

The Trace Paley Wiener Theorem for Schwartz Functions

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Suppose that G is a connected reductive algebraic group over a local field F of characteristic 0. If f is a function in the Schwartz space $\mathcal{C}(G(F))$, and $\pi \in \Pi_{\text{temp}}(G(F))$ is an irreducible tempered representation of $G(F)$, the operator

$$f_G(\pi) = \int_{G(F)} f(x)\pi(x)dx$$

is of trace class. We can therefore map f to the function

$$f_G(\pi) = \text{tr}(\pi(f))$$

on $\Pi_{\text{temp}}(G(F))$. The object of this note is to characterize the image of the map.

Results of this nature are well known. The case of the Hecke algebra on $G(F)$, which is in fact more difficult, was established in [3] and [5]. A variant of the problem for the smooth functions of compact support on a real group was solved in [4]. For the Schwartz space, one has a choice of several possible approaches. We shall use the characterization of the operator valued Fourier transform

$$f \rightarrow \pi(f), \quad f \in \mathcal{C}(G(F)),$$

which was solved separately for real and p -adic groups [2], [9, Part B]. (See also [6, Lemma 5.2].)

Irreducible tempered representations occur as constituents of induced representations

$$\mathcal{I}_P(\sigma) : G(F) \rightarrow \text{End}(\mathcal{H}_P(\sigma)), \quad \sigma \in \Pi_2(M(F)).$$

Here M belongs to the finite subset \mathcal{L} of Levi subgroups of G which contain a fixed minimal Levi subgroup, P belongs to the set $\mathcal{P}(M)$ of parabolic subgroups

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with Levi component M , and $\Pi_2(M(F))$ is the set of (equivalence classes of) irreducible unitary representations of $M(F)$ which are square integrable modulo the center. The irreducible constituents of $\mathcal{I}_P(\sigma)$ in general are determined by projective representations of the R -group R_σ of σ [7], [8]. To convert projective representations to ordinary representations, one takes a finite central extension

$$1 \longrightarrow Z_\sigma \longrightarrow \tilde{R}_\sigma \longrightarrow R_\sigma \longrightarrow 1$$

of the R -group. The usual intertwining operators then give rise to a representation

$$r \longrightarrow \tilde{R}(r, \sigma), \quad r \in \tilde{R}_\sigma,$$

of $\tilde{R}_\sigma = \tilde{R}_\sigma^G$ on $\mathcal{H}_P(\sigma)$ which commutes with $\mathcal{I}_P(\sigma)$. (See [1, §2].) The process singles out a character χ_σ of Z_σ ; following [1], we write $\Pi(\tilde{R}_\sigma, \chi_\sigma)$ for the set of irreducible representations of \tilde{R}_σ whose central character on Z_σ equals χ_σ . Then there is a bijection $\rho \rightarrow \pi_\rho$ from $\Pi(\tilde{R}_\sigma, \chi_\sigma)$ onto the set of irreducible constituents of $\mathcal{I}_P(\sigma)$, with the properties that

$$(1) \quad \text{tr}(\tilde{R}(r, \sigma)\mathcal{I}_P(\sigma, f)) = \sum_{\rho \in \Pi(\tilde{R}_\sigma, \chi_\sigma)} \text{tr}(\rho^\vee(r))\text{tr}(\pi_\rho(f))$$

and

$$(2) \quad \text{tr}(\pi_\rho(f)) = |\tilde{R}_\sigma|^{-1} \sum_{r \in \tilde{R}_\sigma} \text{tr}(\rho(r))\text{tr}(\tilde{R}(r, \sigma)\mathcal{I}_P(\sigma, f)),$$

for any function $f \in \mathcal{C}(G(F))$. (We are writing ρ^\vee for the contragredient of ρ .)

Consider the set $\tilde{T}(G)$ of triplets

$$\tau = (M, \sigma, r), \quad M \in \mathcal{L}, \sigma \in \Pi_2(M(F)), r \in \tilde{R}_\sigma,$$

which are essential in the sense of [1, §3]. (This means that the subgroup of elements $z \in Z_\sigma$ for which zr is conjugate to r lies in the kernel of χ_σ .) The restricted Weyl group W_0^G of G acts on $\tilde{T}(G)$, and we write $T(G)$ for the set W_0^G -orbits in $\tilde{T}(G)$. Set

$$(3) \quad f_G(\tau) = \text{tr}(\tilde{R}(r, \sigma)\mathcal{I}_P(\sigma, f)), \quad f \in \mathcal{C}(G(F)).$$

Then $f_G(\tau)$ depends only on the W_0^G -orbit of τ , and is therefore a function on $T(G)$. There is also a symmetry condition

$$f_G(z\tau) = \chi_\tau(z)^{-1}f_G(\tau), \quad z \in Z_\tau,$$

in which we have written $Z_\tau = Z_\sigma$, $\chi_\tau = \chi_\sigma$ and $z\tau = (M, \sigma, zr)$. Observe that (1) and (2) represent isomorphisms between the two maps $f \rightarrow f_G(\pi)$ and $f \rightarrow f_G(\tau)$. It will therefore be enough for us to characterize the image of the second map.

Before discussing the image, we should recall [1, §3] that $\tilde{T}(G)$ is a disjoint union over all $L \in \mathcal{L}$ of spaces $\tilde{T}_{\text{ell}}(L)$. By definition, $\tilde{T}_{\text{ell}}(L)$ is the set of triplets

$$(M, \sigma, r), \quad M \subset L, \quad r \in \tilde{R}_{\sigma, \text{reg}}^L,$$

in which M is contained in L and where the null space of r , as a linear transformation on the real vector space

$$\mathfrak{a}_M = X(M)_F \otimes \mathbf{R},$$

is the subspace \mathfrak{a}_L . There is an action

$$\tau \longrightarrow \tau_\lambda = (M, \sigma_\lambda, r), \quad \lambda \in i\mathfrak{a}_L^*,$$

of the real vector space $i\mathfrak{a}_L^*$ on $\tilde{T}_{\text{ell}}(L)$. This makes $\tilde{T}(G)$ into a disjoint union of compact tori if F is p -adic, and a disjoint union of Euclidean spaces if F is Archimedean.

We can now define $\mathcal{I}(G(F))$ to be the set of complex valued functions ϕ on $T(G)$ which satisfy the symmetry condition

$$\phi(z\tau) = \chi_\tau(z)^{-1}\phi(\tau), \quad z \in Z_\tau, \quad \tau \in T(G),$$

and which lie in the appropriate space of W_0^G -invariant functions on $\tilde{T}(G)$. That is, ϕ must be in $C_c^\infty(\tilde{T}(G))$ if F is p -adic, and in $\mathcal{S}(\tilde{T}(G))$ if F is Archimedean. In the Archimedean case, we can assume that $F = \mathbf{R}$. Then any representation in $\Pi_2(M(\mathbf{R}))$ can be written uniquely in the form σ_λ , where λ lies in $i\mathfrak{a}_M^*$ and $\sigma \in \Pi_2(M(\mathbf{R}))$ is invariant under the split component $A_M(\mathbf{R})^0$ of the center of $M(\mathbf{R})$. In this case we write μ_{σ_λ} for the linear form that determines the infinitesimal character of σ_λ . Thus, μ_{σ_λ} is a Weyl orbit of elements in the dual of a complex Cartan subalgebra, which we assume is equipped with suitable Hermitian norm $\|\cdot\|$, such that

$$\|\mu_{\sigma_\lambda}\| = \|\mu_\sigma + \lambda\| = \|\mu_\sigma\| + \|\lambda\|.$$

By definition, $\mathcal{S}(\tilde{T}(G))$ is the space of smooth functions ϕ on $\tilde{T}(G)$ such that for each $L \in \mathcal{L}$, each integer n , and each invariant differential operator $D = D_\lambda$ on $i\mathfrak{a}_L^*$, transferred in the obvious way

$$D_\tau\phi(\tau) = \lim_{\lambda \rightarrow 0} D_\lambda\phi(\tau_\lambda), \quad \tau \in \tilde{T}_{\text{ell}}(L),$$

to $\tilde{T}_{\text{ell}}(L)$, the semi-norm

$$\|\phi\|_{L,D,n} = \sup_{\tau \in \tilde{T}_{\text{ell}}(L)} (|D_\tau\phi(\tau)|(1 + \|\mu_\tau\|)^n)$$

is finite. (We are writing $\mu_\tau = \mu_\sigma$ for $\tau = (M, \sigma, r)$.) In both the real and p -adic cases, $\mathcal{I}(G(F))$ has a natural topology.

THEOREM. *The map*

$$\mathcal{T}_G : f \longrightarrow f_G ,$$

defined by (3), is an open, continuous and surjective linear transformation from $\mathcal{C}(G(F))$ onto $\mathcal{I}(G(F))$.

PROOF. As we have already noted, we shall use the characterization [2], [9, Part B] of the operator valued Fourier transform

$$\mathcal{F}_G : \mathcal{C}(G(F)) \longrightarrow \widehat{\mathcal{C}}(G(F))$$

on the Schwartz space. (The discussion of the map \mathcal{T}_G for p -adic groups in [9, Part C, §VI] is incomplete.) The space $\widehat{\mathcal{C}}(G(F))$ consists of smooth operator valued functions

$$\Phi : (P, \sigma) \longrightarrow \Phi_P(\sigma) \in \text{End}(\mathcal{H}_P(\sigma)) , \quad P \in \mathcal{P}(M) , \sigma \in \Pi_2(M(F)) , M \in \mathcal{L} ,$$

which satisfy a symmetry condition and a growth condition. To describe the symmetry condition, we suppose that $M' \in \mathcal{L}$, $P' \in \mathcal{P}(M')$, and that w belongs to the set $W(\mathfrak{a}_M, \mathfrak{a}_{M'})$ of isomorphisms from \mathfrak{a}_M onto $\mathfrak{a}_{M'}$ obtained by restricting elements in W_0^G to \mathfrak{a}_M . Set

$$R_{P'|P}(\tilde{w}, \sigma) = A(\tilde{w})R_{w^{-1}P'|P}(\sigma) ,$$

where $R_{w^{-1}P'|P}(\sigma)$ is the normalized intertwining operator from $\mathcal{H}_P(\sigma)$ to $\mathcal{H}_{w^{-1}P'}(\sigma)$, \tilde{w} is a representative of w in a fixed maximal compact subgroup K , and $A(\tilde{w})$ is the canonical map from $\mathcal{H}_{w^{-1}P'}(\sigma)$ to $\mathcal{H}_{P'}(\tilde{w}\sigma)$. Then Φ must satisfy

$$(4) \quad \Phi_{P'}(\tilde{w}\sigma) = R_{P'|P}(\tilde{w}, \sigma)\Phi_P(\sigma)R_{P'|P}(\tilde{w}, \sigma)^{-1}$$

for all such (M', P', w) . As for the growth condition, observe that the domain of Φ can be identified with a disjoint union of compact tori if F is p -adic. In this case we require simply that Φ have compact support. If F is Archimedean, we require that the semi-norms

$$\sup_{P, \sigma, \delta_1, \delta_2} \|D_\sigma(\Gamma_{\delta_2}\Phi_P(\sigma)\Gamma_{\delta_1})\|(1 + \|\mu_\sigma\|)^n(1 + \|\mu_{\delta_1}\|)^{m_1}(1 + \|\mu_{\delta_2}\|)^{m_2} ,$$

determined by integers n, m_1, m_2 , and differential operators D_σ , be finite. The elements δ_1 and δ_2 here range over irreducible K -types, and Γ_δ stands for the K -invariant projection of $\mathcal{H}_P(\sigma)$ onto the δ isotypical subspace $\mathcal{H}_P(\sigma)_\delta$. As above, D_σ is assumed to come from an invariant differential operator on $i\mathfrak{a}_M^*$. In each case, $\widehat{\mathcal{C}}(G(F))$ becomes a topological vector space, and the Fourier transform

$$\mathcal{F}_G : f \longrightarrow (\mathcal{F}_G f)_P(\sigma) = \mathcal{I}_P(\sigma, f)$$

is a topological isomorphism from $\mathcal{C}(G(F))$ onto $\widehat{\mathcal{C}}(G(F))$.

We define a trace map

$$\widehat{\mathcal{T}}_G : \widehat{\mathcal{C}}(G(F)) \longrightarrow \mathcal{I}(G(F))$$

by setting

$$(\widehat{\mathcal{T}}_G \Phi)(\tau) = \text{tr}(\widetilde{R}(r, \sigma) \Phi_P(\sigma))$$

for any triplet $\tau = (M, \sigma, r)$ in $T(G)$. Our original map \mathcal{T}_G is then the composition of \mathcal{F}_G with $\widehat{\mathcal{T}}_G$. We observe directly from this construction that \mathcal{T}_G maps $\mathcal{C}(G(F))$ continuously into $\mathcal{I}(G(F))$. Moreover, to prove the remaining assertions that \mathcal{T}_G is open and surjective, it suffices to construct a continuous section

$$h_G : \mathcal{I}(G(F)) \longrightarrow \widehat{\mathcal{C}}(G(F))$$

for $\widehat{\mathcal{T}}_G$.

Suppose first that $F = \mathbb{R}$. Then we shall write the representations in $\Pi_2(M(\mathbb{R}))$ in the form

$$\sigma_\lambda, \quad \sigma \in \Pi_2(M(\mathbb{R})/A_M(\mathbb{R})^0), \quad \lambda \in \mathfrak{ia}_M^*,$$

as above. In this case, R_σ is a product of groups $\mathbb{Z}/2\mathbb{Z}$ [8]. Moreover, the cocycle which defines \widetilde{R}_σ splits [8, Theorem 7.1], so we may take $\widetilde{R}_\sigma = R_\sigma$. For any λ , R_{σ_λ} is the subgroup of elements in R_σ which fix λ .

We shall use Vogan's theory of minimal K -types [10], [11], [5, §2.3]. Given $\sigma \in \Pi_2(M(\mathbb{R})/A_M(\mathbb{R})^0)$, let $A(\sigma)$ denote the set of minimal K -types for the representation $\mathcal{I}_P(\sigma)$. If r belongs to R_σ , we shall write

$$R(r, \sigma)_{\min} = \sum_{\delta \in A(\sigma)} R(r, \sigma)_\delta,$$

where $R(r, \sigma)_\delta$ denotes the restriction of $R(r, \sigma)$ to the δ -isotypical subspace $\mathcal{H}_P(\sigma)_\delta$. Then there is a bijection $\rho \rightarrow \delta_\rho$ from the set $\Pi(R_\sigma)$ of (abelian) characters of R_σ onto $A(\sigma)$ with the property that

$$\text{tr}(R(r, \sigma)_{\min} \mathcal{I}_P(\sigma, k)) = \sum_{\rho \in \Pi(R_\sigma)} \rho^\vee(r) \text{tr}(\delta_\rho(k))$$

for each $r \in R_\sigma$ and $k \in K$. This follows from the fact that each δ occurs in $\mathcal{I}_P(\sigma)$ with multiplicity 1, and that moreover any irreducible constituent of $\mathcal{I}_P(\sigma)$ contains exactly one element in $A(\sigma)$. In particular, if $\delta = \delta_\rho$, the operator $R(r, \sigma)_\delta$ is simply equal to the scalar $\rho^\vee(r)$ on $\mathcal{H}_P(\sigma)_\delta$. This suggests that we define operators

$$S_P(r, \sigma) = \sum_{\delta \in A(\sigma)} \text{deg}(\delta)^{-1} R(r, \sigma)_\delta, \quad r \in R_\sigma,$$

on

$$\mathcal{H}_P(\sigma)_{\min} = \bigoplus_{\delta \in A(\sigma)} \mathcal{H}_P(\sigma)_\delta.$$

Then

$$\text{tr}(R(r_1, \sigma) S_P(r_1^{-1}, \sigma)) = \prod_{\rho \in \Pi(R_\sigma)} \rho^\vee(r_1 r_1^{-1}) = \begin{cases} |R_\sigma|, & \text{if } r_1 = r, \\ 0, & \text{otherwise,} \end{cases}$$

for any elements $r_1, r \in R_\sigma$. Suppose that w belongs to $W(\mathfrak{a}_M, \mathfrak{a}_{M'})$, and that λ lies in $i\mathfrak{a}_M^*$. Then the linear transformation $R_{P'|P}(\tilde{w}, \sigma_\lambda)$ in (4) intertwines the action of K on the various spaces $\mathcal{H}_P(\sigma)_\delta$ and $\mathcal{H}_{P'}(\tilde{w}\sigma)_\delta$. Consequently, the operator

$$(5) \quad R_{P'|P}(\tilde{w}, \sigma)^{-1} R_{P'|P}(\tilde{w}, \sigma_\lambda)$$

acts as a scalar on each of the spaces $\mathcal{H}_P(\sigma)_\delta$, and therefore commutes with $S_P(r, \sigma)$. It follows easily that

$$\begin{aligned} & R_{P'|P}(\tilde{w}, \sigma_\lambda) S_P(r, \sigma) R_{P'|P}(\tilde{w}, \sigma_\lambda)^{-1} \\ &= R_{P'|P}(\tilde{w}, \sigma) S_P(r, \sigma) R_{P'|P}(\tilde{w}, \sigma)^{-1} \\ &= S_{P'}(wrw^{-1}, \tilde{w}\sigma), \end{aligned}$$

for any $r \in R_\sigma$.

In order to construct the section h_G , we choose a function $\beta_M^L \in C_c^\infty(i\mathfrak{a}_M^*/i\mathfrak{a}_L^*)$ for each pair of Levi subgroups $M \subset L$ in \mathcal{L} , such that $\beta_M^L(0) = 1$, and such that

$$\beta_{wM}^{wL}(w\lambda) = \beta_M^L(\lambda), \quad \lambda \in i\mathfrak{a}_M^*,$$

for any $w \in W_0^G$. Suppose that ϕ belongs to $\mathcal{I}(G(F))$. The domain of ϕ can be represented as the set of W_0^G -orbits of triplets

$$\{ \tau = (M, \sigma_{\lambda_L}, r) : M \subset L, \sigma \in \Pi_2(M(\mathbf{R})/A_M(\mathbf{R})^0), \lambda_L \in i\mathfrak{a}_L^*, r \in R_{\sigma, \text{reg}}^L \}.$$

We define $h_G(\phi)$ to be the operator valued function

$$\Phi_P(\sigma_\lambda) = |R_\sigma|^{-1} \sum_{L \in \mathcal{L}(M)} \sum_{r \in R_{\sigma, \text{reg}}^L} \beta_M^L(\lambda) \phi(M, \sigma_{\lambda_L}, r) S_P(r^{-1}, \sigma),$$

where λ_L denotes the projection onto $i\mathfrak{a}_L^*$ of the variable $\lambda \in i\mathfrak{a}_M^*$. We shall show that this function lies in $\widehat{\mathcal{C}}(G(\mathbf{R}))$, and that its image under $\widehat{\mathcal{T}}_G$ is ϕ .

Take any $w \in W(\mathfrak{a}_M, \mathfrak{a}_{M'})$ as in the symmetry condition (4). Since β_M^L, ϕ and S_P each satisfy their own symmetry conditions, we see that

$$\begin{aligned} & R_{P'|P}(\tilde{w}, \sigma_\lambda) \Phi_P(\sigma_\lambda) R_{P'|P}(\tilde{w}, \sigma_\lambda)^{-1} \\ &= |R_\sigma|^{-1} \sum_{L \in \mathcal{L}(M)} \sum_{r \in R_{\sigma, \text{reg}}^L} \beta_{wM}^{wL}(w\lambda) \phi(wM, w(\sigma_{\lambda_L}), wrw^{-1}) S_{P'}(wr^{-1}w^{-1}, \tilde{w}\sigma) \\ &= \Phi_{P'}(\tilde{w}\sigma_\lambda). \end{aligned}$$

Therefore the symmetry condition (4) holds. To establish the required growth condition, we use the fact that the infinitesimal character of any of the K -types in $A(\sigma)$ may be bounded linearly in terms of the infinitesimal character of σ ; there is a constant c such that

$$\|\mu_\delta\| \leq c(1 + \|\mu_\sigma\|)$$

for every $\sigma \in \Pi_2(M(\mathbf{R})/A_M(\mathbf{R})^0)$ and $\delta \in A(\sigma)$. The growth condition for ϕ as an element of $\mathcal{I}(G(F))$ then implies the growth condition of $\widehat{\mathcal{C}}(G(F))$ for $\Phi_P(\sigma_\lambda)$. It follows that the function

$$\Phi : (P, \sigma_\lambda) \rightarrow \Phi_P(\sigma_\lambda)$$

belongs to $\widehat{\mathcal{C}}(G(F))$. In fact the estimates imply that $h_G: \phi \rightarrow \Phi$ is a continuous linear map from $\mathcal{I}(G(F))$ into $\widehat{\mathcal{C}}(G(F))$.

Finally, to evaluate $\widehat{\mathcal{T}}_G \Phi$, choose any triplet

$$\tau = (M, \sigma_\lambda, r_1), \quad r_1 \in R_\sigma, \quad r_1 \lambda = \lambda,$$

in $T(G)$. Then

$$\begin{aligned} (\widehat{\mathcal{T}}_G \Phi)(\tau) &= \text{tr}(R(r_1, \sigma_\lambda) \Phi_P(\sigma_\lambda)) \\ &= |R_\sigma|^{-1} \sum_{L \in \mathcal{L}(M)} \sum_{r \in R_{\sigma, \text{reg}}^L} \beta_M^L(\lambda) \phi(M, \sigma_{\lambda_L}, r) \text{tr}(R(r_1, \sigma_\lambda) S_P(r^{-1}, \sigma)). \end{aligned}$$

The operator $R(r_1, \sigma_\lambda) = R_{P|P}(r_1, \sigma_\lambda)$ here depends implicitly on the group $P \in \mathcal{P}(M)$. However, the trace inside the sum does not, so we are free to choose P so that $R(r_1, \sigma_\lambda)$ equals $R(r_1, \sigma)$. Then as we have seen above, a summand will vanish unless r equals r_1 , in which case it equals

$$\beta_M^L(\lambda) \phi(M, \sigma_{\lambda_L}, r_1) |R_\sigma|.$$

If $L_1 \in \mathcal{L}(M)$ is the group for which r_1 lies in $R_{\sigma, \text{reg}}^{L_1}$, then λ lies in $i\mathfrak{a}_{L_1}^*$. Therefore $\lambda_{L_1} = \lambda$ and $\beta_M^{L_1}(\lambda) = \beta_M^{L_1}(0) = 1$. We obtain

$$(\widehat{\mathcal{T}}_G \Phi)(\tau) = \phi(M, \sigma_\lambda, r_1) = \phi(\tau).$$

We have verified that $h_G: \phi \rightarrow \Phi$ is the required section for $\widehat{\mathcal{T}}_G$, thereby establishing the theorem in the case that $F = \mathbb{R}$.

Now suppose that F is a p -adic field. One could use Schwartz-multipliers to establish the theorem, in the spirit of the corresponding result [3] for Hecke algebras. We shall instead follow an argument which is closer to the discussion above. An element $\phi \in \mathcal{I}(G(F))$ is supported on finitely many connected components in $T(G)$. Using an W_0^G -invariant partition of unity, we can assume that ϕ is supported on a small neighbourhood of some fixed point in $T(G)$. More precisely, we assume that $\phi(\tau)$ vanishes unless τ is of the form (M, σ_λ, r) , where (M, σ) belongs to a fixed orbit of W_0^G and λ lies in a small neighbourhood \mathcal{N}_M of 0 in $i\mathfrak{a}_M^*$. One reason for localizing around σ is to ensure that for any $\lambda \in \mathcal{N}_M$, R_{σ_λ} is the subgroup of elements in R_σ which fix λ . We shall impose a second condition on the size of \mathcal{N}_M presently.

Choose an open compact subgroup K_0 of $G(F)$, and let $\widetilde{R}(r, \sigma)_{K_0}$ denote the restriction of $\widetilde{R}(r, \sigma)$ to the subspace $\mathcal{H}_P(\sigma)_{K_0}$ of K_0 -fixed vectors in $\mathcal{H}_P(\sigma)$. The representation

$$r \longrightarrow \widetilde{R}(r, \sigma)_{K_0}, \quad r \in \widetilde{R}_\sigma,$$

of \widetilde{R}_σ on $\mathcal{H}_P(\sigma)_{K_0}$ is equivalent to a direct sum

$$\bigoplus_{\rho \in \Pi(\widetilde{R}_\sigma, \chi_\sigma)} \dim(\pi_{\rho, K_0}) \rho^\vee,$$

where π_{ρ, K_0} denotes the $\mathcal{C}(G(F)//K_0)$ -module of K_0 -fixed vectors in the representation π_ρ . We take K_0 to be so small that π_{ρ, K_0} is nonzero for each ρ . Writing $\tilde{R}(r, \sigma)_{\rho, K_0}$ for the restriction of $\tilde{R}(r, \sigma)$ to the subspace of $\mathcal{H}_P(\sigma)_{K_0}$ corresponding to ρ , we define operators

$$S_P(r, \sigma) = \sum_{\rho \in \Pi(\tilde{R}_\sigma, \chi_\sigma)} \deg(\rho) \dim(\pi_{\rho, K_0})^{-1} \tilde{R}(r, \sigma)_{\rho, K_0}, \quad r \in \tilde{R}_\sigma,$$

on $\mathcal{H}_P(\sigma)_{K_0}$. For any pair of elements $r_1, r \in \tilde{R}_\sigma$, we have

$$\begin{aligned} & \text{tr}(\tilde{R}(r_1, \sigma) S_P(r^{-1}, \sigma)) \\ &= \sum_{\rho \in \Pi(\tilde{R}_\sigma, \chi_\sigma)} \deg(\rho) \text{tr}(\rho^\vee(r_1 r^{-1})) = \begin{cases} |R_\sigma| \chi_\sigma(z), & \text{if } r = r_1 z, z \in Z_\sigma, \\ 0, & \text{if } r \notin r_1 Z_\sigma. \end{cases} \end{aligned}$$

Unlike in the Archimedean case, however, the operator (5) does not generally commute with $S_P(r, \sigma)$. To deal with this complication, we define a function

$$Q_P(\sigma, \lambda) = \sum_{M_1 \in \mathcal{L}} \sum_{w_1 \in W(\mathfrak{a}_M, \mathfrak{a}_{M_1})} \sum_{P_1 \in \mathcal{P}(M_1)} R_{P_1|P}(\tilde{w}_1, \sigma_\lambda)^{-1} R_{P_1|P}(\tilde{w}_1, \sigma),$$

of $\lambda \in i\mathfrak{a}_M^*$. If w and P' are as in (4), we have

$$\begin{aligned} & R_{P'|P}(\tilde{w}, \sigma_\lambda) Q_P(\sigma, \lambda) R_{P'|P}(\tilde{w}, \sigma)^{-1} \\ &= \sum_{M_1, w_1, P_1} R_{P'|P}(\tilde{w}, \sigma_\lambda) R_{P_1|P}(\tilde{w}_1, \sigma_\lambda)^{-1} R_{P_1|P}(\tilde{w}_1, \sigma) R_{P'|P}(\tilde{w}, \sigma)^{-1} \\ &= \sum_{M_1, w_1, P_1} R_{P_1|P'}(\tilde{w}_1 \tilde{w}^{-1}, \tilde{w} \sigma_\lambda)^{-1} R_{P_1|P'}(\tilde{w}_1 \tilde{w}^{-1}, \tilde{w} \sigma), \end{aligned}$$

by the multiplicative properties of the intertwining operators. Changing variables in the sum over w_1 , we see that

$$R_{P'|P}(\tilde{w}, \sigma_\lambda) Q_P(\sigma, \lambda) = Q_{P'}(\tilde{w} \sigma, w \lambda) R_{P'|P}(\tilde{w}, \sigma).$$

Observe that if $\lambda = 0$, $Q_P(\sigma, \lambda)$ is a positive multiple of the identity operator. We assume that the neighbourhood \mathcal{N}_M is so small that the restriction of $Q_P(\sigma, \lambda)$ to $\mathcal{H}_P(\sigma)_{K_0}$ is invertible for every $\lambda \in \mathcal{N}_M$. We can then define

$$S_P(r, \sigma, \lambda) = Q_P(\sigma, \lambda) S_P(r, \sigma) Q_P(\sigma, \lambda)^{-1}, \quad r \in \tilde{R}_\sigma.$$

It follows easily that

$$\begin{aligned} & R_{P'|P}(\tilde{w}, \sigma_\lambda) S_P(r, \sigma, \lambda) R_{P'|P}(\tilde{w}, \sigma)^{-1} \\ &= Q_{P'}(\tilde{w} \sigma, w \lambda) R_{P'|P}(\tilde{w}, \sigma) S_P(r, \sigma) R_{P'|P}(\tilde{w}, \sigma)^{-1} Q_{P'}(\tilde{w} \sigma, w \lambda)^{-1} \\ &= Q_{P'}(\tilde{w} \sigma, w \lambda) S_{P'}(w r w^{-1}, \tilde{w} \sigma) Q_{P'}(\tilde{w} \sigma, w \lambda)^{-1} \\ &= S_{P'}(w r w^{-1}, \tilde{w} \sigma, w \lambda). \end{aligned}$$

Notice also that if $r_1\lambda = \lambda$ for an element $r_1 \in \tilde{R}_\sigma$, and if P is chosen so that $\tilde{R}(r_1, \sigma_\lambda)$ equals $\tilde{R}(r_1, \sigma)$, the operators $\tilde{R}(r_1, \sigma)$ and $Q_P(\sigma, \lambda)$ commute. Therefore

$$\operatorname{tr}(\tilde{R}(r_1, \sigma)S_P(r^{-1}, \sigma, \lambda)) = \begin{cases} |R_\sigma|\chi_\sigma(z), & \text{if } r = r_1z, z \in Z_\sigma, \\ 0, & \text{if } r \notin r_1Z_\sigma. \end{cases}$$

This modification allows us to construct the section h_G as we did in the case $F = \mathbf{R}$. We choose the functions $\beta_M^L \in C_c^\infty(\mathfrak{ia}_M^*/\mathfrak{ia}_L^*)$ as above, with the further stipulation that they each be supported on a small neighbourhood of 0. Given ϕ , we define $h_G(\phi)$ to be the operator valued function

$$\Phi_P(\sigma_\lambda) = |R_\sigma|^{-1} \sum_{L \in \mathcal{L}(M)} \sum_{r \in \tilde{R}_{\sigma, \text{reg}}^L} \beta_M^L(\lambda)\phi(M, \sigma_{\lambda L}, r)S_P(r^{-1}, \sigma, \lambda).$$

Our support conditions on β_M^L and ϕ imply that the right hand side vanishes unless λ belongs to \mathcal{N}_M , and therefore that $S_P(r^{-1}, \sigma, \lambda)$ is well defined. The symmetry condition (4) follows from the remarks above, as in the Archimedean case. The required growth condition is trivial. Consequently, the function

$$\Phi : (P, \sigma_\lambda) \rightarrow \Phi_P(\sigma_\lambda)$$

belongs to $\hat{\mathcal{C}}(G(F))$, and $\phi \rightarrow h_G(\phi) = \Phi$ is a continuous linear map from $\mathcal{I}(G(F))$ into $\hat{\mathcal{C}}(G(F))$. Finally, suppose that τ is any element in $T(G)$. Then $(\hat{\mathcal{T}}_G\Phi)(\tau)$ vanishes unless τ is of the form

$$(M, \sigma_\lambda, r_1), \quad \lambda \in \mathcal{N}_M, r_1 \in \tilde{R}_\sigma, r_1\lambda = \lambda,$$

in which case we deduce that

$$(\hat{\mathcal{T}}_G\Phi)(\tau) = \operatorname{tr}(\tilde{R}(r_1, \sigma_\lambda)\Phi_P(\sigma_\lambda)) = \phi(\tau),$$

again as in the Archimedean case. Therefore, the map $h_G : \phi \rightarrow \Phi$ is the required section for $\hat{\mathcal{T}}_G$. We have established the theorem for arbitrary F . \square

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