Some tempered distributions on semisimple groups of real rank one

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Introduction

The Selberg trace formula leads naturally to the study of certain tempered distributions on reductive groups defined over local fields. An important problem is to calculate the Fourier transforms of these distributions. We shall consider this question for the case that the local field is $\mathbf{R}$ and the group G is semisimple and has real rank one. In this context the notion of the Fourier transform of a tempered distribution has been defined in [1(a)].

A distribution T is said to be *invariant* if

$$T(f^y) = T(f)$$

for every $f \in C_c^\infty(G)$ and $y \in G$, where

$$f^y(x) = f(yxy^{-1}) , \qquad x \in G .$$

The invariant distributions which appear in the trace formula have recently been examined by Sally and Warner. However, the trace formula also contains some interesting noninvariant distributions. In this paper we shall calculate the Fourier transforms of the restriction of these distributions to $\mathcal{C}_0(G)$, the space of cusp forms on G.

For the case that $G = \mathrm{PSL}(2, \mathbf{R})$ these noninvariant distributions have already been dealt with. Here one utilizes the known formula for a matrix coefficient of a discrete series representation of G. This enables one to calculate the required Fourier transform (see, for example, [21]). For the general case, however, new methods are needed.

The basic distributions that we shall consider are the ones which appear in term (9.1) of [1(b)]. They are parametrized by the $\mathbf{R}$-regular points

$$\{a_1 h_t : t \in \mathbf{R},\ t \neq 0,\ a_1 \in A_1\}$$

of the noncompact Cartan subgroup of G, and will be denoted by $T(t, a_1)$. In Section 5 we show that $T(t, a_1)$ satisfies a second order nonhomogeneous differential equation, which becomes homogeneous if we restrict $T(t, a_1)$ to

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$\mathcal{C}_0(G)$. This reduces our problem to a search for boundary conditions.

At first glance, the study of the point $t = \infty$ might seem promising. This in fact would work if we were to calculate the Fourier transform of $T(t, a_1)$ on $\mathcal{C}_1(G)$, the space of functions in $\mathcal{C}(G)$ that are orthogonal to $\mathcal{C}_0(G)$. The boundary condition can be expressed by means of the constant term of the Eisenstein integral. The Fourier transform turns out to be a sum of two components, one which is not invariant and involves the derivative of the constant term, and another which is invariant but rather complicated. We shall give the details in another paper.

However, the asymptotic behavior of functions in $\mathcal{C}_0(G)$ is not yet well enough understood for us to obtain a suitable boundary condition at $t = \infty$. Instead, we take the limit as t approaches 0 of a certain distribution that involves the derivative of $T(t, a_1)$ with respect to t. As we show in Corollary 6.3, this limit defines an invariant distribution. We obtain from it a boundary condition which eventually allows us to compute our Fourier transform on $\mathcal{C}_0(G)$ in Theorem 7.2. In the process we derive a rather curious Jacobian formula (Theorem 6.4).

In Section 4 we introduce a distribution $T(a_1)$ which is closely related to $T(t, a_1)$. We obtain the Fourier transform of $T(a_1)$ in Corollary 7.3. Distributions of the form $T(a_1)$ also appear in the trace formula. They are the noninvariant components of term (9.2) in [1(b)].

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1. Preliminaries

Suppose that G is a connected semisimple Lie group of real rank one. We shall assume that G is contained in $G_{\mathbf{C}}$, a simply connected complexification of G. Let $\mathfrak{g}$ be the Lie algebra of G, and let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be a fixed Cartan decomposition of G with Cartan involution θ. Let K be the analytic subgroup of G corresponding to $\mathfrak{k}$. Then K is compact.

Let $\mathfrak{a}$ be a fixed maximal abelian subspace of $\mathfrak{p}$. Then the dimension of

$\mathfrak{a}$ is one. Let $A = \exp \mathfrak{a}$. Fix an abelian subspace $\mathfrak{a}_1$ of $\mathfrak{k}$ such that

$$\mathfrak{h} = \mathfrak{a}_1 + \mathfrak{a}$$

is a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{m}$ and M be the centralizers of $\mathfrak{a}$ in $\mathfrak{k}$ and K respectively. Then $\mathfrak{a}_1$ is a Cartan subalgebra of $\mathfrak{m}$. If A_1 is the centralizer of $\mathfrak{h}$ in K, A_1 is a Cartan subgroup of M.

Fix compatible ordering on the real dual spaces of $\mathfrak{a}$ and $\mathfrak{a} + i\mathfrak{a}_1$. Let P be the set of positive roots of $(\mathfrak{g}_C, \mathfrak{h}_C)$ with respect to this ordering. Let P_+ be the set of roots in P which do not vanish on $\mathfrak{a}$ and let P_M be the complement of P_+ in P. P_M can be regarded as a set of positive roots of $(\mathfrak{m}_C, \mathfrak{a}_{1C})$.

Let μ' be the linear functional on $\mathfrak{a}$ which equals one half the largest positive restricted root of $(\mathfrak{g}, \mathfrak{a})$. Extend the definition of μ' to $\mathfrak{h}$ by defining it to be zero on $\mathfrak{a}_1$. Let us fix an element H' in $\mathfrak{a}$ such that $\mu'(H') = 1$. Decompose $\mathfrak{g}$ with respect to the adjoint action of $\mathfrak{a}$. Then

$$\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$$

where if $X \in \mathfrak{g}_n$, $-2 \leqq n \leqq 2$,

$$[H', X] = nX .$$

Let

$$r_1 = \dim \mathfrak{g}_1 = \dim \mathfrak{g}_{-1} ,$$
$$r_2 = \dim \mathfrak{g}_2 = \dim \mathfrak{g}_{-2} .$$

Let B be the Killing form of $\mathfrak{g}_C$. The restriction of B to $\mathfrak{h}_C$ is nondegenerate, so we can lift B to the complex dual space of $\mathfrak{h}_C$. Then

$$B(H', H') = B(\mu', \mu')^{-1} = r^2 ,$$

if $r^2 = 2(r_1 + 4r_2)$.

The Cartan involution θ lifts to an automorphism of G which we shall also denote by θ. Let $\mathfrak{n} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and let $N = \exp \mathfrak{n}$. Then

$$\theta(N) = \exp \theta(\mathfrak{n}) = \exp(\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}) .$$

For an element n in N we shall sometimes write $\bar{n}$ for $\theta(n)$. Then

$$n \longrightarrow \bar{n}$$

is an isomorphism from N onto $\bar{N} = \theta(N)$.

From now on we shall write

$$h_t = \exp tH' , \qquad t \in \mathbf{R} .$$

This identifies the group A with the additive real numbers. We shall also write

$$\xi_\alpha(\exp H) = e^{\alpha(H)}\,, \qquad \alpha \in P,\ H \in \mathfrak{h}_{\mathbf{C}}\,.$$

ξ_α is a quasi-character on the Cartan subgroup $\exp \mathfrak{h}_{\mathbf{C}}$ of $G_{\mathbf{C}}$. In particular it defines a quasi-character on $A_1 \cdot A$.

There are two possibilities for $\mathfrak{g}$. Either any Cartan subalgebra of $\mathfrak{g}$ is G-conjugate to $\mathfrak{h}$ or there exists a second G-conjugacy class of Cartan subalgebras, which has a representative contained in $\mathfrak{k}$.

LEMMA 1.1. *There is a Cartan subalgebra* $\mathfrak{b}$ *of* $\mathfrak{g}$ *with* $\mathfrak{b} \subseteqq \mathfrak{k}$ *if and only if there exists a root* β *in* P_+ *which vanishes on* $\mathfrak{a}_1$.

Proof. Let $\varepsilon = \operatorname{rank} \mathfrak{g} - \operatorname{rank} \mathfrak{k}$. Then ε equals 0 or 1 depending on whether $\mathfrak{b}$ exists or not. By the Iwasawa decomposition

$$\dim \mathfrak{g} - \dim \mathfrak{k} = [P_+] + 1\,.$$

For any root α in P_+ define

$$\alpha^\theta(H) = \alpha(\theta H)\,, \qquad H \in \mathfrak{h}\,.$$

Then $-\alpha^\theta$ is also a root in P_+. $\alpha = -\alpha^\theta$ if and only if α vanishes on $\mathfrak{a}_1$. Therefore $[P_+]$ is odd or even, depending on whether a root β in P_+ which vanishes on $\mathfrak{a}_1$ exists or not. Since the number of roots of a reductive Lie algebra is always even,

$$(\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g}) - (\dim \mathfrak{k} - \operatorname{rank} \mathfrak{k})$$

is an even integer. Therefore

$$[P_+] + 1 - \varepsilon$$

is an even integer. This proves the lemma. □

LEMMA 1.2. *Suppose that there exists a root* β *in* P_+ *which vanishes on* $\mathfrak{a}_1$. *Then* $\beta = 2\mu'$.

Proof. Suppose that $\beta = \mu'$. Choose a root α of the form $2\mu' + \alpha_1$ for some linear functional α_1 on $\mathfrak{h}_{\mathbf{C}}$ which vanishes on $\mathfrak{a}_{\mathbf{C}}$. We shall show that α_1 can be chosen to be zero.

Notice that

$$\frac{2B(\alpha, \beta)}{B(\beta, \beta)} = \frac{2B(2\mu' + \alpha_1, \mu')}{B(\mu', \mu')} = \frac{4B(\mu', \mu')}{B(\mu', \mu')} = 4\,.$$

Therefore

$$\mu + \alpha_1,\ \ \alpha_1,\ \ -\mu' + \alpha_1,\ \ -2\mu' + \alpha_1$$

are all roots of $(\mathfrak{g}_C, \mathfrak{h}_C)$. We also have

$$\frac{2B(\alpha, \alpha_1)}{B(\alpha_1, \alpha_1)} = \frac{2B(2\mu' + \alpha_1, \alpha_1)}{B(\alpha_1, \alpha_1)} = 2 .$$

This implies that $\alpha - \alpha_1 = 2\mu'$ is a root of $(\mathfrak{g}_C, \mathfrak{h}_C)$. Therefore 2β is a root of $(\mathfrak{g}_C, \mathfrak{h}_C)$. This is a contradiction. ∎

Let A_1' be the set of semi-regular elements in A_1 of noncompact type; that is, those elements a_1 in A_1 such that $\mathfrak{g}(a_1)'$, the derived algebra of the centralizer of a_1 in $\mathfrak{g}$, is isomorphic to $\mathrm{sl}(2, \mathbf{R})$. Then a_1 is in A_1' if and only if

(i) there is a $\beta \in P_+$ which vanishes on $\mathfrak{a}_1$,

and

(ii) $\xi_\alpha(a_1) \neq 1$ for $\alpha \in P_+ - \{\beta\}$.

Now suppose that β is a root in P_+ that vanishes on $\mathfrak{a}_1$. Choose a root vector X' for β such that

$$-B(X', \theta X') = \frac{1}{2} r^2 .$$

Let $Y' = -\theta X'$. Then

$$[H', X'] = 2X' ,$$
$$[H', Y'] = -2Y' ,$$
$$[X', Y'] = H' .$$

Since β is a real root, X' and Y' are contained in $\mathfrak{g}$. The subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ generated by $\{H', X', Y'\}$ is isomorphic to $\mathrm{sl}(2, \mathbf{R})$. Our notation is consistent with [3(b), § 24]. Following Harish-Chandra's paper, we set

$$Z(A) = \{1, \gamma\} ,$$

where $\gamma = \exp(\pi(X' - Y'))$. Then $A_1 = Z(A)A_1^0$, if A_1^0 is the connected component of A_1.

Fix an element a_1 in A_1'. Let G_1 and $\mathfrak{g}_1$ be the centralizers of a_1 in G and $\mathfrak{g}$ respectively. Then

$$\mathfrak{g}_1 = \mathfrak{l} \oplus \mathfrak{a}_1 .$$

LEMMA 1.3. *G_1 is connected.*

Proof. Suppose that G_1^0 is the connected component of the identity in G_1 and let G_1^+ be any other connected component. Take γ^+ in G_1^+. Notice that

$$(\gamma^+)G_1^0(\gamma^+)^{-1} = G_1^0 .$$

Replacing γ^+ by one of its left G_1^0-translates, all of which lie in G_1^+, we obtain an element in G_1^+ which normalizes $\mathfrak{a}$. Now there is a representative in G_1^0 of the nontrivial element of the restricted Weyl group of $(\mathfrak{g}, \mathfrak{a})$. Therefore we can find an element in G_1^+ which leaves $\mathfrak{a}$ pointwise fixed. In other words, every component of G_1 contains some element in M. On the other hand, since a_1 is semi-regular, the only elements of M which leave a_1 fixed belong to A_1. Therefore

$$G_1 = A_1 \cdot G_1^0 = Z(A) \cdot A_1^0 G_1^0 = Z(A) G_1^0 .$$

Since $Z(A) \subseteqq G_1^0$, G_1 is connected. □

We remark that $\mathfrak{g}_1 = \mathfrak{a}_1 \oplus \mathfrak{l}$ is independent of the semi-regular element a_1 which we chose. G_1 is the analytic subgroup of G corresponding to the Lie algebra $\mathfrak{g}_1$, so G_1 is also independent of a_1.

In the future we shall be dealing with Haar measures on certain subgroups of G. We shall normalize them now for once and for all.

For any X in $\mathfrak{g}$, write

$$|X|^2 = -B(X, \theta X) .$$

This defines a Euclidean norm on $\mathfrak{g}$ and on any subspace of $\mathfrak{g}$. In particular it defines a Euclidean measure dN on $\mathfrak{n}$. We define a normalized Haar measure dn on N by

$$\int_N \phi(n)dn = \int_{\mathfrak{n}} \phi(\exp N)dN , \qquad \phi \in C_c^\infty(N) .$$

More generally for any subspace $\mathfrak{q}$ of $\mathfrak{n}$ we shall define a measure on $\exp \mathfrak{q}$ from the Euclidean measure on $\mathfrak{q}$. Similarly, we define a measure on $\exp \bar{\mathfrak{q}}$ for any subspace $\bar{\mathfrak{q}}$ of $\bar{\mathfrak{n}}$.

We normalize the Haar measure da on A by

$$\int_A \phi(a)da = \int_{\mathbf{R}} \phi(h_t)dt , \qquad \phi \in C_c^\infty(A) .$$

For any compact subgroup H of K, choose the Haar measure on H which assigns to H the volume one.

If a_1 is an element of A_1, let $\mathfrak{g}(a_1)$ and $G(a_1)$ be the centralizers of a_1 in $\mathfrak{g}$ and G respectively. Then

$$G(a_1) = K(a_1) \cdot A \cdot N(a_1)$$

is an Iwasawa decomposition for $G(a_1)$ where $K(a_1) = K \cap G(a_1)$ and $N(a_1) = N \cap G(a_1)$. Let dx_1 be the Haar measure on $G(a_1)$ defined by

$$\int_{G(a_1)} \phi(x_1)dx_1 = \int_{K(a_1) \times N(a_1) \times A} \phi(k_1 n_1 a)dk_1 dn_1 da , \qquad \phi \in C_c^\infty(G(a_1)) .$$

In particular, this defines a Haar measure dx on G.

Finally suppose that H is a unimodular Lie subgroup of G with a distinguished Haar measure dh. Let dx^* be the unique G-invariant measure on G/H defined by

$$\int_G \phi(x)dx = \int_{G/H\times H} \phi(x^*h)dhdx^* , \qquad \phi \in C_c^\infty(G) .$$

We shall normalize all measures according to these conventions without further comment.

2. A function on N

There is an important function on N whose properties we must discuss before we define our distributions.

For any $x \in G$, let x_K, x_N, and x_A be the elements in K, N, and A respectively such that $x = x_K x_N x_A$. Put $H(x) = \log(x_A)$. $H(x)$ is in $\mathfrak{a}$. Define

$$\lambda(\bar{n}) = \mu'(H(\bar{n})) , \qquad n \in N .$$

It is known that $\lambda(\bar{n})$ is a nonnegative real number. It has been computed explicitly by Helgason ([4(b), Th. 1.14]). We describe his formula.

The map

$$(X, Y) \longrightarrow \exp X \cdot \exp Y = \exp(X+Y) , \qquad X \in \mathfrak{g}_1 , \quad Y \in \mathfrak{g}_2 ,$$

is a diffeomorphism of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ onto N. For any Z in $\mathfrak{g}$ recall that $|Z|^2 = -B(Z, \theta Z)$. Then if

$$n = \exp X \cdot \exp Y , \qquad X \in \mathfrak{g}_1 , \quad Y \in \mathfrak{g}_2 ,$$

the formula is

$$\lambda(\bar{n}) = \frac{1}{2} \log \left[\left(1 + \frac{2}{2r^2} |X|^2\right)^2 + \frac{2}{r^2} |Y|^2 \right].$$

Let w be a representative in K of the nontrivial element of the restricted Weyl group of $(\mathfrak{g}, \mathfrak{a})$. Helgason has proved in [4(b), Lemma 1.15], that w can be chosen such that w^2 lies in the center of G. By replacing w with an element of the form $m_0^{-1} w m_0$, for some m_0 in M, we can assume that $\mathrm{Ad}(w)$ maps $\mathfrak{a}_1$ onto itself. Notice that for any X in $\mathfrak{g}$,

$$|w^{-1} X w| = |X| .$$

Since

$$w^{-1}\mathfrak{g}_i w = \mathfrak{g}_{-i} = \theta(\mathfrak{g}_i) , \qquad i = 1, 2 ,$$

we see from the above formula for $\lambda(\bar{n})$ that

$$\lambda(\bar{n}) = \lambda(w^{-1} n w) .$$

We shall refer to the element w later on.

For $X \in \mathfrak{g}_1$, $Y \in \mathfrak{g}_2$

$$\exp X \cdot \exp Y = \exp(X + Y) = \exp Y \cdot \exp X .$$

On the other hand for X_1 and X_2 both in $\mathfrak{g}_1$,

$$\exp X_1 \cdot \exp X_2 = \exp(X_1 + X_2) \cdot \exp\left(\frac{1}{2}[X_1, X_2]\right) .$$

There is a constant a_0 such that

$$|[X_1, X_2]| \leqq a_0 |X_1| \cdot |X_2| , \qquad X_1, X_2 \in \mathfrak{g}_1 .$$

LEMMA 2.1. *There is a constant C such that for $n \in N$, $a_1 \in A_1$ and t sufficiently large,*

$$|\lambda(\bar{n} \cdot a_1 h_t \cdot \bar{n}^{-1} \cdot a_1^{-1} h_t^{-1}) - \lambda(\bar{n})| \leqq Ce^{-t} .$$

Proof. Let $n = \exp X \cdot \exp Y$, $X \in \mathfrak{g}_1$, $Y \in \mathfrak{g}_2$. Then

$$\begin{aligned} &\bar{n} \cdot a_1 h_t \cdot \bar{n}^{-1} \cdot a_1^{-1} h_t^{-1} \\ &\quad = \theta(n \cdot a_1 h_t^{-1} \cdot n^{-1} \cdot a_1^{-1} h_t) \\ &\quad = \theta(\exp X \cdot \exp Y \cdot \exp(-X^{a_1} e^{-t}) \cdot \exp(-Y^{a_1} e^{-2t})) \\ &\quad = \theta\left(\exp(X - e^{-t} X^{a_1}) \cdot \exp\left(Y - e^{-2t} Y^{a_1} - \frac{1}{2} e^{-t}[X, X^{a_1}]\right)\right) , \end{aligned}$$

where

$$Z^{a_1} = \mathrm{Ad}(a_1) Z , \qquad Z \in \mathfrak{g} .$$

Therefore

$$\lambda(\bar{n} \cdot a_1 h_t \cdot \bar{n}^{-1} \cdot a_1^{-1} h_t^{-1}) - \lambda(\bar{n})$$

is one-half the logarithm of

$$\frac{\left(1 + \frac{1}{2r^2} |X - e^{-t} X^{a_1}|^2\right)^2 + \frac{2}{r^2} \left| Y - e^{2t} Y^{a_1} - \frac{1}{2} e^{-t}[X, X^{a_1}]\right|^2}{\left(1 + \frac{1}{2r^2} |X|^2\right)^2 + \frac{2}{r^2} |Y|^2} . \tag{2.1}$$

Replace the numerator of this expression by the sum of

$$\left(1 + \frac{1}{2r^2} |X|^2\right)^2 + \frac{2}{r^2} |Y|^2$$

and a remainder $R(t, a_1, X, Y)$. Now,

$$|X^{a_1}| = |X| , \quad |Y^{a_1}| = |Y| ,$$

and

$$|[X, X^{a_1}]| \leqq a_0 |X|^2 .$$

Therefore we can find a constant C such that

$$|R(t, a_1, X, Y)| \leqq Ce^{-t}\cdot\left[\left(1+\frac{1}{2r^2}|X|^2\right)^2+\frac{2}{r^2}|Y|^2\right].$$

It follows that the expression (2.1) differs from 1 by a function that is bounded by Ce^{-t}. The lemma then follows. □

There is another consequence of the formula for $\lambda(\bar{n})$ which we shall eventually need. For any $n \in N$ we can choose elements $X(n)$ and $Y(n)$ in $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively such that

$$\bar{n} = \bar{n}_K \cdot \bar{n}_A \cdot \exp X(n) \cdot \exp Y(n) .$$

LEMMA 2.2. *Both* $|X(n)|$ *and* $|Y(n)|$ *are bounded functions of* n.

Proof. Suppose that $n = \exp X \cdot \exp Y$, $X \in \mathfrak{g}_1$, $Y \in \mathfrak{g}_2$. We have

$$\bar{n} = \bar{n}_K \cdot \bar{n}_N \cdot \bar{n}_A .$$

Suppose that

$$\bar{n}_N = \nu = \exp R(n) \cdot \exp S(n) , \qquad R(n) \in \mathfrak{g}_1 , \quad S(n) \in \mathfrak{g}_2 .$$

Then

$$n = \bar{n}_K \cdot \bar{\nu}_K \cdot \bar{\nu}_N \cdot \bar{\nu}_A \bar{n}_A^{-1} .$$

Therefore $\bar{\nu}_A = \bar{n}_A$, so that

$$\left(1+\frac{1}{2r^2}|R(n)|^2\right)^2+\frac{2}{r^2}|S(n)|^2=\left(1+\frac{1}{2r^2}|X|^2\right)^2+\frac{2}{r^2}|Y|^2 . \tag{2.2}$$

Now we have

$$\begin{aligned}\exp X(n)\cdot\exp Y(n) &= \bar{n}_A^{-1}\cdot\exp R(n)\cdot\exp S(n)\cdot\bar{n}_A \\ &= \exp\left(e^{-\lambda(\bar{n})}R(n)\right)\cdot\exp\left(e^{-2\lambda(\bar{n})}S(n)\right) .\end{aligned}$$

Therefore

$$|X(n)| = \frac{|R(n)|}{\left[\left(1+\frac{1}{2r^2}|X|^2\right)^2+\frac{2}{r^2}|Y|^2\right]^{1/2}} ,$$

and

$$|Y(n)| = \frac{|S(n)|}{\left[\left(1+\frac{1}{2r^2}|X|^2\right)^2+\frac{2}{r^2}|Y|^2\right]^{1/2}} .$$

A glance at formula (2.2) shows that these functions are bounded. □

3. The distributions $T(t, a_1)$

Assume for the remainder of this paper that there is a root β in P_+ which vanishes on $\mathfrak{a}_1$.

Following standard notation we write

$$\Delta(a_1h_t) = \prod_{\alpha\in P}\left(\xi_{\alpha/2}(a_1h_t) - \xi_{-\alpha/2}(a_1h_t)\right), \qquad a_1 \in A_1,\ t \in \mathbf{R},$$
$$\Delta_M(a_1) = \prod_{\alpha\in P_M}\left(\xi_{\alpha/2}(a_1) - \xi_{-\alpha/2}(a_1)\right), \qquad a_1 \in A_1.$$

For real $t \neq 0$, and $a_1 \in A_1$ define the following distributions:

$$\langle T(t, a_1), f\rangle = \operatorname{sgn} t\cdot\Delta(a_1h_t)\cdot\int_{K\times N} f(kn\cdot a_1h_t\cdot n^{-1}k^{-1})\lambda(\bar{n})dndk,$$
$$\langle F(t, a_1), f\rangle = \operatorname{sgn} t\cdot\Delta(a_1h_t)\cdot\int_{K\times N} f(kn\cdot a_1h_t\cdot n^{-1}k^{-1})dndk, \qquad f \in C_c^\infty(G).$$

These integrals are easily seen to converge absolutely.

The distribution $F(t, a_1)$ is well known. In Harish-Chandra's notation

$$\langle F(t, a_1), f\rangle = F_f(a_1h_t).$$

$F(t, a_1)$ is tempered, and invariant. We mention it only for the sake of comparison. Its Fourier transform can be readily computed in terms of the characters of the principal series. On the other hand, $T(t, a_1)$ is not invariant. Accordingly, its Fourier transform is considerably more complicated. Of course we must first of all show that $T(t, a_1)$ is tempered. This will permit us to regard $T(t, a_1)$ as a continuous linear functional on $\mathcal{C}(G)$, the Schwartz space of G.

Let $\mathcal{C}_0(G)$ be the closed subspace of $\mathcal{C}(G)$ generated by the K-finite matrix coefficients of all the square integrable representations of G. It is known ([3(c)]) that $\mathcal{C}_0(G)$ is nontrivial if and only if there is a compact Cartan subgroup of G. Unlike $F(t, a_1)$, the distribution $T(t, a_1)$ does not vanish on $\mathcal{C}_0(G)$. It will be our goal in this paper to calculate the Fourier transform of the restriction of $T(t, a_1)$ to $\mathcal{C}_0(G)$.

There is a slightly different formula for our distributions which we shall need. Define a function Λ on G as follows: if $x = kna$, $k \in K$, $n \in N$, $a \in A$, let

$$\Lambda(x) = \lambda(\bar{n}).$$

Suppose $w^{-1}nw = \bar{\nu}$. Then $\lambda(\bar{\nu}) = \lambda(\bar{n})$. We also have

$$\begin{aligned} xw &= kw\cdot\bar{\nu}\cdot a^{-1} \\ &= kw\cdot\bar{\nu}_K\bar{\nu}_N\bar{\nu}_A\cdot a^{-1}. \end{aligned}$$

Since

$$\Lambda(x) = \lambda(\bar{n}) = \mu'(H(\bar{\nu}_A)),$$

we have

$$\Lambda(x) = \mu'(H(xw)) + \mu'(H(x)).$$

In particular $\Lambda(xw) = \Lambda(x)$. Λ is actually defined on G/A, so for the normalized measure dx^* on G/A we have

$$\langle T(t, a_1), f\rangle = \operatorname{sgn} t\cdot\Delta(a_1h_t)\cdot\int_{G/A} f(x^*\cdot a_1h_t\cdot x^{*-1})\Lambda(x^*)dx^* .$$

Let $W_{G,\mathfrak{h}}$ be the normalizer of the Cartan subalgebra $\mathfrak{h}$ in G modulo its centralizer in G. $W_{G,\mathfrak{h}}$ operates on the roots of $(\mathfrak{m}_{\mathbf{C}}, \mathfrak{a}_{1\mathbf{C}})$. For any y in $W_{G,\mathfrak{h}}$ let

$$P_M^y = \{\alpha^y\colon \alpha\in P_M\} .$$

Define $\varepsilon_M(y)$ to equal 1 or -1 depending on whether $[P_M^y\cap(-P_M)]$ is even or not. ε_M is a homomorphism of $W_{G,\mathfrak{h}}$ into the multiplicative group $\{1, -1\}$. Now for any y in $W_{G,\mathfrak{h}}$ the map

$$x \longrightarrow xy$$

is a diffeomorphism of G/A that preserves the measure dx^*. Since $\Lambda(xy) = \Lambda(x)$ we obtain the formula

$$T(t^y, a_1^y) = \varepsilon_M(y)T(t, a_1) , \qquad y\in W_{G,\mathfrak{h}} .$$

The same argument applies to the diffeomorphism $x\longrightarrow\theta(x)$ of G/A. It follows that

$$T(-t, a_1) = T(t, a_1) .$$

These formulae also imply that

$$\langle T(t, a_1), f\rangle = \operatorname{sgn} t\cdot\Delta(a_1h_1)\cdot\int_{K\times\bar N} f(k\bar n\cdot a_1h_t\cdot\bar n^{-1}k^{-1})\lambda(\bar n)d\bar n dk .$$

There is a well known Jacobian formula on $\bar N$ which we should mention before proving that $T(t, a_1)$ is tempered. Suppose that $a_1\in A_1$, t is a nonzero real number, and that $\phi\in C_c^\infty(\bar N)$. If

$$\rho = \frac{1}{2}\sum\nolimits_{\alpha\in P_+}\alpha$$

then

$$\begin{aligned}(3.1)\qquad &\operatorname{sgn} t\cdot\Delta(a_1h_t)\cdot\int_{\bar N}\phi(\bar n\cdot a_1h_t\cdot\bar n^{-1}\cdot a_1^{-1}h_t^{-1})d\bar n\\ &\qquad = e^{t\rho(H')}\cdot\Delta_M(a_1)\cdot\int_{\bar N}\phi(\bar n)d\bar n .\end{aligned}$$

This is a consequence of [3(b), Lemmas 11 and 12].

LEMMA 3.1. *For fixed a_1 and $t\neq 0$ the distribution $T(t, a_1)$ is tempered.*

Proof. Since $T(t, a_1)$ is symmetric in t, we may assume $t>0$. For $f\in C_c^\infty(G)$

$$\begin{aligned}&|\langle T(t, a_1), f\rangle|\\ &\quad\leqq|\Delta(a_1h_t)|\cdot\int_{K\times\bar N}|f(k\bar n\cdot a_1h_t\cdot\bar n^{-1}k^{-1})|\cdot\lambda(\bar n)d\bar n dk .\end{aligned}$$

If we set

$$\bar{\nu} = \bar{n}\cdot a_1 h_t \cdot \bar{n}^{-1}\cdot a_1^{-1}h_t^{-1}$$

we can regard $\bar{n}$ as a function of $\bar{\nu}$, t, and a_1. Then from (3.1), $|\langle T(t, a_1), f\rangle|$ is bounded by

$$e^{t\rho(H')}\cdot|\Delta_M(a_1)|\cdot\int_{K\times\bar{N}}|f(k\bar{\nu}\cdot a_1 h_t\cdot k^{-1})|\cdot\lambda(\bar{n})d\bar{\nu}dk\ .$$

We must show that $\lambda(\bar{n})$ is bounded by a suitable function of ν, a_1, and t. Now

$$\begin{aligned}\theta(\nu) = \bar{\nu} &= \bar{n}\cdot a_1 h_t\cdot \bar{n}^{-1}\cdot a_1^{-1}h_t^{-1}\\ &= \theta(n\cdot a_1 h_t^{-1}\cdot n^{-1}\cdot a_1^{-1}h_t)\ .\end{aligned}$$

Suppose that

$$n = \exp X\cdot \exp Y\ , \qquad X\in \mathfrak{g}_1\ ,\quad Y\in\mathfrak{g}_2\ ,$$

and

$$\nu = \exp R\cdot \exp S\ , \qquad R\in \mathfrak{g}_1\ ,\quad S\in\mathfrak{g}_2\ .$$

Then

$$R = X - e^{-t}X^{a_1}\ ,$$

and

$$S = Y - e^{-2t}Y^{a_1} - \frac{1}{2}e^{-t}[X, X^{a_1}]\ .$$

Therefore

$$|X| \leqq |R|\cdot(1 - e^{-t})^{-1}\ ,$$

and

$$\begin{aligned}|Y| &\leqq (|S| + e^{-t}\cdot a_0|X|^2)\cdot(1 - e^{-2t})^{-1}\\ &\leqq |S|\cdot(1 - e^{-t})^{-1} + a_0e^{-t}\cdot(1 - e^{-t})^{-3}\cdot|R|^3\ .\end{aligned}$$

It follows that there exists a constant C_0, independent of t, a_1, R, and S such that

$$\begin{aligned}&\Big(1 + \frac{1}{2r^2}|X|^2\Big)^2 + \frac{2}{r^2}|Y|^2\\ &\qquad\leqq C_0\cdot(1 - e^{-t})^{-6}\cdot\Big[\Big(1 + \frac{1}{2r^2}|R|^2\Big)^2 + \frac{2}{r^2}|S|^2\Big].\end{aligned}$$

Therefore

$$\lambda(\bar{n}) \leqq \log[C_0(1 - e^{-t})^{-6}] + \lambda(\bar{\nu})\ .$$

We have shown that $|\langle T(t, a_1), f\rangle|$ is bounded by

$$e^{t\rho(H')}\cdot|\Delta_M(a_1)|\cdot\int_{K\times\bar{N}}|f(k\bar{\nu}\cdot a_1h_t\cdot k^{-1})| \cdot(\log[C_0(1-e^{-t})^{-6}]+\lambda(\bar{\nu}))\cdot d\bar{\nu}dk\ .$$

For any r, let

$$\|f\|_r=\sup_{x\in G}|f(x)|\,\Xi(x)^{-1}(1+\sigma(x))^r\ .$$

Then for any r,

$$|\langle T(t,a_1),f\rangle|\leqq e^{t\rho(H')}\cdot|\Delta_M(a_1)|\cdot\|f\|_r \cdot\int_{\bar{N}}\Xi(\bar{\nu}\cdot a_1h_t)\cdot(1+\sigma(\bar{\nu}\cdot a_1h_t))^{-r}\cdot(\log[C_0(1-e^{-t})^{-6}]+\lambda(\bar{\nu}))d\bar{\nu}\ .$$

From [1(a), Lemma 11], it follows that $\mu'(H(x))/(1+\sigma(x))$ is a bounded function on G. On the other hand, in [3(c), Lemma 21], Harish-Chandra proves that there is a number r' such that

$$e^{t\rho(H')}\cdot\int_{\bar{N}}\Xi(\bar{\nu}\cdot a_1h_t)\cdot(1+\sigma(\bar{\nu}\cdot a_1h_t))^{-r'}d\bar{\nu}$$

converges and is bounded independently of t and a_1. It follows that $T(t,a_1)$ is tempered. □

From the proof of this lemma we also have

COROLLARY 3.2. *For $f\in\mathcal{C}(G)$ and $t\neq 0$ the integral*

$$\operatorname{sgn}t\cdot\Delta(a_1h_t)\cdot\int_{K\times\bar{N}}f(k\bar{n}\cdot a_1h_t\cdot\bar{n}^{-1}k^{-1})\cdot\lambda(\bar{n})d\bar{n}dk$$

is absolutely convergent and equals $\langle T(t,a_1),f\rangle$. □

If we express the estimates of the lemma more precisely we obtain

COROLLARY 3.3. *For every positive integer d there is a continuous seminorm $\|\ \|'_d$ on $\mathcal{C}(G)$ such that for any $t>0$, $a_1\in A_1$ and $f\in\mathcal{C}(G)$,*

$$|\langle T(t,a_1),f\rangle|\leqq\|f\|'_d\cdot\log(1-e^{-t})\cdot(1+t)^{-d}\ .$$

Proof. Suppose that $\bar{\nu}\in\bar{N}$, $a_1\in A$, and $t>0$. Then

$$(1+\mu'(H(\bar{\nu}a_1h_t)))^{-1}=(1+\lambda(\bar{\nu})+t)^{-1}$$

which is bounded by $(1+t)^{-1}$. Therefore we may take $\|\ \|'_d$ to be a multiple of $\|\ \|_{d+r'+1}$. □

4. The behavior for t near 0

There is another distribution that is also of interest. It may be obtained by examining the behavior of $T(t,a_1)$ as $t\to 0+$.

For $a_1\in A_1$, let $\mathfrak{n}(a_1)$, $\mathfrak{g}_1(a_1)$, $\mathfrak{g}_2(a_1)$, and $N(a_1)$ be the centralizers of a_1 in $\mathfrak{n}$, $\mathfrak{g}_1$, $\mathfrak{g}_2$ and N respectively. Suppose that $\mathfrak{n}(a_1)\neq 0$. Let $P_+(a_1)$ be the set of roots α in P_+ such that $\xi_\alpha(a_1) = 1$. Set

$$\Delta^*(a_1h_t) = \prod_{\alpha\in P-P_+(a_1)} \left(\xi_{\alpha/2}(a_1h_t) - \xi_{-\alpha/2}(a_1h_t)\right) .$$

Let $d\bar{n}$, $d\bar{n}^*$, and $d\bar{n}_1$ be the normalized invariant measures on $\bar{N}$, $\bar{N}/\overline{N(a_1)}$, and $\overline{N(a_1)}$ respectively. Then

$$d\bar{n} = d\bar{n}_1 d\bar{n}^* .$$

For $f\in C_c^\infty(G)$, define

$$\langle T(a_1), f\rangle = \Delta^*(a_1)\cdot\int_{K\times\bar{N}/\overline{N(a_1)}\times\overline{N(a_1)}} f(k\bar{n}^*\cdot a_1\bar{n}_1\cdot \bar{n}^{*-1}k^{-1})\cdot\delta(\bar{n}_1)d\bar{n}_1 d\bar{n}^* dk ,$$

where δ is a function defined on $\overline{N(a_1)}$ as follows:

$$\text{If } \mathfrak{g}_1(a_1) \neq 0 , \quad \delta(\exp(X_1 + Y_1)) = \log|X_1| , \qquad X_1\in\mathfrak{g}_{-1}(a_1) , \quad Y_1\in\mathfrak{g}_{-2}(a_1) ,$$

and

$$\text{if } \mathfrak{g}_1(a_1) = 0 , \quad \delta(\exp Y_1) = \log|Y_1| , \qquad Y_1\in\mathfrak{g}_{-2}(a_1) .$$

Clearly the expression

$$\int_{\overline{N(a_1)}} f(k\bar{n}^*\cdot a_1\bar{n}_1\cdot\bar{n}^{*-1}k^{-1})\cdot\delta(\bar{n}_1)d\bar{n}_1$$

depends only on the $\overline{N(a_1)}$-coset of $\bar{n}^*$.

Now $\langle T(t, a_1), f\rangle$ equals

$$\operatorname{sgn} t\cdot\Delta(a_1h_t)\cdot\int_{K\times\bar{N}/\overline{N(a_1)}\times\overline{N(a_1)}} f(k\bar{n}^* a_1\cdot\bar{n}_1 h_t\bar{n}_1^{-1}\cdot\bar{n}^{*-1}k^{-1})\cdot\lambda(\bar{n}^*\bar{n}_1)d\bar{n}_1 d\bar{n}^* dk .$$

We are going to make a change of variables in this integral. Let

$$\begin{aligned}\bar{\nu} &= \bar{n}_1 h_t \bar{n}_1^{-1} h_t^{-1}\\ &= \theta(n_1 h_t^{-1} n_1^{-1} h_t) = \theta(\nu) .\end{aligned}$$

Suppose that

$$n^* = \exp(X + Y) , \quad n_1 = \exp(X_1 + Y_1) , \quad \nu = \exp(R + S)$$

for $X\in\mathfrak{g}_1$, $Y\in\mathfrak{g}_2$, X_1, $R\in\mathfrak{g}_1(a_1)$, and Y_1, $S\in\mathfrak{g}_2(a_1)$. Then

$$\begin{aligned}\nu &= \exp(X_1 + Y_1)\cdot\exp(-e^t X_1 - e^{-2t}Y_1)\\ &= \exp\left(X_1(1 - e^{-t}) + Y_1(1 - e^{-2t})\right)\end{aligned}$$

so that

$$X_1 = R(1 - e^{-t})^{-1}$$

and

$$Y_1 = S(1 - e^{-2t})^{-1} .$$

It follows that

$$
\begin{aligned}
n^* n_1 &= \exp X \cdot \exp Y \cdot \exp [R(1 - e^{-t})^{-1}] \cdot \exp [S(1 - e^{-2t})^{-1}] \\
&= \exp [X + R(1 - e^{-t})^{-1}] \\
&\quad \cdot \exp \Big[Y + S(1 - e^{-2t})^{-1} + \frac{1}{2}[X, R(1 - e^{-t})^{-1}] \Big] .
\end{aligned}
$$

Therefore $\lambda(\bar{n}^* \bar{n}_1)$ equals one half the logarithm of

$$
\Big(1 + \frac{1}{2r^2} \cdot | X + R(1 - e^{-t})^{-1} |^2 \Big)^2 + \frac{2}{r^2} \cdot \Big| Y + S(1 - e^{-2t})^{-1} + \frac{1}{2}[X, R(1 - e^{-t})^{-1}] \Big|^2 .
$$

Let us write $\lambda(\bar{n}^* : \bar{\nu})$ for $\lambda(\bar{n}^* \bar{n}_1)$, since $\bar{n}_1$ is a function of $\bar{\nu}$ for $t > 0$.

Let $\rho^* = (\sum_{\alpha \in P_+(a_1)} \alpha)/2$. Applying the formula (3.1) to the group $N(a_1)$ we see that $\langle T(t, a_1), f \rangle$ equals

$$
e^{t\rho^*(H')} \cdot \Delta^*(a_1 h_t) \cdot \int_{K \times \bar{N}/\bar{N}(a_1) \times \bar{N}(a_1)} f(k \bar{n}^* a_1 \cdot \bar{\nu} h_t \cdot \bar{n}^{*-1} k^{-1}) \cdot \lambda(\bar{n}^* : \bar{\nu}) d\bar{\nu} d\bar{n}^* dk .
$$

Suppose that $\mathfrak{g}_1(a_1) \neq 0$, and that $R \neq 0$. Then

$$
\begin{aligned}
&\lambda(\bar{n}^* : \bar{\nu}) + \frac{1}{2} \log (1 - e^{-t})^4 \\
&= \frac{1}{2} \log \Big\{ \Big((1 - e^{-t})^2 + \frac{1}{2r^2} | X(1 - e^{-t}) + R |^2 \Big)^2 \\
&\quad + \frac{2}{r^2} (1 - e^{-t})^4 \cdot \Big| Y + S(1 - e^{-2t})^{-1} + \frac{1}{2}[X, R(1 - e^{-t})^{-1}] \Big|^2 \Big\} .
\end{aligned}
$$

The limit of this expression as $t \to 0+$ equals

$$
\frac{1}{2} \log \Big[\frac{1}{2r^2} | R |^2 \Big]^2 .
$$

On the other hand, suppose that $\mathfrak{g}_1(a_1) = 0$ and that $S \neq 0$. Then

$$
\begin{aligned}
\lambda(\bar{n}^* : \bar{\nu}) + \frac{1}{2} \log (1 - e^{-2t})^2 &= \frac{1}{2} \log \Big\{ (1 - e^{-2t})^2 \Big(1 + \frac{1}{2r^2} | X |^2 \Big)^2 \\
&\quad + \frac{2}{r^2} (1 - e^{-2t})^2 \cdot | Y + S(1 - e^{-2t})^{-1} |^2 \Big\} .
\end{aligned}
$$

The limit of this expression as $t \to 0+$ equals

$$
\frac{1}{2} \log \Big[\frac{2}{r^2} | S |^2 \Big] .
$$

For $t \neq 0$ define

$$
S(t, a_1) = T(t, a_1) + \frac{1}{2} \log (1 - e^{-t})^4 \cdot F(t, a_1) , \qquad \text{if } \mathfrak{g}_1(a_1) \neq 0 ,
$$

or

$$S(t, a_1) = T(t, a_1) + \frac{1}{2}\log(1 - e^{-2t})^2 \cdot F(t, a_1)\ , \qquad \text{if } \mathfrak{g}_1(a_1) = 0\ .$$

Then we use the Lebesgue dominated convergence theorem and the above formulae to obtain a formula for $\langle T(a_1), f\rangle$ if $f \in C_c^\infty(G)$. The result is

$$\langle T(a_1), f\rangle = \log(\sqrt{2}\cdot r)\langle F(0, a_1), f\rangle + \frac{1}{2}\lim_{t\to 0+}\langle S(t, a_1), f\rangle\ ,$$
$$\text{if } \mathfrak{g}_1(a_1) \neq 0\ ,$$
$$\langle T(a_1), f\rangle = \log(r/\sqrt{2})\cdot\langle F(0, a_1), f\rangle + \lim_{t\to 0+}\langle S(t, a_1), f\rangle$$
$$\text{if } \mathfrak{g}_1(a_1) = 0\ .$$

We shall have occasion to use the distributions $S(t, a_1)$ to calculate the Fourier transform of $T(t, a_1)$ on $\mathcal{C}_0(G)$. If $\mathcal{C}_0(G) \neq 0$ there is a compact Cartan subgroup B of G, and so by Lemma 1.1 there is a root β whose restriction to $\mathfrak{a}_1$ is trivial. $\beta = 2\mu'$ by Lemma 1.2.

Suppose that a_1 is in A_1', the set of semi-regular elements of non-compact type in A_1, defined in § 1. Then β is the only root in P_+ such that $\xi_\beta(a_1) = 1$. Take a function f in $C_c^\infty(G)$ and put

$$\langle S'(t, a_1), f\rangle = \frac{d}{dt}\langle S(t, a_1), f\rangle\ .$$

We shall find a formula for

$$\lim_{t\to 0+}\{\langle S'(t, a_1), f\rangle - \langle S'(-t, a_1), f\rangle\}\ .$$

In § 1 we introduced the root vector X' in $\mathfrak{g}_2$ corresponding to β. For real x, let

$$n(x) = \exp(xX')\ .$$

For the other roots $\{\alpha\}$ in P_+ let $\{X_\alpha\}$ be a fixed set of root vectors. Put

$$\mathfrak{n}^* = \mathfrak{n} \cap \sum_{\substack{\alpha \in P_- \\ \alpha \neq \beta}} \mathbf{C}X_\alpha\ .$$

Then $\mathfrak{n}^* = \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*$ if $\mathfrak{g}_i^* = \mathfrak{g}_i \cap \mathfrak{n}^*$, $i = 1, 2$. Let $N^* = \exp \mathfrak{n}^*$. We have the diffeomorphism

$$(n^*, x) \longrightarrow n^* \cdot n(x)$$

of $N^* \times \mathbf{R}$ onto N. According to the conventions in § 1, dn^* is the measure on N^* defined by the Euclidean measure on $\mathfrak{n}^*$. For $f \in C_c^\infty(N)$,

$$\int_N f(n)dn = \frac{r}{\sqrt{2}}\cdot\int_{N^*\times\mathbf{R}} f(n^*\cdot n(x))dxdn^*\ ,$$

since $|X'| = r/\sqrt{2}$.

For $X \in \mathfrak{g}_1^*$, $Y \in \mathfrak{g}_2^*$, write

$$n^* = n^*(X, Y) = \exp X \cdot \exp Y .$$

We have

$$\begin{aligned}
&\lambda(\bar{n}^* \overline{n(x(1 - e^{-2t}))}) + \frac{1}{2} \log (1 - e^{-2t})^2 \\
&= \frac{1}{2} \log \left\{\left(1 + \frac{1}{2r^2} \mid X \mid^2\right)^2 + \frac{2}{r^2} \mid Y \mid^2 + x^2(1 - e^{-2t})^{-2}\right\} + \frac{1}{2} \log (1 - e^{-2t})^2 \\
&= \frac{1}{2} \log \{e^{2\lambda(\bar{n}^*)} + x^2(1 - e^{-2t})^{-2}\} + \frac{1}{2} \log (1 - e^{-2t})^2 \\
&= \frac{1}{2} \log \{(1 - e^{-2t})^2 \cdot e^{2\lambda(\bar{n}^*)} + x^2\} .
\end{aligned}$$

Fix $f \in C_c^\infty(G)$. Set $\phi(t\colon n^*\colon x)$ equal to

$$e^{t\rho^*(H')} \cdot \Delta^*(a_1 h_t) \cdot \int_K f(k\bar{n}^* \overline{n(x)} \cdot a_1 h_t \cdot \bar{n}^{*-1} k^{-1}) dk .$$

Then $\langle S(t, a_1), f \rangle$ equals

$$\frac{r}{2\sqrt{2}} \cdot \int_{N^* \times \mathbf{R}} \phi(t\colon n^*\colon x) \cdot \log \{(1 - e^{-2t})^2 e^{2\lambda(\bar{n}^*)} + x^2\} dn^* dx .$$

The function

$$\frac{r}{2\sqrt{2}} \int_{N^* \times \mathbf{R}} \phi(t\colon n^*\colon x) \cdot \log \{x^2\} dn^* dx$$

is differentiable at $t = 0$. Therefore

$$\frac{2\sqrt{2}}{r} \lim_{t \to 0+} \{\langle S'(t, a_1), f \rangle - \langle S'(-t, a_1), f \rangle\}$$

is the limit as $t \to 0+$ of the difference of

$$\int_{N^* \times \mathbf{R}} \phi(t\colon n^*\colon x) \frac{4(1 - e^{-2t}) \cdot e^{-2t} \cdot e^{2\lambda(\bar{n}^*)}}{(1 - e^{-2t})^2 e^{2\lambda(\bar{n}^*)} + x^2} dn^* dx \tag{4.1}$$

and

$$\int_{N^* \times \mathbf{R}} \phi(-t\colon n^*\colon x) \frac{4(1 - e^{+2t}) \cdot e^{+2t} \cdot e^{2\lambda(\bar{n}^*)}}{(1 - e^{+2t})^2 e^{2\lambda(\bar{n}^*)} + x^2} dn^* dx . \tag{4.2}$$

We shall show that the limit of each of these terms exists.

The term (4.1) equals

$$4\int_{N^* \times \mathbf{R}} \phi(t\colon n^*\colon (1 - e^{-2t})x) \frac{e^{-2t} \cdot e^{2\lambda(\bar{n}^*)}}{e^{2\lambda(\bar{n}^*)} + x^2} dn^* dx .$$

Now notice that

$$\begin{aligned}\int_{N^*\times\mathbf{R}}\phi(0\colon n^*\colon 0)\frac{e^{2\lambda(\bar{n}^*)}}{e^{2\lambda(\bar{n}^*)}+x^2}dn^*dx &= \int_{N^*}\phi(0\colon n^*\colon 0)e^{\lambda(\bar{n}^*)}dn^*\cdot\int_{-\infty}^{\infty}\frac{dx}{1+x^2}\\ &= \pi\cdot\int_{N^*}\phi(0\colon n^*\colon 0)e^{\lambda(\bar{n}^*)}dn^*\ .\end{aligned}$$

Since f is of compact support, this last integral is absolutely convergent. Therefore by the Lebesgue dominated convergent theorem, the limit as $t\to 0+$ of (4.1) exists and equals

$$4\pi\int_{N^*}\phi(0\colon n^*\colon 0)e^{\lambda(\bar{n}^*)}dn^*\ .$$

Similarly, the limit as $t\to 0+$ of (4.2) exists and equals

$$-4\pi\int_{N^*}\phi(0\colon n^*\colon 0)e^{\lambda(\bar{n}^*)}dn^*\ .$$

We have proved the following:

LEMMA 4.1. *For $a_1\in A_1'$ and $f\in C_c^\infty(G)$ the limit*

$$\lim_{t\to 0+}\{\langle S'(t, a_1), f\rangle - \langle S'(-t, a_1), f\rangle\}$$

exists and equals

$$2\sqrt{2}\,r\pi\cdot\Delta^*(a_1)\cdot\int_{K\times N^*}f(k\bar{n}^*\cdot a_1\cdot\bar{n}^{*-1}k^{-1})e^{\lambda(\bar{n}^*)}dn^*dk\ . \qquad \square$$

For $f\in C_c^\infty(G)$, write

$$\langle S^*(a_1), f\rangle = \lim_{t\to 0+}\{\langle S'(t, a_1), f\rangle - \langle S'(-t, a_1), f\rangle\}\ .$$

Then $S^*(a_1)$ is a distribution on G. We shall show later that it is both tempered and invariant.

5. A differential equation

The distributions $F(t, a_1)$ satisfy a well-known linear homogeneous differential equation. By studying Harish-Chandra's proof of this fact we shall show that the distributions $T(t, a_1)$ satisfy a linear non-homogeneous differential equation.

Let $\mathcal{G}$ and $\mathcal{H}$ be the universal enveloping algebras of $\mathfrak{g}_C$ and $\mathfrak{h}_C$ respectively. For any X in $\mathfrak{g}_C$ and any g in $\mathcal{G}$, write

$$R_X(g) = gX\ ,$$
$$L_X(g) = Xg\ .$$

For every root α in P fix root vectors X_α and $X_{-\alpha}$ such that

$$B(X_\alpha, X_{-\alpha}) = 1 .$$

Then $[X_\alpha, X_{-\alpha}] = H_\alpha$, where H_α is the vector in $\mathfrak{h}_{\mathrm{C}}$ such that

$$B(H_\alpha, H) = \alpha(H) , \qquad H \in \mathfrak{h}_{\mathrm{C}} .$$

Define

$$\mathfrak{s} = \sum_{\alpha \in P} (\mathrm{C}X_\alpha + \mathrm{C}X_{-\alpha}) .$$

$\mathfrak{s}$ is a subspace of $\mathfrak{g}_{\mathrm{C}}$. Let $\mathfrak{S}$ be the image of the symmetric algebra on $\mathfrak{s}$ under the canonical mapping. $\mathfrak{S}$ is a vector subspace of $\mathfrak{G}$.

Fix a regular element μ in A_1A, the Cartan subgroup of G corresponding to the Cartan subalgebra $\mathfrak{h}$. It is clear that there is a unique linear mapping

$$\Gamma_\mu : \mathfrak{G} \otimes \mathfrak{H} \longrightarrow \mathfrak{G}$$

such that

(i) $\Gamma_\mu(1 \otimes u) = u$, $u \in \mathfrak{H}$

(ii) $\Gamma_\mu(X_1 \cdots X_r \otimes u) = (L_{\mathrm{Ad}(\mu^{-1})X_1} - R_{X_1}) \cdots (L_{\mathrm{Ad}(\mu^{-1})X_r} - R_{X_r})u$,
$X_1, \cdots, X_r \in \mathfrak{s}$, $u \in \mathfrak{H}$.

Harish-Chandra has shown that the restriction of Γ_μ to $\mathfrak{S} \otimes \mathfrak{H}$ maps $\mathfrak{S} \otimes \mathfrak{H}$ bijectively onto $\mathfrak{G}$ ([3(a), Lemma 15]). Let $\mathfrak{S}'$ be the set of elements in $\mathfrak{S}$ of strictly positive degree. Then it is obvious that for any g in $\mathfrak{G}$ there is a unique element $\beta_\mu(g)$ in $\mathfrak{H}$ such that $g - \beta_\mu(g)$ is in $\Gamma_\mu(\mathfrak{S}' \otimes \mathfrak{H})$.

Let $\{H_1\}$ and $\{H_2, \cdots, H_m\}$ be orthonormal bases of $\mathfrak{a}_{\mathrm{C}}$ and $\mathfrak{a}_{1\mathrm{C}}$ respectively, with respect to the Killing form. If $\omega_{\mathfrak{g}}$ is the Casimir operator on G,

$$\omega_{\mathfrak{g}} = H_1^2 + \cdots + H_m^2 + \sum_{\alpha \in P} H_\alpha + 2\sum_{\alpha \in P} X_{-\alpha}X_\alpha .$$

Let ω be the sum of $\omega_{\mathfrak{g}}$ and $\|(1/2)\sum_{\alpha \in P} H_\alpha\|^2 I$, where I is the identity operator. ω is in $\mathfrak{Z}$, the center of $\mathfrak{G}$. We would like to find the element in $\mathfrak{S} \otimes \mathfrak{H}$ whose image under Γ_μ is ω.

Suppose $\alpha \in P$. Then

$$\begin{aligned}
\Gamma_\mu(X_{-\alpha}X_\alpha \otimes 1) &= (L_{\mathrm{Ad}(\mu^{-1})X_{-\alpha}} - R_{X_{-\alpha}})(L_{\mathrm{Ad}(\mu^{-1})X_\alpha} - R_{X_\alpha})I \\
&= (L_{\xi_\alpha(\mu)X_{-\alpha}} - R_{X_{-\alpha}})\big(\xi_\alpha(\mu)^{-1}X_\alpha - X_\alpha\big) \\
&= -\big(\xi_\alpha(\mu)^{-1}X_\alpha - X_\alpha\big)X_{-\alpha} + \xi_\alpha(\mu)X_{-\alpha}\big(\xi_\alpha(\mu)^{-1}X_\alpha - X_\alpha\big) \\
&= X_{-\alpha}X_\alpha\big(1 - \xi_\alpha(\mu) + 1 - \xi_\alpha^{-1}(\mu)\big) + \big(1 - \xi_\alpha^{-1}(\mu)\big)H_\alpha ,
\end{aligned}$$

since

$$X_\alpha X_{-\alpha} = X_{-\alpha}X_\alpha + H_\alpha .$$

Therefore

$$-\big(\xi_{\alpha/2}(\mu) - \xi_{-\alpha/2}(\mu)\big)^{-2}\Gamma_\mu(X_{-\alpha}X_\alpha)$$

equals

$$X_{-\alpha}X_{\alpha} - (1 - \xi_{\alpha}(\mu)^{-1})(\xi_{\alpha/2}(\mu) - \xi_{-\alpha/2}(\mu))^{-2}H_{\alpha} .$$

Since

$$1 + 2(1 - \xi_{\alpha}^{-1}(\mu))(\xi_{\alpha/2}(\mu) - \xi_{-\alpha/2}(\mu))^{-2} = (\xi_{\alpha/2}(\mu) + \xi_{-\alpha/2}(\mu))(\xi_{\alpha/2}(\mu) - \xi_{-\alpha/2}(\mu))^{-1} ,$$

we see that $H_{\alpha} + 2X_{-\alpha}X_{\alpha}$ equals

$$\Gamma_{\mu}\Big(1 \times \frac{\xi_{\alpha/2}(\mu) + \xi_{-\alpha/2}(\mu)}{\xi_{\alpha/2}(\mu) - \xi_{-\alpha/2}(\mu)}\cdot H_{\alpha}\Big) - 2(\xi_{\alpha/2}(\mu) - \xi_{-\alpha/2}(\mu))^{-2}\Gamma_{\mu}(X_{-\alpha}X_{\alpha}) .$$

We have shown that

$$\omega = \Gamma_{\mu}(1 \otimes \beta_{\mu}(\omega)) + \sum\nolimits_{\alpha \in P} a_{\alpha}(\mu)\Gamma(X_{-\alpha}X_{\alpha} \otimes 1) \tag{5.1}$$

where

$$a_{\alpha}(\mu) = -2(\xi_{\alpha/2}(\mu) - \xi_{-\alpha/2}(\mu))^{-2}$$

and $\beta_{\mu}(\omega)$ equals

$$H_1^2 + \cdots + H_m^2 + \sum\nolimits_{\alpha \in P}\Big(\frac{\xi_{\alpha/2}(\mu) + \xi_{-\alpha/2}(\mu)}{\xi_{\alpha/2}(\mu) - \xi_{-\alpha/2}(\mu)}\Big)H_{\alpha} + B(\rho, \rho)I .$$

There is a well-known homomorphism γ of $\mathfrak{Z}$ onto the set of elements in $\mathcal{H}$ invariant under the Weyl group of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$. It is an easy calculation to show that

$$\gamma(\omega) = H_1^2 + \cdots + H_m^2 .$$

See [1(a), p. 6.11]. Harish-Chandra has shown that for any element z in $\mathfrak{Z}$

$$\beta_{\mu}(z) = (\varepsilon_{\mathbf{R}}(\mu)\Delta(\mu))^{-1}\gamma(z) \circ (\varepsilon_{\mathbf{R}}(\mu)\Delta(\mu)) .$$

See [3(a), Theorem 2]. This formula can also be proved directly for $z = \omega$ from our formulae for $\gamma(\omega)$ and $\beta_{\mu}(\omega)$. (The symbol $\varepsilon_{\mathbf{R}}(\mu)$ follows standard notation of Harish-Chandra. We shall also permit ourselves to write $\varepsilon_{\mathbf{R}}(t)$ for

$$\varepsilon_{\mathbf{R}}(h_t) = \operatorname{sgn} t ,$$

whenever t is a nonzero real number.)

Now let us apply (5.1) to the distributions $T(t, a_1)$. For any f in $\mathcal{C}(G)$ and x in G write

$$f(x\mu x^{-1}) = f(x\colon \mu) = f_{x^{-1}}(x\mu) .$$

Then for any X in $\mathfrak{g}$,

$$f(x; X\colon \mu) = \frac{d}{dt}f(x\cdot\exp tX\colon \mu)\Big|_{t=0} ,$$

in the usual notation. This equals

$$\frac{d}{dt}f\big(x\exp(tX)\cdot\mu\cdot\exp(-tX)x^{-1}\big)\Big|_{t=0}$$
$$= \frac{d}{dt}f\big(x\exp(tX)\mu x^{-1}\big)\Big|_{t=0} - \frac{d}{dt}f\big(x\mu\exp(tX)x^{-1}\big)\Big|_{t=0}$$
$$= f_{x^{-1}}\big(x\mu;\, \mathrm{Ad}(\mu^{-1})X - X\big)\ .$$

However,

$$f(x\mu x^{-1};\,\omega) = f_{x^{-1}}(x\mu;\,\omega)$$

since ω lies in the center of $\mathfrak{G}$. By (5.1) and the definition of the map Γ_μ this equals

$$f\big(x\colon\mu;\,\beta_\mu(\omega)\big) + \textstyle\sum_{\alpha\in P} a_\alpha(\mu)f(x;\,X_{-\alpha}X_\alpha\colon\mu)\ .$$

Now suppose that our regular element μ equals a_1h_t, for a_1 in A_1 and $t\neq 0$. As we saw in § 3, $\langle T(t, a_1), f\rangle$ equals

$$\varepsilon_{\mathrm{R}}(t)\Delta(a_1h_t)\int_{G/A} f(x^*a_1h_tx^{*-1})\Lambda(x^*)dx^*\ .$$

Therefore $\langle T(t, a_1), \omega f\rangle$ is the sum of

$$\varepsilon_{\mathrm{R}}(t)\Delta(a_1h_t)\int_{G/A} f\big(x\colon a_1h_t;\,\beta_{a_1h_t}(\omega)\big)\Lambda(x^*)dx^*$$

and

$$\varepsilon_{\mathrm{R}}(t)\Delta(a_1h_t)\textstyle\sum_{\alpha\in P} a_\alpha(a_1h_t)\displaystyle\int_{G/A} f(x^*;\,X_{-\alpha}X_\alpha\colon a_1h_t)\Lambda(x^*)dx^*\ .$$

For any real number s,

$$\mathrm{Ad}(h_s)X_{-\alpha}X_\alpha = \xi_\alpha(h_s)^{-1}X_{-\alpha}X_\alpha = X_{-\alpha}X_\alpha\ .$$

Therefore $f(x^*;\,X_{-\alpha}X_\alpha\colon a_1h_t)$ is well defined for x^* in G/A, so the above integral makes sense.

We shall use the relation between $\beta_{a_1h_t}(\omega)$ and $\gamma(\omega)$. Let ∇^2 be the differential operator

$$H_2^2 + \cdots + H_m^2$$

on A_1. Recall that

$$r^2 = 2(r_1 + 4r_2) = B(H', H') = B(\mu', \mu')^{-1}\ .$$

If ϕ is any function in $C^\infty(A_1A)$,

$$\phi(\mu;\,\gamma(\omega)) = \Big(\nabla^2 + r^{-2}\frac{d^2}{dt^2}\Big)\phi(a_1h_t)\ .$$

Then we have shown that

$$\langle T(t, a_1), \omega f\rangle - \Big(\nabla^2 + r^{-2}\frac{d^2}{dt^2}\Big)\langle T(t, a_1), f\rangle$$

equals

$$\sum_{\alpha\in P} a_\alpha(a_1h_t)\varepsilon_{\mathbf{R}}(t)\Delta(a_1h_t)\int_{G/A} f(x^*;\, X_{-\alpha}X_\alpha\colon a_1h_t)\Lambda(x^*)dx^*\ .$$

For $\alpha\in P$,

$$\int_{G/A} f(x^*;\, X_{-\alpha}X_\alpha\colon a_1h_t)\Lambda(x^*)dx^* = \int_{G/A} f(x^*\colon a_1h_t)\Lambda(x^*;\, X_\alpha X_{-\alpha})dx^*\ .$$

Recall that

$$\Lambda(x) = \mu'(H(x)) + \mu'(H(xw))\ .$$

Then

$$\Lambda(x;\, X_\alpha X_{-\alpha}) = \mu'(H(x;\, X_\alpha X_{-\alpha})) + \mu'(H(x;\, X_\alpha X_{-\alpha}\colon w))\ ,$$

if $H(x\colon w) = H(xw)$.

Therefore

$$\begin{aligned}\Lambda(x;\, X_\alpha X_{-\alpha}) &= \mu'(H(x;\, X_{-\alpha}X_\alpha + H_\alpha)) + \mu'(H(xw;\, \mathrm{Ad}(w^{-1})(X_\alpha X_{-\alpha})))\\ &= \mu'(H(x;\, X_{-\alpha}X_\alpha)) + \mu'(H(x;\, H_\alpha)) + \mu'(H(xw;\, X_{\alpha^w}X_{-\alpha^w}))\ ,\end{aligned}$$

where α^w is the root of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ given by

$$\alpha^w(H) = \alpha(\mathrm{Ad}(w^{-1})H)\ , \qquad H\in\mathfrak{h}_{\mathbf{C}}\ .$$

If α is positive on $\mathfrak{a}$, α^w is negative on $\mathfrak{a}$. If α is zero on $\mathfrak{a}$, so is α^w. In any case if α is in P,

$$\mu'(H(xw;\, X_{\alpha^w}X_{-\alpha^w})) = 0\ .$$

Also

$$\mu'(H(x;\, X_{-\alpha}X_\alpha)) = 0\ .$$

Finally,

$$\mu'(H(x;\, H_\alpha)) = r^{-2}n_\alpha$$

where n_α is an integer which equals 0, 1, or 2 depending on whether $\alpha(H')$ equals 0, 1, or 2 respectively.

Therefore

$$\Lambda(x;\, X_\alpha X_{-\alpha}) = r^{-2}n_\alpha\ .$$

We have proved the following:

THEOREM 5.1. *For a fixed $f\in\mathcal{C}(G)$ and a_1h_t a regular element in A_1A,*

$$\left(\nabla^2 + r^{-2}\frac{d^2}{dt^2}\right)\langle T(t, a_1), f\rangle$$

equals

$$\langle T(t, a_1), \omega f\rangle + 2r^{-2}\cdot\sum_{\alpha\in P} n_\alpha(\xi_{\alpha/2}(a_1h_t) - \xi_{-\alpha/2}(a_1h_t))^{-2}\cdot\langle F(t, a_1), f\rangle\ . \qquad \square$$

The points

$$a_1h_t\ ,\quad a_1 \in A_1\ ,\quad t \neq 0\ ,$$

are sometimes called the **R**-regular elements of A_1A. Theorem 5.1 is certainly true if a_1h_t is only an **R**-regular element in A_1A.

6. An integral formula

As we noted earlier, the distribution $T(t, a_1)$ is not invariant. To proceed further we must associate an invariant distribution with $T(t, a_1)$.

For $f \in C^\infty(G)$, $y \in G$, write

$$f^y(x) = f(yxy^{-1})\ .$$

THEOREM 6.1. *Let $f \in \mathcal{C}(G)$ and $y \in G$. Then for $t \neq 0$,*

$$\langle T(t, a_1), f^y\rangle - \langle T(t, a_1), f\rangle$$

equals

$$\operatorname{sgn} t\cdot\Delta(a_1h_t)\cdot\left\{\int_{K\times N} f(kn\cdot a_1h_t\cdot n^{-1}k^{-1})\cdot\mu'(H(y^{-1}k))dndk\right.$$
$$\left.+\int_{K\times\bar{N}} f(k\bar{n}\cdot a_1h_t\cdot\bar{n}^{-1}k^{-1})\cdot\mu'(H(y^{-1}k))d\bar{n}dk\right\}\ .$$

Proof. Fix $\alpha \in C_c^\infty(A)$ such that $\int_A \alpha(a)da = 1$. Define a function A on G by

$$A(x) = \alpha(x_A)\ ,\qquad x \in G\ .$$

Then $\langle T(t, a_1), f^y\rangle$ equals

$$\operatorname{sgn} t\cdot\Delta(a_1h_t)\cdot\int_{G/A} f(yx^*\cdot a_1h_t\cdot x^{*-1}y^{-1})\cdot\Lambda(x^*)dx^*$$
$$= \operatorname{sgn} t\cdot\Delta(a_1h_t)\cdot\int_{G/A} f(x^*\cdot a_1h_t\cdot x^{*-1})\cdot\Lambda(y^{-1}x^*)dx^*$$
$$= \operatorname{sgn} t\cdot\Delta(a_1h_t)\cdot\int_{K\times N\times A} f(kna\cdot a_1h_t\cdot a^{-1}n^{-1}k^{-1})\Lambda(y^{-1}kna)\cdot A(kna)dkdnda$$
$$= \operatorname{sgn} t\cdot\Delta(a_1h_t)\cdot\int_G f(x\cdot a_1h_t\cdot x^{-1})\Lambda(y^{-1}x)A(x)dx\ .$$

This expression equals

$$\operatorname{sgn} t\cdot\Delta(a_1h_t)\cdot\left\{\int_G f(x\cdot a_1h_t\cdot x^{-1})\cdot\mu'(y^{-1}x))\cdot A(x)dx\right.$$
$$\left.+\int_G f(x\cdot a_1h_t\cdot x^{-1})\cdot\mu'(H(y^{-1}xw))\cdot A(x)dx\right\}\ . \tag{6.1}$$

The first integral in (6.1) equals

$$\int_{K\times N\times A} f(kn\cdot a_1h_t\cdot n^{-1}k^{-1})\cdot\mu'(H(y^{-1}k))\cdot\alpha(a)\cdot dadndk$$
$$+\int_{K\times N\times A} f(kn\cdot a_1h_t\cdot n^{-1}k^{-1})\cdot\mu'(H(a))\cdot\alpha(a)\cdot dadndk\ ,$$

since $H(y^{-1}\cdot kna) = H(y^{-1}k) + H(a)$.

This in turn equals

$$\int_{K\times N} f(kn\cdot a_1h_t\cdot n^{-1}k^{-1})\cdot\mu'(H(y^{-1}k))dndk$$
$$+\int_{K\times N\times A} f(kn\cdot a_1h_t\cdot n^{-1}k^{-1})\cdot\mu'(H(a))\cdot\alpha(a)dadndk\ .$$

The second integral in (6.1) equals

$$\int_{K\times\bar{N}\times A} f(k\bar{n}\cdot a_1h_t\cdot\bar{n}^{-1}k^{-1})\cdot\mu'(H(y^{-1}k)+H(a^{-1}))\cdot A(\bar{n}a)\cdot dad\bar{n}dk\ .$$

Notice that

$$\int_A A(\bar{n}a)da = \int_A \alpha(a)da = 1\ .$$

Therefore, the second integral in (6.1) equals

$$\int_{K\times\bar{N}\times A} f(k\bar{n}\cdot a_1h_t\cdot\bar{n}^{-1}k^{-1})\cdot\mu'(H(a^{-1}))\cdot A(\bar{n}a)\cdot dkd\bar{n}da$$
$$+\int_{K\times\bar{N}} f(k\bar{n}\cdot a_1h_t\cdot\bar{n}^{-1}k^{-1})\cdot\mu'(H(y^{-1}k))\cdot dkdn\ .$$

We have expressed $\langle T(t, a_1), f^y\rangle$ as a sum of four terms, only two of which depend on y. If we set y equal to 1, these two terms vanish, so the sum of the other two terms equals $\langle T(t, a_1), f\rangle$. Therefore,

$$\begin{aligned}\langle T(t, a_1), f^y\rangle &= \langle T(t, a_1), f\rangle\\ &\quad+\operatorname{sgn} t\cdot\Delta(a_1h_t)\cdot\Big\{\int_{K\times N} f(kn\cdot a_1h_t\cdot n^{-1}k^{-1})\cdot\mu'(H(y^{-1}k))dkdn\\ &\quad+\int_{K\times\bar{N}} f(k\bar{n}\cdot a_1h_t\cdot\bar{n}^{-1}k^{-1})\cdot\mu'(H(y^{-1}k))\cdot dkd\bar{n}\Big\}\ .\end{aligned}$$ □

For f in $\mathcal{C}(G)$ let f_0 be the projection of f onto $\mathcal{C}_0(G)$. Define

$$\langle T_0(t, a_1), f\rangle = \langle T(t, a_1), f_0\rangle\ ,\qquad f\in\mathcal{C}(G)\ .$$

COROLLARY 6.2. *$T_0(t, a_1)$ is an invariant distribution.*

Proof. Fix y in G. Since $(f^y)_0 = (f_0)^y$,

$$\begin{aligned}\langle T_0(t, a_1), f^y\rangle &= \langle T(t, a_1), (f_0)^y\rangle\\ &= \langle T_0(t, a_1), f\rangle\\ &\quad+\operatorname{sgn} t\cdot\Delta(a_1h_t)\cdot\Big\{\int_{K\times N} f_0(kn\cdot a_1h_t\cdot n^{-1}k^{-1})\cdot\mu'(H(y^{-1}k))dndk\\ &\quad+\int_{K\times\bar{N}} f_0(k\bar{n}\cdot a_1h_t\cdot\bar{n}^{-1}k^{-1})\cdot\mu'(H(y^{-1}k))d\bar{n}dk\Big\}\ .\end{aligned}$$

For f in $\mathcal{C}(G)$ let

$$\tilde{f}(x) = \int_K f(kxk^{-1})\cdot\mu'(H(y^{-1}k))dk .$$

Then the map

$$f \longrightarrow \tilde{f} , \qquad f \in \mathcal{C}(G) ,$$

is a continuous linear operator on $\mathcal{C}(G)$. If f is in $\mathcal{C}_0(G)$, so is $\tilde{f}$. Therefore

$$\int_N \tilde{f}_0(n\cdot a_1h_t\cdot n^{-1})dn = \int_{\bar{N}} \tilde{f}_0(\bar{n}\cdot a_1h_t\cdot \bar{n}^{-1})d\bar{n} = 0 .$$

It follows that

$$\langle T_0(t, a_1), f^y\rangle = \langle T_0(t, a_1), f\rangle ,$$

which verifies the corollary. □

Suppose that a_1 is a semi-regular element in A_1. In § 4 we defined the distribution

$$\langle S^*(a_1), f\rangle = \lim_{t\to 0+} \{\langle S'(t, a_1), f\rangle - \langle S'(-t, a_1), f\rangle\} , \qquad f \in C_c^\infty(G) .$$

COROLLARY 6.3. *$S^*(a_1)$ is invariant.*

Proof. For any y in G, and $t \neq 0$,

$$\begin{aligned} &\operatorname{sgn} t\cdot\Delta(a_1h_t)\cdot\int_{K\times N} f(kn\cdot a_1h_t\cdot n^{-1}k^{-1})\cdot\mu'(H(y^{-1}k))dndk \\ &= \operatorname{sgn} t\cdot\Delta(a_1h_t)\cdot\int_N \tilde{f}(n\cdot a_1h_t\cdot n^{-1})dn \\ &= e^{t\rho(H')}\cdot\Delta_M(a_1)\cdot\int_N \tilde{f}(a_1h_t\cdot n)dn . \end{aligned}$$

This function is differentiable at $t = 0$. Similarly

$$\operatorname{sgn} t\cdot\Delta(a_1h_t)\cdot\int_{K\times\bar{N}} f(k\bar{n}\cdot a_1h_t\cdot\bar{n}^{-1}k^{-1})\cdot\mu'(H(y^{-1}k))d\bar{n}dk$$

is differentiable at $t = 0$. Therefore $\langle T(t, a_1), f^y\rangle$ and $\langle T(t, a_1), f\rangle$ differ by a function in t which is differentiable at the origin. Since the distribution $F(t, a_1)$ is invariant,

$$\langle S(t, a_1), f^y\rangle - \langle S(t, a_1), f\rangle$$

is also differentiable at $t = 0$. This proves the corollary. □

Recall that we are assuming that there is a root β in P_+ which vanishes on $\mathfrak{a}_1$. Recall that G_1 was the centralizer in G of any element in A_1'. Consider the map

$$\Gamma: (k, n^*) \longrightarrow k\bar{n}^*\cdot G_1$$

of $K \times N^*$ into G/G_1. A consequence of the Iwasawa decomposition for G is that Γ is surjective. Another consequence is that the inverse image under Γ of $e \cdot G_1$ is $K_1 = K \cap G_1$, a compact group.

For any function g in $C_c(G/G_1)$ define

$$T(g) = \int_{K \times N^*} g(k\bar{n}^*) e^{\lambda(\bar{n}^*)} dk dn^* \ .$$

T is a continuous linear functional on $C_c(G/G_1)$. Therefore there is a measure μ on G/G_1 such that

$$\int_{K \times N^*} g(k\bar{n}^*) e^{\lambda(\bar{n}^*)} dk dn^* = \int_{G/G_1} g(x^*) d\mu(x^*) \ , \qquad g \in C_c(G/G_1) \ .$$

Fix $a_1 \in A_1'$ and choose a function ϕ in $C_c^\infty(A_1)$ such that the support of ϕ is contained in A_1' and such that $\phi(a_1) = 1$. Then if $g \in C_c^\infty(G/G_1)$, the function

$$f(x^* \tilde{a}_1 x^{*-1}) = g(x^*) \phi(\tilde{a}_1) \ , \qquad x^* \in G/G_1, \ \tilde{a}_1 \in A_1$$

is in $C_c^\infty(G)$. As we saw in Lemma 4.1,

$$\begin{aligned} \langle S^*(a_1), f \rangle &= 2\sqrt{2} \cdot r \cdot \pi \cdot \Delta^*(a_1) \cdot \int_{K \times N^*} f(k\bar{n}^* \cdot a_1 \cdot \bar{n}^{*-1} k^{-1}) \cdot e^{\lambda(\bar{n}^*)} dk dn^* \\ &= 2\sqrt{2} \cdot r \cdot \pi \cdot \Delta^*(a_1) \int_{G/G_1} g(x^*) d\mu(x^*) \ . \end{aligned}$$

Applying this formula to f^y for $y \in G$, we have

$$\langle S^*(a_1), f^y \rangle = 2\sqrt{2} \cdot r \cdot \pi \cdot \Delta^*(a_1) \cdot \int_{G/G_1} g(yx^*) d\mu(x^*) \ .$$

By Corollary 6.3 we have

$$\int_{G/G_1} g(x^*) d\mu(x^*) = \int_{G/G_1} g(yx^*) d\mu(x^*) \ , \qquad y \in G \ ,$$

so that μ is a G-invariant measure on G/G_1. It is well known that any G-invariant measure on G/G_1 is unique up to a scalar multiple. Therefore, there exists a non-zero constant c such that

$$\int_{G/G_1} g(x^*) dx^* = c \int_{K \times N^*} g(k\bar{n}^*) e^{\lambda(\bar{n}^*)} dk dn \ , \qquad g \in C_c^\infty(G/G_1) \ .$$

We shall prove that $c = 1$.

Suppose that ϕ is any bounded measurable function of compact support on G. Then

$$\begin{aligned} \int_G \phi(x) dx &= \int_{G/G_1 \times G_1} \phi(x^* x_1) dx^* dx_1 \\ &= c \int_{K \times N^* \times G_1} \phi(k\bar{n}^* x_1) e^{\lambda(\bar{n}^*)} dx_1 dk dn^* \\ &= c \int_{K \times N^* \times N_1 \times A \times K_1} \phi(k\bar{n}^* \bar{n}_1 a k_1) e^{\lambda(\bar{n}^*)} e^{2\mu'(H(a))} \cdot dn_1 da dk_1 dk dn^* \ , \end{aligned}$$

where $K_1 = K \cap G_1$, and $N_1 = N \cap G_1$. This equals

$$c\int_{K\times N^*\times N_1\times A\times K_1}\phi(k_1^{-1}k\bar{n}^*\bar{n}_1ak_1)e^{\lambda(\bar{n}^*)}e^{2\mu'(H(a))}dn_1dadk_1dkdn^* .$$

For any $\delta > 0$ let $(\bar{N}A)_\delta$ be the set of elements in $\bar{N}A$ of the form

$$\bar{n}^*\bar{n}_1a , \qquad n^*\in N^*,\ n_1\in N_1,\ a\in A$$

such that

$$|1 - e^{\lambda(\bar{n}^*)}e^{2\mu'(H(a))}| < \delta .$$

$(\bar{N}A)_\delta$ is a neighborhood of the identity in $\bar{N}A$. For any $\delta > 0$ fix a neighborhood G_δ of the identity in G such that

(i) $G_\delta \subseteq K(\bar{N}A)_\delta$,

(ii) $k_1^{-1}G_\delta k_1 = G_\delta$ for every $k_1 \in K_1$.

Let ϕ be the characteristic function of G_δ divided by the volume of G_δ. Then

$$\int_G \phi(x)dx = c\int_{K_1\times K\times N^*\times N_1\times A}\phi(k_1^{-1}k\bar{n}^*\bar{n}_1ak_1)e^{2\mu'(H(a))}e^{\lambda(\bar{n}^*)}dn_1dadk_1dkdn^* .$$

This expression is the sum of

$$c\int_{K_1\times K\times N^*\times N_1\times A}\phi(k_1^{-1}k\bar{n}^*\bar{n}_1ak_1)dn_1dadk_1dkdn^*$$

and

$$-c\int_{K_1\times K\times N^*\times N_1\times A}\phi(k_1^{-1}k\bar{n}^*\bar{n}_1ak_1)\cdot(1 - e^{2\mu'(H(a))}e^{\lambda(\bar{n}^*)})\cdot dn_1dadk_1\cdot dkdn^* .$$

The first term in this sum equals

$$c\int_G \phi(x)dx .$$

Therefore

$$\begin{aligned}|1 - c|\cdot\int_G \phi(x)dx &= |1 - c| \\ &\leq c\int_{K_1\times K\times N^*\times N_1\times A}\phi(k_1^{-1}k\bar{n}^*\bar{n}_1ak_1)\cdot|1 - e^{2\mu'(H(a))}e^{\lambda(\bar{n}^*)}|\cdot dn_1dadk_1dkdn^* \\ &\leq \delta c\cdot\int_G \phi(x)dx = \delta c ,\end{aligned}$$

by the properties of the neighborhood G_δ. Since δ is arbitrary, c equals 1. We have proved

THEOREM 6.4. *For g in $C_c^\infty(G/G_1)$*

$$\int_{G/G_1} g(x^*)dx^* = \int_{K\times N^*} g(k\bar{n}^*)e^{\lambda(\bar{n}^*)}dkdn^* . \qquad \square$$

7. Evaluation of the distributions on $\mathcal{C}_0(G)$

As in [3(b), § 24], let $y = (\pi i/4(X' + Y'))$. y is in $G_{\mathbf{C}}$, and y leaves $\mathfrak{a}_{1\mathbf{C}}$ pointwise fixed. $i(X' - Y')$ equals $(H')^{y^{-1}}$. Define

$$\mathfrak{b}_2 = \mathfrak{l} \cap \mathfrak{h}_{\mathbf{C}}^{y^{-1}} = \mathbf{R}(X' - Y') ,$$
$$\mathfrak{b} = \mathfrak{b}_2 + \mathfrak{a}_1 ,$$

and

$$B = \exp \mathfrak{b} .$$

B is a compact Cartan subgroup of G. If δ equals $\beta^{y^{-1}}$, δ and $-\delta$ are the only roots of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$ that vanish on $\mathfrak{a}_{1\mathbf{C}}$. For $\theta \in \mathbf{R}$, define

$$t_\theta = \exp\left(\theta(X' - Y')\right) .$$

Then $t_\pi = \gamma$ and $t_{2\pi} = 1$. Define

$$\Delta^*(a_1 t_\theta) = \prod_{\substack{\alpha \in P \\ \alpha \neq \beta}} \left(\xi_{\alpha/2}((a_1 t_\theta)^y) - \xi_{-\alpha/2}((a_1 t_\theta)^y)\right) ,$$

and

$$\Delta(a_1 t_\theta) = \prod_{\alpha \in P} \left(\xi_{\alpha/2}((a_1 t_\theta)^y) - \xi_{-\alpha/2}((a_1 t_\theta)^y)\right) .$$

Suppose a_1 belongs to A_1'. Then for a suitably small positive number θ, $a_1 t_\theta$ is a regular element in B. We define, as usual,

$$F_f^B(a_1 t_\theta) = \Delta(a_1 t_\theta) \cdot \int_{G/B} f(\bar{x} \cdot a_1 t_\theta \cdot \bar{x}^{-1}) d\bar{x} , \qquad f \in \mathcal{C}(G) ,$$

and

$$F_g^B(a_1 t_\theta) = (-2i \sin \theta) \cdot \int_{G_1/B} g(u^* \cdot a_1 t_\theta \cdot u^{*-1}) du^* , \qquad g \in \mathcal{C}(G_1) .$$

LEMMA 7.1. *Let a_1 be an element in A_1'. Then for any f in $\mathcal{C}(G)$,*

$$\Delta^*(a_1) \cdot \int_{G/G_1} f(x^* a_1 x^{*-1}) dx^*$$

equals

$$\frac{1}{4\pi i} \lim_{\theta \to 0} \frac{d}{d\theta} F_f^B(a_1 t_\theta) .$$

Proof. This is a special case of a result of Harish-Chandra, ([3(c), Lemmas 23 and 28]). We shall repeat his proof in order to obtain the correct constant. For $x \in G_1$, define

$$g_x(u) = f(xux^{-1}) , \qquad u \in G_1 .$$

Let dx^* and $d\bar{x}$ be the normalized G-invariant measures on G/G_1 and G/B

respectively. Let du^* be the normalized G_1-invariant measure on G_1/B. Then $d\bar{x} = du^*dx^*$. Therefore,

$$F_f^B(a_1t_\theta) = \Delta^*(a_1t_\theta)\int_{G/G_1} F_{x^*}^B(a_1t_\theta)dx^* \; . \tag{7.1}$$

Let B_1 be the Killing form of $\mathfrak{l}_{\mathbf{C}}$, the derived subalgebra of $\mathfrak{g}_{1\mathbf{C}}$. δ and $-\delta$ can be regarded as roots of $(\mathfrak{l}_{\mathbf{C}}, \mathfrak{b}_{2\mathbf{C}})$. Let $H_{1\delta}$ be the element in $\mathfrak{b}_{2\mathbf{C}}$ such that

$$B_1(H_{1\delta}, X) = \delta(X)\,, \qquad X \in \mathfrak{b}_{2\mathbf{C}}\,.$$

Then

$$B_1(H_{1\delta}, i(X' - Y')) = B_1((H_{1\delta})^{\nu^{-1}}, (H')^{\nu^{-1}}) = \beta(H') = 2\,.$$

Since $B_1(H', H') = 8$, we have

$$H_{1\delta} = \frac{i}{4}(X' - Y')\,.$$

From a well known formula on SL(2, **R**) we have for any $g \in \mathcal{C}(G_1)$,

$$\begin{aligned} -\pi g(a_1) &= \lim_{\theta\to 0} F_g^B(a_1t_\theta; H_{1\delta}) \\ &= \frac{i}{4}\lim_{\theta\to 0} F_g^B(a_1t_\theta; X' - Y') \\ &= \frac{i}{4}\lim_{\theta\to 0} \frac{d}{d\theta}F_g^B(a_1t_\theta)\,. \end{aligned}$$

Applying the formula for $g = g_x$, and using (7.1) one obtains the formula

$$\lim_{\theta\to 0}\frac{d}{d\theta}F_f^B(a_1t_\theta) = 4\pi i\cdot\Delta^*(a_1)\int_{G/G_1} g_{x^*}(a_1)dx^*\,.$$

The lemma follows. □

Let $\mathcal{E}_d$ be the set of unitary equivalence classes of square integrable representations of G. Choose a class σ in $\mathcal{E}_d$ corresponding as in [3(c), Theorem 16] to a real linear functional ν on $i\mathfrak{b}$. Fix a function f in $\mathcal{C}_\sigma(G)$, the closed subspace of $\mathcal{C}(G)$ generated by the matrix coefficients of any representation in the class σ. We are going to calculate $\langle T(t, a_1), f\rangle$.

Recall the definitions of $F(t, a_1)$ and $S(t, a_1)$ from § 4. Since f is in $\mathcal{C}_0(G)$, $\langle F(t, a_1), f\rangle = 0$, so that

$$\langle S(t, a_1), f\rangle = \langle T(t, a_1), f\rangle\,.$$

Suppose that a_1 is in A_1'. Then

$$\begin{aligned} &\lim_{t\to 0+}\{\langle T'(t, a_1), f\rangle - \langle T'(-t, a_1), f\rangle\} \\ &= 2\sqrt{2}\cdot\pi r\cdot\Delta^*(a_1)\cdot\int_{K\times N^*} f(k\bar{n}^*\cdot a_1\cdot\bar{n}^{*-1}k^{-1})\cdot e^{\lambda(\bar{n}^*)}dndk \\ &= 2\sqrt{2}\cdot\pi r\cdot\Delta^*(a_1)\cdot\int_{G/G_1} f(x^*a_1x^{*-1})dx^*\,, \end{aligned}$$

by Lemma 4.1 and Theorem 6.4. According to Lemma 7.1 this last expression equals

$$\frac{\sqrt{2}r}{2i}\lim_{\theta\to 0}\frac{d}{d\theta}F_f^B(a_1t_\theta) .$$

On the other hand, a theorem of Harish-Chandra ([3(c), Theorem 14]) asserts that for $f\in\mathcal{C}_\sigma(G)$,

$$F_f^B(a_1t_\theta) = f(1)\cdot d_\sigma^{-1}\cdot\Delta(a_1t_\theta)\cdot\Theta_\sigma(a_1t_\theta) ,$$

where Θ_σ is the character of σ and d_σ is the formal degree.

Now for $b\in B$,

$$\Delta(b)\Theta_\sigma(b) = (-1)^q\cdot\operatorname{sgn}\tilde{\omega}(\nu)\cdot\sum_{s\in W_G}\varepsilon(s)\xi_{s\nu}(b) ,$$

where $\tilde{\omega}(\nu)=\prod_{\alpha\in P_+}B(\nu^y,\alpha)$ and $2q=(\dim\mathfrak{g}_C-\operatorname{rank}\mathfrak{g}_C)-(\dim\mathfrak{h}_C-\operatorname{rank}\mathfrak{h}_C)$. If s is an element in W_G, we can write

$$\xi_{s\nu}(a_1t_\theta) = e^{ik(s\nu)\theta}\cdot\zeta^{s\nu}(a_1) ,$$

for $k(s\nu)$ an integer and $\zeta^{s\nu}$ a character on A_1. Any character ζ on A_1 defines a linear functional $\mu(\zeta)$ on $i\mathfrak{a}_1$. Put $|\zeta|^2=B(\mu(\zeta),\mu(\zeta))$. Notice that

$$\begin{aligned}B(\nu,\nu)-|\zeta^{s\nu}|^2 &= B(s\nu,s\nu)-|\zeta^{s\nu}|^2\\ &= k(s\nu)^2\cdot B(H',H')^{-1}+B(\mu(\zeta^{s\nu}),\mu(\zeta^{s\nu}))-|\zeta^{s\nu}|^2\\ &= k(s\nu)^2\cdot r^{-2} .\end{aligned}$$

Since

$$\Delta_B(a_1t_\theta)\Theta_\sigma(a_1t_\theta)$$

equals

$$(-1)^q\cdot\operatorname{sgn}\tilde{\omega}(\nu)\cdot\sum_{s\in W_G}\varepsilon(s)e^{ik(s\nu)\theta}\cdot\zeta^{s\nu}(a_1) ,$$

we have established the following formula:

$$\begin{aligned}&\lim_{t\to 0+}\{\langle T'(t,a_1),f\rangle-\langle T'(-t,a_1),f\rangle\}\\ &\quad=\frac{\sqrt{2}r}{2}\cdot d_\sigma^{-1}\cdot f(1)\cdot(-1)^q\cdot\Big(\sum_{s\in W_G}\operatorname{sgn}\tilde{\omega}(s\nu)\cdot k(s\nu)\cdot\zeta^{s\nu}(a_1)\Big) .\end{aligned}\tag{7.2}$$

Now let us return to the differential equation. If ω is the differential operator defined in § 5, $\omega f=B(\nu,\nu)f$. Since

$$\langle F(t,a_1),f\rangle = 0 ,$$

the equation of § 5 becomes homogeneous. It is

$$\Big(\nabla^2+r^{-2}\frac{d^2}{dt^2}\Big)\langle T(t,a_1),f\rangle = B(\nu,\nu)\cdot\langle T(t,a_1),f\rangle .$$

Let A_1^* be the dual group of A_1. For $\zeta\in A_1^*$ define

$$\langle T(t, \zeta), f\rangle = \int_{A_1} \langle T(t, a_1), f\rangle \cdot \zeta(a_1^{-1}) da_1 .$$

Then we have differential equation

$$r^{-2}\frac{d^2}{dt^2}\langle T(t, \zeta), f\rangle = (B(\nu, \nu) - |\zeta|^2)\cdot\langle T(t, \zeta), f\rangle .$$

Therefore, there are constants C_ζ^+ and C_ζ^- such that $\langle T(t, \zeta), f\rangle$ equals

$$C_\zeta^-\cdot\exp\{rt\sqrt{B(\nu, \nu) - |\zeta|^2}\} + C_\zeta^-\cdot\exp\{-rt\sqrt{B(\nu, \nu) - |\zeta|^2}\} .$$

Now, by Corollary 3.3, we know that $|\langle T(t, \zeta), f\rangle|$ decreases for large t faster than any power of t. We shall write $A_1^*(\nu)$ for the set of characters ζ in A_1^* such that $|\zeta|^2 < B(\nu, \nu)$. Then if ζ is in $A_1^*(\nu)$, and $t > 0$, C_ζ^+ must be 0. Moreover if ζ is not in $A_1^*(\nu)$ then $C_\zeta^+ = C_\zeta^- = 0$. It follows that there are numbers $\{C(\zeta): \zeta \in A_1^*(\nu)\}$, depending only on f and ν such that for $a_1 \in A_1'$ and $t \neq 0$,

$$\langle T(t, a_1), f\rangle = \textstyle\sum_{\zeta \in A_1^*(\nu)} C(\zeta)\cdot\zeta(a_1)\exp\{-r|t|\sqrt{B(\nu, \nu) - |\zeta|^2}\} . \tag{7.3}$$

From this formula we see that

$$\lim_{t\to 0+}\{\langle T'(t, a_1), f\rangle - \langle T'(-t, a_1), f\rangle\}$$

equals

$$-2r\textstyle\sum_{\zeta \in A_1^*(\nu)} \sqrt{B(\nu, \nu) - |\zeta|^2}\cdot C(\zeta)\cdot\zeta(a_1) . \tag{7.4}$$

Now we are through, because we can compare (7.2) with (7.4) to solve for the numbers $C(\zeta)$. $C(\zeta)$ will be zero unless $\zeta = \zeta^{s\nu}$ for some $s \in W_G$, in which case

$$-2r\cdot C(\zeta^{s\nu})\cdot\sqrt{B(\nu, \nu) - |\zeta^{s\nu}|^2}$$

equals

$$\frac{\sqrt{2}r}{2}\cdot d_\sigma^{-1}\cdot f(1)\cdot(-1)^q\cdot(\operatorname{sgn}\tilde{\omega}(s\nu)\cdot k(s\nu)\cdot\zeta^{s\nu}(a_1)) .$$

Since $B(\nu, \nu) - |\zeta^{s\nu}|^2 = k(s\nu)^2 r^{-2}$, $C(\zeta^{s\nu})$ equals

$$(-1)^{q+1}\cdot\frac{\sqrt{2}r}{4}\cdot\operatorname{sgn}\{k(s\nu)\tilde{\omega}(s\nu)\}\cdot f(1)\cdot d_\sigma^{-1} .$$

We have essentially proved the following, which is our main result.

THEOREM 7.2. *Suppose $f \in \mathcal{C}_\sigma(G)$, and $\sigma \in \mathcal{E}_d$ is associated to the linear functional ν on $i\mathfrak{b}$. Then for $a_1 \in A_1$, and $t \neq 0$, $\langle T(t, a_1), f\rangle$ equals*

$$(-1)^{q+1}\cdot\frac{\sqrt{2}r}{4}\cdot f(1)\cdot d_\sigma^{-1}(\textstyle\sum_{s \in W_G}\operatorname{sgn}\{k(s\nu)\tilde{\omega}(s\nu)\}\cdot e^{-|k(s\nu)t|}\cdot\zeta^{s\nu}(a_1)) .$$

Proof. For $a_1 \in A_1'$ the formula follows by substituting the above value for $C(\zeta^{s\nu})$ in (7.3); but A_1' is dense in A_1 and for fixed $t \neq 0$, $\langle T(t, a_1), f\rangle$ is a smooth function of a_1. Therefore the formula is true for all a_1 in A_1. □

COROLLARY 7.3. *In the notation of the theorem, $\langle T(a_1), f\rangle$ equals*

$$\varepsilon^{-1}\cdot(-1)^{q+1}\cdot\frac{\sqrt{2}r}{4}\cdot f(1)\cdot d_\sigma^{-1}\cdot\left(\sum_{s\in W_G}\operatorname{sgn}\{k(s\nu)\tilde{\omega}(s\nu)\}\cdot\zeta^{s\nu}(a_1)\right),$$

where ε equals 1 *or* 2, *depending on whether* $\mathfrak{g}_1(a_1) = 0$ *or not.*

Proof. $\langle F(t, a_1), f\rangle$ equals zero for all a_1 and all t. Therefore by a formula in § 4,

$$\varepsilon\langle T(a_1), f\rangle = \lim_{t\to 0+}\langle T(t, a_1), f\rangle .$$

The corollary follows from the theorem. □

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