $\Sigma_G(G') \times \Gamma_G(G)$ that vanishes unless δ' is an image of γ (in the language of [9, (1.3)]). The Langlands-Shelstad transfer conjecture in this context asserts that if f lies in $\mathcal{C}(G)$, there is a function in $\mathcal{C}(G')$ whose stable orbital integrals are given by the values of f'.

If G is quasisplit, G itself is an element in $\mathcal{E}_{\text{ell}}(G)$. Then $f^G(\delta)$ equals the stable orbital integral of f at $\delta \in \Sigma_G(G)$, at least up to a constant multiple. An invariant tempered distribution S on $G(F)$ is said to be *stable* if it vanishes on the kernel of the map $f \to f^G$. If this is so, there is a tempered distribution $\widehat{S}$ on $\Sigma_G(G)$ (a notion that is not hard to make precise) such that

$$\widehat{S}(f^G) \ = \ S(f)\,, \qquad f \in \mathcal{C}(G).$$

Returning to the general case, we assume in what follows that the Langlands-Shelstad transfer conjecture holds for any $G' \in \mathcal{E}_{\text{ell}}(G)$. If S' is a stable tempered distribution on $G'(F)$, $\widehat{S}'(f')$ is then defined for any $f \in \mathcal{C}(G)$.

The conjectural transfer properties of $I_M(\gamma, f)$ are best stated in terms of adjoint transfer factors. We assume that for each $G' \in \mathcal{E}_{\text{ell}}(G)$, we have been able to identify ξ' with an L-embedding ${}^LG' \to {}^LG$. This is possible, for example, if the derived group of G is simply connected. The same property then holds for each of the endoscopic data $M' \in \mathcal{E}_{\text{ell}}(M)$. We shall also discuss only the case that γ lies in the subset $\Gamma_{G,\text{ell}}(M)$ of M-elliptic conjugacy classes in $\Gamma_G(M)$. Parallel to $\Gamma_{G,\text{ell}}(M)$, we have the "endoscopic" set $\Gamma^{\mathcal{E}}_{G,\text{ell}}(M)$, which consists of the M-isomorphism classes of pairs

$$(M', \delta') \qquad M' \in \mathcal{E}_{\text{ell}}(M),\ \delta' \in \Sigma_{G,\text{ell}}(M').$$

We can also identify $\Gamma^{\mathcal{E}}_{G,\text{ell}}(M)$ with a disjoint union of orbits

$$\coprod_{M' \in \mathcal{E}_{\text{ell}}(M)} \big(\Sigma_{G,\text{ell}}(M') \,/\, \text{Out}_M(M')\big)\,,$$

where as in [6, §2], $\text{Out}_M(M')$ is the group of outer automorphisms of the endoscopic datum M'. Since $\Delta_M(\delta', \gamma)$ is invariant under the action of $\text{Out}_M(M')$ on δ', the transfer factors for M can be combined into a function on $\Gamma^{\mathcal{E}}_{G,\text{ell}}(M) \times \Gamma_{G,\text{ell}}(M)$. We then introduce the adjoint transfer factor

$$\Delta_M(\gamma, \delta') \ = \ |\mathcal{K}_\gamma|^{-1}\overline{\Delta_M(\delta', \gamma)}$$

on $\Gamma_{G,\mathrm{ell}}(M) \times \Gamma^{\mathcal{E}}_{G,\mathrm{ell}}(M)$ as in [6, (2.3)].

We shall attach a family of endoscopic data for G to an endoscopic datum for M. Consider an element $M' \in \mathcal{E}_{\mathrm{ell}}(M)$. We choose a representative $(M', \mathcal{M}', s'_M, \xi'_M)$ within the given equivalence class so that $\mathcal{M}'$ is a subgroup of ${}^L M$, and so that the embedding ξ'_M is the identity. Then s'_M is a semisimple element in $\widehat{M}$ which stabilizes $\mathcal{M}'$. Suppose that s' is an element in the set $s'_M Z(\widehat{M})^\Gamma$, where $Z(\widehat{M})^\Gamma$ denotes the subgroup of elements in the center of $\widehat{M}$ that are invariant under $\Gamma = \mathrm{Gal}(\overline{F}/F)$. Let $\widehat{G}'$ be the connected centralizer of s' in $\widehat{G}$. Then $\mathcal{G}' = \widehat{G}'\mathcal{M}'$ is a subgroup of ${}^L G$, and is a split extension of W_F by $\widehat{G}'$. Taking ξ' to be the identity embedding of $\mathcal{G}'$ into ${}^L G$, we obtain an endoscopic datum $(G', \mathcal{G}', s', \xi')$ for G. We shall write $\mathcal{E}_{M'}(G)$ for the set of such s', taken modulo the subgroup $Z(\widehat{G})^\Gamma$ of $Z(\widehat{M})^\Gamma$, for which the corresponding endoscopic datum for G is elliptic. Following the earlier convention, we shall represent a given element in $\mathcal{E}_{M'}(G)$ by its endoscopic group G'. We are not actually taking isomorphism classes of endoscopic data here, so different elements in $\mathcal{E}_{M'}(G)$ could give the same element in $\mathcal{E}_{\mathrm{ell}}(G)$. However, the ellipticity condition we have imposed means at least that there are only finitely many elements in $\mathcal{E}_{M'}(G)$. We can identify M' with a Levi subgroup of any given $G' \in \mathcal{E}_{M'}(G)$. For each such G', we define a coefficient

$$\iota_{M'}(G, G') \;=\; |Z(\widehat{M}')^\Gamma / Z(\widehat{M})^\Gamma| |Z(\widehat{G}')^\Gamma / Z(\widehat{G})^\Gamma|^{-1} .$$

The quasisplit case plays a special role in the conjecture we are about to state. If one of the groups G or M is quasisplit, so is the other, in which case we shall say that (G, M) is quasisplit.

Conjecture 4.1. *There are stable distributions*

$$S^G_M(\delta, f)\,, \qquad\qquad f \in \mathcal{C}(G),$$

defined for quasisplit pairs (G, M) and elements $\delta \in \Sigma_{G,\mathrm{ell}}(M)$, such that for any (G, M) and any element $\gamma \in \Gamma_{G,\mathrm{ell}}(M)$, the endoscopic expression

$$I^{\mathcal{E}}_M(\gamma, f) \;=\; \sum_{(M',\delta') \in \Gamma^{\mathcal{E}}_{\mathrm{ell}}(M)} \Delta_M(\gamma, \delta') \sum_{G' \in \mathcal{E}_{M'}(G)} \iota_{M'}(G, G') \widehat{S}^{G'}_{M'}(\delta', f') \tag{4.1}$$

equals $I_M(\gamma, f)$.

Remarks. 1. The conjecture includes the existence of new distributions $S^G_M(\delta, f)$ and $I^{\mathcal{E}}_M(\gamma, f)$. These objects are to be regarded as stable and endoscopic analogues of the invariant distributions $I_M(\gamma, f)$.

2. Implicit in the assertion is that the outer summands in (4.1) depend only on the image of (M', δ') in $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(M)$. It is not hard to show inductively from Lemma 3.3 that this actually holds for each of the terms $\widehat{S}^{G'}_{M'}(\delta', f')$.

3. We have been assuming the existence of an L-isomorphism $\mathcal{G}' \to {}^L G'$ for each G'. This is only for simplicity. In general, one must choose a central extension $\widetilde{G}' \to G'$ for each G' by a suitable torus $\widetilde{Z}'$, as in [9, (4.4)]. (See also [6, §2].) One can then choose an L-injection $\widetilde{\xi}'\colon \mathcal{G}' \to {}^L \widetilde{G}'$ that plays the role of the L-isomorphism above. The definitions and conjecture are easily modified to include the general case.

4. If F is p-adic, the transfer factors $\Delta_M(\gamma, \delta')$ and $\Delta_M(\delta', \gamma)$ satisfy adjoint relations [6, Lemma 2.2] that provide an inversion of the formula (4.1). In the quasisplit case, this inversion gives an inductive definition of $S^G_M(\delta, f)$ in terms of the distributions $I_M(\gamma, f)$. If F is archimedean, however, the adjoint relations fail, essentially because the set $\Gamma_{G,\mathrm{ell}}(M)$ is too small. It is possible to place the archimedean case on an equal footing with the p-adic case by embedding $\Gamma_{G,\mathrm{ell}}(M)$ in a larger set. A natural extension of the conjecture asserts that $I^{\mathcal{E}}_M(\gamma, f)$ vanishes if γ lies in the complement of $\Gamma_{G,\mathrm{ell}}(M)$.

5. We have assumed that γ lies in the subset $\Gamma_{G,\mathrm{ell}}(M)$ of $\Gamma_G(M)$. Again, this was just for simplicity. Parallel to $\Gamma_G(M)$, one can introduce the endoscopic set $\Gamma^{\mathcal{E}}_G(M)$ as in [6, §2]. The conjecture can then be stated for γ and δ' in

the larger sets $\Gamma_G(M)$ and $\Gamma^{\mathcal{E}}_G(M)$. Unlike the special case of $\Gamma^{\mathcal{E}}_{G,\mathrm{ell}}(M)$, however, an element $\delta' \in \Gamma^{\mathcal{E}}_G(M)$ can lie in the image of several of the sets $\Sigma_G(M')$. In particular, δ' does not determine a unique element $M' \in \mathcal{E}_{\mathrm{ell}}(M)$. To establish that the summands of (4.1) are independent of the choice of M', or more generally, that they depend only on the image of (M', δ') in $\Gamma^{\mathcal{E}}_G(M)$, we would need to establish descent formulas for the new distributions analogous to [3, Corollary 8.3].

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