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Mathematics

A theorem on the Schwartz space of a reductive Lie group

(associated parabolic subgroups/Plancherel measure/Fourier transform on a reductive Lie group)

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ABSTRACT **The purpose of this paper is to define the Fourier transform of an arbitrary tempered distribution on a reductive Lie group. To this end we define a topological vector space, $\mathcal{C}(\hat{G})$, in terms of the classes of irreducible unitary representations of G, which plays the role of a dual Schwartz space. Our main theorem then asserts that the usual L^2 Fourier transform, when restricted to functions in the Schwartz space, $\mathcal{C}(G)$ defined by Harish-Chandra, provides a topological isomorphism from $\mathcal{C}(G)$ onto $\mathcal{C}(\hat{G})$.**

Let G be a reductive Lie group with Lie algebra $\mathfrak{g}$. We make the three assumptions on G stated in (ref. 1, § 3), and adopt the conventions and terminology of (ref. 1, § 2 and § 3). In particular K is a maximal compact subgroup of G. Furthermore if P is a parabolic subgroup we have the decomposition

$$P = NAM,$$

where N is nilpotent, M is reductive, and A is a vector group. Fix a minimal parabolic subgroup

$${}^{(0)}P = {}^{(0)}N \cdot {}^{(0)}A \cdot {}^{(0)}M$$

of G. A parabolic subgroup $P = NAM$ is said to be standard if it contains ${}^{(0)}P$. When this is so, $\mathfrak{a}$, the Lie algebra of A, is contained in ${}^{(0)}\mathfrak{a}$, the Lie algebra of ${}^{(0)}A$. A subspace of ${}^{(0)}\mathfrak{a}$ obtained in this way is called a distinguished subspace. Denote the restricted Weyl group of $\mathfrak{g}$ on ${}^{(0)}\mathfrak{a}$ by Ω. If $\mathfrak{a}$ and $\mathfrak{a}'$ are distinguished subspaces of ${}^{(0)}\mathfrak{a}$, let $\Omega(\mathfrak{a},\mathfrak{a}')$ be the set of distinct mappings from $\mathfrak{a}$ onto $\mathfrak{a}'$ that can be obtained by restricting transformations in Ω to $\mathfrak{a}$. Recall that the standard parabolic subgroups P and P' corresponding to $\mathfrak{a}$ and $\mathfrak{a}'$ are said to be *associated* if $\Omega(\mathfrak{a},\mathfrak{a}')$ is not empty.

Given a standard P, we write $\mathcal{E}_2(M)$ for the (possibly empty) set of equivalence classes of irreducible unitary square integrable representations of M. Fix $\omega \in \mathcal{E}_2(M)$. Let $L_\omega^2(M,K)$ be the space of complex valued measurable functions

$$\psi\colon (k_1,m,k_2) \to \psi(k_1{:}m{:}k_2), \qquad k_1,k_2 \in K, \qquad m \in M,$$

such that

(1) $\psi(k_1{:}k_1'mk_2'{:}k_2) = \psi(k_1k_1'{:}m{:}k_2'k_2), \quad k_1',k_2' \in K \cap M,$

(2) $\displaystyle \|\psi\|^2 = \int_{K\times K}\int_M |\psi(k_1{:}m{:}k_2)|^2\, dm\, dk_1\, dk_2 < \infty,$

(3) for almost all k_1 and k_2, the function

$$m \to \psi(k_1{:}m{:}k_2), \qquad m \in M,$$

belongs to the closed subspace of $L^2(M)$ generated by the matrix coefficients of ω. Suppose that σ is a representation in the class ω which acts on the Hilbert space H_σ. Let $\mathcal{H}(\sigma)$ be the Hilbert space of functions ϕ from $NA\backslash G$ to H_σ such that

$$\phi(mx) = \sigma(m)\phi(x), \qquad m \in M, \qquad x \in G,$$

and

$$\|\phi\|_2{}^2 = \int_K |\phi(k)|^2 dk < \infty.$$

Then $L_\omega^2(M,K)$ is canonically isomorphic to the space of Hilbert-Schmidt operators on $\mathcal{H}(\sigma)$. For any $\lambda \in \mathfrak{a}_{\mathbf{C}}$ we have the usual induced representation $\pi_{\omega,\lambda}$ of the group G, as well as the convolution algebra $C_c^\infty(G)$, on $\mathcal{H}(\sigma)$. By means of the above isomorphism and the map $f \to \pi_{\omega,\lambda}(f)$, we obtain a map

$$f \to \hat{f}(\omega,\lambda), \qquad f \in C_c^\infty(G),$$

from $C_c^\infty(G)$ to $L_\omega^2(M,K)$.

Suppose that $P' = N'A'M'$ is associated to P and that $s \in \Omega(\mathfrak{a},\mathfrak{a}')$. s determines a unique coset in $K/K \cap M$, from which we can define a map

$$\omega \to s\omega, \qquad \omega \in \mathcal{E}_2(M),$$

from $\mathcal{E}_2(M)$ to $\mathcal{E}_2(M')$. For fixed ω there is a unique map

$$M(s{:}\lambda)\colon L_\omega^2(M,K) \to L_{s\omega}^2(M',K),$$

which depends meromorphically on $\lambda \in \mathfrak{a}_{\mathbf{C}}$, such that for any $f \in C_c^\infty(G)$,

$$\hat{f}(s\omega,s\lambda) = M(s{:}\lambda)\hat{f}(\omega,\lambda).$$

$M(s{:}\lambda)$ is unitary if λ is purely imaginary.

Let $Cl(G)$ be the set of equivalence classes of associated standard parabolic subgroups of G. For any $\mathcal{P} \in Cl(G)$, define $L_{\mathcal{P}}^2(\hat{G})$ to be the space of measurable functions

$$(\omega,\lambda) \to a_{\mathcal{P}}(\omega,\lambda), \qquad P \in \mathcal{P},\ \omega \in \mathcal{E}_2(M),\ \lambda \in i\mathfrak{a},$$

with values in $L_\omega^2(M,K)$, which satisfy the following two conditions:

(*1*) if $P,P' \in \mathcal{P}$, $s \in \Omega(\mathfrak{a},\mathfrak{a}')$, $\omega \in \mathcal{E}_2(M)$, and $\lambda \in i\mathfrak{a}$, then

$$a_{\mathcal{P}}(s\omega,s\lambda) = M(s{:}\lambda)a_{\mathcal{P}}(\omega,\lambda);$$

(*2*) the expression

$$\|a_{\mathcal{P}}\|^2 = \sum_{P\in\mathcal{P}} \sum_{\omega\in\mathcal{E}_2(M)} \int_{i\mathfrak{a}} \|a_{\mathcal{P}}(\omega,\lambda)\|^2 \mu_\omega(\lambda)d\lambda$$

is finite. Here $d\lambda$ is a fixed Haar measure on $i\mathfrak{a}$, and $\mu_\omega(\lambda)$ is the Plancherel density, an analytic function on $i\mathfrak{a}$ which depends on the measure $d\lambda$. Harish-Chandra has computed $\mu_\omega(\lambda)$ explicitly.

Define $L^2(\hat{G})$ to be the direct sum over all $\mathcal{P} \in Cl(G)$ of the spaces $L_{\mathcal{P}}^2(\hat{G})$. Suppose that f is a function in $C_c^\infty(G)$. Define $\hat{f}_{\mathcal{P}}$ to be the function whose value at $P \in \mathcal{P}$, $\omega \in \mathcal{E}_2(M)$, and $\lambda \in i\mathfrak{a}$ is the vector

$$\hat{f}(\omega,\lambda) = \hat{f}_{\mathcal{P}}(\omega,\lambda)$$

in $L_\omega^2(M,K)$ introduced above. Define

$$\hat{f} = \oplus_{\mathcal{P}} \hat{f}_{\mathcal{P}}.$$

The following lemma is a reformulation of Harish-Chandra's Plancherel formula (ref. 1, Lemma 11).

LEMMA. *The map*

$$f \to \hat{f}, \qquad f \in C_c(G),$$

extends to an isometry from $L^2(G)$ onto $L^2(\hat{G})$.

The Schwartz space, $\mathcal{C}(G)$, is a dense subspace of $L^2(G)$. It is a natural problem to characterize the image of $\mathcal{C}(G)$ under the above map. To do this, we must define a family of seminorms on $L^2(\hat{G})$.

First of all, fix a bi-invariant metric on K, and let Z_K be the Laplace Beltrami operator. Each class $\eta \in \mathcal{E}(K)$ determines an eigenspace of Z_K. We define the *absolute value*, $|\eta|$, of η to be the absolute value of the corresponding eigenvalue. Next, suppose that $P = NAM$ is a standard parabolic subgroup of G, belonging to the associated class $\mathcal{P}$. Let ω be an element in $\mathcal{E}_2(M)$. Let us call an element $\psi \in L_\omega^2(M,K)$ simple if there are two classes $\eta_1(\psi)$ and $\eta_2(\psi)$ in $\mathcal{E}(K)$ such that for each $m \in M$, the function

$$(k_1,k_2) \to \psi(k_1{:}m{:}k_2), \qquad k_1,k_2 \in K,$$

belongs to the subspace of $L^2\ (K \times K)$ determined by $[\eta_1(\psi), \eta_2(\psi)]$. Denote the set of simple *unit* vectors in $L_\omega^2(M,K)$ by $U(\omega)$. Now, suppose that n is a positive integer and that $D = D_\lambda$ is a differential operator with constant coefficients on $i\mathfrak{a}$. For $a_{\mathcal{P}} \in L_{\mathcal{P}}^2(\hat{G})$, we set $||a_{\mathcal{P}}||_{D,n} = \infty$ if for some $\omega \in \mathcal{E}_2(M)$, and some $\psi \in U(\omega)$, the function

$$\lambda \to (a_{\mathcal{P}}(\omega,\lambda),\psi), \qquad \lambda \in i\mathfrak{a},$$

is not differentiable. Otherwise, we define $||a||_{D,n}$ to be the supremum over all $\lambda \in i\mathfrak{a}$, all $\omega \in \mathcal{E}_2(M)$, and all vectors ψ in $U(\omega)$ of

$$|D_\lambda(a_{\mathcal{P}}(\omega,\lambda),\ \psi)|(1 + |\lambda|^2)^n[1 + |\eta_1(\psi)|^2]^n[1 + |\eta_2(\psi)|^2]^n.$$

Let $\mathcal{C}_{\mathcal{P}}(\hat{G})$ be the set of those $a_{\mathcal{P}} \in L_{\mathcal{P}}^2(\hat{G})$ such that for each $P \in \mathcal{P}$, and all D and n, $||a_{\mathcal{P}}||_{D,n}$ is finite. $\mathcal{C}_{\mathcal{P}}(\hat{G})$, together with the above family of seminorms, becomes a topological vector space. Define $\mathcal{C}(\hat{G})$ to be the direct sum over all $\mathcal{P}$ of the spaces $\mathcal{C}_{\mathcal{P}}(\hat{G})$.

THEOREM. *The map*

$$\mathfrak{F}: f \to \hat{f}, \qquad f \in \mathcal{C}(G),$$

is a topological isomorphism from $\mathcal{C}(G)$ onto $\mathcal{C}(\hat{G})$.

The proof of this theorem is quite long. The easier half is to show that the image of $\mathcal{C}(G)$ is contained in $\mathcal{C}(\hat{G})$. The techniques for proving the other half, namely, that the inverse image of $\mathcal{C}(\hat{G})$ is contained in $\mathcal{C}(G)$, are entirely due to Harish-Chandra. They are his asymptotic estimates, introduced first in ref. 2, and later in ref. 3. For the proof of this theorem in case G has real rank one, see ref. 4.

The theorem allows us to define the Fourier transform of a tempered distribution on G. Let $\mathcal{C}'(G)$ and $\mathcal{C}'(\hat{G})$ be the topological dual spaces of $\mathcal{C}(G)$ and $\mathcal{C}(\hat{G})$. They become topological vector spaces when endowed with the weak topology. An immediate consequence of the theorem is

COROLLARY. *The transpose*

$$\mathfrak{F}': \mathcal{C}'(\hat{G}) \to \mathcal{C}'(G)$$

of $\mathfrak{F}$ is a topological isomorphism.

The details of the results announced above appear in the notes cited in the footnote.* I would like to thank my thesis advisor Robert Langlands for his advice and encouragement. This work was partially supported by National Science Foundation Grant GP-33893.

1. Harish-Chandra (1971) "On the theory of the Eisenstein integral," in *Lecture Notes in Mathematics* (Springer-Verlag, New York), Vol. 266, pp. 123–149.
2. Harish-Chandra (1958) "Spherical functions on a semisimple Lie group II," *Am. J. Math.* **80**, 553–613.
3. Harish-Chandra (1966) "Discrete series for semisimple Lie groups II," *Acta Math.* **116**, 1–111.
4. Arthur, J. (1970) "Harmonic analysis of tempered distributions on semisimple Lie groups of real rank one," Thesis, Yale; to appear in *University of Chicago Lecture Notes*.

* "Harmonic analysis on the Schwartz space of a reductive Lie group I, II," mimeographed notes, Yale University Mathematics Department, New Haven, Conn.