Associahedron

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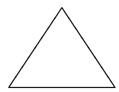
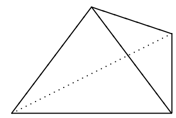
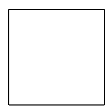
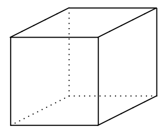
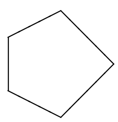
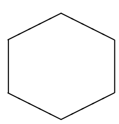
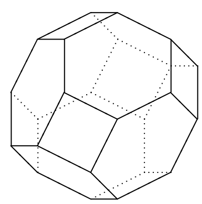
Polytopes

simplex			$\cdots$
cube			$\cdots$
?		?	??
permutohedron			$\cdots$
$n =$	2	3	$\cdots$

Permutohedron := convex hull of $(n+1)!$ points

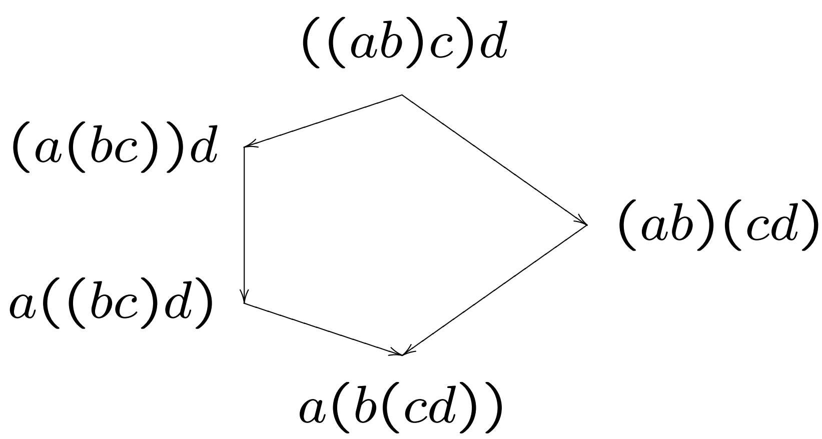
$$(\sigma(1), \ldots, \sigma(n+1)) \in \mathbb{R}^{n+1}$$

Parenthesizing

X= topological space with product $(a, b) \mapsto ab$

Not associative but associative up to homotopy

With four elements:



We suppose that there is a homotopy between the two composite paths, and so on.

Jim Stasheff

Staheff's result (1963): There exists a cellular complex such that
– vertices in bijection with the parenthesizings
– edges in bijection with the homotopies
– 2-cells in bijection with homotopies of composite homotopies
– etc,

and which is homeomorphic to a ball.

Problem: construct explicitely the Stasheff complex in any dimension.

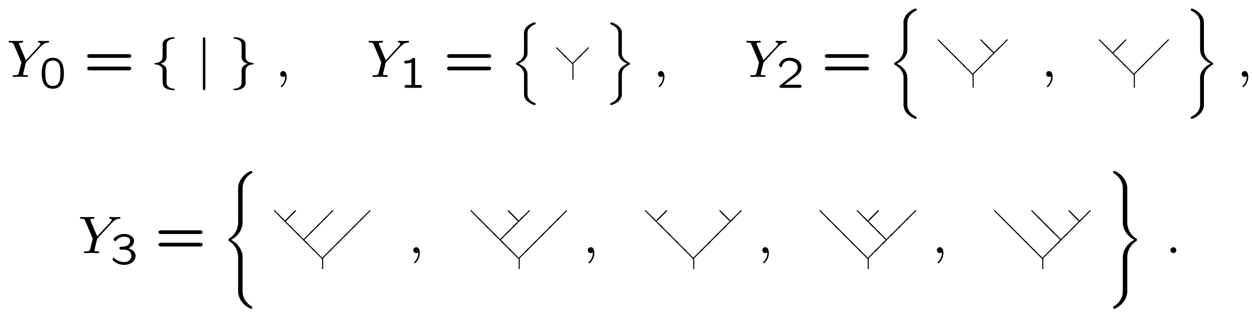
??

$n =$ 0 1 2 3

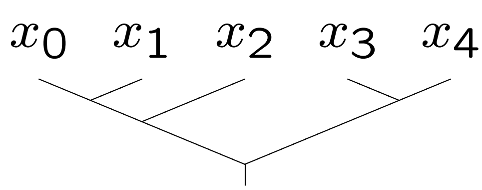
Planar binary trees

(see R. Stanley's notes p. 189)

Planar binary trees with $n+1$ leaves, that is n internal vertices:

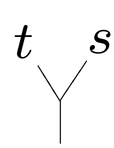


Bijection between planar binary trees and parenthesizings:



$$(((x_0x_1)x_2)(x_3x_4))$$

The notion of *grafting*

$t \vee s =$ 

Associahedron

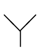
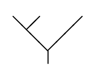
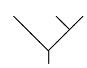
To $t \in Y_n$ we associate $M(t) \in \mathbb{R}^n$:

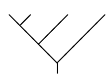
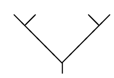
$$M(t) := (a_1 b_1, \cdots, a_i b_i, \cdots, a_n b_n) \in \mathbb{R}^n$$

$a_i =$ # leaves on the left side of the ith vertex

$b_i =$ # leaves on the right side of the ith vertex

Examples:

$M($  $) = (1)$, $M($  $) = (1,2)$, $M($  $) = (2,1)$,

$M($  $) = (1,2,3)$, $M($  $) = (1,4,1)$.

For the tree corresponding to $(((x_0x_1)x_2)(x_3x_4))$:

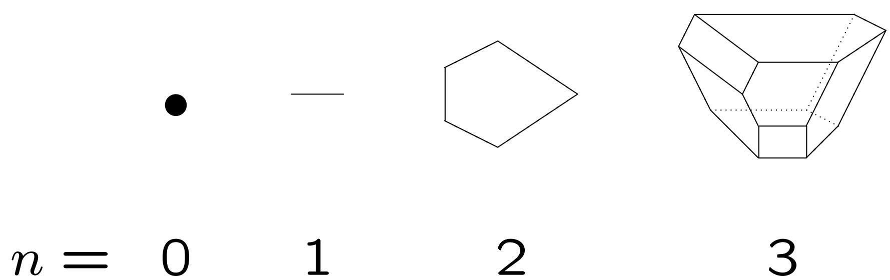
$$(1 \times 1, 2 \times 1, 3 \times 2, 1 \times 1) = (1,2,6,1)$$

Definition of the **associahedron**:

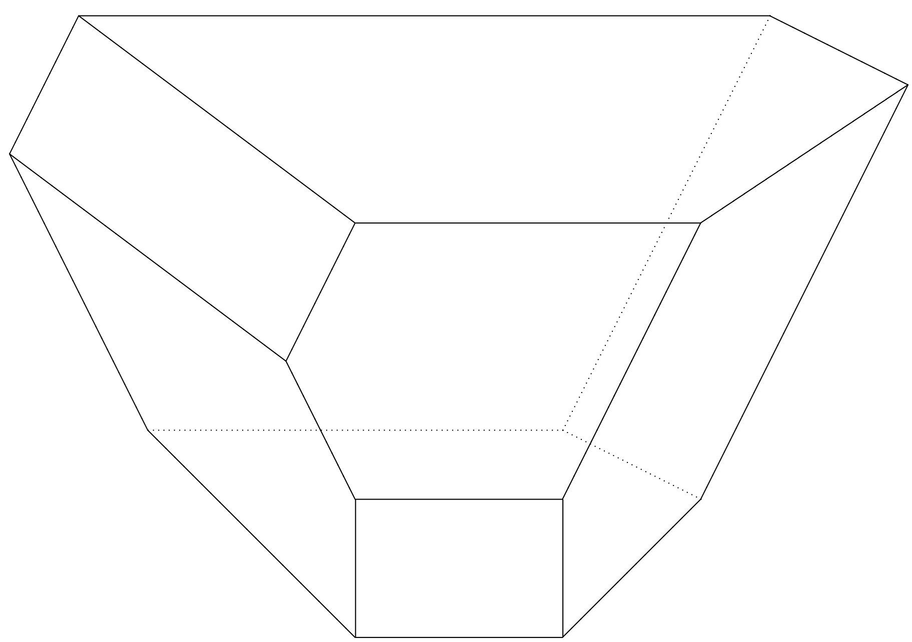
$$\mathcal{K}^{n-1} := \text{ convex hull of } M(t), t \in Y_n$$

Stasheff polytope

Theorem The associahedron is isomorphic to the Stasheff complex as a cellular complex.



$\mathcal{K}^3$



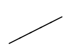
$\mathcal{K}^3$

Construction of $\mathcal{K}^{n+1}$ out of $\mathcal{K}^n$

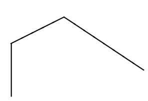
- Start with $\mathcal{K}^n$, boundary = cellular sphere
- cells of the boundary of the form $\mathcal{K}^p \times \mathcal{K}^q$ where $p+q=n-1$
- enlargement of $\mathcal{K}^p \times \mathcal{K}^q$, make it $\mathcal{K}^p \times \mathcal{K}^{q+1}$
- take the cone over the resulting space
- check that this is $\mathcal{K}^{n+1}$.

Example $n=1$:

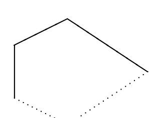
– $\mathcal{K}^1$



– $\mathcal{K}^1$ enlarged

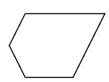


– Cone over $\mathcal{K}^1$ enlarged $= \mathcal{K}^2$

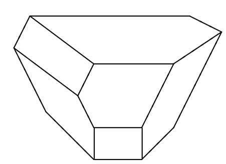


Example $n = 2$:

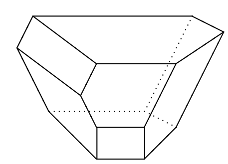
– $\mathcal{K}^2$



– $\mathcal{K}^2$ enlarged



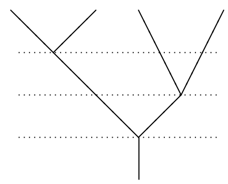
– Cone over $\mathcal{K}^2$ enlarged $= \mathcal{K}^3$



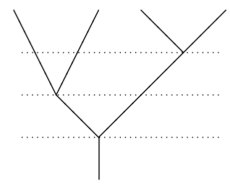
Exercise: # of simplices in $\mathcal{K}^n$ is $(n + 1)^{n-1}$.

Associahedron and permutohedron

$\tilde{Y}_n =$ set of p.b. *leveled* trees with $n+1$ leaves



vs

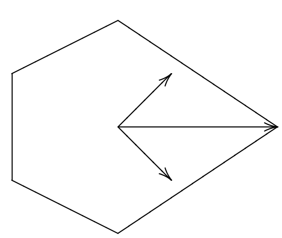


$$\phi : S_n \cong \tilde{Y}_n \longrightarrow Y_n$$

(forget the levels)

Proposition *Let* $C =$ *center of* $\mathcal{P}^{n-1}$
$C = (\frac{n+1}{2}, \ldots, \frac{n+1}{2})$. *Then on has*

$$\overrightarrow{CM(t)} = \sum_{\sigma \in \phi^{-1}(t)} \overrightarrow{CM(\sigma)} \ .$$



Inversion of power series

$$f(x) = x + a_1 x^2 + a_2 x^3 + \cdots + a_n x^{n+1} + \cdots$$

$$g(x) = x + b_1 x^2 + b_2 x^3 + \cdots + b_n x^{n+1} + \cdots$$

such that $f(g(x)) = x$

$b_n =$ polynomial in the coefficients $a_i, 1 \leq i \leq n$

$$\begin{aligned}
b_1 &= -a_1 \\
b_2 &= 2a_1^2 - a_2 \\
b_3 &= -5a_1^3 + 5a_1 a_2 - a_3 \\
b_4 &= 14a_1^4 - 21a_1^2 a_2 + 6a_1 a_3 + 3a_2^2 - a_4 \\
\cdots &= \cdots \\
b_n &= \sum (-1)^{\sum n_i} \lambda(n_1, \ldots, n_k) a_1^{n_1} \cdots a_k^{n_k}
\end{aligned}$$

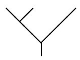
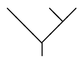
where $n_1 + 2n_2 + \cdots + kn_k = n$

Claim: $\lambda(n_1, \ldots, n_k) = \#$ cells in $\mathcal{K}^{n-1}$ isomorphic to $(\mathcal{K}^0)^{n_1} \times \cdots \times (\mathcal{K}^{k-1})^{n_k}$

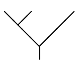
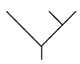
Examples: $\lambda(0, \ldots, 0, 1) = 1$
$\lambda(n) =$ Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$

Poset structure

Partial order on the set Y_n of p.b. trees

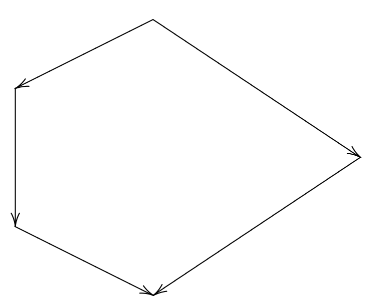
In Y_2:  $\longrightarrow$ 

In Y_n: change, locally in the tree t,

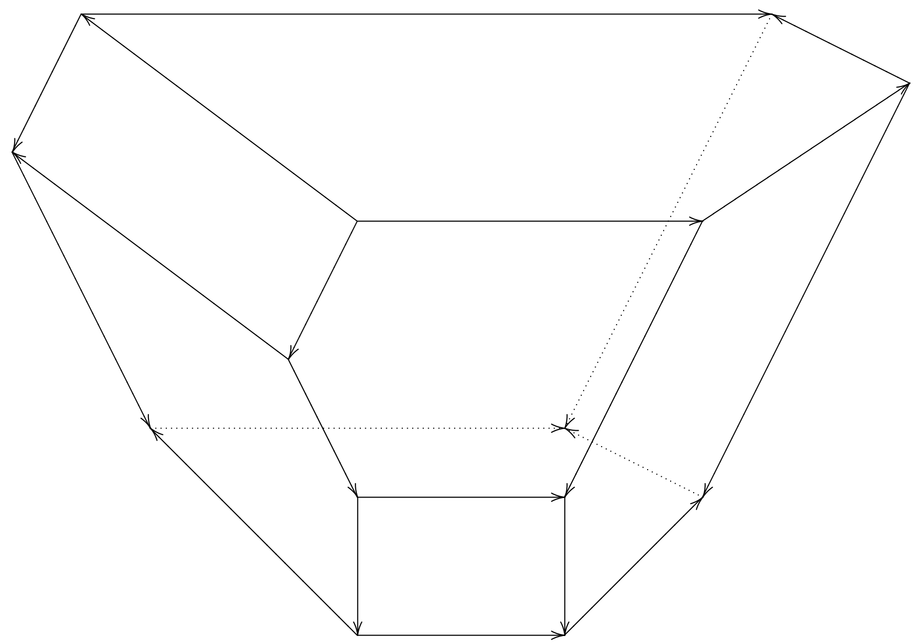
 into  to get s

covering relation: $t \to s$

Examples:



Poset structure of Y_4 on $\mathcal{K}^3$



Algebraic structure

$K[Y_n]$ = vector space over K spanned by p.b.trees having n vertices

Define inductively an operation on $\bigoplus_{n\geq 0} K[Y_n]$,

$$t * s := t^l \vee (t^r * s) + (t * s^l) \vee s^r, \qquad | = 1$$

Example:

Prop *The operation $*$ is associative and unital*

Theorem

$$t * s = \sum_{t/s \leq x \leq t\backslash s} x$$

t/s "over" operation, $t\backslash s$ "under" operation

Dendriform algebras

Define $t \prec s := t^l \vee (t^r * s)$ and $t \succ s := (t * s^l) \vee s^r$, so $t * s = t \prec s + t \succ s$

Prop *The operations $\prec$ and $\succ$ satisfy the following relations*

$$\begin{cases} (x \prec y) \prec z &= x \prec (y * z), \\ (x \succ y) \prec z &= x \succ (y \prec z), \\ (x * y) \succ z &= x \succ (y \succ z). \end{cases}$$

Definition A *dendriform algebra* is a vector space A over K equipped with two operations $\prec$ and $\succ$ satisfying the three relations above.

Theorem *The dendriform algebra* $(\bigoplus_{n \geq 0} K[Y_n], \prec, \succ)$ *is the free dendriform algebra on one generator, namely the tree* $\curlyvee$ *.*

Hint: $t \vee s = t \succ \curlyvee \prec s$

Applications of dendriform algebras

The dendriform algebras are involved in many topics:

- shuffles and noncommutative shuffles,

- preLie and brace algebras (algebraic topology),

- Hopf algebras, noncommutative version of Connes and Kreimer (theoretical physics),

- combinatorics (nc symmetric functions)

- arithmetic of trees (arithmetree)

- series indexed by trees (differential equations)

Dendriform and preLie

Definition preLie algebra: $(A, \circ)$ such that
$(x \circ y) \circ z - x \circ (y \circ z) = (x \circ z) \circ y - x \circ (z \circ y)$

Claim 1: $[x, y] := x \circ y - y \circ x$ is a Lie bracket

Claim 2: $x \circ y := x \prec y - y \succ x$ is a preLie product

$$\begin{array}{ccc} Dend & \longrightarrow & preLie \\ \downarrow & & \downarrow \\ As & \longrightarrow & Lie \end{array}$$

Proof.

$$x \prec y + x \succ y$$

$$x * y - y * x = \qquad\qquad = x \circ y - y \circ x$$

$$-y \succ x - y \prec x$$

Series indexed by trees

Power series:
$f(x) = a_1 x + a_2 x^2 \cdots + a_n x^n + \cdots , \quad n \in \mathbb{N}$

Dendriform series:
$f(x) = a_1 x + \cdots + a_t x^t + \cdots , \quad t \in Y_\infty$

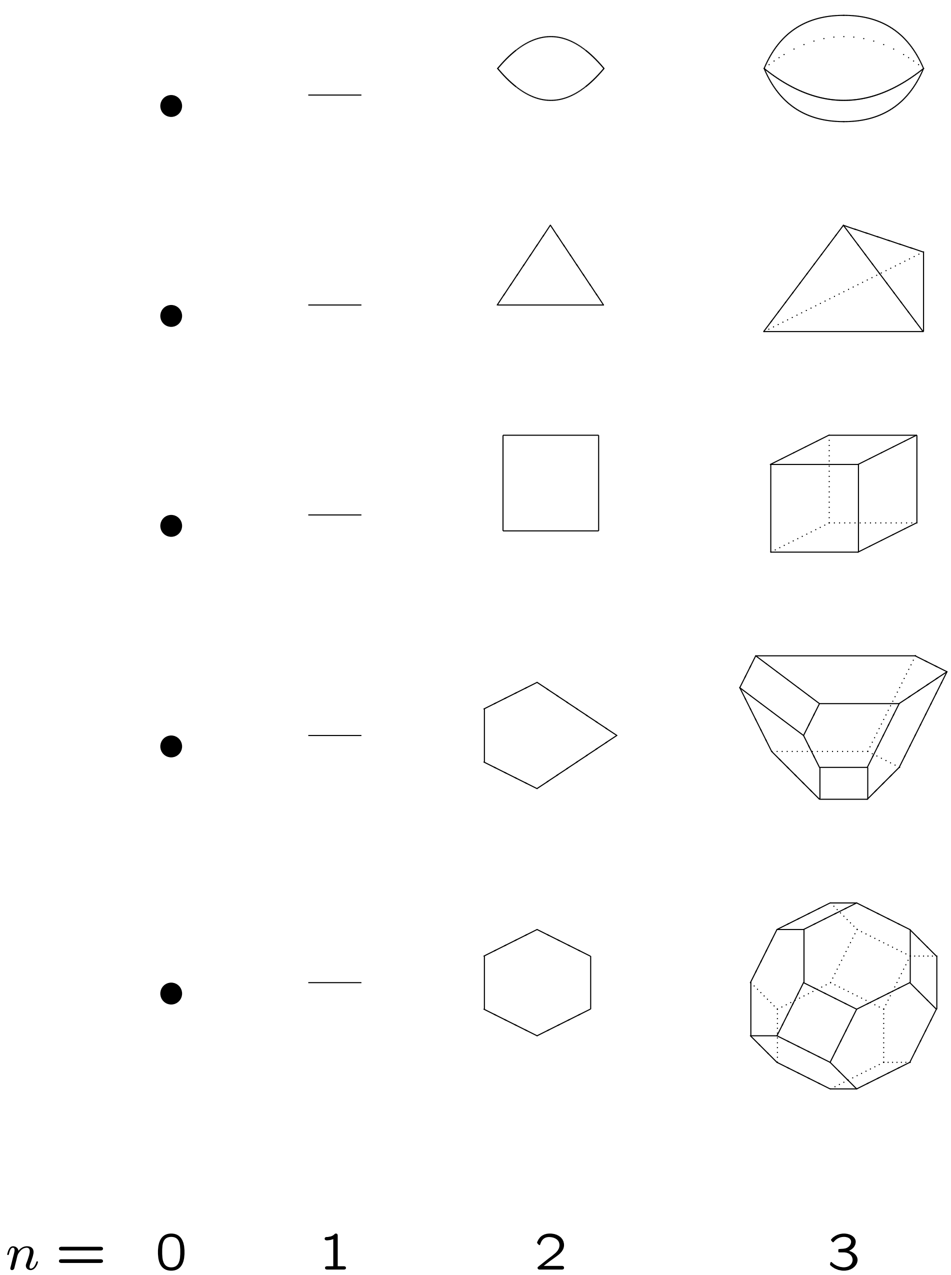
• Addition: OK (term by term),

• Multiplication: $x^t x^s = x^{t*s}$

• Composition: $f(g(x)) =$? consequence of the Theorem about freeness:

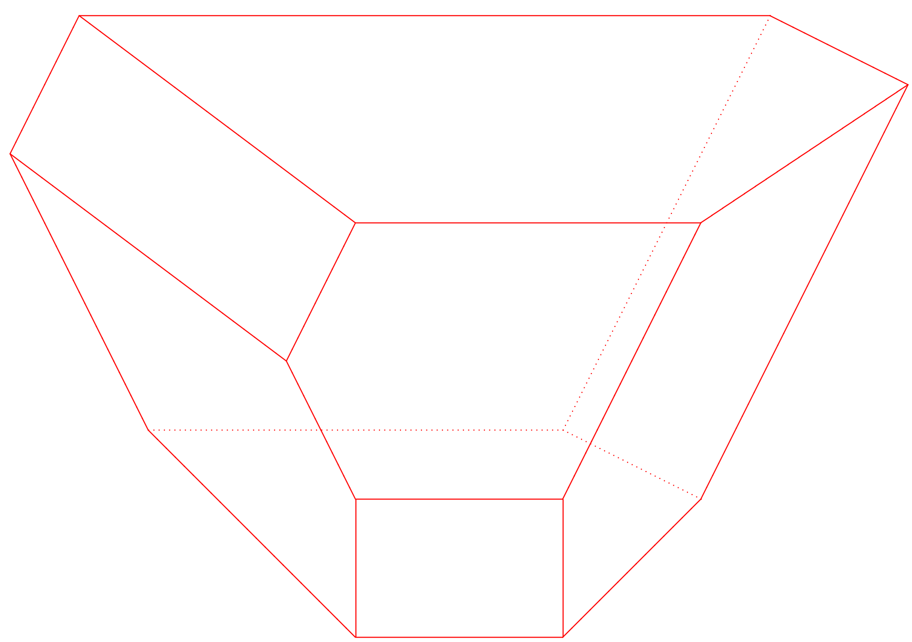
what is $g(x)^t$ for a p.b. tree t ? Write t as (generalized) product of the generator tree $\curlyvee$, then replace $\curlyvee$ by $g(x)$ and compute.

Families of polytopes



End

Many thanks for your attention !



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Associahedron

Jean-Louis Loday