Clay Research Conference

Alex Eskin, (University of Chicago)
Dynamics of rational billiards

Eskin called his talk "a short and extremely biased survey of recent developments in the study of rational billiards and Teichmüller dynamics."

David Fisher (Indiana University)
Coarse differentiation and quasi-isometries of solvable groups

In the early 1980s Gromov initiated a program to study finitely generated groups up to quasi-isometry. This program was motivated by rigidity properties of lattices in Lie groups. A lattice T in a group G is a discrete subgroup where the quotient G/T has finite volume. Gromov's own major theorem in this direction is a rigidity result for lattices in nilpotent Lie groups.

In the 1990s, a series of dramatic results led to the completion of the Gromov program for lattices in semisimple Lie groups. The next natural class of examples to consider are lattices in solvable Lie groups, and even results for the simplest examples were elusive for a considerable time. Fisher's joint work with Eskin and Whyte in which they proved the first results on quasi-isometric classification of lattices in solvable Lie groups was discussed. The results were proven by a method of coarse differentiation, which was outlined.

Some interesting results concerning groups quasi-isometric to homogeneous graphs that follow from the same methods will also be described.

Satellite Workshop at the Clay Mathematics Insitute May 16–17

A satellite workshop held at the Clay Mathematics Institute in the days following the Conference consisted of more detailed talks on recent progress in higher dimensional algebraic geometry. On this occasion, Christopher Hacon and James McKernan spoke on the existence of flips and MMP scaling to an audience of advanced graduate students in the field.

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The Clay Research Awards

Below is a brief account of the mathematics of the work for which each of the three Clay Research Awards were given. – jc

1. Minimal Models in Algebraic Geometry

Let X be projective algebraic variety over the complex numbers, that is, the set of common zeroes of a system of homogeneous polynomial equations. The meromorphic functions on X form a field, the *function field* of X. For the Riemann sphere (the projective line $\mathbf{CP}^1$) this field is $\mathbf{C}(t)$, the field of rational functions in one variable. For an elliptic curve $y^2 = x^3 + ax + b$, it is the field obtained by adjoining the algebraic function $y = \sqrt{x^3 + ax + b}$ to $\mathbf{C}(x)$. Two varieties are *birationally equivalent* if they have isomorphic function fields.

The *birational equivalence problem* is a fundamental one in algebraic geometry. Given two varieties X and Y, how do we recognize whether they are birationally equivalent? In the case of elliptic curves, there is an easy answer: the fields are isomorphic if and only if the quantity b^2/a^3 is the same in both cases. What can we say about other varieties? On what data does the birational equivalence class of a variety depend?

Consider first the case of complex dimension one. Every algebraic curve is birational to a smooth one, its normalization. Thus two curves are birational if and only if their smooth models are isomorphic. Consequently, the birational equivalence problem is the same as the moduli problem. Take, for example, the algebraic curves defined by the affine equations $x+y = 1$, $x^2+y^2 = 1$ and $x^2+y^2+x^3+y^3 = 0$.

The first two curves are smooth and isomorphic to the Riemann sphere, as one sees by stereographic projection. The last curve has one singular point, but its smooth model is the Riemann sphere, as we see by the parametrization

$$x = -(1 + t^2)/(1 + t^3), \; y = -t(1 + t^2)/(1 + t^3).$$

Thus all three varieties are birationally equivalent, with function field $\mathbf{C}(t)$.

Varieties of higher dimension are birationally equivalent to a smooth one by Hironaka's resolution of singularities theorem. Nonetheless, this powerful result does not answer the birational equivalence problem. To see why, consider a smooth algebraic

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surface X and a point p on it. One may replace the point by the set of tangent lines through p to obtain a new surface Y. The set of tangent lines is an algebraic curve E isomophic to one-dimensional projective space $\mathbf{CP}^1$. Since $X - \{p\}$ and $Y - E$ are isomorphic dense open sets in X and Y, respectively, the latter two varieties have isomorphic function fields. In algebraic geometry we say that Y is obtained from X by blowing up p. In more topological language, we say that Y is obtained from X by surgery: cut out the point p, and glue in the projective line E. What is important here is that the surgery is an operation on algebraic varieties.

More generally, we can (and will) consider surgeries of the form "cut out a subvariety A and paste in a variety B." More formally, we have varieties X and Y such that $X - A$ is isomorphic to $Y - B$, where we say that Y is obtained from X by surgery. Since $X - A$ and $Y - B$ are dense open sets, the function fields of X and Y are isomorphic. The partially defined map $X \dashrightarrow Y$ induces the isomorphism of function fields.

The curve E obtained by blowing up p is a projective line with self-intersection number -1. Such curves are known in the trade as "(-1) curves." Any time one finds a (-1) curve on a surface Y, one can construct a smooth surface X and a map $f : Y \longrightarrow X$ that maps E to a point. This operation is called "blowing down," or "contracting E." By successively contracting all the (-1) curves in sight, one can construct from any algebraic surface S a smooth variety S_{min} devoid of such curves. Let us call S_{min} a *classical minimal model* for S. Existence of classical minimal models was proved by Castelnuovo and Enriques in 1901. They also showed that as long as S and S' are not *uniruled*, they are birational if and only if S_{min} and S'_{min} are isomorphic. In the non-uniruled case a classical minimal model X_{min} is topologically the simplest: its second Betti number is smaller than that of any smooth surface birationally equivalent to it.

A variety X is uniruled if there is a map $\mathbf{CP}^1 \times Y \longrightarrow X$ whose image contains an open dense set. Thus, there is a curve birational to a projective line passing through almost every point of X.

A ruled surface, that is, a $\mathbf{CP}^1$ bundle over a curve, is uniruled. So is $\mathbf{CP}^2$. For uniruled surfaces, the minimal model is not unique. For example, blow up two points a and b on $\mathbf{CP}^2$. The proper transform of the line joining them is a (-1) curve. Blow it down to obtain a new surface. It is isomorphic to $\mathbf{CP}^1 \times \mathbf{CP}^1$. Both $\mathbf{CP}^2$ and $\mathbf{CP}^1 \times \mathbf{CP}^1$ are classically minimal, and both represent the purely transcendental function field $\mathbf{C}(x, y)$.

What can one say in dimension greater than two? The conjecture of Mori-Reid (see [4]) states the following:

> $(*)$ *Let X be an algebraic variety of dimension n which is not uniruled. Then (a) it has a minimal model X_{min} and (b) it has a Kähler metric whose Ricci curvature is ≤ 0.*

In the Mori-Reid conjecture, minimality is defined in a different way, as a kind of algebro-geometric positivity condition. We will discuss this notion in greater detail below. For surfaces, it coincides with the classical one: there are no (-1) curves. For higher dimensional varieties, minimality as positivity signaled a major change in the way mathematicians viewed the birational equivalance problem. The new line of investigation, initiated by Shigefumi Mori, developed further by Kawamata, Kollár, Mori, Reid, and Shokurov, culminated in 1988 with Mori's proof of (a) for varieties of dimension three [8]. For this result, the goal of the "minimal model program," Mori received a Fields Medal in 1990. Although refereeing is still in process, it now appears that (a) is also a theorem for all dimensions. An algebraic approach has been given by Birkar, Cascini, Hacon, and M^cKernan [1] and an analytic approach has been give by Siu [9].

In the remainder of this article we explain the modern notion of minimality and how it relates to the classical one. We then touch on just one of the crucial parts of the proof. This is the existence of *flips* and *flops*. These are surgeries that alter a variety in codimension two. Flops leave the positivity of the canonical bundle unchanged, whereas flips make it it more positive, transforming the variety to one that is closer to minimal. One cost of introducing a flip is that certain mild singularities must be admitted. These are the so-called "terminal singularities." A consequence of working with singular varieties is that the natural intersection numbers, while well-defined, can be rational numbers.

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The technical issues thus raised can all be successfully dealt with; indeed, working in the larger category of varieties with terminal singularities is crucial to the success of the minimal model program.

Another cost associated with flips is the difficulty of proving that they exist in sufficient generality. One can construct motivating examples (see below), but even these are somewhat complicated. Important special cases were proved by Tsunoda, Shokurov, Mori, and Kawamata. Finally, Mori proved the general existence theorem for flips in dimension three [8], [7, p. 268]. The Clay Research Award was given to Hacon and McKernan for their proof of the existence of flips in dimension n assuming termination of flips in dimension $n-1$. See [3]. This result suggests that one can prove the existence of minimal models inductively, a program carried out in [1].

Let us now discuss minimality and flips in a more substantitve way. To say that X is minimal is to say that its canonical bundle is *numerically effective*, or *nef*. The canonical bundle is the line bundle K whose local sections are holomorphic n-forms A line bundle L is nef if the integral

$$L \cdot C = \int_C \omega$$

is positive for all algebraic curves C on X, where ω is a differential form representing the first Chern class of L. This integral represents the intersection number, which, as noted, may be a rational number.

What is the relation of the new definition of minimality to that of the Castelnuovo-Enriques theory? By the adjunction formula, the value of the above integral, i.e., the intersection number $K \cdot C$, is nonnegative. Thus a variety of dimension two whose canonical class is nef has no (-1) curves. Consequently, surfaces that are minimal are classically minimal.

Following [7, Example 12.1], we give an example, first observed by Atiyah, of a flop. We then modify the example to give a flip. Consider the affine variety C_0 given by the quadratic equation $xy - uv = 0$. Let C be its closure in $\mathbf{CP}^4$; it is the cone over a quadric $\mathbf{CP}^1 \times \mathbf{CP}^1$. Blow up the vertex of the cone to obtain a variety C_{12}. The resulting exceptional set $F = F_1 \times F_2$ is isomorphic to $\mathbf{CP}^1 \times \mathbf{CP}^1$. It is possible to blow down all the fibers $\{ x \} \times F_2$ of F to obtain a variety C_1 with a subvariety $E_1 \cong F_1 \cong \mathbf{CP}^1$. Likewise, we can blow down the fibers $F_1 \times \{ y \}$ to obtain a variety C_2 with a subvariety $E_2 \cong F_2 \cong \mathbf{CP}^1$. Thus $C_1 - E_1 \cong C_{12} - F \cong C_2 - E_2$. The birational map $C_2 \dashrightarrow C_1$ is the flop obtained by the surgery "cut out E_2 and glue in E_1." If $q_i : C_{12} \to C_i$ is the canonical projection, then the birational map $C_2 \dashrightarrow C_1$ is just the composition $q_1 q_2^{-1}$. In this case, the piece cut out and the piece glued in are both projective lines. Flops do not affect the intersection number with the canonical divisor.

To understand this flop better, consider the family of planes P_λ on C given by $x = \lambda u$, $v = \lambda y$. There is a corresponding family of projective planes in C and in C_{12}, and a corresponding family of hypersurfaces $\tilde{P}_\lambda$ in C_{12}. Each $\tilde{P}_\lambda$ is a projective plane with a point blown up. The blowup of the point is one of the fibers $\{ x \} \times F_2$.

Consider now the surfaces $q_1(\tilde{P}_\lambda) \subset C_1$. Obtained by blowing down $\{ x \} \times F_2$, these surfaces are disjoint for distinct λ, isomorphic to $\mathbf{CP}^2$, and each one meets E_1 in a single point. Thus $q_1(\tilde{P}_\lambda) \cdot E_1 = 1$. A more subtle computation yields $q_2(\tilde{P}_\lambda) \cdot E_2 = -1$. Thus the flop changes the intersection number of $q_i(P_\lambda)$ with the surgery loci E_i. In the example, the canonical bundle of C is defined and trivial at the vertex of the cone and so also trivial on the exceptional set F. Therefore $K \cdot E_i = 0$: the flop does not affect the positivity of the canonical bundle.

For the second example, we follow [7, Example 12.5]. Factor C_0 by the $\mathbf{Z}_2$ action given by the map $(x, y, u, v) \longrightarrow (x, -y, u, -v)$. The $\mathbf{Z}_2$ actions make sense on C_1, C_2, and C_{12}. However, there is a symmetry that is broken. In addition to the family of planes P_λ there is a family of planes Q_λ defined by $x = \lambda v$, $u = \lambda y$. They are interchanged by the map permuting u and v. This global symmetry is the reason why $C_1 \cong C_2$. However, the permutation of coordinates does not commute with the $\mathbf{Z}_2$ action. As a result, one finds that $C_1/\mathbf{Z}_2$ and $C_2/\mathbf{Z}_2$ are not isomorphic. Indeed, the intersection of the canonical class with $E_2/\mathbf{Z}_2$ is negative whereas its intersection with E_1/Z_2 is positive. The natural birational map $C_2/\mathbf{Z}_2 \dashrightarrow C_1/\mathbf{Z}_2$ can be described as "cut out $E_2/\mathbf{Z}_2$ and replace it by $E_1/\mathbf{Z}_1$." The quotients $E_i/\mathbf{Z}_2$ are projective lines. Thus, as in the first example, the surgery is obtained by removing a projective line and glueing it back in a different way. But in this case, the geometry of the variety subjected to surgery changes and positivity of the canonical bundle increases.

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For part (b) of the Mori-Reid conjecture, the differential geometric form of positivity for minimal models, consider first the case of algebraic curves. Via the uniformization theorem, every Riemann X has as universal cover which is a space of constant curvature $+1$, 0, or -1: the sphere in the case of genus zero, the plane in the case of genus one, and the upper half plane in the case of higher genus. For genus $g \geq 0$, the non-uniruled case, the curvature 0 or -1 condition is a strong differential-geometric form of nefness. In higher dimension, suppose that X is a smooth minimal variety of general type. By the base-point-free theorem, its canonical bundle is semi-ample, and so the first Chern class of the canonical bundle is represented by a semi-positive $(1, 1)$ form. It follows from the solution of the Calabi Conjecture, independently given by Aubin and Yau in their work on the Monge-Ampère equation, that X has a metric whose Ricci curvature is negative (≤ 0). Alternatively, the Ricci form of the metric is positive-semidefinite. As noted, one generally has to admit terminal singularities in the minimal model. The best results for (b) in the general case are due to Eyssidieux, Guedj, and Zeriahi, generalizing the work of Aubin and Yau on Monge-Ampre equations.

Acknowledgments. The author relied heavily on [2], [4], *and* [7] *for this article, and would like to thank Herb Clemens and János Kollár for their help in its preparation.*

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2. The Sato-Tate Conjecture

Since the time of the Greeks, the study of Diophantine equations has driven some of the most important developments of mathematics. Of particular significance are the cubic equations in two variables, which we may put in the form $(*)$ $y^2 = x^3+ax+b$. The set of complex solutions of $(*)$ plus the point at infinity is a torus; in general the solution set, including the point at infinity, is called an "elliptic curve," written E. The name is something of a historical accident having do with the problem of computing the lengths of arcs on ellipses. Its solution leads to integrals of the form $\int dx/y$, where $y = \sqrt{x^3 + ax + b}$. Of course the differential form dx/y plays a leading role in the modern theory of elliptic curves, e.g., by determining its Hodge structure.

It was Gauss, in his *Disquisitiones Arithmetica*, who formally introduced the idea of studying Diophantine equations by examining their reduction modulo a prime p. The notion certainly predates Gauss, however, and goes back at least as far as Fermat. The central problem is to count the number of points N on E modulo p. By N we mean the number of solutions of the equation $(*)$ modulo p, plus one for the point at infinity. A classical theorem of Hasse tells us how many solutions to expect:

$$|N - (p + 1)| \leq 2\sqrt{p}.$$

Consider, for example, the elliptic curve defined by $y^2 = x^3-x-1$. Modulo 3 there is just one solution, namely the point at infinity. Modulo 5 there are eight. Further experiment reveals the behavior of N as a function of p to be quite random, suggesting a statistical interpretation: the expected value of N is $p + 1$ and the standard deviation is proportional to $\sqrt{p}$. To make more precise statement, consider the quantity

$$\delta = \frac{N - (p + 1)}{2\sqrt{p}}.$$

According to Hasse's theorem, this normalized measure of the deviation of the number of solutions modulo p from its expected value is a number in the range $[-1, 1]$. A deeper question, then, is the nature of the probability law governing the distribution of the numbers δ.

Around 1960, Mikio Sato and John Tate independently conjectured that the probability law for elliptic curves without complex multiplication ("extra symmetry") is given by the function $f(\delta) = (2/\pi)\sqrt{1-\delta^2}$. Sato was led to the conjecture by experimental evidence. Although the documentary record is sparse, there is still extant a letter from John Tate to Jean-Pierre Serre dated August 5, 1963, about Tate's thoughts on the conjecture. In this letter Tate adds, "Mumford tells me that Sato has found $f(\theta) = (2/\pi)sin^2(\theta)$ experimentally on one curve with thousands of p." The angle θ is $\cos^{-1}\delta$; the formulations in terms of δ and θ are equivalent. See [11] for computations now accessible to anyone.

Tate was led to the conjecture on theoretical grounds having to do with the connection between alebraic cycles and the zeroes and poles of L-functions. Starting from the Hasse-Weil function $L(s, E)$ of the elliptic curve, J.-P. Serre defined [6] a natural sequence of functions $L(s, E, sym^n)$ associated to the irreducible representations of $SU(2)$. When $n = 1$, the $L(s, E, sym^n) = L(s, E)$. These functions were variants of those considered by Tate [7]. Serre showed, as Tate had predicted, that the Sato-Tate conjecture would follow from the assertion that $L(s, E, sym^n)$ has an extension to an analytic function in the half-plane $\Re(s) \geq n + 1/2$ and is non-vanishing there. The work of Wiles [10] and of Taylor and Wiles [9] on Fermat's last theorem, and finally the work of Breuil, Conrad, Diamond, and Taylor, [1] established the crucial fact that the function $L(s, E)$ extends to an analytic function in the half-plane $\Re(s) \geq 3/2$ and is non-vanishing there.

One important ingredient in the proof is an extension of Wiles' technique for identifying L-functions of elliptic curves with L-functions of modular forms. This was begun in the paper [2] by Clozel, Harris, and Taylor and completed in later paper by Taylor [8]. Another is an extension of an idea of Taylor for proving meromorphic continuation of L-functions associated to two-dimensional Galois representations by applying to moduli spaces an approximation theorem of Moret-Baily. Execution of this plan relies on the existence of a suitable moduli space. Harris, Shepherd-Barron, and Taylor [4] found one suitable for studying n-dimensional representations for any even n. It is a twisted form of the moduli space for Calabi-Yau manifolds originally studied by Dwork in certain cases and now an important part of string theory.

The kind of probability distribution proved for elliptic curves is conjecturally far more general. See Barry Mazur's article in Nature [5].

Acknowledgments. The author relied heavily on the account in [3] *in preparing this article.*

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3. Virtual Quasi-Isometric Rigidity of Sol

In his 1983 ICM address, Mikhael Gromov proposed a program for studying finitely generated groups as geometric objects [6]. The story begins with the *Cayley graph*, a metric space $C_S(\Gamma)$ associated to a group Γ and a set of generators S. The set of vertices is Γ itself. Two vertices are connected by an edge if right multiplication by some generator maps one to the other. There is a natural action of the group on this graph given by left

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translation. The Cayley graph has a natural metric where each edge is isometric to a unit interval, and the distance between points is given by the length of a shortest path joining them.

The Cayley graph depends on the choice of generating set and so its geometry is not intrinsic to the group Γ. However, it turns out that there is a natural equivalence relation, quasi-isometry, relating graphs defined by different sets of generators. We say that two metric spaces X and Y are *quasi-isometric* if there is a map $f : X \to Y$ that does not distort distances too much. To make a precise statement, let d_X and d_Y denote the metrics. Suppose that there are constants $K \geq 1$ and $C \geq 0$ such that for every $x_1, x_2 \in X$

$$\frac{1}{K} d_X(x_1, x_2) - C \leq d_Y(f(x_1, f(x_2)$$

and

$$d_Y(f(x_1, f(x_2) \leq K d_X(x_1, x_2) + C$$

and such that the C-neighborhood of $f(X)$ is all of Y. Such a map is a (K, C) quasi-isometry. Two spaces are said to be quasi-isometric if there is a (K, C) quasi-isometry between them for some K and C.

The central question that Gromov raised was the classification of groups up to quasi-isometry, i.e., the enumeration and characterization of the quasi-isometry classes of finitely generated groups.

A modest beginning is to note that all finite groups are quasi-isometric, that is, quasi-isometric to a point. Thus Gromov's theory is a theory of infinite groups. For nontrivial examples, consider a group Γ that acts properly discontinuously by isometries on a nice space X, such as a connected Riemannian manifold. Suppose further that the quotient X/Γ is compact. (We say that Γ is "co-compact.") Then Γ and X are quasi-isometric. For example, the lattice $\mathbf{Z}^d$ in $\mathbf{R}^d$ is quasi-isometric to $\mathbf{R}^d$ — which is not quasi-isometric to a point. More generally, consider a lattice Γ in a Lie group G: a discrete subgroup such that the quotient G/Γ has finite volume relative to a left invariant Haar measure on G. If Γ is co-compact, then it is quasi-isometric to G, when G is equipped with any left invariant distance function. Moreover, if K is a compact subgroup of G and $X = G/K$ is the quotient, then Γ is quasi-isometric to X. Any cocompact lattice in $SO(1, n)$ or $PSO(1, n)$, for example, is quasi-isometric to real hyperbolic n-space. Likewise, any two such lattices are quasi-isometric to each other. In particular the fundamental groups Γ and Γ' of compact Riemann surfaces S and S' are quasi-isometric, so long as both surfaces have genus at least two.

Since any finitely generated group isomorphic to a cocompact lattice in a Lie group G is quasi-isometric to G, it is natural to ask whether the converse is true: whether any group quasi-isometric to G is a cocompact lattice. However, this fails for trivial reasons because passing to finite index subgroups or finite extensions does not change the quasi-isometry class of a group. If we say that two groups are *weakly commensurable* if they are the same, modulo applying a finite sequence of these two operations, then the most one can hope to show is that a finitely generated group quasi-isometric to G is weakly commensurable to a cocompact lattice. Proving this statement is the *quasi-isometric rigidity problem* for G. For G a semisimple Lie group, quasi-isometric rigidity holds: a deep theorem that is the result of the work of many people, including Sullivan, Tukia, Gromov, Pansu [9], Casson-Jungreis, Gabai, Schwartz [10], Kleiner-Leeb [7], Eskin [2], and Eskin-Farb [4].

A landmark in the development of Gromov's program was his polynomial growth theorem [5]. To state it, let $N(r)$ be the number of group elements within distance r of the identity element relative to the word metric. For the group $\mathbf{Z}^d$, the function $N(r)$ is bounded by a constant times r^d. A group for which $N(r) \leq Cr^d$ is said to have polynomial growth. The least integer d for which the preceding estimate holds is independent of the generating set, so this notion depends on the group alone. It is not hard to prove that a nilpotent group, as an iterated extension of abelian groups, has polynomial growth. For example, the group of matrices

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

with integer coefficients, has polynomial growth of degree 4, one greater than the the dimension of the corresponding nilpotent Lie group (the Heisenberg group) that contains it. Gromov's theorem gave a converse: that any group of polynomial growth is virtually nilpotent, that is, is weakly commensurable with a cocompact lattice in a nilpotent Lie group. This theorem can then be applied to prove quasi-isometric rigidity for many nilpotent groups. The classification of general nilpotent groups up to quasi-isometry is still wide a open problem.

The tractability of semisimple Lie groups comes from their curvature properties: they act on non-positively curved spaces, and for such manifolds powerful tools have been forged. Nilpotent groups are tractable for a different reason. Their asymptotic structure is well approximated by simple scale invariant models which are also nilpotent Lie groups. On the other hand, solvable Lie groups can fail to have either of these simplifying characteristics. The easiest such example is the group *Sol*, given by 3×3 matrices

$$\begin{pmatrix} e^{z/2} & x & 0 \\ 0 & 1 & 0 \\ 0 & y & e^{-z/2} \end{pmatrix}.$$

This group, with the invariant metric

$$ds^2 = e^{-z}dx^2 + e^z dy^2 + dz^2$$

gives one of the seven geometries in Thurston's geometrization program. It also represented a key obstacle in Gromov's program. Indeed, the question of whether a group quasi-isometric to a latice in *Sol* is virtually a lattice in *Sol* became known as the Farb-Mosher conjecture.

The Farb-Mosher conjecture was proved in the affirmative by recent work of Alex Eskin, David Fisher, and Kevin Whyte [3]. One of the main tools was the notion of *coarse differentiation*, which has found application to other areas, e.g., the geometry of Banach spaces [1] and combinatorics [8], was introduced by Eskin to the problem around 2005. Coarse differentiation may be viewed as a coarse variant of the theorem of Rademacher, which states that a Lipschitz function $R^n \longrightarrow R$ is differentiable almost everywhere; instead of stating that at almost every point a Lipschitz function has linear behavior on small scales, coarse differentation says that in a quantitative sense, Lipschitz functions $Sol \rightarrow R$ have linear behavior at large scales, at many points. Proving such a result requires one to develop a quantitative version of the Rademacher theorem.

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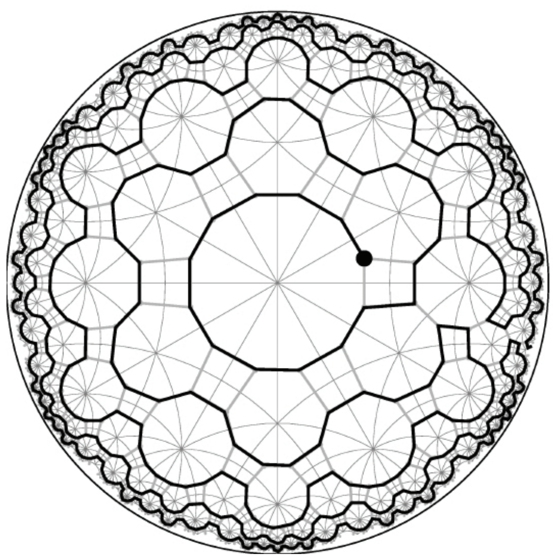

The group [6,4] is the triangle group generated by reflections in a right triangle with angles π/6 and π/4. The figure above shows a Hamiltonian path in the Cayley graph of this group relative to the standard set of generators. The heavy line segments, both solid and dashed, represent the Cayley graph; the light lines show the triangle tessellation. The Hamiltonian path consists of the solid line segments. Figure and text credit: Douglas Dunham, "Creating Repeating Hyperbolic Patterns—Old and New," Notices of the AMS, Volume 50, Number 4, April 2003, p. 453.