Julia Robinson and Hilbert's Tenth Problem: Conference and Film

From top to bottom: Julia Robinson, courtesy of Constance Reid; Yuri Matiyasevich, photo by George Csicsery; David Hilbert, courtesy AK Peters, Ltd.

On March 15 and 16, 2007, CMI held a small conference at its Cambridge office on Hilbert's Tenth Problem. Participants included Martin Davis, Hilary Putnam, Yuri Matiyasevich, and Constance Reid, sister of Julia Robinson. The conference was coupled with a screening at the Museum of Science in Boston of a pre-release version of George Csicsery's film *Julia Robinson and Hilbert's Tenth Problem.*

The Problem

At the 1900 International Congress of Mathematicians in Paris, David Hilbert presented a list of twenty- three problems that he felt were important for the progress of mathematics. Tenth on the list was a question about Diophantine equations. These are polynomial equations like

$$x^2 + y^2 = z^2 \qquad \text{or} \qquad 3x^3 + 4y^3 + 5z^3 = 0$$

that have integer coefficients and for which we seek integer solutions. The first equation, which comes from the Pythagorean theorem, was known to the Babylonians and the Greeks. It has infinitely many solutions, of which the smallest is 3, 4, 5. The second, which defines an "elliptic curve," is the kind of object that played a crucial role in Wiles' proof of Fermat's last theorem and which is also important in modern cryptography: elliptic curves help keep your credit card data safe. In 1957, Selmer showed that the second important equation has no integer solutions.

Hilbert, in posing his Tenth Problem, asked whether it was possible "to devise a process according to which it can be determined in a finite number of operations whether the equation is solvable in rational integers." What is sought is a general method applicable to all Diophantine equations, not just to specific equations like those above, or even specific classes of equations such as

$$ax^2 + by^2 + cz^2 + dz^2 = 0. \qquad (*)$$

Today we would ask whether "the solubility of Diophantine equations is decidable." That is, we ask whether there is an algorithm or computer program which, given the equation, runs for a finite amount of time and then prints out the answer "yes, it is soluble" or "no, it is not soluble."

The Solution

The story of the solution of Hilbert's Tenth Problem is one of great intellectual challenge, adventure, and accomplishment. Hilbert himself worked on it, and probably thought that it could be solved in the affirmative. He knew, of course that the solvability of equations of the form (*) could be determined by an algorithm of his colleague Minkowski. Nevertheless, the first real progress came in the 1930s with the work of Gödel on the undecidability of arithmetic. This work, which gave a negative solution of Hilbert's First Problem, presaged the solution of the Tenth Problem. Later in that decade came the work of Alan Turing and a group of logicians: Church, Kleene, Post, and Rosser. A key discovery was the existence of sets of numbers that could be listed by a Turing machine but for which no Turing machine could answer the problem "is x an element of the set" for all x.

The discovery of listable but non-computable sets set the stage for the contributions of Martin Davis and Hilary Putnam, and later Julia Robinson. By the late 1940s Davis had made substantial progress, and he formulated a bold conjecture relating listable sets with those defined by Diophantine equations. Julia Robinson, working independently, had been studying a seemingly simple question of the great

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Polish logician Alfred Tarski: can the set of powers of two be defined via Diophantine equations? Robinson was not able to solve the problem, but in a 1952 paper, she reduced it to the question of whether there was a set of pairs of numbers (a,b) that (i) grows exponentially (but not too fast) and (ii) is definable by Diophantine equations. Dubbed "JR" by Martin Davis, this hypothesis was to play a decisive role. Indeed, in a 1961 paper, Davis, Hilary Putnam, and Robinson reduced the solution of Hilbert's Tenth Problem to the problem of proving JR.

The last, crucial step was taken by Yuri Matiyasevich shortly after New Year's Day, 1970. As a sophomore at Leningrad State University, Matiyasevich had taken up Hilbert's Tenth Problem, but after several years of frustration, set it aside, vowing never to look at it again. Nevertheless, when asked to review a new paper by Julia Robinson, he saw almost immediately a way of proving JR, and within a few days had done so.

With Matiyasevich's work, Hilbert's Tenth Problem was at last solved. Despite the difficulties of communication during the Cold War, the good news quickly traveled from the USSR to the USA. Robinson and Matiyasevich exchanged letters, and thus began a long, fruitful, and generous collaboration among Davis, Matiyasevich, Putnam, and Robinson. For more about both the history and the mathematics, see "Decidability in Number Theory," by Bjorn Poonen, *The Notices of the AMS*, November 2008.

The Film

The threads of this story form the warp and weave of the film *Julia Robinson and Hilbert's Tenth Problem*, produced and directed by George Csicsery with major support from CMI and Will Hearst III. All the main players – Davis, Matiyasevich, Putnam, and Robinson — appear, as does Julia Robinson's sister, Constance Reid, author of the well-known biography of David Hilbert.

CMI organized a screening of a preliminary version of the film at the Museum of Science in Boston on March 15, in conjunction with a two-day conference, held March 15 and 16, on Hilbert's Tenth Problem. Following the film was a panel discussion moderated by Jim Carlson; panelists were George Csicsery, Kirsten Eisentrager, Martin Davis, Yuri Matiyasevich, Hilary Putnam, and Constance Reid.

Panelists George Csicsery and Constance Reid

Julia Robinson and Hilbert's Tenth Problem has now been released and is available on DVD from AK Peters (www.akpeters.com). It was shown to an enthusiastic audience at the winter meeting of the American Mathematical Society in San Diego on January 6, 2008.

The Conference

Held at CMI's offices at One Bow Street in Cambridge, the conference brought together all the living participants in the solution of the problem: Yuri Matiyasevich, Martin Davis, and Hilary Putnam. The talks given were as follows:

Constance Reid, *Genesis of the Hilbert Problems*
George Csicsery, *Film clip on life and work of Julia Robinson*
Bjorn Poonen, *Why number theory is hard?*
Yuri Matiyasevich, *My collaboration with Julia Robinson*
Martin Davis, *My collaboration with Hilary Putnam*
Yuri Matiyasevich, *Hilbert's Tenth Problem: What was done and what is to be done*
Bjorn Poonen, *Thoughts about the analogue for rational numbers*
Alexandra Shlapentokh, *Diophantine generation, horizontal and vertical problems, and the weak vertical method*
Yuri Matiyasevich, *Computation paradigms in the light of Hilbert's Tenth Problem*
Gunther Cornelisson, *Hard number-theoretical problems and elliptic curves*
Kirsten Eisentrager, *Hilbert's Tenth Problem for algebraic function fields*