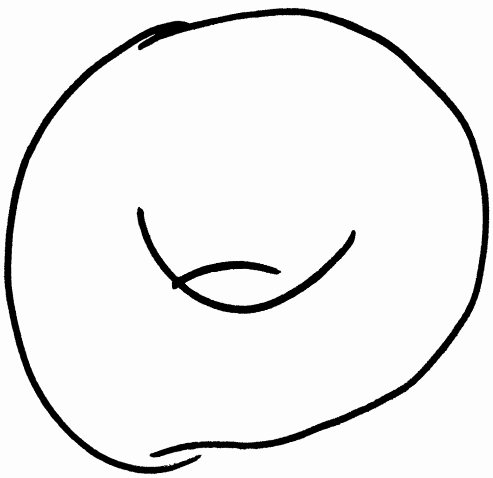
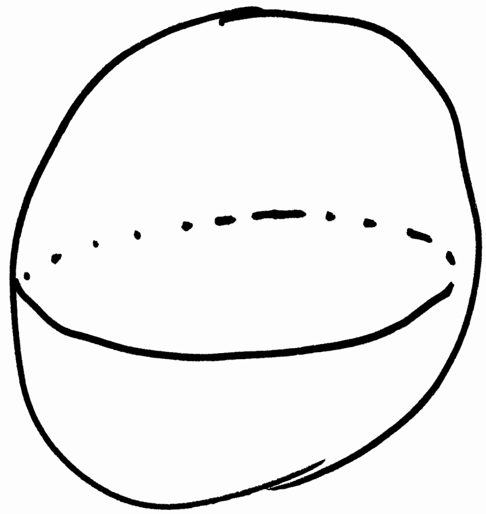
HOMOTOPY





T^2 $\qquad$ S^2

$$Y \underset{f_1}{\overset{f_0}{\rightrightarrows}} X \qquad \text{homotopic}$$

$$\iff \exists\, f : Y \times [0,1] \longrightarrow X$$

with $f_i(y) = f(y, i)$

$i = 0, 1$

$X \sim Y$ (same homotopy type)

$\Longleftrightarrow \exists\, X \underset{g}{\overset{f}{\rightleftarrows}} Y$

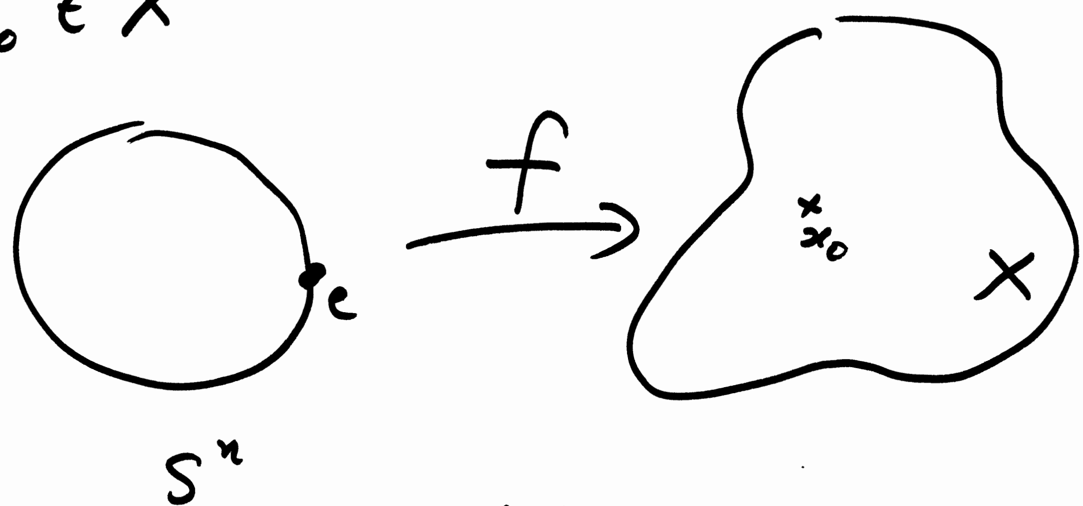
s.t. $f \circ g \sim Id_Y$

$g \circ f \sim Id_X$

classifying homotopy types is a difficult problem!

HOMOTOPY GROUPS

$x_0 \in X$



$f(e) = x_0$

$\Rightarrow \pi_n(X, x_0) =$ set of hty classes of such maps

set if $n = 0$
group if $n = 1$
abelian group if $n > 1$

Sometimes, we write $\pi_n(X)$

Computing hty groups is a difficult task!!

$$\pi_n(S^2) = 0 \quad n < 2$$
$$= \mathbb{Z} \quad n = 2$$
$$= ? \quad n > 2$$

Partial results (Serre)

1) $\pi_n(S^2)$ is a f.g. group

2) $\pi_{4n-1}(S^{2n}) = \mathbb{Z} \oplus$ finite gp

3) $\pi_i(S^n)$ is finite if $i > n$ except case 2)

$GL_n(\mathbb{C})$ = gp of invertible $n \times n$ complex matrices

$GL_n(\mathbb{R})$ = same (real) matrices

<u>Thm (Bott)</u> (n large enough)

$$\pi_i(GL_n(\mathbb{C})) \simeq \pi_{i+2}(GL_n(\mathbb{C}))$$

$$(n > \tfrac{i}{2} + 1)$$

$$\pi_i(GL_n(\mathbb{R})) \simeq \pi_{i+8}(GL_n(\mathbb{R}))$$

$$(n > i + 8)$$

"homogeneous spaces"

$$G/H$$

$$g_1 \sim g_2 \iff g_2^{-1} g_1 \in H$$

$$GL_n(\mathbb{C}) / GL_n(\mathbb{R})$$

$$GL_{2n}(\mathbb{R}) / GL_n(\mathbb{C})$$

$$i : GL_n(\mathbb{C}) \longrightarrow GL_{2n}(\mathbb{R})$$

$$A + iB \longmapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

$BG =$ classifying space of G

Case $GL_n = G$

Take the union of "Grassmanians"

or take a Hilbert space H

$p : H \to H$, $p^2 = p$, $rk\,p = n$

$\Omega\left(GL(\mathbb{C})/GL(\mathbb{R})\right) \sim \mathbb{Z} \times BGL_\infty(\mathbb{R})$

etc

K-theory

A ring (with unit)

$\mathcal{P}(A)$ = category of f.g. projective A-modules

$E \in \mathrm{ob}\, \mathcal{P}(A) \Leftrightarrow \exists n \; \exists E'$ s.t.

$$E \oplus E' \simeq A^n$$

Denote $M = \Phi(A)$ the set of isom. classes of such modules

M is a semi. group.

$$\dot{E} + \dot{F} = \overset{\bullet}{\overline{E \oplus F}}$$

Take now its associated group

$$M \times M / \sim \; = S(M)$$

$$(a,b) \sim (c,d) \iff \exists e \mid a+d+e = b+c+e$$

<u>Def</u> $\quad K(A) = S(\Phi(A))$

<u>Note</u> $\quad K(A)$ is a functor

Remark $p \in M_2(A)$, $p^2 = p$

$\Rightarrow A^2 = \operatorname{Im} p \oplus \operatorname{Im}(1-p)$

Ex. 1 $A = \mathbb{Z}[X]/(1+X+\dots+X^{p-1})$

p prime

ring of cyclotomic integers

$\Longrightarrow K(A) = \mathbb{Z} \oplus \mathcal{C}$

finite group

Kummer If $\mathcal{C}_p = 0$

$\Longrightarrow$ FLT for the prime p

Ex 2

$A = C(X)$, X compact

= ring of continuous functions

$f : X \longrightarrow k$ ($\mathbb{R}$ or $\mathbb{C}$)

$$K(A) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq H^{ev}(X; \mathbb{Q}) \quad k = \mathbb{C}$$

$$\simeq H^{4*}(X; \mathbb{Q}) \quad k = \mathbb{R}$$

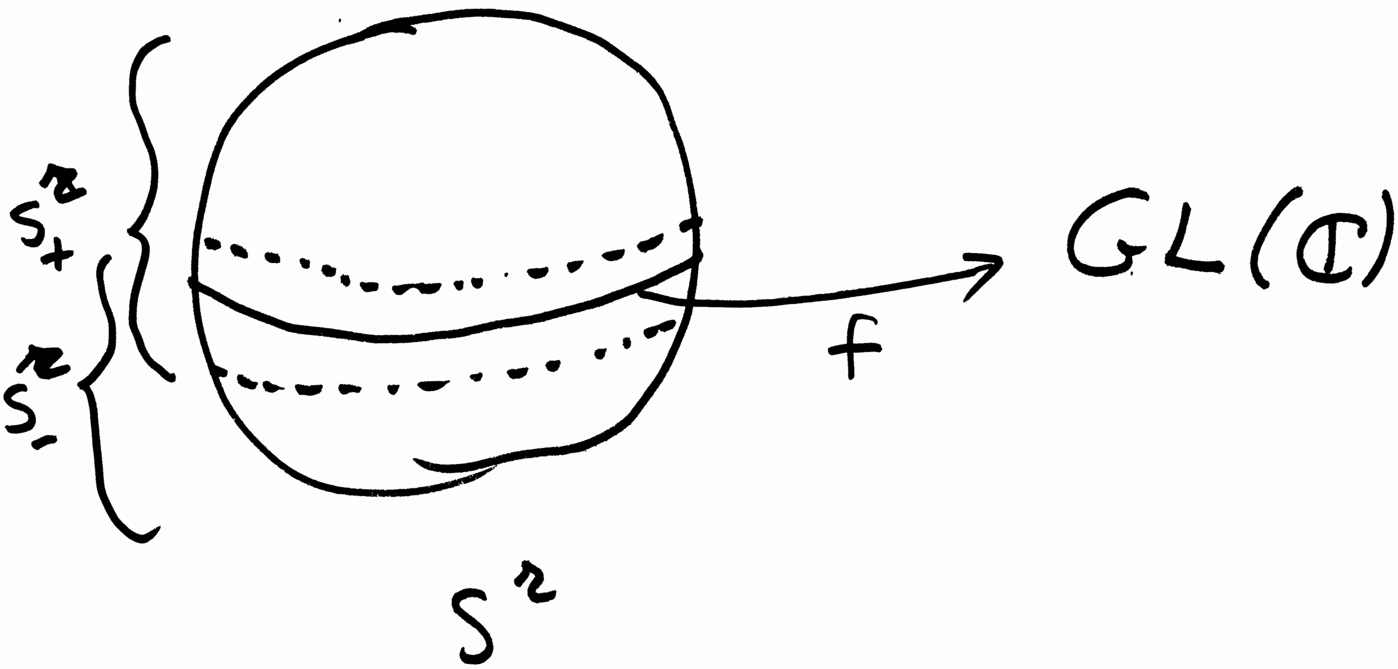
$K(C(X))$ is a <u>hty invariant</u>

Note $K(A) = \mathbb{Z} \oplus \tilde{K}(A)$

if A is commutative

Thm ($\mathbb{C}$)

$$\tilde{K}(C(S^n)) \simeq \pi_{n-1}(GL(\mathbb{C}))$$

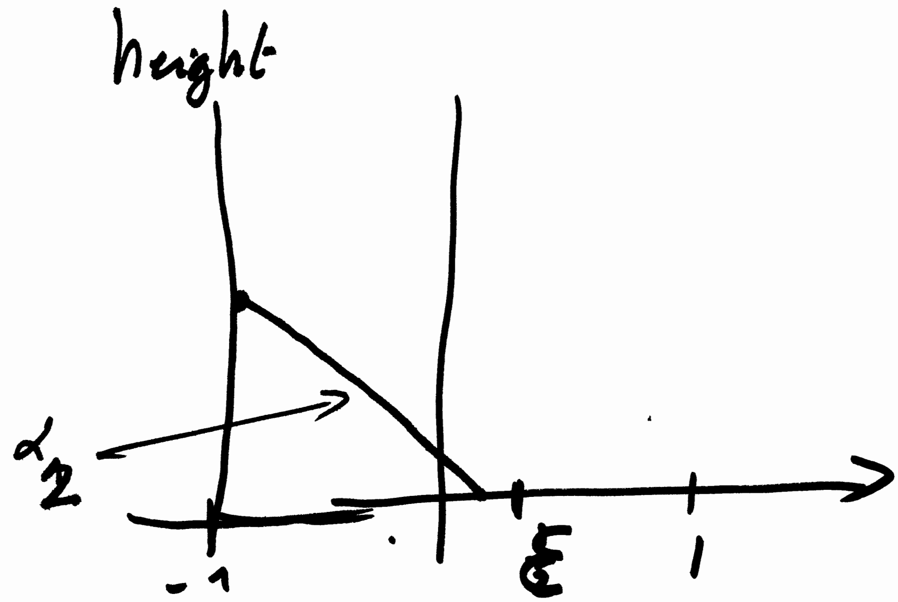


$$S^n_+ \cap S^n_- \simeq S^{n-1} \times \,]-\varepsilon, +\varepsilon[$$

$$\alpha_1 : S^2_+ \longrightarrow [0,1]$$

$$\alpha_2 : S^2_- \longrightarrow [0,1]$$

$$\alpha_1 + \alpha_2 = 1$$



$$\alpha_1 = 1 - \alpha_2$$

$$\beta_i = \frac{\alpha_i}{\sqrt{\alpha_1^2 + \alpha_2^2}} \qquad 2n \times 2n$$

$$p = \left(\begin{array}{c|c} \beta_1^2 & \beta_1 \beta_2 f \\ \hline \beta_1 \beta_2 f^{-1} & \beta_2^2 \end{array}\right) = p^2$$

$$f \longmapsto \operatorname{Im} p$$

Extension of K-theory to rings without unit.

A k-algebra

$$A^+ = A \times k$$

$$(a,\lambda)(a',\lambda') = (aa' + \lambda a' + \lambda' a, \lambda\lambda')$$

$$A^+ \longrightarrow k$$

<u>Def</u> $K(A) = \operatorname{Ker}\left(K(A^+) \longrightarrow K(k)\right)$

<u>ex</u> X loc. compact space

$$A = C_0(X) \Rightarrow A^+ = C(X^\bullet)$$

<u>Thm</u> $K(A)$ is a $\frac{1}{2}$ exact functor

$$0 \to A' \to A \to A'' \to 0$$

(A' 2-sided ideal, $A'' = A/A'$)

$$\Rightarrow \,?\, \to K(A') \xrightarrow{\alpha} K(A) \xrightarrow{\beta} K(A'') \to \,?$$

is an exact sequence

(this means $\operatorname{Im}\alpha = \operatorname{Ker}\beta$)

Q] what can we put at the right and at the left?

Assume now A is a <u>Banach algebra</u>

1) ring
2) there is a norm $a \mapsto \|a\|$

$$d(a,b) = \|a-b\|$$

complete metric space

3) $\|ab\| \leq \|a\| . \|b\|$

<u>ex 1</u> $A = C(X)$

$$\|f\| = \sup_{t \in X} |f(t)|$$

<u>ex 2</u> $A = \mathcal{B}(H)$, H Hilbert-space

$$\|u\| = \sup_{x \neq 0} \frac{\|u(x)\|}{\|x\|}$$

X loc. compact

$$A(X) = \left\{ f : X \to A \;\middle|\; \begin{array}{l} f(x) \to 0 \\ \text{if } x \to \infty \end{array} \right\}$$

Thm (Homology) $\exists !$ $K_n(A)$
$n \geq 0$

1) exactness

$$\to K_{n+1}(A'') \xrightarrow{\partial} K_n(A') \to K_n(A) \to$$
$$\to K_n(A'') \to K_{n-1}(A')$$

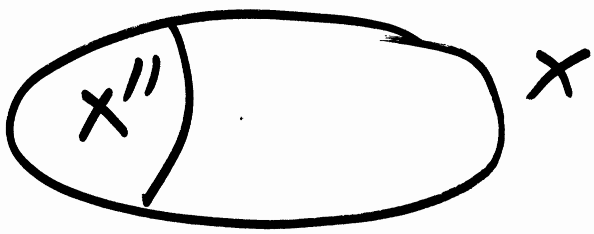
2) Homotopy ($I = [0,1]$)

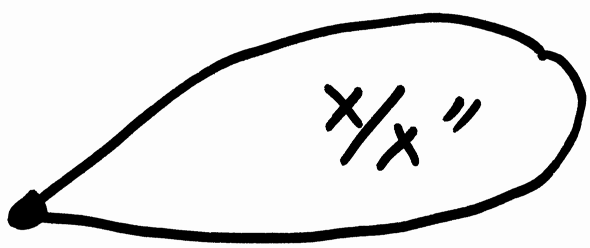
$$K_n(A) \simeq K_n(A(I))$$

3) Normalization $\quad K_0(A) = K(A)$

→ Translation in the category of l. compact spaces

$$X'' \subset X \longrightarrow X/X''$$





$$A'' = C(X''), \quad A = C(X)$$

$$A' = C_0(X - X'').$$

(Note $C(X/X'') = C(X - X'')^+$

Define $\qquad$ Atiyah-Hirzeb.

$$K^{-n}(X,Y) = K_n(C_0(X-Y))$$

$\Rightarrow$ axioms of a cohomology theory. (Note for $n \geq 0$)

<u>Thm</u> $\quad K_n(A) = K_0(A(\mathbb{R}^n))$

<u>Thm</u> $\quad$ A (complex) Banach algebra

$$K_n(A) \simeq K_{n+2}(A)$$

(real)

$$K_n(A) \simeq K_{n+8}(A)$$

Cup-product

$$A \times C \xrightarrow{\varphi} B \quad \text{bilinear}$$

$$\varphi(aa', cc') = \varphi(a,c)\,\varphi(a',c')$$
+ continuity

$\Longrightarrow$ This induces

$$K_n(A) \times K_p(\mathbb{C}) \longrightarrow K_{n+p}(B)$$

3rd version of Bott periodicity

$$K_2(\mathbb{C}) \simeq \mathbb{Z} \qquad u_2 \text{ generator}$$

$$K_n(A) \xrightarrow{\ \smile u_2\ } K_{n+2}(A)$$

Scheme of the proof

- Assume we have defined K_n for $n < 0$
- Assume all previous formal properties extend
- Assume $\exists\, u_{-2} \in K_{-2}(\mathbb{C})$

such that

$$u_2 \smile u_{-2} = 1 \in K_0(\mathbb{C})$$

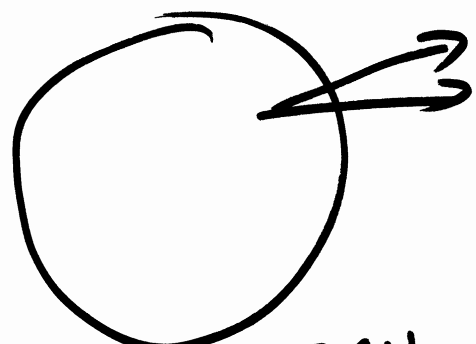
$$\begin{tikzcd} K_n \arrow[r, "\beta"] \arrow[rr, bend right, dashed, "\mathrm{Id}"'] & K_{n+2} \arrow[r, "\beta'"] & K_n \end{tikzcd}$$

$$(x \smile u_2) \smile u_{-2} = x \smile (u_2 \smile u_{-2})$$
$$= x \smile 1 = x$$

Applications

$n \geq 1$ $\quad S^n$ is a topological group $\Rightarrow n = 1, 3, 7$

S^{t-1}



Q] What is the maximum $\overset{\rho(t)}{\#}$ of independent lin fields ?

t odd $\Rightarrow \# = 0$

t even $= (2\alpha - 1)\, 2^{\beta}$, $\beta = \gamma + 4\delta$

Then $\rho(t) = 2^{\gamma} + 8\delta - 1$

$t = 8 \Rightarrow \rho(t) = 7$