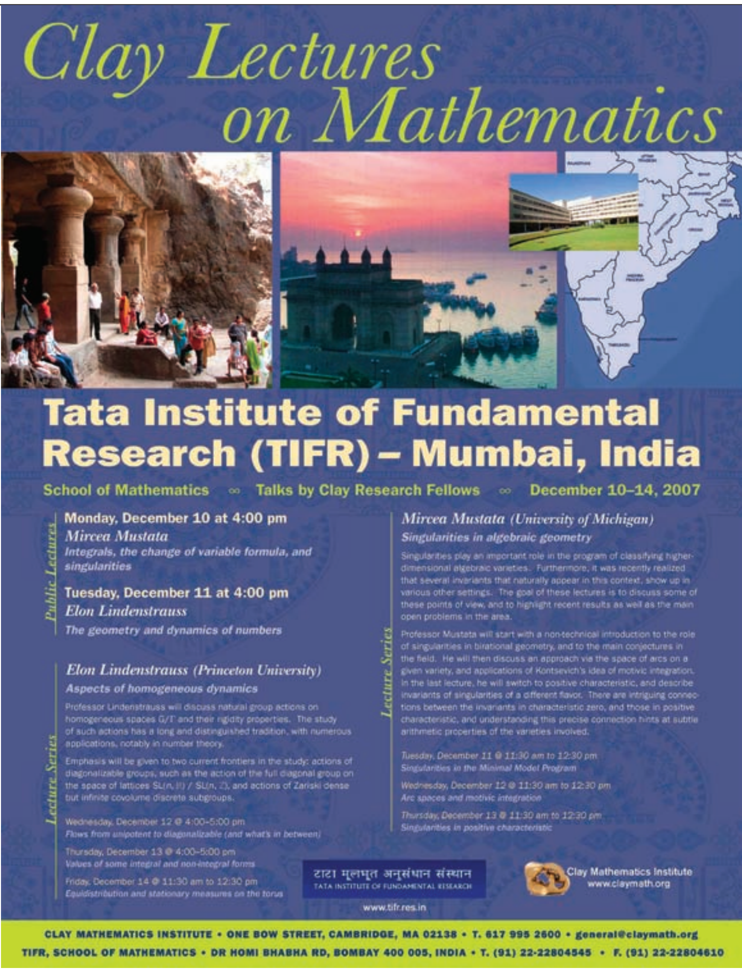
Clay Lectures on Mathematics at the Tata Institute of Fundamental Research



Clay Lectures on Mathematics, Mumbai

The 2007 Clay Lectures on Mathematics were hosted by the School of Mathematics, Tata Institute of Fundamental Research (TIFR) in India from December 10 through 14, 2007.

The lecturers were Mircea Mustaţă of the University of Michigan and Elon Lindenstrauss of Princeton University.

Mustaţă delivered a series of three lectures:

- Singularities in the Minimal Model Program
- Arc Spaces and Motivic Integration
- Singularities in Positive Characteristic

and one public lecture

- Integrals, the Change of Variable Formula, and Singularities.

Lindenstrauss delivered a parallel series of three lectures:

- Flows from Unipotent to Diagonalizable (and what is in between)
- Values of some Integral and Non-integral Forms
- Equidistribution and Stationary Measures on the Torus

and one public lecture

- The Geometry and Dynamics of Numbers.

The following is a summary of Mustaţă's and Lindenstrauss's lectures.

1. Singularities in Algebraic Geometry: Mircea Mustaţă

The role of singularities in the program of classifying higher-dimensional algebraic varieties is well-known. The public lecture gave an introduction to some results relating invariants of singularities (like the log canonical threshold) with various integration theories. The other lectures covered various aspects of invariants of singularities, discussing some of the recent results, as well as the main open problems and conjectures in the field. The content of the talks was roughly the following:

Integrals connected to singularities (complex powers, p-adic zeta functions): Let K be a field with an absolute value $| \, . \, |$ and a measure. Consider the corresponding product measure on K^n. Roughly speaking, the goal is to relate the singularities of a polynomial $f \in K[x_1, \cdots, x_n]$ with the asymptotic behavior of $\mu(\{x \in K^n | \, |f(x)| < \epsilon\})$, when ϵ goes to zero. One way of encoding this asymptotic behavior is by studying certain integrals. The key is to use a log resolution of singularities for f and some version of the Change of Variable Formula. The cases $K = \mathbb{C}$, $K = \mathbb{Q}_p$ (or more general p-adic fields) and $K = \mathbb{C}((t))$ are the main examples. The first case that was discussed was the Archimedean case (that is, when $K = \mathbb{R}$ or $\mathbb{C}$), which goes back to a question of Gelfand, subsequently proved by Atiyah, Bernstein and Gelfand. Mustaţă then gave an overview of the p-adic side of the theory. There was a discussion of p-adic integration, the Igusa zeta function and the main result

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of Igusa about the rationality of the zeta function. In this context, he emphasized the connection between the largest pole of the zeta function and the log canonical threshold.

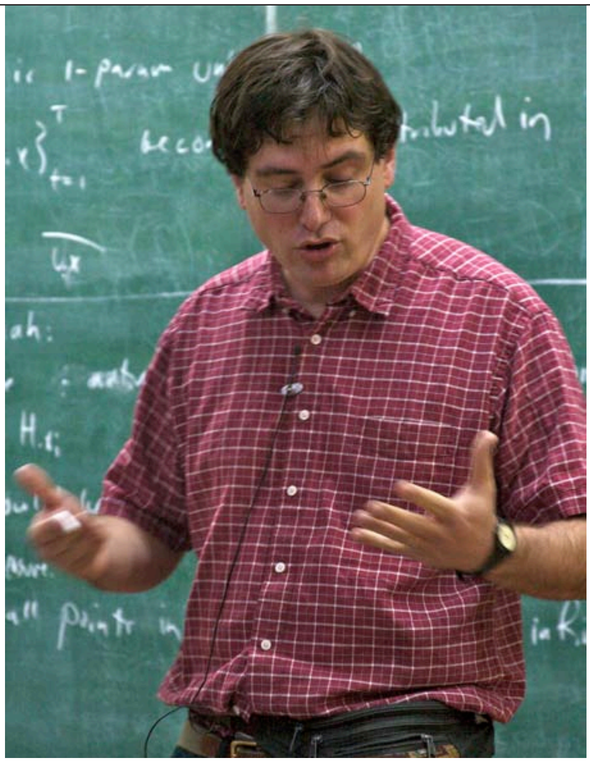
Invariants of singularities (the log canonical threshold) in birational geometry: In this lecture Mustaţă discussed the role that singularities play in higher dimensional birational geometry. The first part covered the role of vanishing theorems and the canonical divisor. This motivated the definition of various classes of singularities. After a brief overview of the basic setup in the Minimal Model Program, and of the recent progress in this field, Mustaţă explained a conjecture of Shokurov and its relevance in this setting.

Spaces of arcs and singularities: The talk covered some basic facts about spaces of arcs, with an emphasis on geometric aspects and applications to singularities. One result that was discussed was a theorem of Kontsevich relating the spaces of arcs of two smooth varieties X and Y, with a proper birational morphism between them. This result is the key ingredient in the theory of motivic integration. In the rest of the talk, Mustaţă explained how this result can be used to relate the log canonical threshold to the codimension of certain subsets in spaces of arcs. This result can be considered an analogue in the context of spaces of arcs of Igusa's results in the p-adic setting.

Invariants of singularities in positive characteristic: The invariants of singularities that appear in birational geometry (in characteristic zero) are defined via valuations, or equivalently, via resolution of singularities. In positive characteristic, one can define invariants in a very algebraic way, exploiting the Frobenius morphism. In this talk. Mustaţă discussed analogies, results, and conjectures relating the invariants such as the log canonical threshold with algebraic invariants in positive characteristic.

2. Aspects of Homogeneous Dynamics

The Geometry and Dynamics of Numbers: Lindenstrauss began his public lecture by explaining Minkowski's idea that the study of lattice points in $\mathbb{R}^n$ implies many deep results about number fields. Given the determinant, some lattices behave more "unboundedly" than others in the sense that they contain very short vectors. The group $SL(n,\mathbb{R})$ acts on the space of lattices in $\mathbb{R}^n$ while preserving the determinant. The key observation here is that orbits of actions of large enough diagonal subgroups of $SL(n,\mathbb{R})$ have to either contain an "unbounded" lattice or be periodic. Two theorems from number theory, Diophantine approximation and the existence of solutions to Pell's equation, can be demonstrated as elegant examples of these two situations. After mentioning the three dimensional setting (Dirichlet Theorem), Lindenstrauss discussed Cassels and Swinnerton-Dyer conjecture, which was reformulated by Margulis in the following way: in the moduli space of n-dimensional lattices, $SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})$, any orbit of the full diagonal subgroup of $SL(n,\mathbb{R})$ is either unbounded or periodic. It was orbserved by Cassels and Swinnerton-Dyer(1955) and Margulis (1997) that this conjecture would imply the famous Littlewood conjecture: $\forall(\alpha,\beta)\in\mathbb{R}^2$, $\inf_{n\in\mathbb{N}} n\,\|n\alpha\|\ \|n\beta\| = 0$, where $\|x\| = \min_{z\in\mathbb{Z}}|x-z|$. By studying dynamics on $SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})$, Einsiedler, Katok, and Lindenstrauss recently proved that the exceptional set of Littlewood's conjecture is of Hausdorff dimesion 0. Lindenstrauss also talked about applications of geometry of numbers to the study of number fields.



Elon Lindenstrauss delivering one of his lectures at TIFR.

Rigidity of unipotent and diagonal flows: The general setting is always that of a homogeneous space $\Gamma\backslash G$ where G is a linear algebraic group and Γ one of its discrete subgroups. From Ratner's work, it is well known that unipotent actions are rigid: if H is a subgroup of G generated by unipotent one-parameter subgroups, then any H-invariant probability measure on $\Gamma\backslash G$ is homogeneous and any orbit of a one-parameter unipotent subgroup is equidistributed in the closure of the H orbit. A result of Mozes and Shah claims that for a sequence $\{\mu_i\}$ of natural measures on periodic H-orbits, there would be a subsequence converging to a homogeneous measure unless all points in their orbits escape to infinity. There remain several challenges in the domain of unipotent dynamics: spaces $\Gamma\backslash G$ of infinite volume, polynomial trajectories and effective estimates. Lindenstruass discussed these challenges, and some of the recent progress that has been made.

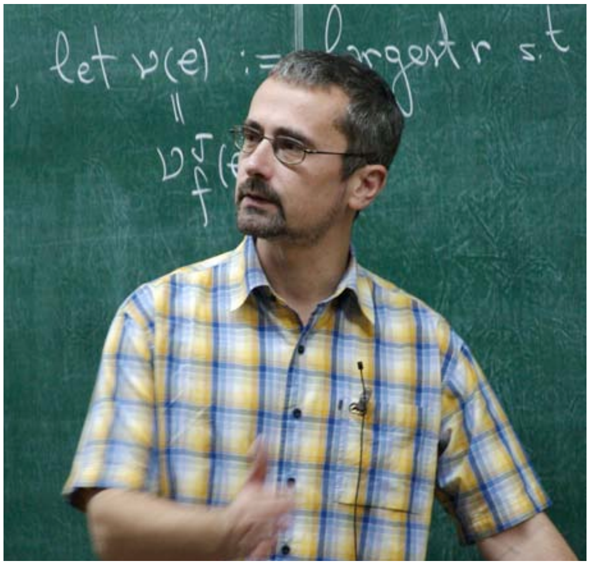
The second part of the lecture focused on the rigidity of diagonal flows. Although the linearization technique does *not* work in this case, there is a weaker analogue called isolation, that essentially says that any point close enough to (but not inside) a periodic orbit has an unbounded orbit itself. However, isolation techniques are proven only for $G = SL(n, \mathbb{R})$. Let A be the full diagonal subgroup; when $G = SL(2, \mathbb{R})$ the orbits are known to behave in a nasty way. But for $G = SL(n, \mathbb{R}), n \geq 3$, it is conjectured that any ergodic A-invariant measure is homogeneous. Lindenstruass discussed high entropy techniques, as well as the recent work of Maucourant and its relation to this conjecture.

Values of integral and non-integral forms: The main focus of this lecture was on homogeneous polynomials F that can be written in the form $F = \prod_{i=1}^{n}(\sum_{i=1}^{n} h_{ij}x_j)$ where $H = (h_{ij})$ is a non-degenerate $n \times n$ real matrix. Lindenstrauss discussed the example of such forms arising as the norm of an element in a number field of degree n given an integral basis. The objective is to study the distribution of $F(\mathbb{Z}^n)$, which is preserved under the action of $GL(n, \mathbb{Z})$ given by $\gamma.F(x) = F(x\gamma)$. One observes that the dynamics of $GL(n, \mathbb{Z})$ on the space Y_n of equivalence classes of F up to scaling is closely related to that of A on

$$PGL(n, \mathbb{Z})\backslash PGL(n, \mathbb{R}) = SL(n, \mathbb{Z})\backslash SL(n, \mathbb{R}).$$

Conditions on the distribution of $F(\mathbb{Z}^n)$ can be translated into conditions on the A-orbit of $[H] \in PGL(n, \mathbb{Z})\backslash PGL(n, \mathbb{R})$. By such translation, the reformulation of Margulis mentioned in his public lecture is equivalent to the original Cassels-Swinnerton-Dyer conjecture. Lindenstrauss explained his work with Einsiedler and Katok on the Hausdorff dimension of the exceptional set as well as more recent work with Einsiedler, Michel, and Venkatesh that gives an upper bound $\#\{orbit\ GL(n, \mathbb{Z}).F \subset Y_n\ of\ determinant\ D:\ F(\mathbb{Z}^n\backslash\{0\}) \cap [-\delta, \delta] = \emptyset\} <<_{\epsilon,\delta} D^{\epsilon}, \forall \epsilon, \delta, D$.

The Torus: Equidistribution and Stationary Measures: Suppose S is the multiplicative semigroup generated by two multiplicatively independent postive integers a and b. Let S act naturally on $T = \mathbb{R}/\mathbb{Z}$. The Furstenberg theorem asserts that any closed S-invariant subset of T is either full or finite. Furstenburg also conjectured that any ergodic S-invariant probability measure μ is either Lebesgue or finitely supported. Though the conjecture is still open, Rudolph and Johnson showed if $h(\mu, a) > 0$ then μ has to be Lebesgue measure. Lindenstrauss discussed how an effective Rudolph-Johnson theorem can be used to prove an effective version of Furstenberg's theorem. He then discussed how higher-rank analogues of these problems on $(\mathbb{R}/\mathbb{Z})^d, d > 1$ diverge into two cases. The first type of problems deal with two commuting matrices $A, B \in SL(d, \mathbb{Z})$ acting on $(\mathbb{R}/\mathbb{Z})^d$, there are results by Berend, Kalinin-Katok-Spatzier and Einsiedler-Lindenstrauss. The other extreme is a pair of non-commuting matrices A and B generating a Zariski-dense subgroup S in $SL(d, \mathbb{Z})$.



Mustata delivering one of his lectures at TIFR.