A Tribute to Euler by William Dunham

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With acclaim normally reserved for matinee idols, Leonhard Euler has recently been basking in the mathematical limelight. The cause of this spike in publicity was his tercentenary. Euler was born in Switzerland in 1707, and thus 2007 provided the perfect opportunity for mathematicians to celebrate his life and work. Discussions, conferences, and special events were held in his honor. The Mathematical Association of America published not one, not two, but *five* books about his remarkable career. And, in October of 2008 ("tercentenary plus one"), Euler was the subject of my Clay Public Lecture at Harvard University. His story, in both its personal and scholarly dimensions, is one of the great tales from the history of mathematics.

Leonhard Euler (1707—1783)

As a youth, Euler showed such signs of genius that he was mentored by Johann Bernoulli, who was then on the faculty at the University of Basel. Euler would later recall his sessions with the illustrious Bernoulli as an exciting, if daunting, experience. For his part, Johann recognized the special talent of his young student. Indeed, Bernoulli—a person not naturally self-effacing—would later write these laudatory words to Euler: "I present higher analysis as it was in its childhood, but you are bringing it to man's estate."

When he was 15, Euler graduated from the University of Basel, and by the age of 20 he had won a prize from the Paris Academy. In those days, the Academy would challenge the mathematicians of Europe with specific, and often quite difficult, problems. In this instance, the problem required a mathematical analysis of the placement of masts on a sailing ship. Euler's submission received what amounted to a second prize, an achievement all the more remarkable because of his Swiss— i.e., landlocked—upbringing. (He would win the Academy's first prize a dozen times over the course of his career.)

On the heels of this success, Euler applied for a faculty position at his alma mater. To his dismay, the job went to Benedict Staehelin, an individual who thereby earned the distinction of being perhaps the worst hiring choice in history. But Euler's fortunes improved with an offer from the St. Petersburg Academy in Russia. His appointment came through the influence of Daniel Bernoulli, son of Johann, who had himself secured a position at St. Petersburg a few years before.

And so Euler bade farewell to Switzerland and moved to St. Petersburg in 1727. He stayed until 1741 when he accepted a call to the rival Berlin Academy. There he worked under Frederick the Great until friction between them proved too much. Euler returned to St. Petersburg in 1766, where he remained until his death in 1783.

It was during his first Russian stint that he married Katharina Gsell, and the Eulers would eventually have 13 children. However, child mortality took a dreadful toll in those days, and only five of their children would survive to adolescence. The accompanying sorrow defies comprehension.

Meanwhile, Euler faced a physical challenge of his own. By his early 30s, he had lost vision in his right eye. A modern diagnosis, insofar as such a thing is possible, attributes this to an ocular infection that was untreatable at the time. Visual limitations aside, Euler continued his research unabated and maintained his productivity up to the year 1771, when he lost sight in his other eye. This was due to a cataract. Such a malady, easily corrected today, was a most serious matter back then. Euler's doctors tried eye surgery to save his vision (and no one wants to contemplate the horrors of eye surgery in the 18th century), but the procedure was unsuccessful. By 1771, Euler was essentially blind.

It is tempting to conclude that this marked the end of his career, but Euler would not be stopped. He instructed his assistants to read aloud the newly arrived books and journals, and he in turn dictated his ideas to a tableful of scribes working furiously to keep up. It is said that Euler could create mathematics faster than most people can write it, and he daily put his assistants to the test. A case in point: in 1775, when he was blind, he produced a paper a week! Like Beethoven, who wrote music that he never heard, Euler created mathematics that he never saw. This triumph in the face of adversity makes Euler's the most inspirational story in the history of mathematics.

Such a biography, although compelling, might be forgotten had the results he produced been of minor interest. But nothing could be further from the truth. If one measures a mathematician's impact along the three "axes" of quantity, diversity, and significance, then Euler is pretty much off the charts on all three. Let me address each in turn.

In terms of *quantity*, Euler has no peer. Indeed, a major challenge for those who sought to publish his collected works was the "simple" task of locating them all. This was complicated by the fact that Euler published 228 papers *after he died*, making the deceased Euler one of history's most prolific mathematicians.

In any case, by the dawn of the 20th century, the scholar Gustav Eneström had identified a total of 866 books and papers that Euler produced over his long career. Eneström briefly described each of these in a catalogue that itself ran to 388 pages. With this massive document as its guide, the Swiss Academy of Sciences began publishing Euler's collected works —his *Opera Omnia*—in 1911, when the first hefty volume appeared. Thereafter, the books kept coming and coming … and coming. At the moment, there are 75 volumes in print, totaling over 25,000 pages, but the project is not yet complete. By the time all of the papers and letters and notebooks are in print, Euler will have kept his publishers busy for more than a century. There is nothing else like this in all of mathematics.

In terms of its *diversity*, Euler's work covers a range of subject matter that can only be described as "universal." Consider the following dichotomies:

Pure/Applied: Euler, of course, made innumerable contributions to pure mathematics, but he was also the leading applied mathematician of his day. In fact, a good half of those 75 volumes of the *Opera Omnia* treat subjects like mechanics, acoustics, and optics —subjects that are today classified under physics or applied math.

Continuous/Discrete: Euler was as comfortable working in the continuous realm (e.g., calculus and differential equations) as he was working in the discrete one (e.g., number theory and combinatorics). Such breadth has become a rarity in our age of specialization.

Advanced/Elementary: Euler certainly contributed to the advanced mathematics of his time, but he was also successful writing about elementary topics. For instance, in 1738 he published his *Rechenkunst*, an arithmetic text for the schools, and his best-selling work of all was *Letters to a German Princess* of 1768, a survey of popular science written for the layperson.

Old/New: Euler made some remarkable discoveries in the venerable subject of plane geometry, discoveries that would have been accessible to old Euclid himself. Yet Euler also worked in fields so new that he was making them up as he went along.

But quantity and diversity do not fully account for Euler's mathematical reputation. There is one additional dimension of excellence that is surely the greatest of all: the *significance* of his work. It is remarkable how many seminal ideas in our discipline can be traced back to him. Consider, for instance:

The concept of function. It was Euler who elevated the "function" into its starring role in analysis. Prior to that, people had applied calculus to the "curve," a quasi-precise idea rooted in, and limited by, geometrical understanding. In his classic 1748 text, *Introductio in Analysin Infinitorum*, Euler emphasized functions and introduced the special types—polynomial, exponential, logarithmic, trigonometric, and inverse trigonometric—that still occupy center stage in analysis.

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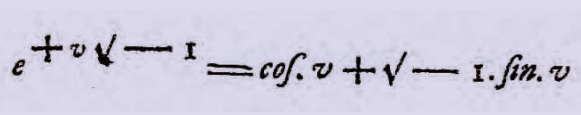


Euler Identity from his 1748 classic *Introductio in Analysin Infinitorum*

The Euler Identity. It was Euler who gave us the formula $e^{ix} = \cos x + i\sin x$. Those who encounter this for the first time are apt to regard it as a typo, so peculiar is its fusion of the exponential and the trigonometric, the real and complex. It was from this that Euler deduced such strange consequences as $i^i = \dfrac{1}{\sqrt{e^{\pi}}}$, about which the Harvard mathematician Benjamin Peirce is reported to have said, "Gentlemen, we have not the slightest idea of what this equation means, but we may be certain that it means something very important."

The Euler Polyhedral Formula. In a 1752 study of polyhedra, Euler observed that $V + F = E + 2$, where V is the number of vertices, F is the number of faces, and E is the number of edges of a solid figure. Because of the utter simplicity of this relationship, Euler confessed that, "I find it surprising that these general results in solid geometry have not previously been noticed by anyone, so far as I am aware. " Of course, no previous mathematician had had Euler's penetrating insight.

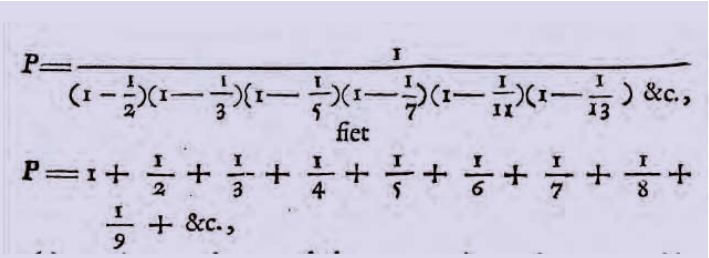
The Basel Problem. In the late 17th century, Jakob Bernoulli had challenged the mathematical community to find the *exact* sum of the infinite series $\sum_{k=1}^{\infty} 1/k^2$. This remained an open question for a generation until Euler, then a young and still relatively unknown mathematician, stunned the world by finding the sum to be $\pi^2/6$. As much as anything, this discovery made Euler famous.

The Euclid-Euler Theorem. Four volumes of the *Opera Omnia* address the theory of numbers, and Euler made untold contributions to this ancient and challenging field. One of these harks back to Book IX of the *Elements*, where Euclid had demonstrated that a whole number will be perfect (i.e., the sum of its proper divisors) if it is of the form $N = 2^{k-1}(2^k - 1)$, where the rightmost factor is prime. There matters stood for two millennia until Euler proved that this sufficient condition is also necessary for an *even* number to be perfect. Taken together, these results characterize even perfect numbers in the so-called "Euclid-Euler theorem," surely one of the most illustrious hyphenations in the history of mathematics. (By the way, Euler suggested that the matter of odd perfect numbers was likely to be "most difficult"—an indisputably accurate assessment.)

The Euler Product-Sum Formula. In 1737, Euler proved that $\sum_{k=1}^{\infty} 1/k = \prod_{p} \dfrac{1}{1-1/p}$, where the product on the right is taken over all the primes. Of course, he was here equating a divergent series (the harmonic) with a divergent product, so one might dismiss it as so much drivel. But Euler saw how to exploit his formula to establish the divergence of $\sum_{p} 1/p$.

This non-trivial theorem, which employed techniques of analysis to attack questions of number theory, prompted the 20th century mathematician André Weil to comment, "One may well regard these investigations as marking the birth of analytic number theory."

Such spectacular results notwithstanding, we have barely scratched the surface. Consider this partial list of other Eulerian "hits": The Bridges of Königsberg (1736); the original partitioning theorem for whole numbers (1740); the Euler line of a triangle (1767); the first textbook on the calculus of variations (1744); the analysis of Greco-Latin squares (1782); the landmark study of continued fractions (1744); the gamma function (1729); and the influential mechanics text of 1736 that cast Newton's physics in the language of Leibniz's calculus.



$$P = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1-\frac{1}{13})\ \&c.,}$$

fiet

$$P = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \&c.,$$

Euler product-sum formula from his 1748 classic *Introductio in Analysin Infinitorum*

And, as if these achievements were not enough, there are some downright quirky results scattered among his papers. For instance, he sought four different whole numbers, the sum of any pair of which is a perfect square. (If you think this is easy, try it.) Euler came up with this fearsome foursome: 18530, 38114, 45986, and 65570.

From all of this, it should be clear why I chose to focus my Clay Public Lecture on Euler and his triumphs (those interested can find a video of the talk at www.claymath.org/video). It should be equally clear why the mathematical community so enthusiastically celebrated Euler's 300th birthday. For, if anyone stands as the mathematical counterpart of Shakespeare or Rembrandt or Bach, it is the incomparable master, Leonhard Euler.

Other pertinent texts (in English):

Emil Fellmann, *Leonhard Euler*, (trans., E. and W. Gautschi), Birkhauser, 2007.

Andreas and Alice Heyne, *Leonhard Euler: A Man to be Reckoned With*, Birkhauser, 2007. [This is the "Euler Comic Book" I mentioned in the lecture. It's actually pretty good!]

Euler in translation: *Leonhard Euler, Introduction to Analysis of the Infinite* (2 vols.), (trans. John Blanton), Springer-Verlag, 1988.

Leonhard Euler, *Foundations of Differential Calculus*, (trans. John Blanton) Springer-Verlag, 2000.

Leonhard Euler, *Elements of Algebra*, (trans. John Hewlett), Springer-Verlag, 1840 (reprint).

Surveys:

Edward Sandifer, *The Early Mathematics of Leonhard Euler*, MAA, 2007.

Edward Sandifer, *How Euler Did It*, MAA, 2007.

William Dunham, *Euler: The Master of Us All*, MAA, 1999.

William Dunham (ed.), *The Genius of Euler*, MAA, 2007.

The Mathematical Association of America named William Dunham as the recipient of the 2008 Beckenbach Book Prize for *Euler: The Master of Us All*, MAA, 1999.

"Mathematician William Dunham has written a superb book about the life and amazing achievements of one of the greatest mathematicians of all time. Unlike earlier writings about Euler, Professor Dunham gives crystal clear accounts of how Euler ingeniously proved his most significant results, and how later experts have stood on Euler's broad shoulders. Such a book has long been overdue. It will not need to be done again for a long long time." *—Martin Gardner*

"William Dunham has done it again! In *Euler: The Master of Us All*, he has produced a masterful portrait of one of the most fertile mathematicians of all time. With Dunham's beautiful clarity and wit, we can follow with amazement Euler's strokes of genius which laid the groundwork for most of the mathematics we have today." *—Ron Graham, Chief Scientist, AT&T*

Euler the Master of Us All is available through the MAA at the following website: www.maa.org.