Homogeneous Flows, Moduli Spaces and Arithmetic, Pisa, Italy

Aerial view of Pisa. © Peter Adams.

Clay Mathematics Institute 2007 Summer School on Homogeneous Flows, Moduli Spaces and Arithmetic

The Centro di Ricerca Matematica Ennio De Giorgi in Pisa, Italy provided a wonderful setting for the 2007 Clay Mathematics Institute Summer School. The school was designed to serve as a comprehensive introduction to the theory of flows on homogeneous spaces, moduli spaces and their many applications. These flows give concrete examples of dynamical systems with highly interesting behavior and a rich and powerful theory. They are also a source of many interesting problems and conjectures. Furthermore, understanding the dynamics of such concrete system lends to numerous applications in number theory and geometry regarding equidistributions, diophantine approximations, rational billiards and automorphic forms. The program was built around three foundation courses:

(1) *Unipotent flows and applications* by Alex Eskin and Dmitry Kleinbock;

(2) *Diagonalizable actions and arithmetic applications* by Manfred Einsiedler and Elon Lindenstrauss;

(3) *Interval exchange maps and translation surfaces* by Jean-Christophe Yoccoz.

These were supplemented by various Short Courses and Advanced Mini Courses and Lectures:

- *Equidistribution and L-Functions* by Gergely Harcos;
- *Reveiw of Vatsal's work on equidistribution and non-vanishing L-functions* by Nicolas Templier;

Organizers: David Ellwood (CMI), Manfred Einsiedler (Ohio State), Alex Eskin (Chicago), Dmitry Kleinbock (Brandeis), Elon Lindenstrauss, chair (Princeton), Gregory Margulis (Yale), Stefano Marmi (La Scuola Normale Superiore di Pisa), Peter Sarnak (Princeton), Jean-Christophe Yoccoz (Collège de France), Don Zagier (MIT)

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- *Homogeneous flows, buildings and tilings* by Shahar Mozes;
- *Fuchsian groups, geodesic flows on surfaces of constant negative curvature and symbolic coding of geodesics* by Svetlana Katok
- *Chaoticity of the Teichmuller flow* by Artur Avila;
- *Eigenfunctions of the laplacian: a semi-classical study* by Nalini Anantharaman;
- *Equidistribution on homogeneous spaces and the analytic theory of L-functions* by Akshay Venkatesh;
- *Counting and equidistribution on homogeneous spaces, via mixing and unipotent flows* by Hee Oh;
- *Informal introduction to unipotent flows* by Gregory Margulis;
- *Modular shadows* by Yuri Manin;
- *On the regularity of solutions of the cohomological equation for IET's and translation flows Modular Shadows* by Giovanni Forni;
- *The Distribution of free path lengths in the Periodic Lorentz Gas* by J. Marklof;
- *Uniform spectral gap bounds and arithmetic applications* by A. Gamburd;
- *Multi-valued Hamiltonians and Birkhoff sums over rotations and IET* by C. Ulcigrai;
- *Random hyperbolic surfaces and measured laminations* by M. Mirzakhani.

One way to orientate oneself within the formidable mathematical landscape explored in the school is to consider the familiar space $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$: to a hyperbolic geometer, it is the unit tangent bundle of a hyperbolic surface; to a number theorist, it is the space of elliptic curves; to a low-dimensional topologist it is the moduli space of flat metrics (with an associated vector field) on genus 1 surfaces; to those who study Diophantine approximation, it is the space of unimodular lattices in $\mathbb{R}^2$; and to Lie theorists it is a motivating example of a finite volume homogeneous space G/Γ, that is, of a lattice Γ inside a Lie group G.

Two principal generalizations provided the setting for much of the material presented at the school:

Dynamics on the space of lattices: Let $X_n = SL(n,\mathbb{R})/SL(n,\mathbb{Z})$. This is the space of unimodular lattices in $\mathbb{R}^n$, as well as a homogeneous space G/Γ. On our motivating example X_2 there are two important dynamical systems arising from the left action of one-parameter subgroups: the *geodesic flow*, given by action of the diagonal subgroup

$$A = \left\{ g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right\}_{t\in\mathbb{R}};$$

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and the *horocycle flow*, given by action of the unipotent subgroup

$$U = \left\{ h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right\}_{t \in \mathbb{R}}.$$

Two sets of lectures focused primarily on generalizations of each of these actions: Alex Eskin and Dmitry Kleinbock (with a two lecture prequel by Grigorii Margulis) delivered a lecture series on the action of unipotent subgroups H on homogeneous spaces G/Γ, exploring the results of Dani, Margulis, and others on non-divergence of orbits; Margulis's use of unipotent dynamics to prove the Oppenheim conjecture on the values of quadratic forms; and Ratner's classification of orbit closures and invariant measures. Here, even in the case $n = 2$, the situation is quite rigid: for example, all orbits of the group U are either periodic or dense.

In contrast, the action of A for $n = 2$ is remarkably chaotic: given any $1 \leq \alpha \leq 3$, one can produce an orbit whose closure has Hausdorff dimension α. However, for $n \geq 3$, there are conjectures of Margulis on the rigidity of the action of the diagonal subgroup. Two of the main contributors to this field, Manfred Einsiedler and Elon Lindenstrauss, gave lectures on the progress made toward these conjectures, focusing on the theory of entropy; the applications to the Littlewood conjecture on simultaneous diophantine approximation; and the theory of quantum unique ergodicity.

Building on these lectures, Nalini Anantharaman, Gergely Harcos, Hee Oh, and Akshay Venkatesh gave shorter series of more advanced lectures, indicating applications to quantum chaos, automorphic forms, and counting points on varieties. In addition, a special session on Diophantine approximation was organized by Dmitry Kleinbock, allowing many of the younger participants to present their recent results in the subject.

Dynamics on the moduli space of flat surfaces: Instead of considering the space X_n of lattices in higher dimensions, we can consider the moduli spaces $\mathcal{H}$ of flat metrics (with associated vector fields) on surfaces of higher *genus* $g \geq 2$. There is a natural $SL(2, \mathbb{R})$ action on this space, coming from the linear action on $\mathbb{R}^2$. The action of the subgroup A is called *Teichmüller geodesic flow*, and the orbit of a point $x \in \mathcal{H}$ under this flow yields information about the ergodic properties of the associated vector field.

Elon Lindenstrauss conducting a session in Pisa.

A seemingly unrelated family of dynamical systems are *interval exchange maps*: given a partition of the unit interval into n labeled subintervals, rearrange them according to a permutation $\pi \in S_n$. However, if we take a first return map for the flow associated to a vector field $x \in \mathcal{H}$ to a transverse interval, we obtain exactly one of these exchange maps.

In the last foundational course, Jean-Chrstophe Yoccoz explored this connection from the perspective of combinatorics and dynamics of interval exchanges. In particular, he showed how to use the ergodicity of Teichmüller flow and associated renormalization procedures on the space of interval exchanges to show the resolution (by Masur & Veech) of the Keane conjecture that almost every interval exchange map is uniquely ergodic.

Following Yoccoz's lectures, Artur Avila, Giovanni Forni, and Maryam Mirzakhani gave further talks on Teichmüller dynamics, exploring applications to the ergodic theory of polygonal billiards, and studying closely related spaces of foliations and laminations on surfaces.

All the lecturers made a special effort to ensure their presentations would be accessible to *all* the participants in the summer school, from beginning graduate students on upwards. Ninety-seven young mathematicians participated, from Europe, Asia, the Middle East, and the Americas. In addition to bringing many of the participants in these fields together to exchange ideas, the summer school has hopefully helped spark the interest of a new generation of mathematicians in these beautiful areas.