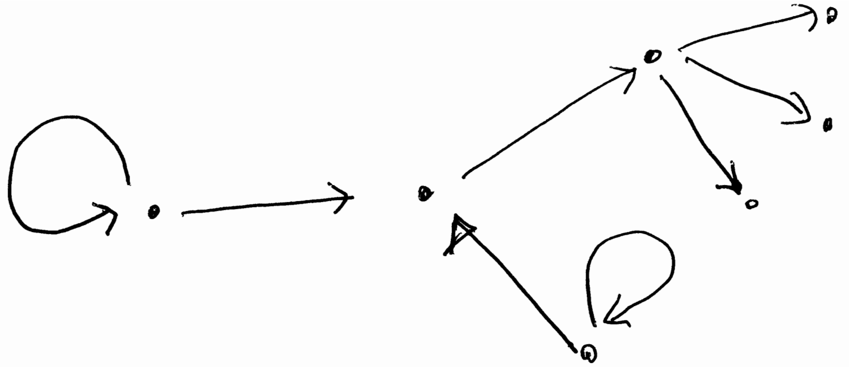


Q

Hall(Q)

Basis $=$ $\mathrm{Reps}(Q)$ in $\mathbb{F}_q$ vect spaces

$$A * B = \sum_{C} r_{AB}^{C} \cdot C \,, \qquad A, B, C \in \mathrm{Reps}(Q)$$

$$r_{AB}^{C} = \#\{ B' \subset C \mid B' \simeq B \,,\ C/B' \simeq A \}$$

Hall(Q) associative, not necessarily commutative

- $r^{C}_{A,B} = r^{C}_{A,B}(q)$ polynomial in q.

- $C/B \simeq A \Rightarrow \dim C = \dim A + \dim B$

Q without loops $\rightsquigarrow$ Lie algebra Lie(Q)

A_n $\bullet \to \bullet \to \bullet \cdots \to \bullet$ n vertices $\rightsquigarrow$ $(n+1) \times (n+1)$ matrices trace $= 0$

<u>Thm.</u> $\mathrm{Hall}(Q)|_{q=1} \cong$ Upper triangular part Lie(Q)

$$\begin{pmatrix} 0 & & * \\ & 0 & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}$$ upper triangular part

$n = 2$ $\quad e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$$e_{12} = [e_1, e_2] = e_1 e_2 - e_2 e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

e_{12} central

$$e_1 \cdot e_{12} = e_{12}\, e_1$$
$$e_2 \cdot e_{12} = e_{12} \cdot e_2$$

q-analogue $\quad \boxed{e_1 \cdot e_{12} = q \cdot e_{12} \cdot e_1}$

$Q \qquad \bullet \longrightarrow \bullet$

$\mathbb{F} = \mathbb{F}_q$

Irreps:

$e_1 : \quad \overset{\mathbb{F}}{\bullet} \longrightarrow \overset{0}{\bullet} \qquad\qquad e_2 : \quad \overset{0}{\bullet} \longrightarrow \overset{\mathbb{F}}{\bullet}$

$$e_1 * e_2 = \overset{\mathbb{F}}{\bullet} \xrightarrow{\sim} \overset{\mathbb{F}}{\bullet} \quad + \quad \overset{\mathbb{F}}{\bullet} \xrightarrow{0} \overset{\mathbb{F}}{\bullet}$$

$$C/e_2 \simeq e_1$$

$$e_2 * e_1 = \overset{\mathbb{F}}{\bullet} \xrightarrow{0} \overset{\mathbb{F}}{\bullet}$$

$$C/e_1 \simeq e_2$$

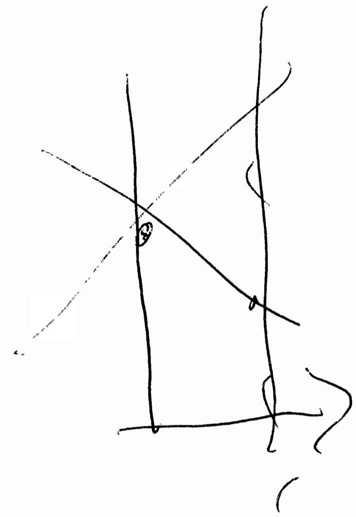
$$e_1 * e_2 - e_2 * e_1 = \overset{\mathbb{F}}{\bullet} \xrightarrow{\sim} \overset{\mathbb{F}}{\bullet} \quad =: e_{12}$$

$$e_{12} * e_1 = \overset{\mathbb{F}^2}{\bullet} \overset{f}{\twoheadrightarrow} \overset{\mathbb{F}}{\bullet}$$

$$C/e_1 \simeq e_{12}$$ ← surjective quotient

$$e_1 = \overset{\mathbb{F}}{\bullet} \longrightarrow \overset{0}{\bullet} = \overset{\mathrm{Ker} f}{\bullet} \longrightarrow \overset{0}{\bullet}$$

$$e_1 * e_{12} = \begin{tikzcd} \overset{\mathbb{F}^2}{\bullet} \arrow[r, two heads, "f"] & \overset{\mathbb{F}}{\bullet} \\ \overset{\mathbb{F}}{\bullet} \arrow[u, phantom, "\cup" description] \arrow[r, "\sim"] & \overset{\mathbb{F}}{\bullet} \arrow[u, phantom, "\parallel" description] \end{tikzcd}$$



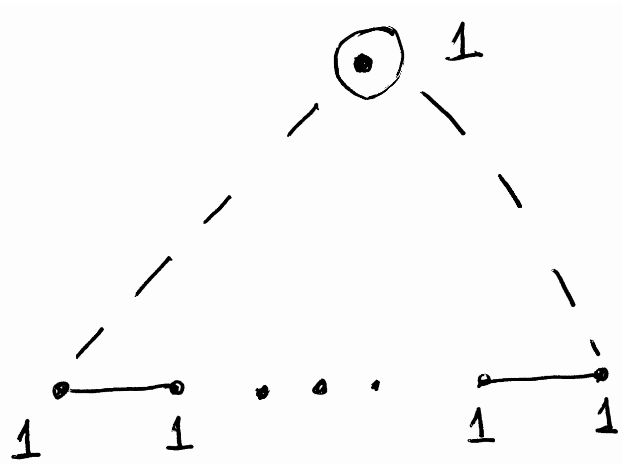
$$C/e_{12} \simeq e_1$$

surjective sub rep

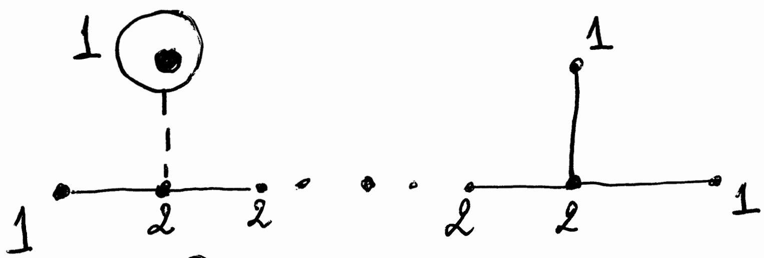
$$\# \text{ subreps} = \# \begin{matrix} \text{lines in } \mathbb{F}^2 \\ \neq \mathrm{Ker} f \end{matrix} = q$$

$$\boxed{e_1 * e_{12} = q \cdot e_{12} * e_1}$$

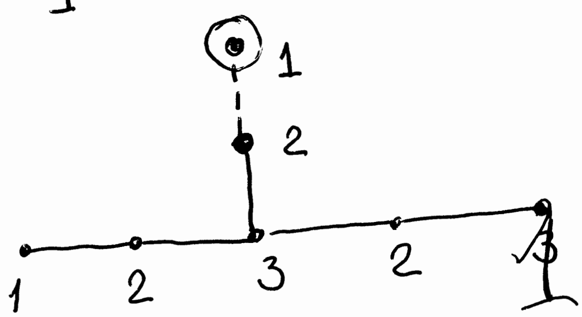
A_n



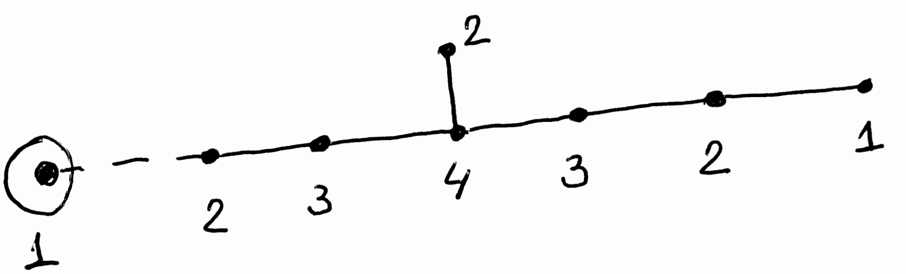
D_n



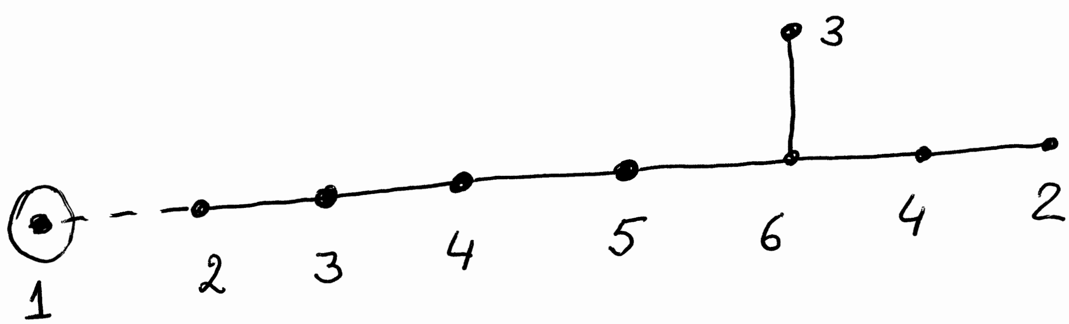
E_6



E_7



E_8



$$a_{ij} = \#\{\text{edges between } i \,\&\, j\}$$

Adjacency matrix $\quad A = (a_{ij})$

Cartan matrix

$$C = \begin{pmatrix} 2 & & & 0 \\ & 2 & & \\ & & \ddots & \\ 0 & & & 2 \end{pmatrix} - A$$

$$C = (c_{ij}) \qquad c_{ij} = c_{ji}$$

$$\mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}$$

$$q(x, y) = \sum_{ij} c_{ij}\, x_i\, y_j$$

<u>Thm.</u> connected graph is Dynkin $\iff$ $q(x,x) > 0$ for any $x \neq 0$.

or extended Dynkin $\iff$ $q(x,x) \geq 0$.

<u>Lemma</u> Let $d = (d_1, \dots, d_m) \in \mathbb{R}^m$

- $d_i > 0$ for all i.
- $q(x,d) = 0$ for all $x \in \mathbb{R}^m$

Then

$$q(x,x) \geq 0 \qquad \forall x \in \mathbb{R}^m.$$

Pf:

$$q(x,x) = \sum_{i \neq j} a_{ij} \frac{d_i d_j}{4} \left(\frac{x_i}{d_i} - \frac{x_j}{d_j} \right)^2$$

$$SL_2(\mathbb{C}) = \left\{ \begin{array}{c} \text{matrices } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \det(g) = ad - bc = 1 \end{array} \right\}$$

$$\cup$$

G finite subgroup

$$L = \mathbb{C}^2 \circlearrowleft G$$

vertices = $\mathrm{Irreps}(G)$

$$= \dim \mathrm{Hom}_G(V_i, L \otimes V_j)$$

<u>Claim</u> For McKay graph have

$$q(x, d) = 0 \quad \forall x \in \mathbb{R}^m$$

<u>Schur lemma.</u>

<u>Corollary</u> For McKay graph $q(\cancel{x}) \geq 0$

$$\boxed{\text{McKay graphs}} = \boxed{\text{Extended Dynkin graphs}}$$

Example: A_{n-1}

$$\varepsilon^n = 1$$

$$G = \left\{ \begin{pmatrix} \varepsilon^k & 0 \\ 0 & \varepsilon^{-k} \end{pmatrix}, \quad \begin{array}{l} \varepsilon = \text{primitive } n\text{-th root of } 1 \\ k = 0, 1, \ldots, n-1. \end{array} \right\}$$

$$G \simeq \mathbb{Z}/n\mathbb{Z}$$

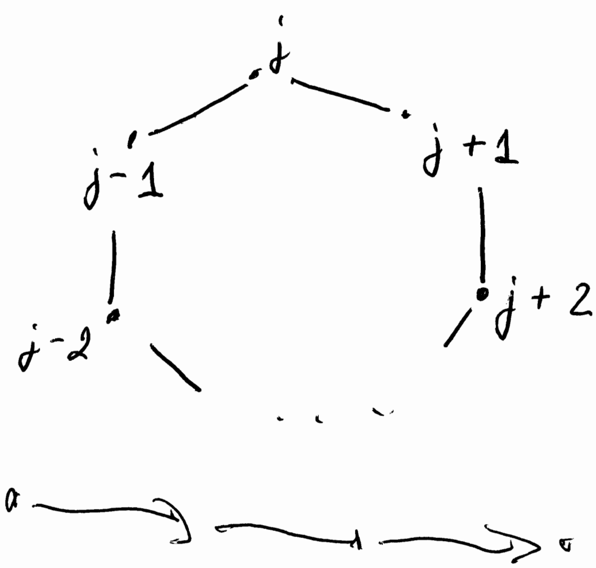
$\mathrm{Irreps}(G) =$ n-th roots of unity $= \{0, \ldots, n-1\}$

$$V_i : \begin{pmatrix} \varepsilon & \\ & \varepsilon^{-1} \end{pmatrix} \longmapsto \text{multiplication by } \varepsilon^i$$

$$L = V_1 \oplus V_{-1} \qquad (V_{-1} = V_{n-1})$$

$$L \otimes V_j = V_{j+1} \oplus V_{j-1}$$

$$\mathrm{Hom}(V_i, V_{j+1} \oplus V_{j-1})$$



$$M = \mathbb{C}^2/G \qquad \dim_{\mathbb{C}} M = 2$$

$$\mathrm{Fun}(M) = \left\{ \begin{array}{l} f \in \mathbb{C}[u,v] \\ f \text{ constant on } G\text{-orbits} \end{array} \right\}$$

<u>Ex</u> A_{n-1} type

$$\begin{pmatrix} \varepsilon^k & 0 \\ 0 & \varepsilon^{-k} \end{pmatrix} : \quad (u,v) \longmapsto (\varepsilon^k u, \varepsilon^{-k} v)$$

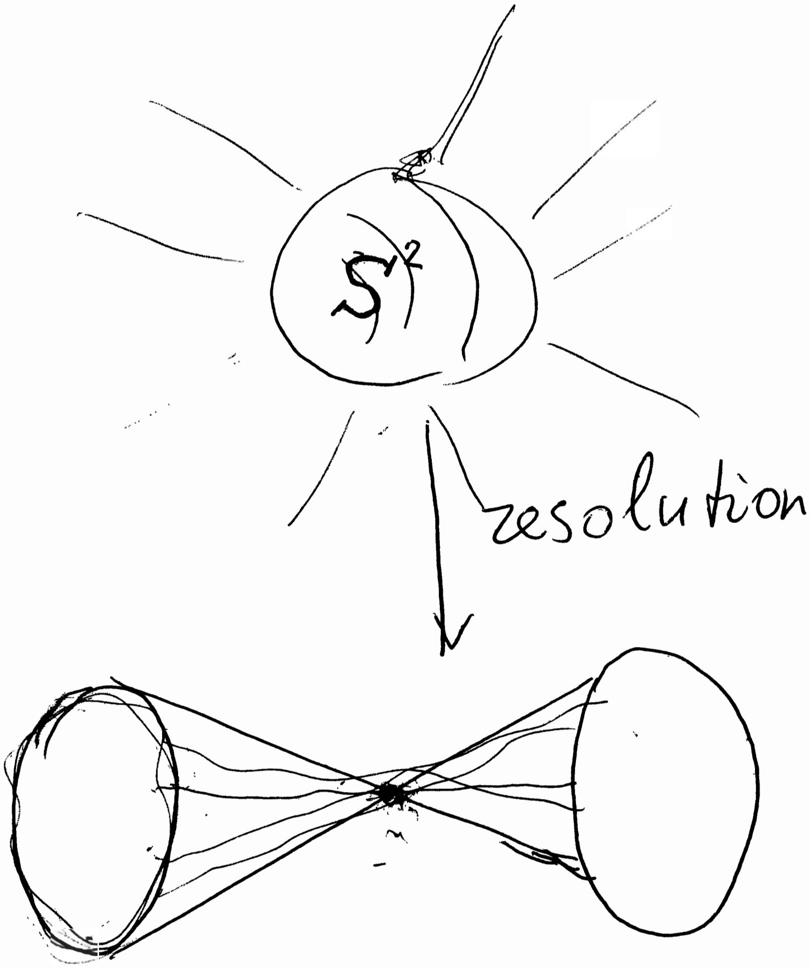
$$f(u,v) = f(\varepsilon u, \varepsilon^{-1} v) \qquad \forall\, u,v \in \mathbb{C}$$

$$x := u^n, \quad y = v^n, \quad z = uv$$

$$f = f(x,y,z) \qquad \boxed{z^n = xy}$$

$$\mathrm{Fun}(M) = \mathbb{C}[x,y,z]/(z^n - xy)$$

$$M = \{ (x,y,z) \in \mathbb{C}^3 \mid z^n = xy \}$$



$= \tilde{M}$

$= M \subset \mathbb{C}^3$

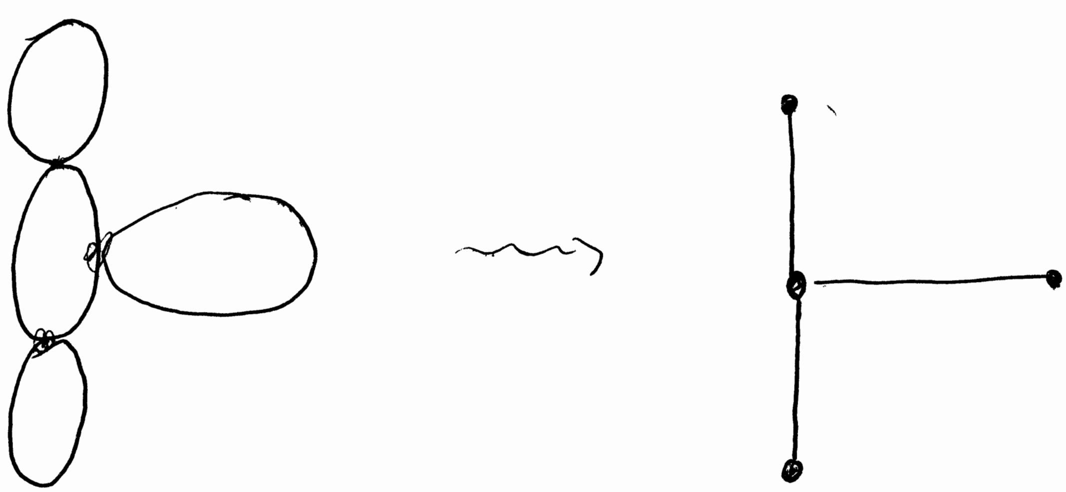
$n = 2$: $\qquad z^2 = xy$

General case

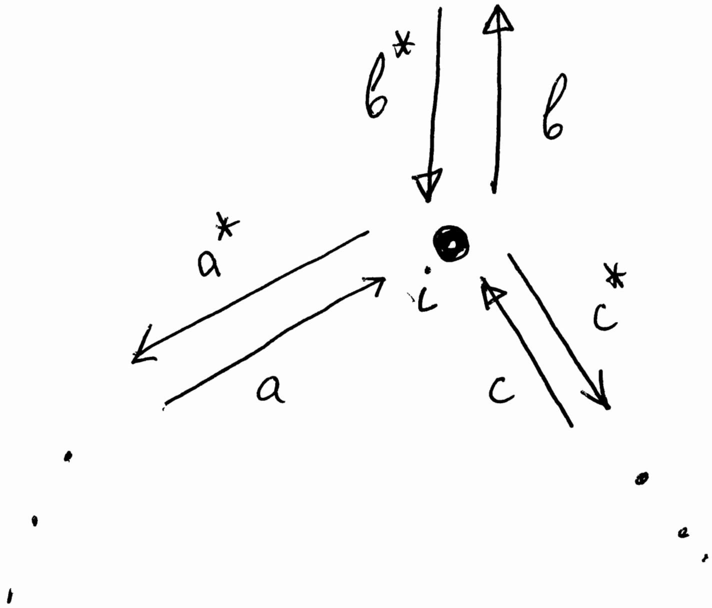
$$\begin{array}{ccc} \tilde{M} \setminus \begin{pmatrix}\text{zero}\\ \text{fiber}\end{pmatrix} & \subset & \tilde{M} \quad \text{resolution} \\ \wr \big\downarrow & & \big\downarrow \\ M \setminus \{0\} & \subset & M = \mathbb{C}^2/G \end{array}$$

~~deformations~~

zero fiber



$G \longmapsto$ nonextended Dynkin graph.



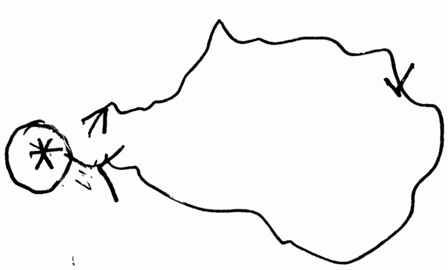
Quiver Q

$$P_i = -aa^* + b^*b - cc^* \in \mathrm{Path}(Q)$$

$$\mathbb{A} = \frac{\mathrm{Path}(Q)}{\{P_i,\ i \in I\}}$$ ← two-sided ideal

$\cup$

$\mathbb{B} = \{$ paths with source and end at $*$ = extending vertex $\}$



Thm.

$\mathrm{Fun}(M) \simeq \mathbb{B}$