Clay Mathematics Institute

Contents

2006

James Carlson.

Dear Friends of Mathematics,

For the past five years, the annual meeting of the Clay Mathematics Institute has been a one-afternoon event, held each November in Cambridge, Massachusetts, devoted to presentation of the Clay Research Awards and to talks on the work of the recipients. The award recognizes major breakthroughs in mathematical research. Awardees receive flexible research support for one year and the bronze sculpture "Figureight Knot Complement vii/CMI" by Helaman Ferguson. Past awardees, in reverse chronological order, are Manjul Bhargava, Nils Dencker, Ben Green, Gérard Laumon and Bao-Châu Ngô, Richard Hamilton, Terence Tao, Oded Schramm, Manindra Agrawal, Edward Witten, Stanislav Smirnov, Alain Connes, Laurent Lafforgue, and Andrew Wiles.

Beginning in 2007, the annual meeting will be held in May, alternating between Harvard and MIT as in the past, with an expanded two-day program of talks on recent research developments in addition to presentation of the awards. The aim is to offer a series of high-quality expository lectures that will inform mathematicians regardless of specialty.

Another major change this year concerns the editorial board for the Clay Mathematics Institute Monograph Series, published jointly with the American Mathematical Society. Simon Donaldson and Andrew Wiles will serve as editors-in-chief, while I will serve as managing editor. Associate editors are Brian Conrad, Ingrid Daubechies, Charles Fefferman, János Kollár, Andrei Okounkov, David Morrison, Cliff Taubes, Peter Ozsváth, and Karen Smith. The Monograph Series publishes selected expositions of recent developments, both in emerging areas and in older subjects transformed by new insights or unifying ideas. The next volume in the series will be *Ricci Flow and the Poincaré Conjecture*, by John Morgan and Gang Tian. Their book will appear in the summer of 2007.

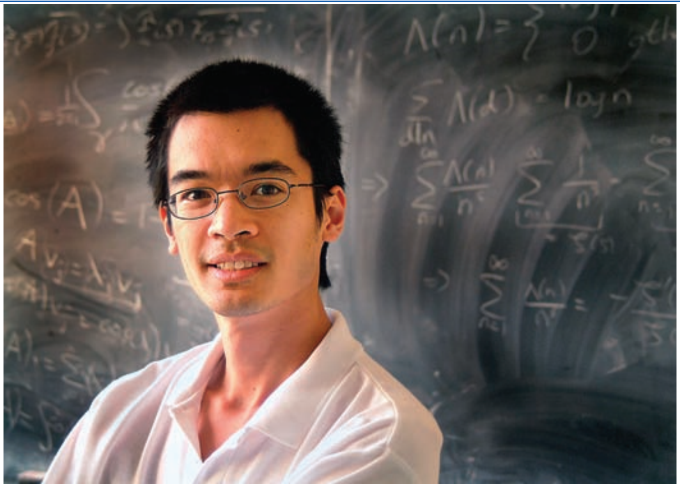
In related publishing news, the Institute has had the complete record of the Göttingen seminars of Felix Klein, 1872–1912, digitized and made available on the web. Part of this project, which will play out over time, is to provide online annotation, commentary, and translations to complement the original source material. The same will be done with the results of an earlier project to digitize the 888 AD copy of Euclid's *Elements*. See www.claymath.org/library/historical.

Mathematics has a millennia-long history during which creative activity has waxed and waned. There have been many golden ages, among which have figured the schools of Greece and Göttingen. The current period, with the resolution of so many long-standing problems, among which are Fermat's last theorem, the Sato-Tate conjecture, arithmetic progressions in the primes, and the Poincaré conjecture, is arguably one of these. In any case, we live in exciting times for mathematics.

Sincerely,

James A. Carlson
President

Fields Medal Winner Terence Tao



Terence Tao. Courtesy Reed Hutchinson/UCLA.

Terence Tao, a Clay Research Fellow from 2000 to 2004, was one of four recipients of the Fields Medals awarded August 22, 2006. The citation read: "for his contributions to partial differential equations, combinatorics, harmonic analysis and additive number theory." The other awardees were Andrei Okounkov, Grigori Perelman and Wendelin Werner.

Tao, born in 1975, is a native of Adelaide, Australia. He began learning calculus as a seven-year-old high school student and by age eleven was well known in international math competitions. After graduating from Flinders University in Australia with a Masters Degree, Tao earned his Ph.D. from Princeton University under the direction of Elias Stein. He then joined UCLA's faculty, where he became full professor at age twenty-four. Tao has also held professorships at the Mathematical Sciences Institute and Australian National University in Canberra.

Among Tao's many awards are the Salem Prize in 2000, the Bôcher Prize in 2002, and the Clay Research Award in 2003. He is also the recipient of a MacArthur Fellowship.

Tao's work is astonishing not only in its depth and originality, but also in its quantity and breadth. He is the author of more than 140 papers, about three-quarters of which have been written with one or more of fifty collaborators. While the core of his work to date has been concentrated in harmonic analysis and partial differential equations, it ranges from dynamical systems to combinatorics, representation theory, algebraic geometry, number theory, and compressed sensing, a new area of applied mathematics. Of special note is his joint work with Ben Green, a Clay Research Fellow from 2005 through 2007. In their 2004 paper, "The primes contain arbitrarily long arithmetic progressions," the authors answered in the affirmative a long-standing conjecture that had resisted many attempts. Vinogradov settled the case of arithmetic progressions of length 3 in 1939. Since then, however, progress had stalled, and even the case of progressions of length 4 was unresolved. For this and other work, Tao was awarded the Australian Mathematical Society Medal.

Tao is also unusual in the attention that he gives to the exposition of mathematics. At his website, one will find, among other things, a list of courses taught with an online edition of his textbook on partial differential equations (Math 251B, Spring 2006), and an archive of over ninety notes on topics ranging from a survey of harmonic analysis to the Black-Scholes formula in mathematical finance. More recently Tao started a blog. It makes for excellent reading and includes many of his recent lectures, including the Simons Lectures at MIT on structure and randomness, the Ostrowski lecture in Leiden on the uniform uncertainty principle and compressed sensing, and the ACM Symposium talk on the condition number of randomly perturbed matrices. There are also detailed notes on other lectures, e.g., Shing-Tung Yau's, *What is a Geometric Structure*, in the Distinguished Lecture Series at UCLA, as well as talks and notes intended for more general audiences: *The cosmic distance ladder*, a talk given to UCLA chapter of the Pi Mu Epsilon society, and *Advice on mathematical careers, and mathematical writing*. Tao has also written an article on Perelman's recent spectacular work.

On the web:

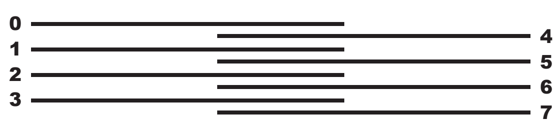
Home page: www.math.ucla.edu/~tao
Blog: terrytao.wordpress.com
Article on Perelman's work: arXiv:math/0610903
Interview: www.claymath.org/library

Mathematics and Magic Tricks

Persi Diaconis
Department of Mathematics and Statistics
Stanford University

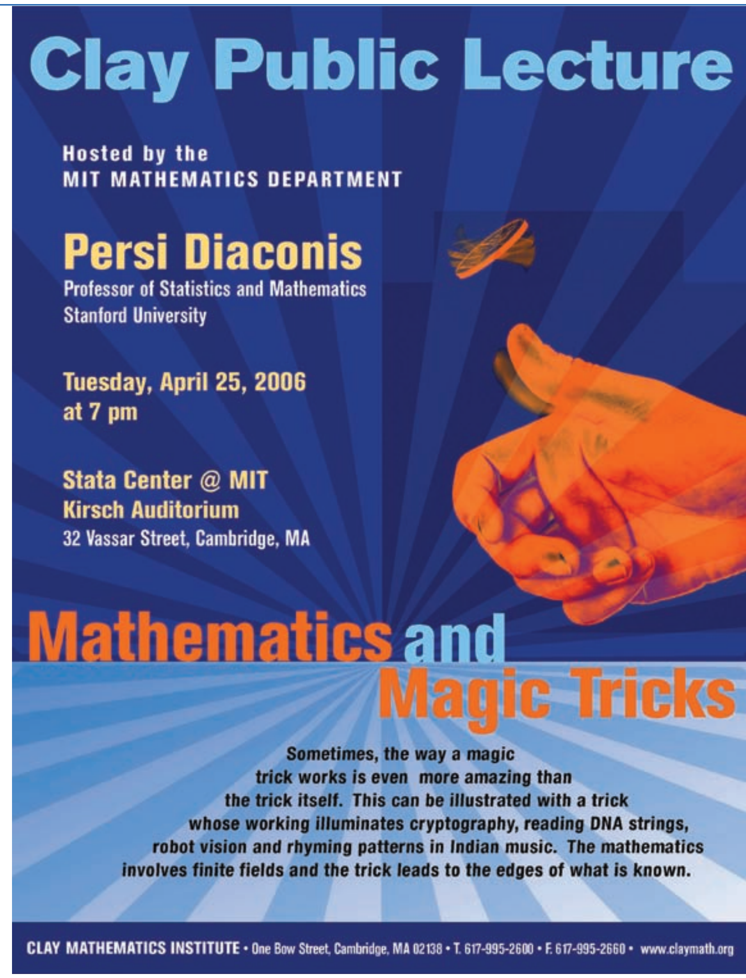
The way that a magic trick works can be just as amazing as the trick itself. My favorite way of illustrating this is to talk about shuffling cards. In this article, I will try to explain how there is a direct connection between shuffling cards and the Riemann Hypothesis — one of the Clay Mathematics Institute's Millennium Prize Problems.

Let us begin with perfect shuffles. Magicians and gamblers can take an ordinary deck of cards, cut it exactly in half, and shuffle the two halves together so that they alternate perfectly as in figure one, which shows a perfect shuffle of an eight-card deck.



If the shuffle is repeated eight times with a fifty-two card deck, the deck returns to its original order. This is one reason that perfect shuffles interest magicians. To see why gamblers are interested, suppose that the deck begins with four aces on top. After one perfect shuffle, the top of the deck is Ace, X, Ace, X, Ace, X, Ace, X, where X is an indifferent card. After two perfect shuffles, the aces are four cards apart. Thus, if four hands of poker are dealt, the dealer's accomplice gets the aces. This motivates the study of just what can be done with perfect shuffles. Magicians and gamblers (along with a few mathematicians) have been thinking about such things for at least three hundred years.

To see the connections with mathematics, consider the problem of how many times a deck should be shuffled to recycle it. The answer is eight for a fifty-two card deck. The answer is fifty-two for a fifty-four card deck and six for a sixty-four card deck. The number of perfect shuffles needed to recycle various size decks is shown in table one.



deck size $2n$	2	4	6	8	10	12	14	16	18	20	22	24	26	
$ord_2\,(2n-1)$	1	2	4	3	6	10	12	4	8	18	6	11	20	
deck size $2n$	28	30	32	34	36	38	40	42	44	46	48	50	52	54
$ord_2\,(2n-1)$	18	28	5	10	12	36	12	20	14	12	23	21	8	52

Can the reader see any pattern in these numbers? Some people find it surprising that sometimes larger decks recycle after fewer shuffles. Decks that have size power of two can be seen to recycle particularly fast. To understand this better, label a deck of size $2n$, starting from the top, as 0, 1, 2, 3, ..., $2n$ - 1. Observe that after one perfect shuffle, the card in position i moves to position $2i$. This statement is true even when i is greater than $n/2$, provided we take the remainder of $2i$ when divided by $2n$ - 1. Or, as mathematicians say, we compute $2i$ modulo $2n$ - 1. Thus, when $2n = 52$, the card in position 0 stays there. The card in position 1 moves to position

2 (because we label things starting at zero, position 2 means the third card from the top). The card in position 28 goes to position 56 modulo 51, that is, to position 5. And so on.

Returning to the problem of a deck of arbitrary size n, we see that after one shuffle, card i moves to position $2i$ modulo $(2n - 1)$. After two shuffles, it moves to position $4i$ modulo $(2n - 1)$. After three shuffles, it moves to position $8i$ modulo $(2n - 1)$. Therefore, the deck recycles after k shuffles, where k is the smallest power of two such that 2 raised to the kth power is one modulo $2n - 1$. For example, when $2n = 52$, $2n - 1 = 51$, and the various powers of two modulo $2n - 1$ are

0 2 4 8 16 32 13 26 1

That is, $2^8 = 256 = 1$ modulo 51, so fifty-two cards recycle after eight shuffles. For a fifty-four card deck, 2 raised to the power 52 is 1 modulo 53. One sees that fifty-two shuffles, but no fewer, are required. For a sixty-four card deck, 2 raised to the power 6 is 1 modulo 63. One sees that in six shuffles, but no fewer, the deck is recycled.

Persi Diaconis delivering his talk at MIT.

From these observations, it is natural to wonder what the longest recycling times are. Fermat's little theorem shows that the worst that can happen is that the deck recycles after $2n - 2$ shuffles. Does this happen for arbitrarily large decks? No one knows. It is a conjecture, due to Emil Artin, with antecedents in the work of Gauss, that 2 is a primitive root for arbitrarily large primes. (See [1, 6] and [3, 4]). This would yield an affirmative answer to the card shuffling problem. It has been rigorously proved that if the generalized Riemann Hypothesis holds, then the Artin conjecture holds. But, alas, a proof of the Riemann Hypothesis, even in its original form, continues to elude the efforts of the world's mathematicians.

The audience assisting in one of Persi's tricks.

I find these connections wonderful. It is inspiring, indeed awe-inspiring, that a simple card-shuffling question that fascinated me as a kid of thirteen can lead to the edge of mathematics and beyond. If you want to know more about shuffling cards and its connections to all sorts of mathematics, see [2].

References

1. Artin, E., letter to Helmut Hasse, September 27, 1929. Recorded in the diary of Helmut Hasse.

2. Diaconis, P. and Graham, R.L. (2007) "The Solutions to Elmsley's Problem." *Math Horizons*, Feb. 2007, pg. 22 - 27. See maa.org.

3. Gauss, C-F., *Disquisitiones Arithmetica*, articles 315-317. 1801.

4. Li, Shuguang and Pomerance, C., *Primitive roots: a survey*, http://www.math.dartmouth.edu/~carlp/PDF/primitiverootstoo.pdf

5. Moree, P., Artin's primitive root conjecture, http://www.math.tau.ac.il/~rudnick/dmv/moree.ps

6. Wikipedia, Artin's conjecture on primitive roots, http://en.wikipedia.org/wiki/Artin's_conjecture_on_primitive_roots

Clay Lectures at Cambridge University

On November 28, 2006, the Clay Institute launched the Clay Lectures in Mathematics, an annual series of talks given by CMI's past or current research fellows. The talks, extending over a period of four days, feature three research talks and one public lecture by each of two fellows.

The lecture series is aimed at young mathematicians, as well as experts from other fields, and aims to develop a theme related to the research fellows' interests.

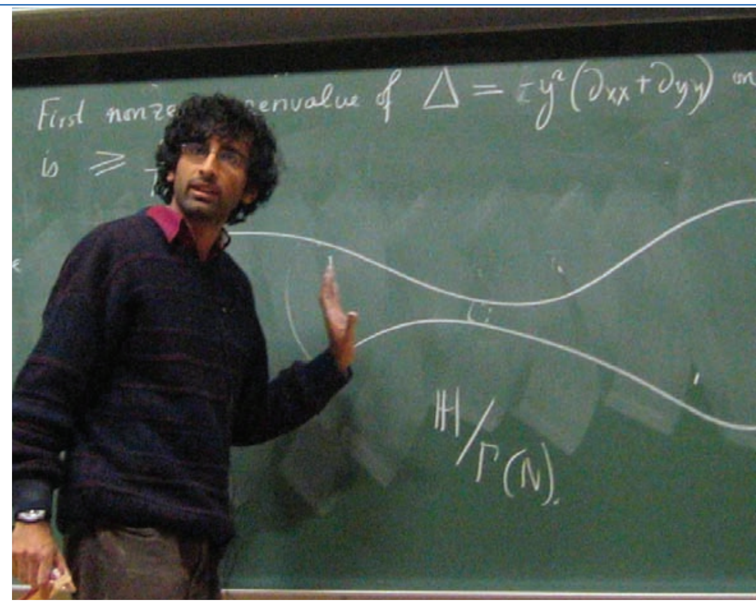
The Cambridge lectures were delivered November 28–December 1, by fellows Ben Green (2005–07) and Akshay Venkatesh (2004–06). Green is now at Cambridge University and Venkatesh is at the Courant Institute of Mathematics (New York University).

Venkatesh gave three lectures entitled *Abelian and Nonabelian Symmetry in Analytic Number Theory,* and a public lecture *Adding Square Numbers*. The operation of adding together square numbers (1, 4, 9, 16, 25, ...) gives rise to complex and beautiful patterns, that have motivated mathematicians from ancient times to the present.

Venkatesh's talks began with a discussion of harmonic analysis on the circle and one of its early triumphs in the 1918 paper of Hardy-Ramanujan, which gave an exact formula for the number of partitions of an integer (e.g., 4 = 3+1 = 2+1+1 = 2+2 = 1+1+1+1). Modular forms already made their appearance in the Hardy-Littlewood paper; behind them lurks the nonabelian group $SL_2(\mathbf{R})$.

Green gave three lectures entitled *Themes in Additive Combinatorics* and a public lecture, *Adding Prime Numbers*. While it has been noted that it is more natural to multiply primes than to add them, many famous open problems in number theory are concerned with adding primes. The study of these problems has led to some fascinating mathematics, including the question of the existence and abundance of arithmetic progressions in the primes. The latter question was resolved by the recent work of Green and Tao.

Green's lectures on additive combinatorics dealt with additive properties of sets of integers. If a set *A* is somewhat closed under addition, what is the structure of *A*? What do we need to know about *A* in order to be able to locate very regular structures, such as arithmetic progressions, inside *A*? How does the Fourier transform of *A* reflect the additive structure of *A*?



Akshay Venkatesh delivering one of the Clay Lectures at DPMMS.

Akshay Venkatesh (Courant Institute)

Lecture Series:

Abelian and Nonabelian Symmetry in Analytic Number Theory

- Some theorems of Hardy, Littlewood and Ramanujan. Partitions and sums of squares
- Some theorems of Linnik, Duke and Iwaniec
- A survey of modern developments

Public Lecture:

Adding Square Numbers

Ben Green (University of Cambridge / CMI)

Lecture Series:

Themes in Additive Combinatorics

The structure theory of set addition. Freiman's theorem

Gowers norms and nilsequences

The idempotent theorem: an application of additive combinatorics to harmonic analysis

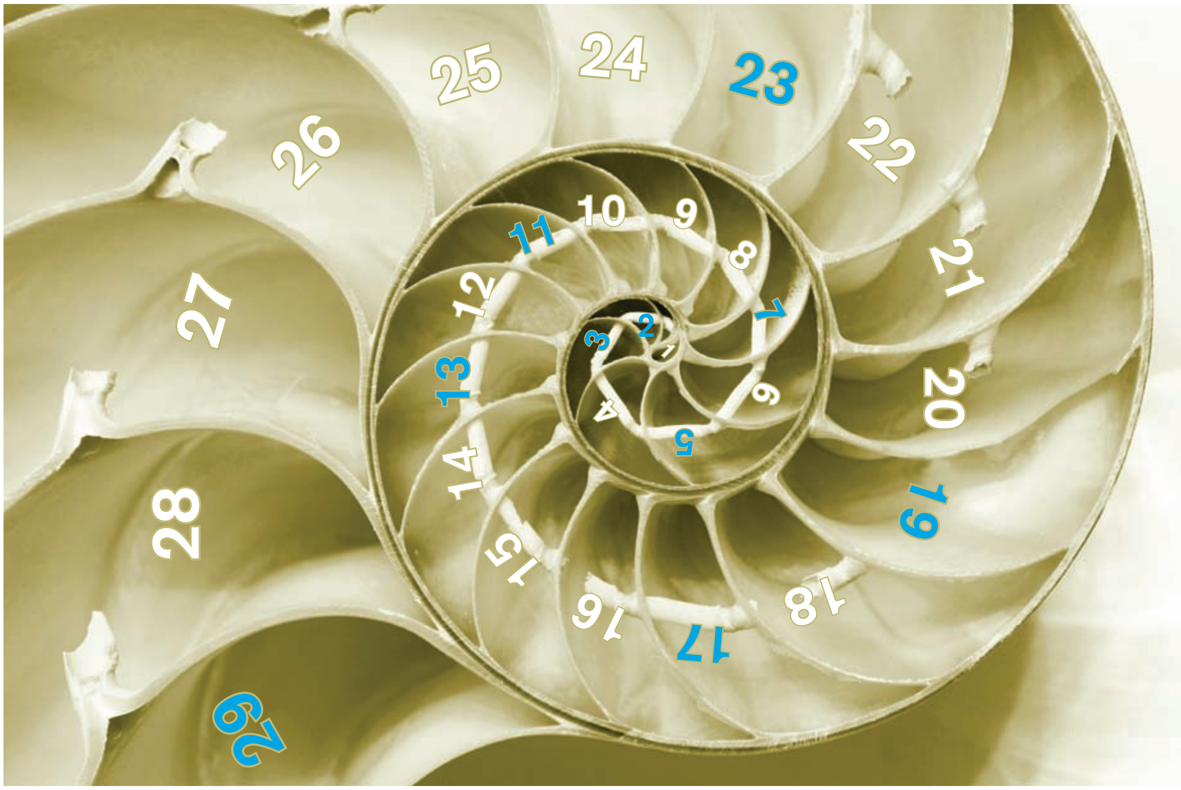
Public Lecture:

Adding Prime Numbers

Ben Green delivering one of the Clay Lectures at DPMMS.

The public lectures bring recent research developments to the educated general public. For the Cambridge event, the Centre for Mathematical Sciences converted its central atrium into a massive lecture hall. A capacity crowd, with many of the Centre's 900 undergraduate mathematics majors, attended.

The 2007 lectures will be held at the Tata Institute in Mumbai, India, with the talks to be given by fellows Elon Lindenstrauss (2003–05) and Mircea Mustata (2001–04). Lindenstrauss and Mustata are at Princeton University and the University of Michigan, respectively.



Summary of 2006 Research Activities

Clay Research Fellow Samuel Payne.

The activities of CMI researchers and research programs are described below. Researchers and programs are selected by the Scientific Advisory Board (see inside back cover).

Clay Research Fellows

Artur Avila began his three-year appointment in July 2006. He is currently working at IMPA in Rio de Janeiro, Brazil, where he received his Ph.D.

Samuel Payne graduated from the University of Michigan and is working at Stanford University. He has a four-year appointment that began in June 2006.

Sophie Morel graduated from Université de Paris-Sud, where she is currently conducting her work. She began her five-year appointment in October 2006 at the Institute for Advanced Study in Princeton.

Avila, Payne, and Morel joined CMI's current group of research fellows Daniel Biss (University of Chicago), Maria Chudnovsky (Columbia University), Ben Green (MIT), Bo'az Klartag (Princeton University), Ciprian Manolescu (Columbia University), Maryam Mirzakhani (Princeton University), David Speyer (University of Michigan), András Vasy (Stanford) and Akshay Venkatesh (Courant Institute).

Research Scholars

Wolfgang Ziller (University of Pennsylvania). September 1, 2005—June 30, 2006 at IMPA, Brazil.

Yaroslav Vorobets (Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of Ukraine). September 1, 2005—August 31, 2006 at Texas A&M University.

Tom Coates (Harvard University). February 1—May 31, 2006. Research on Gromov-Witten Theory at MSRI.

Dihua Jiang (University of Minnesota). May 1—May 31, 2006. Working with Bao-Châu Ngô in Orsay.

Alan Carey (Australian National University). May 1—July 30, 2006 at the Erwin Schrodinger Institute in Vienna.

Ludmil Katzarkov (University of California, Irvine). June 1– June 30, 2006 at the University of Miami.

Mihalis Dafermos (University of Cambridge). December 31, 2006 – December 30, 2007.

Senior Scholars

Yongbin Ruan (University of Wisconsin, Madison). January—May 2006. MSRI program on New Topological Methods in Physics.

Jean-Louis Colliot-Thélène (Université de Paris-Sud). January 9—May 19, 2006. MSRI program on Rational and Integral Points on Higher-Dimensional Varieties.

Robion Kirby (Stanford University). June 25–July 15, 2006. PCMI program on Low-Dimensional Topology.

Clay Research Fellow Artur Avila.

Yakov Eliashberg (Stanford University). June 25–July 15, 2006. PCMI program on Low-Dimensional Topology.

Peter Newstead (University of Liverpool). October 2006. Tufts–BU joint semester on Vector Bundles.

John Milnor (SUNY, Stony Brook). June 25, 2006–July 2, 2006. PCMI program on Low-Dimensional Topology.

Book Fellows

Appointed in 2006 were Ralph Greenberg (University of Washington), who began working on the monograph *Topics in Iwasawa Theory*, and John Morgan (Columbia University) and Gang Tian (Princeton and MIT), who collaborated on the monograph *Ricci Flow and the Poincaré Conjecture*.

Liftoff Fellows

CMI appointed nineteen Liftoff Fellows for the summer of 2006. Clay Liftoff Fellows are recent Ph.D. recipients who receive one month of summer salary and travel funds before their first academic position. See www.claymath.org/liftoff.

Research Programs Organized and Supported by CMI

February 1–May 31. Eigenvarieties program at Harvard University.

February 24–27. Conference on Lie Groups, Dynamics, Rigidity and Arithmetic at Yale University.

April 6–12. Workshop on Additive Combinatorics at CRM (Montreal).

April 26. Public Lecture by Persi Diaconis.

May 10–15. Eigenvarieties Workshop at CMI.

May 14–16. Conference on Automorphic Forms and *L*-Functions at Weizmann Institute of Science (Tel Aviv).

May 17–22. Conference on Global Dynamics Beyond Uniform Hyperbolicity at Northwestern University.

June 19–24. Conference on Hodge Theory at Venice International University (Italy).

June 19–July 14. Workshops on Affine Hecke Algebras and Langlands Program at CIRM (Luminy, France).

July 17–August 11. CMI Summer School on Arithmetic Geometry at Göttingen, Germany.

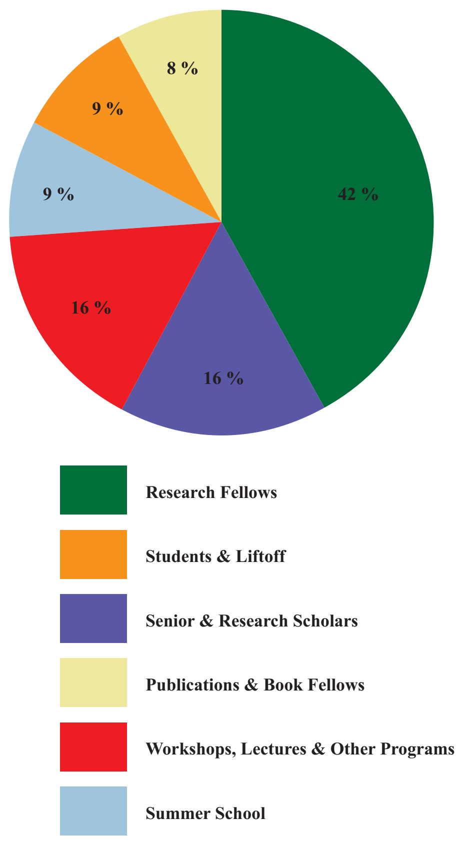
October 5–10. Moduli Spaces of Vector Bundles at CMI.

October/November. Clay Research Conference.

Program Allocation

Estimated number of persons supported by CMI in selected scientific programs for calendar year 2006:

Research Fellows, Research Awardees, Senior Scholars, Research Scholars, Book Fellows and Public Lecturers	30
Summer School participants and faculty	135
Student Programs, participants and faculty	100
CMI Workshops, Liftoff program	60
Participants attending joint programs and the Independent University of Moscow	> 1000



Research Expenses for Fiscal Year 2006
(comparative allocations change annually based on scientific merit)

Profile

Interview with Research Fellow Ben Green

Ben Green was born in 1977 in Bristol, England, and educated at Trinity College, Cambridge, first as an undergraduate and later as a research student of Fields Medalist Tim Gowers. Since 2001 he has been a Fellow of Trinity College, and in that time he has made extended research visits to Princeton, the Rényi Institute in Budapest, the University of British Columbia, and the Pacific Institute of Mathematics (PIMS), where he was a postdoctoral fellow. In February 2005 Green was named a Clay Research Fellow. In January 2005, he took up a Chair in Pure Mathematics at the University of Bristol. He began his appointment as a Clay Research Fellow in July 2005, the first year of which he spent at MIT. Ben also spent from February to March of 2006 at CMI working with his student Tom Sanders. In the Spring of 2007, Ben and his student Julia Wolf visited CMI for two weeks.

What first drew you to mathematics? What are some of your earliest memories of mathematics?

I was always very interested in numbers as a small child — my mother tells me that I used to demand "sums" from the age of about 3 and I took an interest in such things as car registration plates and distances on signs which would not, perhaps, be regarded as normal for a young boy. Apparently the head teacher of my primary school (ages 5–11 in the UK) used me as an example of why it is not a good idea to try to teach your children at home, since I had learnt to subtract "the wrong way" (I don't recall the method I was using but, in my parents' defense, it was one I had discovered myself). I first started discovering "real" mathematics around the age of thirteen. The Olympiad movement — taking part in national competitions — was very important to me in this respect. However, I also started paying regular visits to the city library in Bristol, which contained a surprisingly large, if somewhat eccentric, collection of mathematics books. Thankfully, my father could always be persuaded to take me there so that he could indulge his interest in obscure folk and blues music at the same time. Two books which particularly influenced me were Richard Guy's *Unsolved Problems in Number Theory* and Albert Beiler's *Recreations in the Theory of Numbers*.

Could you talk about your mathematical education in the UK? What experiences and people were especially influential? Can you comment on your experiences at Cambridge as an undergraduate? Is there something special in the college system that had a particular impact on your development?

As I said above, the Olympiad movement was very important to me. I was very lucky that there were two teachers at my secondary school, Julie Kirby and Frank Burke, who took an interest in my mathematical development and ensured that I was entered for the national competitions. They (and I) were rather surprised when I obtained the highest mark in one of these competitions (for students under the age of thirteen). My school is currently ranked somewhere around 2000th in the UK academically so we were quite pleased to have scored this very minor victory over the famous schools like Eton and St Paul's. This was when I realized that I had a particular aptitude for mathematics and started taking it more seriously. Subsequently I took part in more senior mathematics competitions and twice represented the UK at the International Mathematical Olympiad. In doing this I made many lasting friends and was influenced by several wonderful teachers. Among these I would single out Tony Gardiner, Christopher Bradley and David Monk who would regularly send me sets of interesting problems by post. At the time the training system in the UK was delightfully low-key and personal, and refreshingly non-intensive. There was a long weekend at Trinity College, Cambridge, but nothing like the "hothouse" training camps some other countries employ.

However, I also started paying regular visits to the city library in Bristol, which contained a surprisingly large, if somewhat eccentric, collection of mathematics books. Thankfully, my father could always be persuaded to take me there so that he could indulge his interest in obscure folk and blues music.

Cambridge is an excellent place to be an undergraduate in mathematics. The course is hard and interesting, and moreover one is surrounded by other good and serious students. Essentially all of my close friends at university have gone on to tenured positions in mathematics of one kind or another. One aspect of the Cambridge education that I like personally is the fact that it is quite hands-off. The example sheets contain tough problems, and one is expected to bash one's head against them repeatedly as one would a research problem. You won't generally find Cambridge supervisors (people who conduct tutorials) giving away the key to the more interesting problems on a sheet unless the student has made a real effort.

At the time the training system in the UK was delightfully low-key and personal, and refreshingly non-intensive. There was a long weekend at Trinity College, Cambridge, but nothing like the "hothouse" training camps some other countries employ.

The collegiate system gives students the opportunity to come in close contact with world-class mathematicians. When I was a first-year undergraduate I was taught as one of a pair by both Tim Gowers and Bela Bollobas, eight times each: that keeps you on your toes, and exposes you to some pretty interesting mathematics.

Did you have a mentor? Who helped you develop your interest in mathematics, and how?

I've mentioned a few great teachers that I had whilst at school. When at university I was heavily influenced by Tim Gowers, who later became my thesis advisor. Towards the end of my thesis I gained a lot by talking to Imre Ruzsa in Budapest – I found we were interested in exactly the same types of questions.

What attracted you to the particular problems you have studied?

I very nearly opted to do a Ph.D. in algebraic number theory, but some somewhat negative experiences of this area in my last year as an undergraduate, coupled with the recent award of a Fields Medal to Tim Gowers, persuaded me to work under Gowers in the area now known as additive combinatorics. The area is appealing in that the problems may be stated quite easily to a general mathematical audience. A particular attraction for me was that I could embark on research straight away – I did not need to go and read Hartshorne, let alone SGA.

It is hard to say exactly what it is that attracts me to a problem nowadays. I am particularly fond of instances in which it is possible to extract "rigid" structure from rather soft information – in fact most of the questions I am working on right now have this kind of flavor. A theorem of this type that I very much admire (though I don't quite know how to prove, I'm ashamed to say) is Marina Ratner's theorem on the closures of orbits of unipotent flows. She related these to exact subgroups — that is, she took soft information (in this case a dynamical system) and found algebraic structure in it. Terry Tao and I are working on Freiman's theorem and on inverse theorems for the so-called Gowers norms — in both of these one starts with something very combinatorial and produces an algebraic object from it.

Another thing we try and do is make "robust" versions of algebraic results. What is meant by an approximate group? An approximate homomorphism? How do these relate to the corresponding "exact" structures? Often much can be gained by enlarging one's universe to include these approximate algebraic objects, provided one is able to handle the requisite approximate algebra.

Of course I am also motivated by the desire to prove results on the basic questions in number theory, say about prime numbers. But my results with Tao in this area have really come out of an attempt to understand the underlying structures in a more general context.

Can you describe your research in accessible terms? Does it have applications to other areas?

Right now I am working with Tao on generalizing the Hardy-Littlewood method for primes as far as we can. Using this method, Vinogradov proved in 1937 that every large odd number N can be written as the sum of three primes. We have a program which should eventually allow us to count solutions to a more-or-less arbitrary system of linear equations in primes (an example that we have already dealt with is the system $p_1 + p_3 = 2p_2$, $p_2 + p_4 = 2p_3$, which defines an arithmetic progression of four primes). There is one important exception — we do not have a feasible plan for handling certain "degenerate" systems, which include the system $p_1 - p_2 = 2$ (twin primes) and $p_1 + p_2 = N$ (Goldbach conjecture).

Although people seem to like results about the primes, from a mathematician's point of view the underlying methods are much more interesting. Our work, together with the work of many other people, has hinted at deep connections between several areas of mathematics: analytic number theory, graph theory, ergodic theory and Lie groups.

Although people seem to like results about the primes, from a mathematician's point of view the underlying methods are much more interesting. Our work, together with the work of many other people, has hinted at deep connections between several areas of mathematics: analytic number theory, graph theory, ergodic theory and Lie groups.

What research problems and areas are you likely to explore in the future?

There is plenty of work left to be done on the program I have just described, and a really serious amount of work to be done on the general area of "rigidity" results in additive combinatorics and their applications. A proper quantitative understanding of three main types of result in this vein (Freiman-type theorems, inverse theorems for Gowers-type norms and Ratner's theorem) is probably decades away. In the longer term I want to become more competent with "non-abelian" tools and questions, that is to say the theory of "multiplicative combinatorics". Who knows what may be brought to bear here — given the prevalence of Fourier-analytic methods in additive combinatorics, it seems likely that representation theory will have a major role to play. I also have quite a long list of miscellaneous problems that I would like to think about at some point.

Could you comment on collaboration versus solo work as a research style? Are certain kinds of problems better suited to collaboration? What do you find most rewarding or productive?

I just noticed, looking at my webpage, that almost all of my first ten papers had just me as an author, whereas my ten latest are all coauthored. I have never written a three-author paper, but have found collaboration in pairs very productive. It took me a while to realize that collaboration works best when both parties are completely open to sharing their best ideas — when I was a Ph.D. student I was terrified that people might steal my ideas, or jump in on a paper that I had 95 percent finished. That attitude was probably fairly sensible at that stage, but with the luxury of a tenured job I take a much more open position. My joint paper with Tao on arithmetic progressions of primes was a memorable example of collaboration (it was mostly done in a rapid-fire exchange of emails). I am sure Terry would agree that this result could never have been proved by either of us individually.

You have taken on thesis students at a very early stage in your career. Was that a conscious decision? How did you first start working with research students?

Does working with students have rewards as well as responsibilities?

I currently have three Ph.D. students and also talk quite a bit to other students in additive combinatorics at Cambridge. I started working with Tim Gowers' student Tom Sanders about four years ago, largely because he bugged me quite persistently with questions about the projects he was thinking about. After a while I came to realize that I rather enjoyed these discussions and resolved to take on a few good students should any come my way. I have a theory that having two children is less work than having one, as they can play with one another (I currently have none, so this hasn't been tested very thoroughly). I believe that this carries over in a reasonably obvious way to graduate students — we hold regular reading seminars as a group and they can talk amongst themselves when I am not available.

How has the Clay Fellowship made a difference for you?

It allowed me to spend the whole academic year 2005–06 at MIT, which was handy since my girlfriend is doing a Ph.D. at Harvard. I was also able to bring Tom Sanders over for a few months during this time, and we had a very productive period leading to an Annals paper that I'm very happy with. There is no doubt that the Clay Research Fellowship has some of the best conditions of any postdoc out there — no teaching duties, excellent funds for travel, and so on — and this allows the Fellow to work very intensively on research.

What advice would you give to young people starting out in math (i.e., high school students and young researchers)?

A few tips that I have found handy, in no particular order: 1. At high school, it's good to have the experience of tackling really hard problems (and failing, more often than not). Real mathematics is not as "safe" as Olympiad mathematics in that you don't have an *a priori* upper bound for the difficulty of the problem. I've listed a few books that I enjoyed reading at school in one of my answers below. 2. Follow your nose, not necessarily what other people tell you, when you choose what questions you work on. I have worked on some questions which even people in my own subject would probably think uninteresting. I've certainly written papers on questions that nobody (before me) asked. Naturally, over the course of a career (and to get a job) you want to have some results that a lot of people *are* interested in. Let me just say, however, that I can trace my line of thought that eventually led to my joint paper on arithmetic progressions of primes back to a paper Ruzsa and I wrote in answer to a question of Jacques Verstraete: how many of the subsets of $\mathbf{Z}/p\mathbf{Z}$ have the form $A + A$, for some set A in $\mathbf{Z}/p\mathbf{Z}$? I think most people would think of that question as more of a "puzzle" than a serious problem. 3. Check the ArXiv every day and use MathSciNet obsessively. The latter is a wonderful resource — all the papers in mathematics (certainly all those in the last 60 years) are indexed, cross-linked and reviewed.

What advice would you give laypersons who would like to know more about mathematics — what it is, what its role in our society has been and is, etc.? What should they read? How should they proceed?

Well, I find it hard to do better than recommend my advisor Tim Gowers' little book entitled *Mathematics, A Very Short Introduction*, the aim of which is pretty much to answer those questions. A couple of books that I really enjoyed as a teenager, long before I had any real understanding of what mathematics was about, are *The Mathematical Experience* by Davis and Hersh and *Game, Set and Math: Enigmas and Conundrums* by Ian Stewart. Both of these books do have *some* mathematics in them but they are certainly accessible to bright high-school students. Concerning the history of mathematics, I recall getting a lot from *Makers of Mathematics* by Stuart Hollingdale. Maybe some of these choices are eccentric — perhaps they were just the books that Bristol library had in stock — but I certainly enjoyed them myself.

There was a TV program in Britain about Wiles' proof of Fermat's last theorem which gave a wonderful insight into the personalities and mode of working of mathematicians.[1] I don't know how widely available it is.

1. Ben refers to the BBC documentary *Fermat's Last Theorem* that was written and produced by Simon Singh and John Lynch. Later, the same documentary (reversioned for American audiences and renamed *The Proof*) aired on PBS as part of the NOVA series. For more information, see http://www.pbs.org/wgbh/nova/proof/.

To get some sense of the way mathematicians talk to one another, it could be fun to check out one of the increasing number of mathematicians' blogs. Terry Tao has recently created one which attracts a lot of attention, and I have followed Luca Trevisan's "In Theory" for a while.

And of course the Clay Institute has some pretty interesting and accessible lectures linked from its website.

How do you think mathematics benefits culture and society?

Though this question seems like an invitation to say something wildly pretentious, I'll try and avoid doing so. I think one only needs to look at the attractiveness of mathematics graduates on the job market to realize that the mathematician's way of thinking is something that can be extremely useful in many areas of society. I doubt that most jobs require a specific knowledge of homological algebra (say) but the ability to think creatively within the confines of logic and to think "out of the box" are clearly very important everywhere. Let me stop before I start sounding like a management consultant.

I personally find that mathematics is a wonderful way of breaking down cultural barriers. For example I spent several months working in Hungary even though I speak (almost) no Hungarian. I doubt that would have been possible in many other walks of life.

Please tell us about things you enjoy when not doing mathematics.

Unlike quite a lot, possibly even most, other mathematicians, I almost completely avoid activities like chess, bridge or computer programming. When I'm not doing mathematics I like to do something that doesn't use my brain so intensively. I'm a keen cyclist and outdoor enthusiast, I enjoy playing cricket (in the summer) and I play jazz saxophone to a rather mediocre standard.

You were recently appointed a full professor at Cambridge. Congratulations! What are you planning next?

Well I was very pleased to get the job at Cambridge and I don't anticipate moving on for at least ten years or so. I want to develop a group of students and postdocs here, a seminar series, and graduate courses. I'm very happy with the way my career has gone so far but it is important to avoid burnout. I believe that diversity in research is the key to that — I always like to feel that one of my projects could be completely taken away (solved by someone else or studied from a totally new perspective that I don't understand, say) and I'd still have a decent portfolio of research projects.

Tom Sanders and Ben Green at the Clay Mathematics Institute.

Recent Research Articles

"Linear Equations in primes," with Terence Tao, to appear in *Annals of Math.*

"A quantitative version of the idempotent theorem in harmonic analysis," with T. Sanders, to appear in *Annals of Math.*

"Freiman's theorem in finite fields via extremal set theory," with Terence Tao, arXiv:math/0703668

"A note on the Freiman and Balog-Szemeredi-Gowers theorems in finite fields," with Terence Tao, arXiv: math/0701585

"New bounds for Szemeredi's theorem, II: A new bound for $r_4(N)$," with Terence Tao, arXiv:math/0610604

Normal Numbers are Normal

By Davar Khoshnevisan
Department of Mathematics
University of Utah

Abstract. A number is *normal* in base b if every sequence of k symbols in the letters $0, 1, \ldots, b-1$ occurs in the base-b expansion of the given number with the expected frequency b^{-k}. From an informal point of view, we can think of numbers normal in base 2 as those produced by flipping a fair coin, recording 1 for heads and 0 for tails. Normal numbers are those which are normal in every base. In this expository article, we recall Borel's result that almost all numbers are normal. Despite the abundance of such numbers, it is exceedingly difficult to find specific exemplars. While it is known that the Champernowne number $0.123456789101112131415\cdots$ is normal in base 10, it is (for example) unknown whether $\sqrt{2}$ is normal in any base. We sketch a bit of what is known and what is not known of this peculiar class of numbers, and we discuss connections with areas such as computability theory.

1. Introduction

Let x be a real number between zero and one. We can write it, in binary form, as $x = 0.x_1x_2\cdots$, where each x_j takes the values zero and one. We are interested first of all in "balanced" numbers—numbers x such that half of their binary digits are zeros and the remaining half are ones. More precisely, we wish to know about numbers x that satisfy

$$\lim_{n\to\infty} \frac{\#\{1 \le j \le n : x_j = 1\}}{n} = \frac{1}{2}, \tag{1}$$

where # denotes cardinality.

Equation (1) characterizes some, but not all, numbers between zero and one. For example, $x = 0$ and $x = 1$ do not satisfy (1), whereas the following do: $0.\overline{10}, 0.\overline{01}, 0.\overline{001011}$. The last three examples are eventually periodic. It is therefore natural to ask whether there are numbers that satisfy (1) whose digits are *not* periodic. Borel's normal number theorem gives an affirmative answer to this question. In fact, Borel's theorem implies, among other things, that the collection of non-normal numbers has zero length. Surprisingly, this fact is intimately connected to diverse areas in mathematics (probability, ergodic theory, b-adic analysis, analytic number theory, and logic) and theoretical computer science (source coding, random number generation, and complexity theory).

In this article, we describe briefly a general form of Borel's normal-number theorem and some of its consequences in other areas of mathematics and computer science. Our discussion complements some related papers by Berkes, Philipp, and Tichy [3], Harman [15], and Queffélec [21].

2. Borel's theorem

Given an integer $b \ge 2$ and a number x between zero and one, we can always write $x = \sum_{j=1}^{\infty} x_j b^{-j}$, where the x_j's take values in $\{0, \ldots, b-1\}$. This representation is unique for all but b-adic rationals; for those we opt for the representation for all but a finite number of digits x_j are zero.

We may think of $\{0, \ldots, b-1\}$ as our "alphabet," in which case a "word" of length m is nothing but the sequence $\sigma_1 \ldots \sigma_m$, where each σ_j can take any of the values $0, \ldots, b-1$.

continued on page 27

CMI–Göttingen Library Project

Feature article

The Felix Klein Protocols Digitized by Eugene Chislenko

Two plain shelves in Göttingen, in the entrance room of the mathematics library, hold one of the best-kept secrets in the history of mathematics. In this locked *Giftschrank*, or poison cabinet, stand several hundred volumes, largely handwritten and mostly unique, that form an extensive record of one of the world's most important mathematical centers, the home of Gauss, Riemann, Dirichlet, Klein, Hilbert, Minkowski, Courant, Weyl, and other leading mathematicians and physicists of the 19th and early 20th centuries. A recent *Report on the Göttingen Mathematical Institute Archive* cites "a range of material unrivalled in quantity and quality: No single archive is even remotely comparable," not only because Göttingen was "the leading place for mathematics in the world," but also because "no other community has left such a detailed record of its activity — usually we are lucky to have lecture lists, with no indication of the contents." The collection runs from early handwritten lectures by Riemann and Clebsch through almost 100 volumes by Hilbert to volumes of Minkowski on number theory and Max Born on quantum mechanics. But the largest and richest of its centerpieces is the Seminar-Protokolle of Felix Klein: a detailed handwritten record, spanning over 8,000 pages in 29 volumes, of 40 years of seminar lectures by him, his colleagues and students, and distinguished visitors.

The record begins in 1872, when the 23-year-old Klein began his new professorship at Erlangen with the announcement of his revolutionary Erlangen program, unifying the various geometries of the time by classifying them by their corresponding groups of transformations. He had recently proved that non-Euclidean geometry is consistent if and only if Euclidean geometry is, and he would go on to do ground breaking work in many other areas, becoming, along with Hilbert and Poincaré, the last of the mathematicians who could claim to have a grasp of the entire field. Klein then moved to Münich, Leipzig and finally Göttingen. His energy and administrative talent made him the central figure in Germany's leading mathematics department at Göttingen, the nation's leading mathematics journal *Mathematische Annalen*, its first national association of mathematicians, and a program of reforms in higher education that became known as the Klein reforms. His influence on all aspects of mathematical life was unmistakable, even in his wife's wedding dress, patterned with arabesques from Kummer surfaces.

Klein's impact was especially strong in the United States. By 1875, in the first century after the Revolution, the growing network of American universities had only managed to award six doctoral degrees in mathematics, an average of less than one per decade. As programs finally began to expand and to look to Europe for inspiration, Klein took up the challenge, making repeated trips to the United States to present the latest in modern mathematics to his eager listeners. His series of lectures in Evanston, Illinois, held in conjunction with the World's Fair in Chicago and now known as the Evanston Colloquium, had a legendary influence, as did his tours of the universities on the East Coast. Klein himself soon became convinced of the potential of American mathematics, and worked to organize funding for the brightest American students to study in Göttingen. He was soon supporting a steady stream of enthusiastic American visitors. Harry Walter Tyler from MIT wrote, "I know of no one who can approach him as a lecturer…. He's certainly acute, fertile in resource, not only understands other people, but makes them

Mathematisches Institut Georg-August-Universität, Göttingen.

understand him, and seems to have a very broad firm grasp of the philosophical relations and bearings of different subjects, as well as great versatility and acquaintance with literature."

Tyler was one of many Americans to be marked by the breadth and power of Klein's teaching, and to leave their own mark, first in his *Protokolle* and then in the world at large: six of the American Mathematical Society's early presidents and two of the University of Chicago's first three mathematics professors were students of Klein.

The *Protokolle* cover every aspect of his astonishingly wide-ranging activity. The first volume alone includes presentations not only on Lie groups, icosahedra, Riemann, and Abel's Theorem, but also on heat distribution, crystals, comets, and the theory of the Northern Lights. From an early emphasis on geometry, group theory, and function theory, the other volumes expand into number theory, probability theory, mechanics, astronomy, geodesy, hydrodynamics, electricity, elasticity theory, and, in Klein's last years before his retirement in 1912, the psychology and teaching of mathematics. The meetings were small and on a high level. Participants included the young Pauli and Zermelo, Planck and Hurwitz, Prandl and Bernstein. Many of the later seminars were organized jointly with Hilbert and Minkowski, whom Klein had attracted to Göttingen and who shared his commitment to a close tie between mathematics and physics. Presentations made in the seminar were painstakingly recorded in the Seminar-Protokolle books, just as Göttingen mathematics lectures were recorded in other notebooks and placed in the library for students' reference. These notebooks have continued to astonish those who see them, and they remain the most complete record of a great era of mathematical creativity.

To make these volumes more widely available, CMI and Professor Yuri Tschinkel have organized a digitization initiative, using the latest in scanning technology to digitize the complete *Protokolle* in November of 2006. They are now being published for the first time, in a digital edition available online at www.claymath.org/library/historical. The full resolution scans are available for study by scholars at CMI and at the Göttingen Mathematical Institut at www.librarieswithoutwalls.org/klein.html.

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The Klein Protokolle

Modern technology makes possible as never before access for everyone to the classics of mathematics. The Clay Mathematics Institute has undertaken several initiatives in cooperation with other institutions to digitize and disseminate significant historical mathematical works. The first project, entirely funded by CMI, was the digitization of the oldest extant copy of Euclid's *Elements*. This is the d'Orville manuscript, dated to 888 AD, when it was copied in Constantinople by Stephen the Clerk for Arethas, later bishop of Caesarea Cappadociae. The manuscript has been in the collection of the Bodleian Library since 1804. The photography, directed by Chet Grycz of Octavo and Richard Ovenden of the Bodleian, took place at Oxford in the fall of 2004. From it resulted a set of 386 digital images, one per spread of the mansucript, each with a resolution of 639 pixels per inch and a file size of 254 megabytes. CMI, the Bodleian Library, and Octavo.com maintain copies of the original images for use by any interested person. Online copies are available at CMI and the non-profit organization Libraries Without Walls.

The next two projects took place in Göttingen with the help of Yuri Tschinkel of the Mathematisches Institut. Bernhard Riemann's 1859 manuscript "On the number of primes below a fixed bound," was photographed in 2005 by the Niedersächsische Staats- und Universitätsbibliothek Göttingen in 2005 with the assistance of Helmut Rolfing, curator of manuscripts.

Much greater in scope was the digitization of the Klein *Protokolle* at the Mathematische Institut in Göttingen — twenty-nine volumes comprising 8600 pages. The work was carried out by Libraries without Walls under the direction of Chet Grycz, again with CMI funding. Ardon Bar Hama, the photographer, used a Leaf Aptus 75 camera with a digital back and was able to complete the job in three days of round-the-clock work at the Matematische Institut. The images were captured as high resolution camera Raw DNG files for magnification and close inspection by scholars using a careful non-intrusive handling procedure developed specifically for rare and delicate bound material.

Eugene Chislenko, CMI Senior Research Assistant on the project, has been facilitating digitization of the Klein *Protokolle* and other historical volumes. He is now editing and annotating the digitized volumes and is engaged in researching the history of mathematics with this material as a primary source.

There is much more of value to be digitized at the Mathematische Institut, for long the home of many of the world's best-known mathematicians, from Gauss to Hilbert. A complete catalogue of their manuscript holdings was prepared by Jeremy Gray of the Open University in a research project supported by CMI.

The most recent CMI digitization project, currbibliothek, is the preservation of portions of Riemann's *Nachlass* at the Staats- und Universitätsbibliothek.

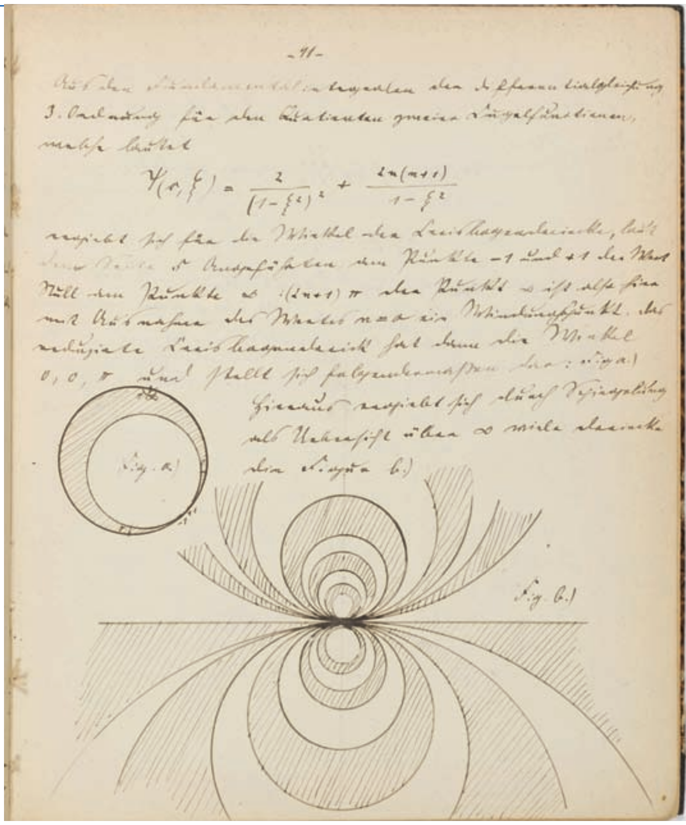
Portrait of Felix Klein, Courtesy Mathematisches Institut Georg-August-Universität, Göttingen.

Websites:

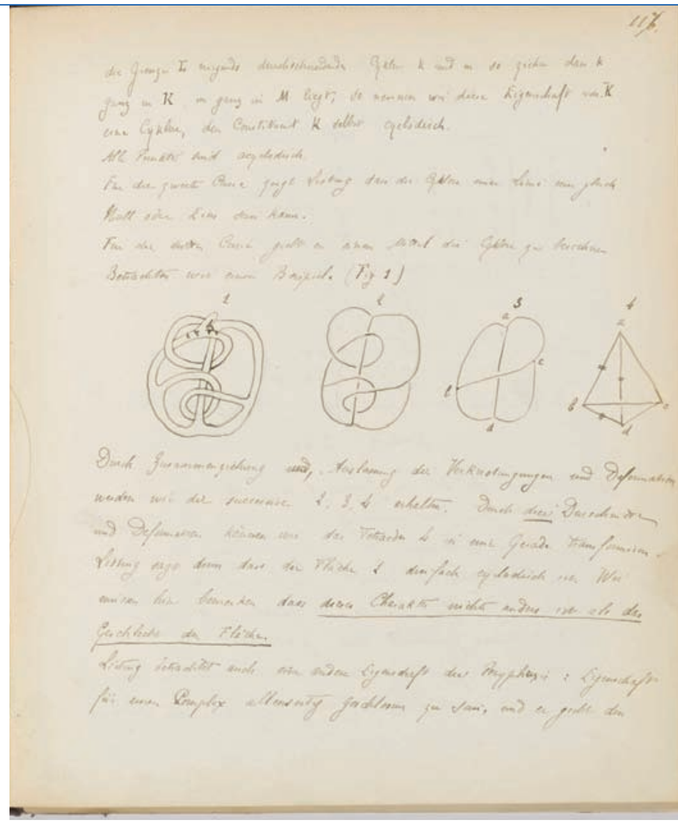
www.claymath.org/library/historical

www.librarieswithoutwalls.org

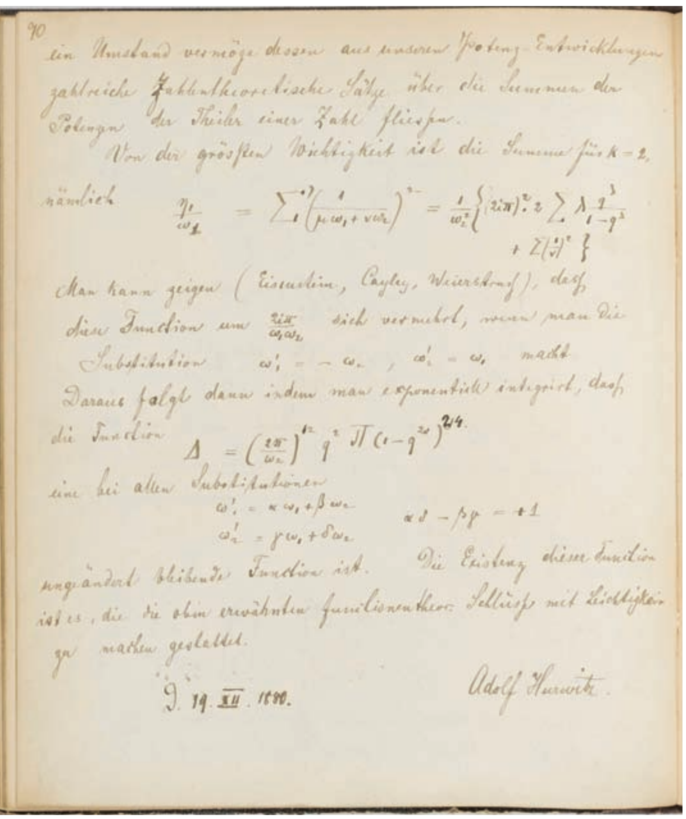
www.librarieswithoutwalls.org/klein.html



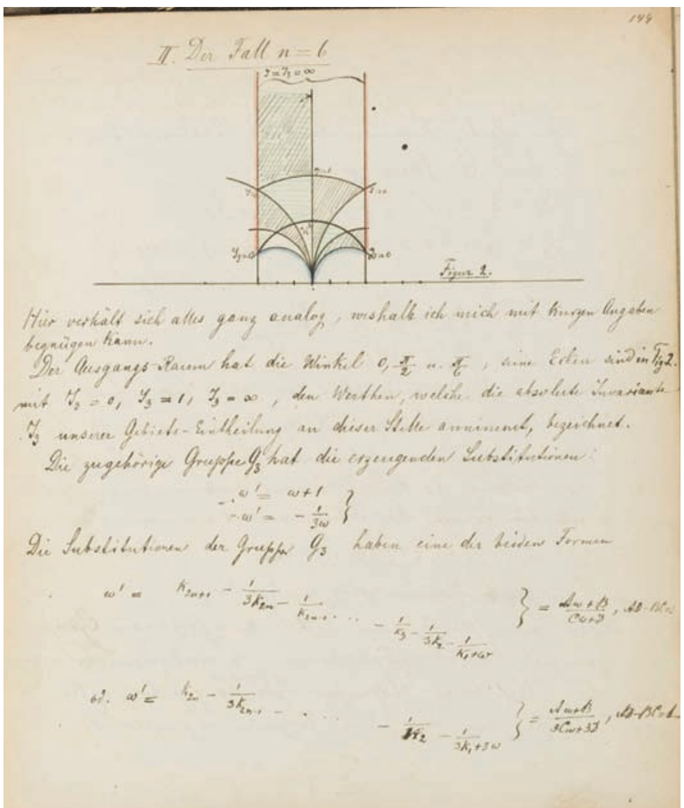
Felix Klein, Protokolle, Vol V, p 11, 7 May 1883



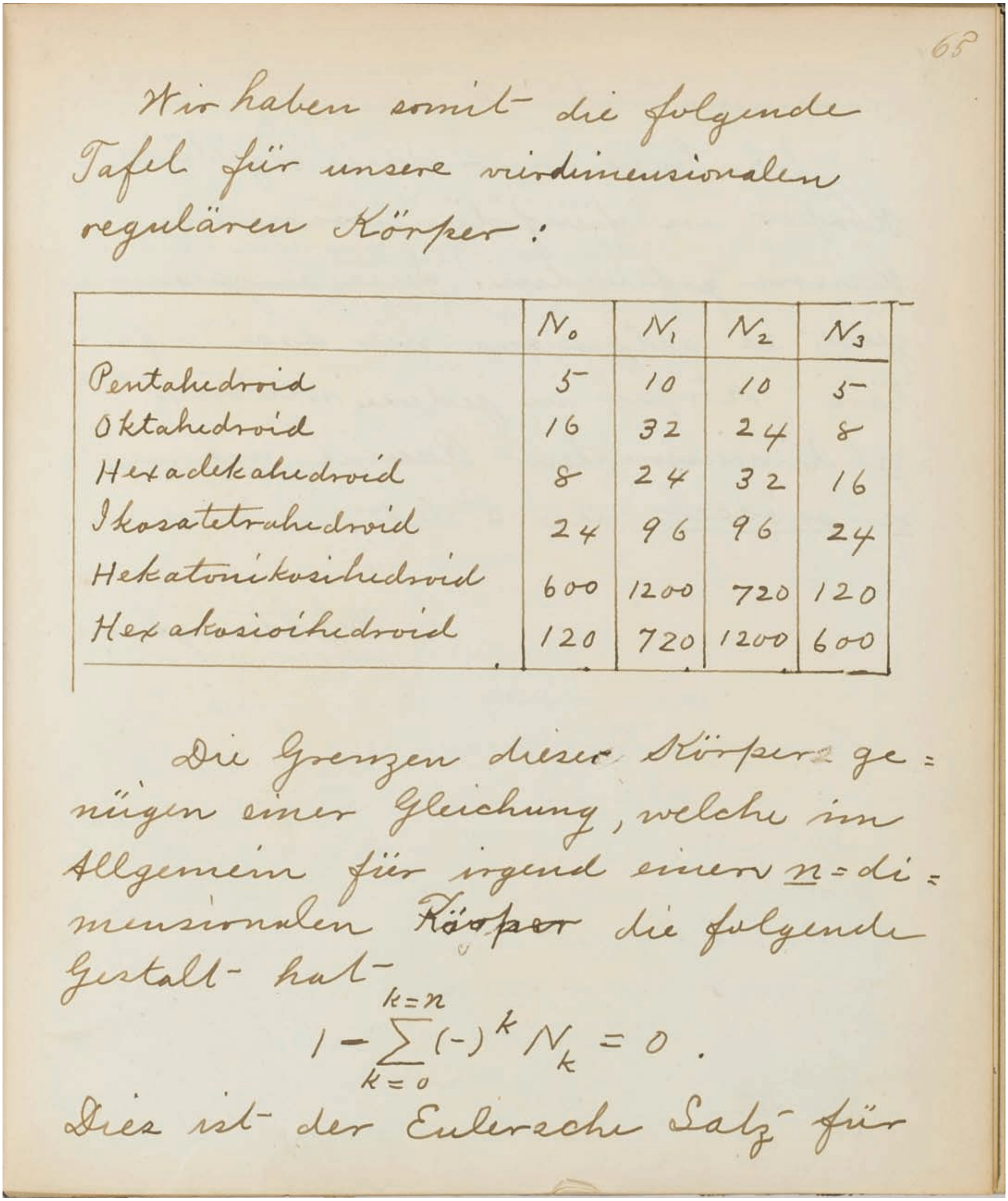
"Ueber de Analysis Situs," Protokolle, Vol II, p. 114, Monday, 31 January 1881



Adolf Hurwitz, "Über die Bildung der Modul-Functionen," Protokolle, Vol II, p. 70, Monday, 6 December 1880



Adolf Hurwitz, "Über eine Reihe neuer Functionen ...," Protokolle, Vol II, p. 144. Monday, 21 February, 1887


65

Wir haben somit die folgende Tafel für unsere vierdimensionalen regulären Körper:

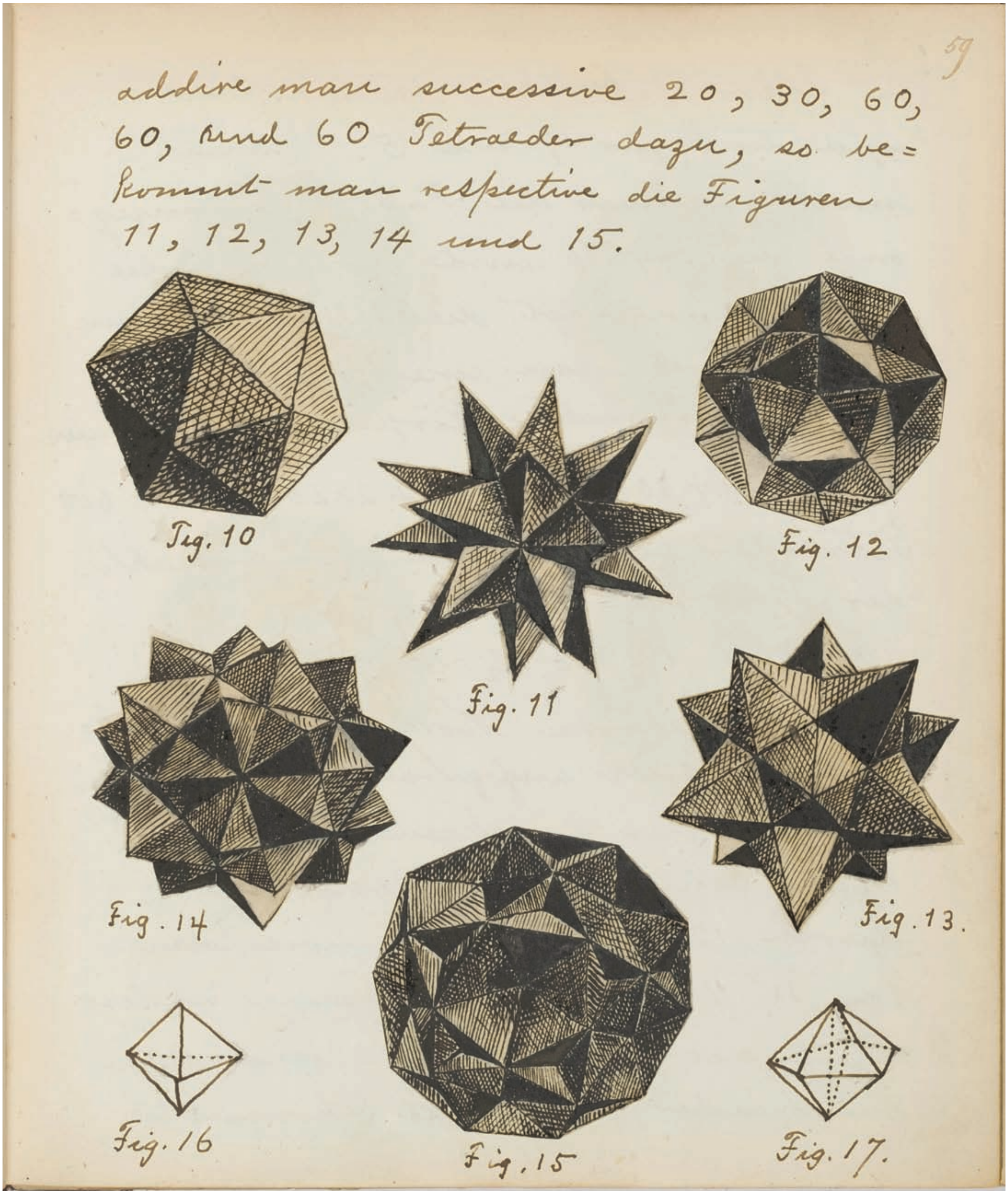
	N_0	N_1	N_2	N_3
Pentahedroid	5	10	10	5
Oktahedroid	16	32	24	8
Hexadekahedroid	8	24	32	16
Ikosatetrahedroid	24	96	96	24
Hekatonikosihedroid	600	1200	720	120
Hexakosioihedroid	120	720	1200	600

Die Grenzen dieser Körper genügen einer Gleichung, welche im Allgemein für irgend einen <u>n</u>-dimensionalen ~~Körper~~ die folgende Gestalt hat

$$1-\sum_{k=0}^{k=n}(-)^k N_k = 0 .$$

Dies ist der Eulersche Satz für

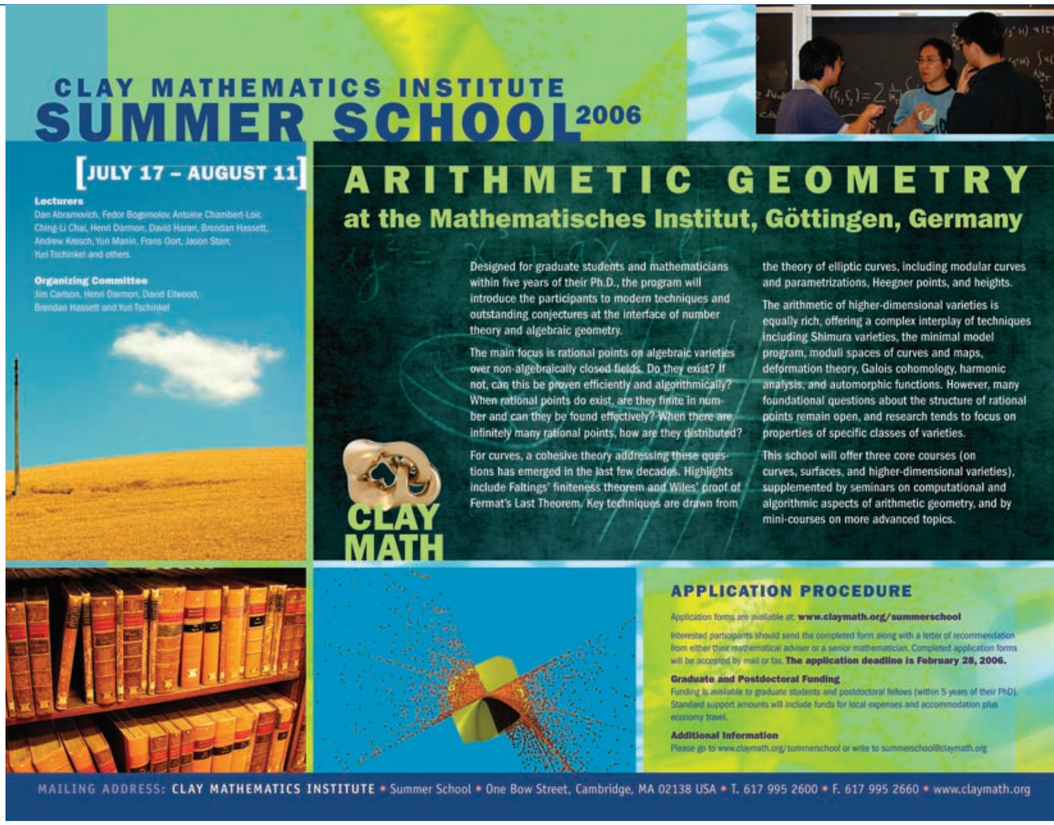
"Ueber reguläre Körper im vier-dimensionalen Raum," W.I. Stringham, Felix Klein, Protokolle, Vol . II, p 65, Monday, 29 November 1880.


59

addire man successive 20, 30, 60, 60, und 60 Tetraeder dazu, so bekommt man respective die Figuren 11, 12, 13, 14 und 15.

“Ueber reguläre Körper im vier-dimensionalen Raum,” W.I. Stringham, Felix Klein, Protokolle, Vol. II, p 59, Monday, 29 November 1880.

Arithmetic Geometry at the Mathematisches Institut, Göttingen, Germany



The 2006 CMI Summer School was held at the Mathematisches Institut of Georg-August-Universität, Göttingen, Germany. A leading scientific center since the time of Gauss, Göttingen became a Mecca for mathematics in the 20th century, with Hilbert, Klein and Minkowski, the first three chairholders at the Institute. The Mathematics Institute continues to be a leading international center for mathematical research, and the 107 participants[1] at the school enjoyed the excellent facilities and hospitality of the Institute from July 17 through August 11.

The main focus of the school was on rational points on algebraic varieties over non-algebraically closed fields. Do they exist? If not, can this be proven efficiently and algorithmically? When rational points do exist, are they finite in number and can they be found effectively? When there are infinitely many rational points, how are they distributed?

The school was organized around three core coures on *Curves, Surfaces, and Higher-dimensional Varieties*, supplemented by seminars on *Computational and Algorithmic aspects of Arithmetic Geometry*, and by mini-courses on more advanced topics. For *Curves*, a cohesive theory addressing these questions has emerged in the last few decades. Highlights include Faltings' finiteness theorem and Wiles' proof of Fermat's Last theorem. Key techniques are drawn from the theory of elliptic curves, including modular curves and parametrizations, Heegner points and heights. Henri Darmon gave five lectures on *Key Finiteness Theorems* (Mordell-Weil theorem, Faltings' theorem, Modular curves and Mazur's theorem, Fermat curves and Wiles' theorem), followed by a more specialized set of lectures, focusing on elliptic curves and their rational points with special emphasis on the Heegner point construction arising from modularity and the theory of complex multiplication.

Brendan Hassett, Andrew Kresch and David Harari gave courses on *the Arithmetic of Surfaces*. Hassett lectured on the geometry of rational surfaces, with a view toward arithmetic applications. Kresch lectured on the theory of descent and the Brauer-

1. In addition to the 107 participants funded by Clay, about 50 participants attended using their own funding.

Brendan Hassett's course.

Manin obstruction to the Hasse principle and weak approximation, and Harari presented concrete applications of the theory of descent. The arithmetic of higher-dimensional varieties is equally rich, offering a complex interplay of techniques including Shimura varieties, the minimal model program, moduli spaces of curves and maps, deformation theory, Galois cohomology, harmonic analysis, and automorphic functions. Yuri Tschinkel gave eight lectures on the distribution of rational points with respect to heights, focusing on varieties closely related to linear algebraic groups, e.g., equivariant compactifications of groups and homogeneous spaces. Topics covered included the circle method and hypersurfaces, toric varieties, height zeta functions of toric varieties, flag varieties, compactifications of additive groups, spherical varieties and conjectures on rational and integral points. Boris Moroz lectured on the classical application of the circle method to the Waring problem, and then explained how Deligne's estimates on exponential sums lead to Heath-Brown's theorem on cubic forms in ten variables. Jason Starr gave three lectures, one on *the Tsen-Lang Theorem*, one on *Arithmetic over Function Fields of Curves* and one on *Arithmetic over Function Fields of Surfaces*. Dan Abramovich lectured on *Birational Geometry for Number Theorists* (Kodaira dimension and the birational classification of varieties, the minimal model program, the conjectures of Lang and Vojta, Campana's program, and applications to specific number-theoretic problems). Finally, Antoine Chambert-Loir lectured on the distribution of points of "small" height on arithmetic varieties. Topics covered included equidistribution on the projective line, Arakelov geometry and equidistribution, and Equidistribution on Berkovich spaces

The school also included a three week-workshop on *Computational Aspects of Arithmetic Algebraic Geometry*, as well as advanced mini-courses on *Moduli of Abelian Varieties and p-Divisible Groups* (Frans Oort and Ching-Li Chai), *Zink's Theory of Displays and Crystalline Dieudonné Theory* (William Messing), *Non-commutative Cartier Isomorphism and Hodge-to-de Rham Degeneration* (Dmitry Kaledin), *Classical and Iterated Shimura Symbols* (Yuri Manin), *Geometry over Finite Fields* (Fedor Bogomolov), *André-Oort Conjectures* (Emmanuel Ullmo), and *Varieties over Finite Fields* (Bjorn Poonen).

Downtown Göttingen, photo courtesy Ulrich Derenthal.

The Ross Program at Ohio State University

www.math.ohio-state.edu/ross

Program overview

"Think deeply of simple things"

In 2006, the Clay Mathematics Institute continued its support of summer programs for talented high school students who excel in math by sponsoring, in part, the Ross Program at Ohio State University and PROMYS at Boston University. Both of these programs are distinguished for offering the best pre-college learning experiences available to American students with a special aptitude for mathematics by immersing them in the creative world of mathematical discovery.

The Ross program at Ohio State University is an eight-week intensive summer course in mathematics for bright young students. Spurred by the launch of Sputnik and the subsequent surge of interest in science education, Dr. Arnold Ross founded his program at Notre Dame in 1957. The program moved with Dr. Ross to Ohio State in 1964 and has run every summer since then.

The central goal of this program has always been to instruct and encourage students in the art of abstract thinking and to inspire them to discover for themselves that abstract ideas are valuable and important. Beginning students who do well are invited back for a second summer, and may return as junior counselors or counselors in subsequent summers. Returning students and counselors also take advanced courses, which vary from year to year.

For the past several years, CMI employed instructors and counselors in the Ross Program. This made it possible for the University to recruit top-ranking mathematics professors and graduate students to teach and coach the work of the enrolled students. In 2006, 35 students were involved in the number theory course (23 first-year students, 8 second-year students, and 4 undergraduates). These participants were assisted in their mathematical work by eight Junior Counselors and seven Counselors.

The first-year students (mostly 14 to 18 years old) take the basic course in number theory, which Daniel Shapiro reported to be "elementary but fast-paced." Each summer's session starts with the Euclidean algorithm and congruences, then moves on to prime factorization, Gaussian integers, quadratic reciprocity, Möbius inversion, polynomial rings, geometry of numbers, etc. Students are expected to work through these ideas (with proofs), guided by the extensive problem sets.

"To discuss the number theory problems, students broke into three seminars taught by retired Ohio State University professor Robert Gold and his colleague Jim Brown, a postdoc at Ohio State, and Stefan Patrikis, one of the senior counselors. Students with a bit more experience also participated in a Topics Seminar designed and run by the Counselors. Junior Counselors and Counselors attended the two advanced courses: Combinatorics taught by Professor Kenneth Supowit, and Sums of Square taught by me," explained Shapiro.

In addition to these eight-week courses and seminars, there were several "colloquium style" lectures. The lecturers in 2006 were Ray Pierrehumbert (University of Chicago, geophysical sciences), Paul Pollack (Ph.D. student at Dartmouth), Tom Weston (University of Massachusetts–Amherst), David Pollack (Wesleyan University), Glen Whitney (Renaissance Technologies), and Susan Goldstine (St. Mary's College of Maryland).

For each of the past several years, the Ross Program has also offered a three-week component for in-service high school mathematics teachers. These teacher-participants join the others in the number theory lecture, but have separate workshops and seminars. A geometry course was introduced in 2006 for alumni of the teacher program. There were 19 teacher-participants involved in the number theory course, and 8 joined the geometry course. These efforts were supported by funds from the Park City Math Institute and the Math Department's VIGRE grant from the National Science Foundation.

"Next summer we will host a 50th Anniversary Reunion/Conference, held on July 20–22, 2007," Shapiro reported. "We will use this event to help demonstrate how influential the Ross Program has been on the American mathematical community. It should also provide us with some fund-raising opportunities."

Daniel B. Shapiro is Professor and Vice Chair of the Department of Mathematics at Ohio State University. He was a student in the program in the 1960s and took over as director when Dr. Ross stepped down in 2000 at the age of 94.

PROMYS at Boston University

http://math.bu.edu/people/promys

Since 1999, the Clay Mathematics Institute has sponsored a variety of advanced seminars and research projects for returning students enrolled in the Program in Mathematics for Young Scientists (PROMYS) at Boston University.

Now in its nineteenth year, PROMYS is a six-week summer program that was developed by BU Professor Glenn Stevens with the aim of engaging ambitious high school students in intensive mathematics research. Young students who excel in math are invited to explore the creative world of mathematics in a supportive community of peers, counselors, research mathematicians, and visiting scientists.

Students are selected from around the United States based on their interest and ability in mathematics. PROMYS moves well beyond the high school curriculum by offering students the opportunity to participate in the process of scientific research. First-year participants engage in intensive problem solving in elementary number theory. Returning participants study more advanced topics. These participants are divided into “lab groups” of two to four students. Each group works together on open-ended exploratory projects that they will present to other PROMYS participants at the end of the program. Throughout the summer, several research mathematicians serve as mentors to the advanced students. Mentors pose new research problems at the start of each summer and provide guidance for the students. Their assistance includes hints for getting started and references to the pertinent literature.

Behind the scenes, a group of counselors, who are also participants in the program, maintain an intensive level of interaction with the high school participants. Counselors are undergraduate math majors recruited from the country’s top universities, who live and work alongside the younger participants, aiding them in their research. “It is no exaggeration to say that the success of PROMYS depends primarily on the dedication and expertise of the counselor staff. They bring an enthusiastic attitude to PROMYS that is easily transmitted to the participants. They are the main channel by which the *esprit de corps*, so vital to PROMY’s success, is conveyed,” says Glenn Stevens. Counselors share their knowledge and expertise with the high school participants by grading their daily homework, engaging in informal discussions, and offering mini-courses on themes of their choosing.

To ensure that returning students and counselors find their experience intellectually stimulating, the CMI/PROMYS’s partnership offers a variety of advanced seminars and research projects each summer. Past seminars have included V*alues of the Riemann zeta function, Hyperbolic Geometry; Random Walks on Groups, Dirichlet Series, Mathematics of Computer Graphics, Graphs and Knots*, and *The Mathematics of Algorithms*. This year, PROMYS and the Clay Mathematics Institute are offering advanced seminars in *Geometry and Symmetry, Modular Forms,* and *Abstract Algebra*.

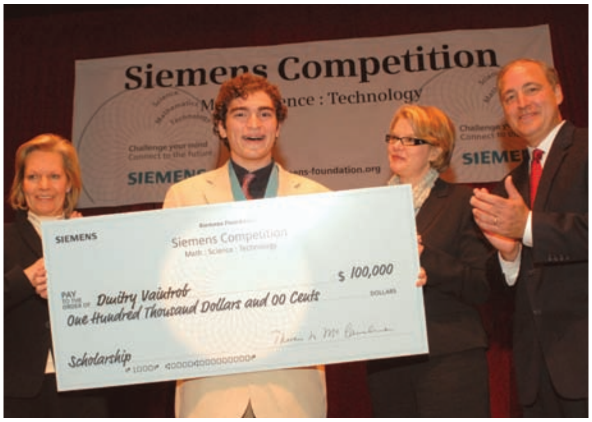
In 2006, three research mathematicians — Jonathan Kanke (Duke University), Kiran Kedlaya (MIT), and Paul Gunnells (University of Massachusetts at Amherst) — were invited to serve as mentors to work with students on advanced research projects. Topics for their research projects in the summer of 2006 were: *Quaternion Algebras*, proposed by Jonathan Hanke; *Combinators*, proposed by Ira Gessel; *Quadratic Forms and Quadratic Fields*, proposed by Jonathan Hanke; *Finiteness Theorems for Quadratic Forms*, proposed by Jonathan Hanke; and *Purely Periodic Continued Fractions*, proposed by Kiran Kedlaya.

Since 1989, Glenn Stevens has directed Boston University’s Program in Mathematics for Young Scientists (PROMYS). Professor Stevens is a Professor of Mathematics at Boston University, where he has taught and conducted research since 1984.

Awards & Honors

Dmitry Vaintrob wins Siemens Competition

On December 4, 2006, former Clay Research Academy student Dmitry Vaintrob from Eugene, Oregon, won top honors in the 2006–07 Siemens Competition in Math, Science and Technology, the nation's premier high school science competition. The Siemens Competition, a signature program of the Siemens Foundation, is administered by the College Board. The awards were presented by U.S. Secretary of Education Margaret Spellings at New York University, host of the 2006–07 Siemens Competition national finals.



From left to right: Bettina von Siemens; Siemens Competition Individual Winner Dmitry Vaintrob; U.S. Secretary of Education Margaret Spellings; George Nolen, President and CEO of Siemens Corporation Academy. Photo courtesy the Siemens Foundation.

Dmitry won the $100,000 Grand Prize scholarship in the individual category for exciting research in a new area of mathematics called string topology. His mentor for the competition was MIT mathematics professor Pavel Etingof, who coached Dmitry over a session of the Clay Research Academy in 2004. Pavel shares his impressions of Dmitry and recounts how such a young student came to win such an honor:

"Mitka is an amazing mathematical talent. At 18, he knows as much mathematics as graduate students at good universities in the beginning or even middle of their graduate studies. He is extremely creative, and extraordinarily gifted. He was in my representation theory group in the 2004 Clay Research Academy and did extremely well. In the summer of 2006 Mitka worked on a project at the Research Science Institute at MIT, under the joint supervision of Aaron Tievsky (an MIT mathematics graduate student) and myself. This was the most sophisticated mathematical research project by a high school student that I have ever seen. I suggested it to Mitka in June 2006. The project was to calculate explicitly the Hochschild cohomology of the group algebra of the fundamental group of a closed oriented surface (as a Batalin-Vilkovisky algebra) in terms of a certain Lie algebra of loops introduced by Goldman. This project could have been a part of a Ph.D. thesis in our graduate program, and requires a deep knowledge of graduate-level topology. Normally it would have been insane to give such a project to a high school student. But knowing Mitka's exceptional talent and accomplishments, I decided to give it a try, and the results were even better than I had expected. Mitka not only gave a complete solution to the problem, but took the initiative to generalize it from the case of surfaces to the case of higher-dimensional closed aspherical manifolds. In this case, he found that the answer is expressible via the the so-called string topology of the manifold, introduced in 1999 by Moira Chas and Dennis Sullivan. Thus in his work Mitka obtained original results, which will no doubt be of considerable interest to experts working in the area and are publishable in a high quality mathematics journal. These are all reasons to expect that he will become a major research mathematician."

Dmitry attending Pavel's course at the 2004 Clay Research Academy.

continued from page 15

Let w be a fixed word of finite length m, and choose and fix integers $n \geq m$, as well as a real number $x \in [0\,,1]$. We can then define $N_n^b(x\,;w)$ to be the number of times the word w appears continguously among $(x_1\,,\ldots,x_n)$. The reader is invited to verify that $N_n^{10}(0.5\,,\{5\}) = N_n^2(0.5\,,\{1\}) = 1$ for all $n \geq 1$.

A number x is said to be *simply normal in base* b if

$$\lim_{n\to\infty} \frac{N_n^b(x\,;\{j\})}{n} = \frac{1}{b} \qquad \text{for all letters } j \in \{0\,,\ldots,b-1\}. \tag{2}$$

That is, x is simply normal in base b when, and only when, all possible letters in the alphabet $\{0\,,\ldots,b-1\}$ are distributed equally in the b-ary representation of x. Balanced numbers are simply normal in base 2.

More generally, a number x is said to be *normal in base* b if given any finite word w with letters from the alphabet $\{0\,,\ldots,b-1\}$,

$$\lim_{n\to\infty} \frac{N_n^b(x\,;w)}{n} = \frac{1}{b^{|w|}}, \tag{3}$$

where $|w|$ denotes the length of the word m. The number $a = 0.101010\cdots$ is simply normal, but not normal, in base 2. This can be seen, for example, by inspecting the two-letter word "11." Still more generally, we say that $x \in [0\,,1]$ is *simply normal* if it is simply normal in all bases $b \geq 2$, and [absolutely] *normal* if it is normal in all bases $b \geq 2$. These definitions are all due to Borel [4].

The first nonperiodic numbers which are normal in some base b were constructed by Champernowne [9] in 1933. These were the numbers $C_2 = 0.1011011100101001110010111 0111\ldots$, $C_{10} = 0.1234567891011121314\ldots$. etc., obtained by concatenating the base b numerals in their natural order. Champernowne also conjectured that $0.13571113171923\ldots$, obtained by concatenating all primes, is simply normal in base 10. His conjecture was verified in 1946 by Copeland and Erdős [10].

It is possible to construct numbers that are simply normal in one base, but not in another. For example, the simply normal binary number $a = 0.101010\cdots$ is not normal in base 10, since $a = 2/3 = 0.\bar{6}$ in decimal notation.

The Champernowne numbers are admittedly artificial. Are there "natural" normal numbers? Although nothing is known, there are several conjectures. The first of these [5], due to Borel in 1950, states that all irrational algebraic numbers are normal; see also Mahler's 1976 lectures [19] wherein he proved, among other things, that Champernowne's number is transendental. Unfortunately, not much further progress has been made in this direction. For example, it is not known whether household numbers such as e, π, $\ln 2$, or $\sqrt{2}$ are simply normal in any given base. ($x > b$ is said to be [simply] normal in base b when x/b is [simply] normal in base b.) We do not even know if $\sqrt{2}$ has infinitely-many 5's [say] in its decimal expansion!

I hasten to add that there are compelling arguments that support the conjecture that e, π, $\sqrt{2}$, and a host of other nice algebraic irrationals, are indeed normal; see Bailey and Crandall [1].

The preceding examples, and others, were introduced in order to better understand the remarkable *normal number theorem* of Borel [4] from 1909:

Theorem 2.1 (Borel)**.** *Almost every number in* $[0\,,1]$ *is normal.*

The veracity of this result is now beyond question. However, to paraphrase Doob [11, p. 591], Borel's original derivation contains an "unmendably faulty" error. Borel himself was aware of the gap in his proof, and asked for a complete argument. His plea was answered a year later by Faber [14, p. 400], and also later by Hausdorff [16].

Theorem 2.1 suggests that it should be easy to find normal numbers. But I am not aware of any easy-to-describe numbers that are even simply normal. Recently, Becker and Figueira [2] have built on a constructive proof of Theorem 2.1, due to Sierpiński [25], to prove the existence of computable normal numbers. Their arguments suggest possible ways for successively listing out the digits of some normal numbers. But a direct implementation of this program appears to be at best arduous.

Borel's theorem is generally considered to be one of the first contributions to the modern theory of mathematical probability; a fact of which Borel himself was aware [4]. In order to describe this connection to probability, let us select a number X uniformly at random from the interval $[0\,,1]$. The key feature of this random selection process is that for all Borel sets $A \subseteq [0\,,1]$,

$$\mathrm{P}\{X \in A\} = \text{Lebesgue measure of } A, \tag{4}$$

where P denotes probability.

We can write X in b-ary form as $\sum_{j=1}^{\infty} X_j b^{-j}$. Borel's central observation was that $\{X_j\}_{j=1}^{\infty}$ is a collection of *independent* random variables, each taking the values 0, 1, ..., $b-1$ with equal probability. Then he proceeded to [somewhat erroneously] prove his *strong law of large numbers*, which was the first of its kind. Borel's law of large numbers states that for all letters $j \in \{0\,,\ldots,b-1\}$,

$$\mathrm{P}\left\{\lim_{n\to\infty} \frac{\mathbf{1}_{\{X_1=j\}} + \cdots + \mathbf{1}_{\{X_n=j\}}}{n} = \frac{1}{b}\right\} = 1, \tag{5}$$

where $\mathbf{1}_A$ denotes the characteristic function of A. It follows readily from (5) that with probability one X is simply normal in base b. Because there are only a countable number of integers $b \geq 2$, this proves that X is simply normal. Normality of X is proved similarly, but one analyses blocks of digits in place of single digits at a time.

Let $\mathscr{N}_b$ denote the collection of all numbers normal in base b. The preceding argument implies that $\mathrm{P}\{X \in \cap_{b=2}^{\infty} \mathscr{N}_b\} = 1$. This and (4) together imply Theorem 2.1.

We conclude this section by making a few more comments:

(1) In 1916 Weyl [27] described a tantalizing generalization of Theorem 2.1 that is nowadays called Weyl's equidistribution theorem. In this connection, we mention also the thesis of Wall [26]. (2) Riesz [22] devised a slightly more direct proof of Theorem 2.1. His derivation appeals to Birkhoff's ergodic theorem in place of Borel's (or more generally, Kolmogorov's) strong law of large numbers. But the general idea is not dissimilar to the proof outlined above. (3) The probabilistic interpretation of Theorem 2.1 has the following striking implication:

$$\text{Finite-state, finite-time random number generators do not exist.} \tag{6}$$

Of course, this does not preclude the possibility of generating a random number one digit at a time. But it justifies our present day use of *psuedo* random-number generators; see Knuth [17] for more on this topic. Remarkably, a complexity theory analogue to (6) completely characterizes all normal numbers; see Schnorr and Stimm [24] and Bourke, Hitchcock, and Vinochandran [6]. In this general direction, see also the interesting works of Chaitin [8] and Lutz [18].

(4) The proof of Borel's theorem is more interesting than the theorem itself, because it identifies the digits of a uniform random variable as independent and identically distributed. Such sequences have interesting properties that are not described by Theorem 2.1. Next we mention one of the many possible examples that support our claim.

Let $R_n(x)$ denote the length of the largest run of ones in the first n binary digits of x. [A run of ones is a continguous sequences of ones.] Then, according to a theorem of Erdős and Rényi [13] from

1970,

$$\lim_{n\to\infty} \frac{R_n(x)}{\log_2(n)} = 1 \qquad \text{for almost every } x \in [0\,,1]. \tag{7}$$

Because this involves words of arbitrarily large length, it is not a statement about normal number per se. There are variants of (7) that are valid in all bases, as well.

3. **Unbiased sampling**

As was implied earlier, one of the perplexing features of normal numbers is that they are abundant (Theorem 2.1), and yet we do not know of a single concrete number that is normal. This has puzzled many researchers, but appears to be a fact that goes beyond normal numbers, or even the usual structure of the real line.

Next we present an example that examines an analogous problem in a similar setting. This example suggests the following general principle: *Quite often, schemes that involve taking "unbiased samples from large sets" lead to notions of normality that are hard to pinpoint concretely.* I believe that this principle explains our inability in deciding whether or not a given number is normal. But I have no proof [nor disproof].

Let us consider the ternary Cantor set C, which we can think of as all numbers $x \in [0\,,1]$ whose ternary expansion $\sum_{j=1}^\infty x_j 3^{-j}$ consists only of digits $x_j \in \{0\,,2\}$.

In order to take an "unbiased sample" from C, it is necessary and sufficient to find a probability measure on C that is as "flat" as possible. [We are deliberately being vague here.] There are many senses in which the most flat probability measure on C can be identified with the restriction m_C of the usual $\log_3(2)$-dimensional Hausdorff measure to C. That is, m_C is the Cantor–Lebesgue measure. Now it is not difficult to show that m_C can be defined directly as follows:

$$m_C(A) := \mathrm{P}\left\{ \sum_{j=1}^\infty \frac{X_j}{3^j} \in A \right\} \qquad \text{for all Borel sets } A \subseteq [0\,,1], \tag{8}$$

where $X_1, X_2, \ldots$ are independent random variables, taking the values zero and two with probability $1/2$ each. A ready application of the strong law of large numbers then reveals that the following holds for m_C-almost every $x \in C$:

$$\lim_{n\to\infty} \frac{N_n^3(x\,;w)}{n} = \frac{1}{2^{|w|}} \qquad \text{for all words } w \in \bigcup_{k=1}^\infty \{0\,,2\}^k. \tag{9}$$

We say that a number $x \in C$ is normal in the Cantor set C if it satisfies (9). Although m_C-almost every number in C is normal in C, I am not aware of any concrete examples. On the other hand, I point out that we do not know very many concrete numbers in C at all—be they normal or otherwise. By analogy, this suggests the sightly uncomfortable fact that we do not know very many numbers—normal as well as non-normal—in $[0\,,1]$.

4. **Non-normal numbers**

At first glance, one might imagine that because normal numbers are so complicated, non-normal numbers are not. Unfortunately, this is not the case. We conclude this article by mentioning two striking results that showcase some of the complex beauty of non-normal numbers.

4.1. **Eggleston's theorem.** Let us choose and fix a base $b \geq 2$ and a probability vector $\boldsymbol{p} := (p_0, \ldots, p_{b-1})$; that is, $0 \leq p_j \leq 1$ and $p_0 + \cdots + p_{b-1} = 1$. Consider the set

$$\mathscr{E}(\boldsymbol{p}) := \left\{ x \in [0,1] : \lim_{n\to\infty} \frac{N_n^b(x\,; \{j\})}{n} = p_j \text{ for all } j = 0, \ldots, b-1 \right\}. \tag{10}$$

Note that if any one of the p_j's is different from $1/b$, then all elements of $\mathscr{E}(\boldsymbol{p})$ are non-normal. In 1949, Eggleston [12] confirmed a conjecture of I. J. Good by deriving the following result.

Theorem 4.1 (Eggleston)**.** *The Hausdorff dimension of $\mathscr{E}(\boldsymbol{p})$ is precisely the thermodynamic entropy*

$$H(\boldsymbol{p}) := -\sum_{j=0}^{b-1} p_j \log_b(p_j), \tag{11}$$

where $0 \times \log_b(0) := 0$.

This theorem is true even if $p_0 = \cdots = p_{b-1} = 1/b$, but yields a weaker result than Borel's theorem in that case. Ziv and Lempel [29] developed related ideas in the context of source coding.

4.2. **Cassels's theorem.** For the second, and final, example of this article we turn to a striking theorem of Cassels [7] from 1959:

Theorem 4.2 (Cassels)**.** *Define the function* $f : [0,1] \to \mathbf{R}$ *by*

$$f(x) := \sum_{j=1}^{\infty} \frac{x_j}{3^j}, \tag{12}$$

where $x_1, x_2, \ldots$ denote the binary digits of x. Then, for almost every $x \in [0,1]$, $f(x)$ is simply normal with respect to every base b that is not a power of 3.

It is manifestly true that Cassels's $f(x)$ is not normal in bases 3, 9, etc. Hence, non-normal numbers too have complicated structure. We end our discussion by making two further remarks:

(1) Cassels's theorem answered a question of Hugo Steinhaus, and was later extended by Schmidt [23]. See Pollington [20] for further developments.

(2) Because $2f$ is a bijection between $[0,1]$ and the Cantor set C, Cassels's theorem constructs an uncountable number of points in $\frac{1}{2}C$ that are simply normal with respect to every base b that is not a power of 3. Not surprisingly, we do not have any concrete examples of such numbers.

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Nominations, Proposals and Applications

The Clay Mathematics Institute at One Bow Street in Harvard Square (winter).

Nominations for Senior and Research Scholars are considered four times a year at our Scientific Advisory Board (SAB) meetings. Principal funding decisions for Senior Scholars are made at the September SAB meeting. Additional nominations will be considered at other times as funds permit. Clay Liftoff and Clay Research Fellow nominations are considered once a year and must be submitted according to the schedule below:

Nomination Deadlines

Senior Scholars: August 1
Research Fellows: October 30
Liftoff Scholars: February 15

Send to the attention of CMI's Program Manager at nominations@claymath.org

Nominations can also be mailed to

Clay Mathematics Institute
One Bow Street
Cambridge, MA 02138

The mathematical library of Raoul Bott at One Bow Street.

The Clay Mathematics Institute invites proposals for conferences and workshops. Proposals will be judged on their scientific merit, probable impact, and potential to advance mathematical knowledge.

Proposals should be accompanied by a budget and the standard coversheet. This material should be submitted one month prior to the board meeting at which it will be considered. Please send to the attention of CMI's Program Manager at proposals@claymath.org

Proposals can also be mailed to

Clay Mathematics Institute
One Bow Street
Cambridge, MA 02138

Please find more information and the standard cover sheet at

http://www.claymath.org/proposals

Noteworthy proposals will be considered at other times. However, most funding decisions will be made with respect to the deadlines below.

Proposal Deadlines

Workshops & Conferences: August 1, February 15
Bow Street Workshops: 6 months prior

Selected Articles by Research Fellows

ARTUR AVILA

"Reducibility or non-uniform hyperbolicity for quasiperiodic Schrödinger cocycles," with R. Krikorian. *Annals of Mathematics* 164 (2006), 911–940.

"Exponential mixing for the Teichmüller flow," with S. Gouëzel and J.-C. Yoccoz. *Publications Mathématiques de l'IHÉS* 104 (2006), 143–211.

DANIEL BISS

"Large annihilators in Cayley-Dickson algebras," with Dan Dugger & Dan Isaksen. arXiv.org/abs/math.RA/0511691.

"Large annihilators in Cayley-Dickson algebras II," with Dan Christensen, Dan Dugger, and Dan Isaksen. arXiv.org/abs/math.RA/0702075

MARIA CHUDNOVSKY

"The Strong Perfect Graph Theorem," with N. Robertson, P. Seymour and R. Thomas. *Annals of Mathematics* 164 (2006), 51–229.

"The Roots of the Independence Polynomial of a Clawfree Graph," with Paul Seymour. To appear in *Journal of Combinatorial Theory*, Ser. B. Lecture Note Series, vol. 327.

BEN GREEN

"Linear Equations in primes," with Terence Tao. To appear in *Annals of Mathematics*.

"A quantitative version of the idempotent theorem in harmonic analysis," with T. Sanders. To appear in *Annals of Mathematics*.

BO'AZ KLARTAG

"A central limit theorem for convex sets." *Invent. Math.* 168, (2007), 91–131.

"An example related to Whitney extension with almost minimal Cm norm," with Charles Fefferman. Manuscript.

CIPRIAN MANOLESCU

"A combinatorial description of knot Floer homology," with P. Ozsváth and S. Sarkar. arXiv.org/abs/math.GT/0607691. Submitted to *Annals of Mathematics*.

"An unoriented skein exact triangle for knot Floer homology." To appear in *Mathematical Research Letters*.

MARYAM MIRZAKHANI

"Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces." *Invent. Math.* 167 (2007), no. 1, 179–222.

"Random Riemann surfaces and measured laminations." To appear in *Proceedings of the Ahlfors-Bers Colloquium*.

SAMUEL PAYNE

"Equivariant Chow cohomology of toric varieties." *Mathematical Research Letters* 13 (2006), no. 1, 29–41.

"Ehrhart polynomials and stringy Betti numbers," with M. Mustata. *Annuals of Mathematics* 333 (2005), no. 4, 787–795.

DAVID SPEYER

"A Broken Circuit Ring," with Nick Proudfoot. *Beiträge zur Algebra und Geometrie* 47 (2006), no. 1, 161–166.

"Computing Tropical Varieties with T. Bogart," with A. N. Jensen, B. Sturmfels and R. R. Thomas. *Journal of Symbolic Computation* 42, Issue 1-2 (January 2007) 54–73.

ANDRÁS VASY

"Propagation of singularities for the wave equation on edge manifolds," with R. Melrose, and J. Wunsch. Submitted for publication. Available online at arXiv.org/abs/math.AP/0612750.

"Propagation of singularities for the wave equation on manifolds with corners." To appear in *Annals of Mathematics*.

AKSHAY VENKATESH

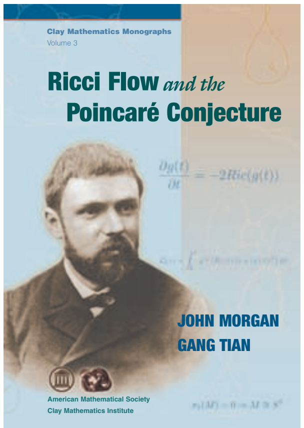
"Reflection principles and bounds for class group torsion," with Jordan Ellenberg. To appear in *International Mathematics Research Notices*.

"The distribution of periodic torus orbits on homogeneous spaces," with M. Einsiedler, E. Lindenstrauss, and P. Michel. arXiv.org/abs/math.DS/0607815.

Publications

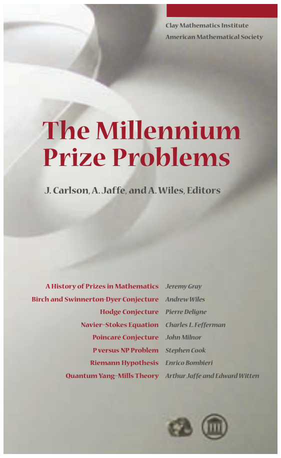
Books & Videos

Analytic Number Theory; A Tribute to Gauss and Dirichlet; Editors: William Duke, Yuri Tschinkel. This volume contains the proceedings of the Gauss-Dirichlet Conference held in Göttingen, June 20–24 in 2005, commemorating the 150th anniversary of the death of Gauss and the 200th anniversary of Dirichlet's birth. It begins with a definitive summary of the life and work of Dirichlet by J. Elstrodt and continues with thirteen papers by leading experts on research topics of current interest within number theory that were directly influenced by Gauss and Dirichlet.

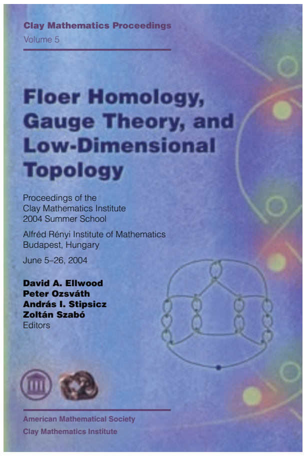


Ricci Flow and the Poincaré Conjecture; Authors: John Morgan, Gang Tian. This book presents a complete and detailed proof of the Poincaré Conjecture. This conjecture was formulated by Henri Poincaré in 1904 and has remained open until the recent work of Grigori Perelman. The arguments given in the book are a detailed version of those that appear in Perelman's three preprints.

The Millennium Prize Problems; Editors: James Carlson, Arthur Jaffe, Andrew Wiles. CMI/AMS, 2006, 165 pp. www.claymath.org/publications/Millennium_Problems. This volume gives the official description of each of the seven problems as well as the rules governing the prizes. It also contains an essay by Jeremy Gray on the history of prize problems in mathematics.



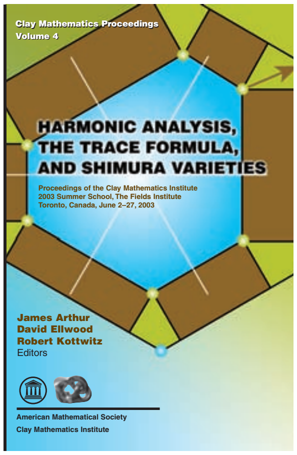
Floer Homology, Gauge Theory, and Low-Dimensional Topology; Proceedings of the 2004 CMI Summer School at Rényi Institute of Mathematics, Budapest. Editors: David Ellwood, Peter Ozsváth, András Stipsicz, and Zoltán Szábo. CMI/AMS, 2006, 297pp. www.claymath.org/publications/Floer_Homology. This volume grew out of the summer school that took place in June of 2004 at the Alfréd Rényi Institute of Mathematics in Budapest, Hungary. It provides a state-of-the-art introduction to current research, covering material from Heegaard Floer homology, contact geometry, smooth four-manifold topology, and symplectic four-manifolds.



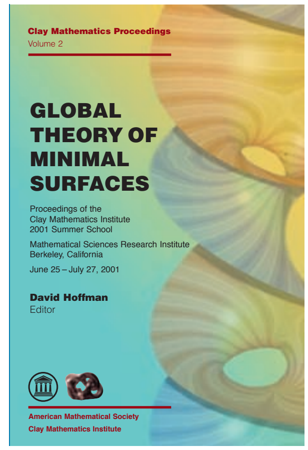
Lecture Notes on Motivic Cohomology; Authors: Carlo Mazza, Vladimir Voevodsky, Charles Weibel. CMI/AMS, 2006, 210 pp. http://www.claymath.org/publications/Motivic_Cohomology. This book provides an account of the triangulated theory of motives. Its purpose is to introduce the reader to Motivic Cohomology, develop its main properties and finally to relate it to other known invariants of algebraic varieties and rings such as Milnor K-theory, étale cohomology and Chow groups.

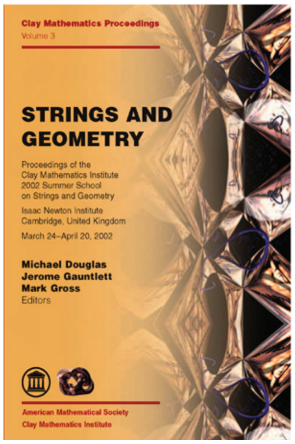
Surveys in Noncommutative Geometry; Editors: Nigel Higson, John Roe. CMI/AMS, 2006, 189 pp. www.claymath.org/publications/Noncommutative_Geometry. In June 2000 a summer school on Noncommutative Geometry, organized jointly by the American Mathematical Society and the Clay Mathematics Institute, was held at Mount Holyoke College in Massachusetts. The meeting centered around several series of expository lectures intended to introduce key topics in noncommutative geometry to mathematicians unfamiliar with the subject. Those expository lectures have been edited and are reproduced in this volume.

Harmonic Analysis, the Trace Formula and Shimura Varieties; Proceedings of the 2003 CMI Summer School at Fields Institute, Toronto. Editors: James Arthur, David Ellwood, Robert Kottwitz. CMI/AMS, 2005, 689 pp. www.claymath.org/publications/Harmonic_Analysis. The subject of this volume is the trace formula and Shimura varieties. These areas have been especially difficult to learn because of a lack of expository material. This volume aims to rectify that problem. It is based on the courses given at the 2003 Clay Mathematics Institute Summer School. Many of the articles have been expanded into comprehensive introductions, either to the trace formula or the theory of Shimura varieties, or to some aspect of the interplay and application of the two areas.



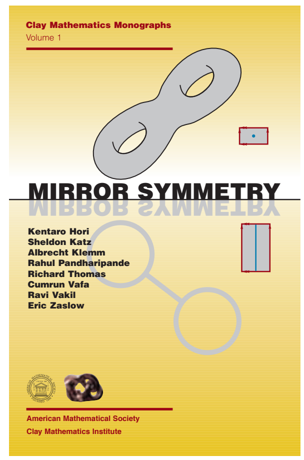
Global Theory of Minimal Surfaces. Proceedings of the 2001 CMI Summer School at MSRI. Editor: David Hoffman. CMI/AMS, 2005, 800 pp. www.claymath.org/publications/Minimal_Surfaces. This book is the product of the 2001 CMI Summer School held at MSRI. The subjects covered include minimal and constant-mean-curvature submanifolds, geometric measure theory and the double-bubble conjecture, Lagrangian geometry, numerical simulation of geometric phenomena, applications of mean curvature to general relativity and Riemannian geometry, the isoperimetric problem, the geometry of fully nonlinear elliptic equations, and applications to the topology of three-manifolds.





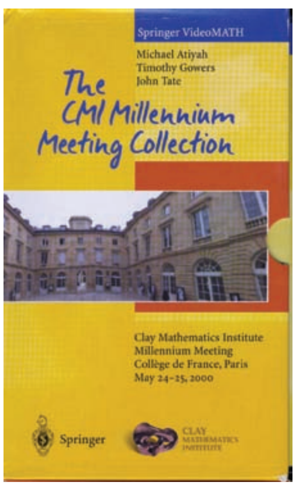
Strings and Geometry. Proceedings of the 2002 CMI Summer School held at the Isaac Newton Institute for Mathematical Sciences, UK. Editors: Michael Douglas, Jerome Gauntlett and Mark Gross. CMI/AMS publication, 376 pp., Paperback, ISBN 0-8218-3715-X. List: $69. AMS Member: $55. Order code: CMIP/3. To order, visit www.ams.org/bookstore.

Mirror Symmetry. Authors: Kentaro Hori, Sheldon Katz, Albrecht Klemm, Rahul Pandharipande, Richard Thomas, Ravi Vakil. Editors: Cumrun Vafa, Eric Zaslow. CMI/AMS publication, 929 pp., Hardcover. ISBN 0-8218-2955-6. List: $124. AMS Members: $99. CMIM/1. To order, visit www.ams.org/bookstore.



Strings 2001. Authors: Atish Dabholkar, Sunil Mukhi, Spenta R. Wadia. Tata Institute of Fundamental Research. Editor: American Mathematical Society (AMS), 2002, 489 pp., Paperback, ISBN 0-8218-2981-5, List $74. AMS members: $59. Order code: CMIP/1. To order, visit www.ams.org/bookstore.

Video Cassettes



The CMI Millennium Meeting Collection. Authors: Michael Atiyah, Timothy Gowers, John Tate, François Tisseyre. Editors: Tom Apostol, Jean-Pierre Bourguignon, Michele Emmer, Hans-Christian Hege, Konrad Polthier. Springer VideoMATH, © Clay Mathematics Institute, 2002. Box set consists of four video cassettes: The CMI Millennium Meeting, a film by François Tisseyre; The Importance of Mathematics, a lecture by Timothy Gowers; The Millennium Prize Problems, a lecture by Michael Atiyah; and The Millennium Prize Problems, a lecture by John Tate. VHS/NTSC or PAL. ISBN 3-540-92657-7, List: $119, EUR 104.95. To order, visit www.springer-ny.com (in the United States) or www.springer.de (in Europe).

These videos document the Paris meeting at the Collège de France where CMI announced the Millennium Prize Problems. For anyone who wants to learn more about these seven grand challenges in mathematics.

Videos of the 2000 Millennium event are available online and in VHS format from Springer-Verlag. To order the box set or individual tapes, visit www.springer.com.

2007 Institute Calendar

JANUARY FEBRAURY MARCH	Semester Long Program in Symplectic Topology at MIT. January, Spring Semester Senior Scholar Gang Tian at MSRI: Program on Geometric Evolution Equations. January–March School and Workshop in the Geometry and Topology of Singularities at CIMAT. January 8–27 Senior Scholar Peng Lu at MSRI: Program on Geometric Evolution Equations. January 8–March 30 Diophantine and Analytic Problems in Number Theory Conference at Moscow Lomonosov University. January 29–Feb 2 Homological Mirror Symmetry and Applications Conference at IAS. January 1–April 30 Loday and Stanley Workshop on Hopf Algebras and Props at CMI. March 5–9 Conference on Hilbert's 10th Problem at CMI, including a preview screening of George Csicsery's film on Julia Robinson at the Museum of Science, Boston. March 15–16 Motives and Algebraic Cycles: A Conference dedicated to the Mathematical Heritage of Spencer J. Bloch at the Fields Institute. March 19–23
APRIL MAY	Noncommutative Geometry at IHES in Paris. April 2–7 Workshop on Symplectic Topology at CMI. April 20–22 Clay Public Lecture by Ingrid Daubechies: Surfing with Wavelets. Stata Center at MIT, Kirsch Auditorium. April 10 Advances in Algebra and Geometry conference at MSRI in Berkeley, CA. April 29–May 5 Clay Research Conference, Harvard University Science Center, Cambridge. May 14–15
JUNE JULY AUGUST	Geometry and Imagination Conference at Princeton University. June 7–11 Summer School on Serre's Modularity Conjecture at CIRM (Marseille, France). June 7–20 Dynamics and Number Theory CMI Summer School in Pisa, Italy. June 11–July 6 Senior Scholar Andrei Okounkov at PCMI Program on Statistical Mechanics. July 1–21 Srinivasa Varadhan at PCMI Program on Statistical Mechanics. July 1–21 Infinite Dimensional Algebras and Quantum Integrable Systems II Conference, University of the Algarve, Faro, Portugal. July 23–27 Conference On Certain *L*-Functions at Purdue University. July 30–August 3 Alex Eskin, MSRI Program on Teichmuller Theory and Kleinian Groups. August 10–December 14
SEPTEMBER OCTOBER NOVEMBER DECEMBER	Solvability and Spectral Instability at CMI. September 18–21 Clay Public Lecture by Terence Speed (Department of Statistics, UC Berkeley and Division of Genetics and Bioinformatics, Walter and Eliza Hall Institute of Medical Research, Melbourne, Australia) at the Harvard Science Center. October 30 Workshop on Geometry of Moduli Spaces of Rational Curves with applications to Deophantine Problems over Function Fields at CMI. November Clay Lecture Series at the Tata Institute of Fundamental Research (TIFR) in Mumbai, India. December 11–14