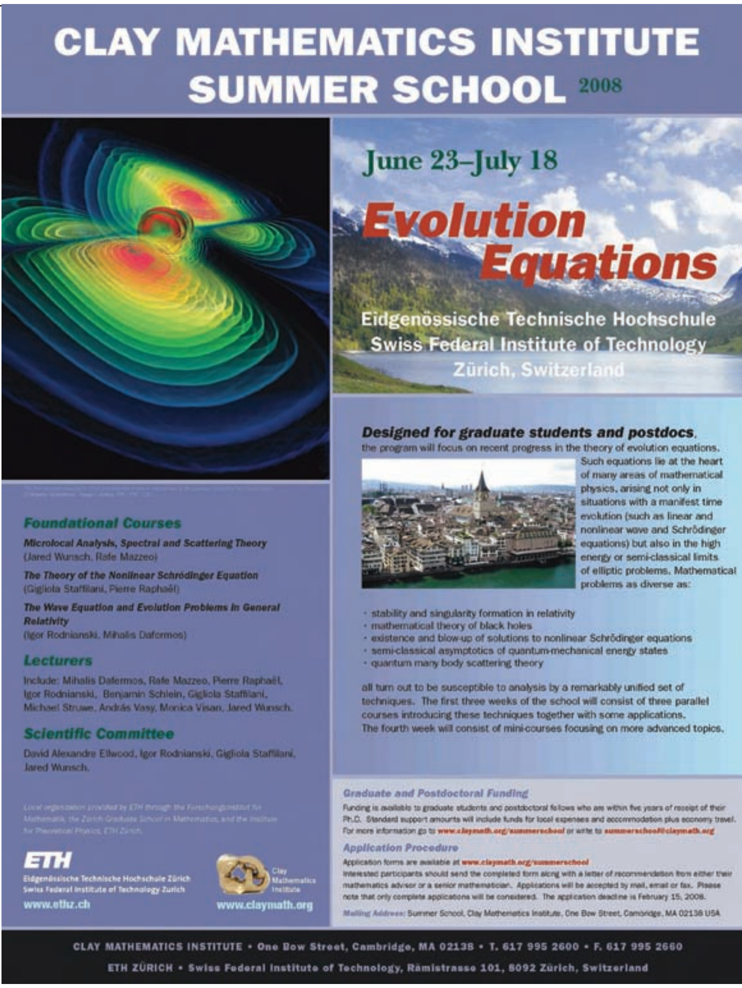
Evolution Equations at the Swiss Federal Institute of Technology, Zürich, Switzerland



by Dean Baskin (Stanford), Jacques Smulevici (Cambridge), and Vedran Sohinger (MIT)

The Clay Mathematics Institute 2008 Summer School took place in the wonderful setting provided by the Eidgenössische Technische Hochschule Zürich and focused on recent progress in the theory of evolution equations. These equations lie at the heart of many areas of mathematical physics, arising not only in situations with a manifest time evolution but also in the high energy or semi-classical limits of elliptic problems.

The study of evolution equations has a long history but its rich and varied landscape make it an ever renewed field with many interesting open questions and conjectures waiting to be solved.

The program was built around three foundational courses:

- Microlocal Analysis, Spectral and Scattering Theory by Jared Wunsch and Rafe Mazzeo
- The Theory of the Nonlinear Schrödinger Equation by Gigliola Staffilani and Pierre Raphaël
- Wave Equation and Evolution Problems in General Relativity by Igor Rodnianski and Mihalis Dafermos

These courses were supplemented by several mini-courses:

- Derivation of Effective Evolution Equations from Microscopic Quantum Dynamics by Benjamin Schlein
- Nonlinear Schrödinger Equations at Critical Regularity by Monica Visan
- Wave Maps With and Without Symmetries by Michael Struwe
- Quantum N-body Scattering, Diffraction of Waves, and Symmetric Spaces by András Vasy

One of the fundamental bricks of evolution equations is the homogeneous scalar wave equation on Minkowski spacetime $\Box\phi = 0$. Generalizations include inhomogeneous terms, non flat-geometries, non-linearity, higher dimensions, and so forth. For the scalar homogeneous wave equation in Minkowski space, one may easily obtain an explicit representation of the solution using either spherical means or in terms of a Fourier decomposition.

It was therefore natural that one of the foundational courses addressed in particular a generalization of Fourier analysis using microlocal tools and their applications to evolution equations and scattering theory. Jared Wunsch began with an axiomatic treatment of pseudodifferential operators and wavefront sets, which generalize differential operators and singular sets, respectively. As an example of their applications, he provided a proof of the Duistermaat-Hörmander propagation of singularities theorem for operators of real principal type. He then constructed the solution operator for the wave equation and the wave trace via a geometric optics construction. This construction provided a motivation for an axiomatic treatment of the calculus of Fourier integral operators, and the construction of the wave operator within this calculus.

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Rafe Mazzeo continued the microlocal lectures with a discussion of scattering theory, both time-independent and time-dependent. The former is broadly a study of the continuous spectrum of the Laplacian on noncompact spaces, i.e. the study of solutions of $(-\Delta - \lambda^2)\ u = 0$. Taking the Fourier transform in λ changes the problem into time-dependent scattering, i.e. into the study of the asymptotic behavior of wave evolution.

Building on the constructions in the foundational course, András Vasy's lectures outlined the construction of the resolvent $(-\Delta + V - \lambda)^{-1}$ for N-body potentials V via a geometric resolution. He then illustrated the geometric similarity between this construction and the construction of the resolvent $(-\Delta - \lambda)^{-1}$ on symmetric spaces of noncompact type.

Another foundational course was devoted to the study of hyperbolic wave motion and its applications to general relativity. Igor Rodnianski started with a derivation of wave motions from the equations of different theories of physics such as electromagnetism, acoustics, and general relativity. The representations of the solution of the wave equation in Minkowski spacetime easily provide quantitative estimates, e.g., L^∞ to L^∞ estimates and energy estimates. However, they rely on many features of the Minkowski space and the linear property of the equation.

The method of compatible currents was thus introduced to study more complex systems of hyperbolic equations arising, such as the Euler-Lagrange equations of given Lagrangian. For instance, for the scalar wave equation on a Lorentzian manifold $\Box_g \phi = 0$, one can contract the energy-momentum tensor $T_{\mu\nu}(\phi)$ arising from Noether's theorem with any timelike vector field X^μ. The resulting vector field $J^\mu = T_{\mu\nu} X^\nu$ enjoys several remarkable properties. First, J^μ and its divergence only depend on the 1-jet of ϕ. Moreover, the integral of $J^\mu n_\mu$ over a spacelike hypersurface with normal n_μ controls $\nabla\phi$ in L^2. If we choose X^μ to a be Killing vector, Stokes's theorem gives us a conservation law and other choices of vector fields give us different energy estimates. These methods were applied to obtain global existence results for several non-linear wave equations such as the Yang-Mills equations.

Views of Zürich (above) and the ETH Chemistry building, where courses took place.

Mihalis Dafermos continued the course with an analysis of the wave equation on different black hole spacetimes. His lectures introduced the geometry of one of the simplest family of explicit solutions of the Einstein equations, the so-called Schwarzschild family. This one-parameter family of solutions lives as a subfamily of a two-parameter family known as the Kerr family. Both cases are models of black hole spacetimes. Black hole spacetimes are characterized by the presence of an event horizon, a global geometric property. One may try to use a compatible current with a timelike Killing vector field in order to obtain decay for the wave equation on Schwarzschild. In contrast with Minkowski space, this Killing field becomes null on the event horizon and so the weights in the energy estimates degenerate on the horizon. To bypass this difficulty, one needs to capture the celebrated *red-shift effect* with the introduction of a new vector field. To prove decay, it is also necessary to understand the geometry of the so-called *photonsphere* where trapped null geodesics accumulate. Due to the more complex geometry of the Kerr solution, controlling trapping here requires delicate analysis. Mihalis Dafermos presented the recent results concerning boundedness and decay for Kerr spacetimes sufficiently close to Schwarzchild.

In his series of lectures, Michael Struwe presented a study of the wave map problem. He focused in particular on the application of Strichartz estimates and geometric arguments in physical space for the Cauchy problem. Using these methods, he provided a proof of global existence for two symmetric cases, either when the target manifold is a surface of revolution of dimension 2 satisfying appropriate conditions or when it is a smooth compact Riemannian manifold without boundary and the initial data has radial symmetry.

Benjamin Schlein proved in his mini course that these non-linear dispersive equations, such as the Hartree equation with bounded or Coulomb potentials, arise naturally from microscopic quantum dynamics. The proofs involve in particular the study of the time evolution of the marginal density associated with the wave function describing the quantum system. He then presented the recent results concerning the evolution of Bose-Einstein condensates and the derivation of the Gross-Pitaevskii equation.

The third foundational course was devoted to the nonlinear Schrödinger equation. In this class, we introduced the power-nonlinearity semilinear Schrödinger equation $i \cdot u_t + \frac{1}{2}\Delta u = \lambda \cdot |u|^{p-1} \cdot u$. In her lectures, Gigliola Staffilani studied the *defocusing equation* given by $\lambda = 1$. We first observed the fundamental conservation laws of mass, energy, and momentum for our equation. From the scaling heuristic, we defined the concepts of subcritical, critical, and supercritical nonlinearities that led into the well-posedness theory for our equation. We then proved local well-posedness of the H^1-subcritical nonlinear Schrödinger equation, and, in the case of the defocusing equation, we deduced that the well-posedness was global.

Continuing with our study of the defocusing equation, we arrived at the concept of *scattering*. Our approach to this problem was based on the notion of Morawetz estimates. After recapitulating the standard Morawetz estimate, we turned to the Interaction Morawetz estimate from which we derived scattering for the cubic, defocusing, nonlinear Schrödinger equation in $\mathbb{R}^3$. Having presented the global well-posedness and scattering in the H^1-subcritical case, we set out to learn the analogue of the result in the H^1-critical case.

Pierre Raphaël's class emphasized the focusing equation, where $\lambda = -1$. The goal was to describe the long-time behavior of solutions, and in particular, the singularity formation in the space H^1. In studying this problem, we also considered the related problem of Soliton Stability. We began by proving H^1-global well-posedness in the L^2-subcritical case and turned to the orbital stability of solitons. We used *concentration compactness* as a remedy for the failure of compactness of $L^q \hookrightarrow H^1$. The main topic of the class was the L^2-critical equation. From the soliton characterization

The Summer School barbecue held on the ETH Hönggerberg Campus.

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ETH Summer School participants joined by organizers CMI Research Director David Ellwood and ETH Professor Gian Michele Graf (center, front row).

in this case, one can obtain the sharp constant in the *Gagliardo-Nirenberg inequality*. From this fact, we deduced global well-posedness for the L^2-critical problem as long as the mass of the initial data is less than that of the ground state soliton.

At this point, we proved that this theorem was sharp by applying the mass-preserving *Pseudo-conformal transformation*. In particular, this gave us a concrete example of a blow-up solution and led to the first *Liouville (Rigidity) Theorem* of the class. We then turned to the study of the blow-up phenomenon and which blow-up rates occur. We considered a solution by writing it in terms of *modulational parameters*, which we substituted into the equation to obtain the evolution equations for the parameters. For the remainder of the course, we studied explicit blow-up regimes for solutions with small super-critical data.

During the conference, Monica Visan talked about the NLS in the case of critical regularity. She discussed global existence and scattering for the mass-critical and for the energy-critical equations. In particular, we proved an extension of a result of Carlos Kenig and Frank Merle, removing the assumption of radial initial data in dimension ≥ 5.

All lecturers included topics of interest to advanced students, but also took care to provide concrete examples that were accessible to non-experts.

View from Zürich's Lindenhof across the Limmat river to the ETH campus.