The Mandelbrot set, Farey tree, and Fibonacci sequence

- How to count
- How to add

math.bu.edu/DYSYS
math.bu.edu/people/bob
ITERATE

\[ x^2 + c \]

\( C \uparrow \) REAL OR COMPLEX

GIVEN SEED \( x_0 \), WHAT IS THE FATE OF ORBIT

\[
\begin{align*}
x_1 &= x_0 + c \\
x_2 &= x_1 + c \\
x_3 &= x_2 + c \\
&\vdots
\end{align*}
\]

ORBIT OF \( x_0 \)
\[ x^2 + c \]

FATE OF ORBIT OF 0?

\( c = 1 : x^2 + 1 \)

\[ 0 \rightarrow 1 \rightarrow 2 \rightarrow 5 \rightarrow 26 \rightarrow \text{BIG} \rightarrow \text{BIGGER} \ldots \rightarrow \text{TO} \rightarrow \infty \]

\( c = 0 : x^2 \)

\[ 0 \rightarrow 0 \rightarrow 0 \rightarrow \ldots \rightarrow \text{FIXED} \]

\( c = -1 \)

\[ 0 \rightarrow -1 \rightarrow 0 \rightarrow -1 \ldots \rightarrow 2 \text{ CYCLE} \]
# Summary

<table>
<thead>
<tr>
<th>$c$</th>
<th>Fate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\rightarrow \infty$</td>
</tr>
<tr>
<td>0</td>
<td>Fixed PT</td>
</tr>
<tr>
<td>-1</td>
<td>2-cycle</td>
</tr>
<tr>
<td>-1.3</td>
<td>4-cycle</td>
</tr>
<tr>
<td>-1.38</td>
<td>8-cycle</td>
</tr>
<tr>
<td>-1.75</td>
<td>3-cycle</td>
</tr>
<tr>
<td>-1.9</td>
<td>Chaos</td>
</tr>
</tbody>
</table>

Bif'n diagram = Spine of the Mandelbrot Set.
Fundamental Dichotomy

Sometimes orbit of $0 \to \infty$
\( (c \notin [-2, .25]) \)

Sometimes it does not
- fixed pt
- cycle
- chaotic
IF C IS COMPLEX

ORBIT IS A SEQUENCE IN THE PLANE

EX: $x^2 + i$

EVENTUAL 2-CYCLE
ex. $x^2 + 2i$

$0 \rightarrow 2i \rightarrow -4 + 2i \rightarrow 12 - 14i \rightarrow \cdots \rightarrow \infty$

SAME DICHOTOMY

EITHER

• ORBIT OF $0 \rightarrow \infty$

• ORBIT OF $0 \rightarrow \infty$

- TEND TO FIXED PTS,
  CYCLES, CHAOTIC,...

- POSSIBLE FATES
  STILL NOT KNOWN*
Mandelbrot Set:

All c-values for which orbit of 0 does not go to \( \infty \)

\[ i \in M-set \]

\( 2i \in M-set \)

\([-2, .25] \subset M-set \)
ALGORITHM

• CHOOSE GRID
  IN C-PLANE

• GIVEN C IN GRID
  COMPUTE ORBIT OF
  0 UNDER X^2 + C

• ORBIT OF 0 ESCAPES
  THEN COLOR C
  (NOT IN M-SET)

• ORBIT OF 0 BOUNDED
  THEN LEAVE C BLACK
  (IN M-SET)
COLORS INDICATE # OF ITERATIONS TO EXCEED BOUND

RED ← FAST ESCAPE
ORANGE
YELLOW
GREEN
BLUE
VIOLET ← SLOW ESCAPE
Complicated Mess

How understand?
(FILLED) JULIA SET

Fix C.
Let seed vary

(FILLED) J-set = all seeds whose orbit does not go to \( \infty \)

Different J-set for each C-value
$\text{Filled J-set is all seeds on, inside } |z| = 1$
IMPORTANT DISTINCTION

M-SET
- PARAMETER PLANE
- VARY C
- COMPUTE ORBIT OF O

J-SET
- DYNAMICAL PLANE
- FIX C
- VARY SEEDS
THM

C in M-set \iff J-set is connected

C not in M-set \iff J-set is totally disconnected (Cantor set)
ROTATION NUMBERS

HOW TO COUNT....
How to Add

To get largest bulb in between....
Add numerators + add denominators

\[ \frac{1}{2} + \frac{1}{3} = \frac{2}{5}! \]
Farey Tree

\[
\begin{align*}
\frac{0}{1} & \quad \frac{1}{1} \\
\frac{1}{2} &
\end{align*}
\]

 Farey Neighbors

 Farey Parents

 Farey Child

\* Fraction between parents with smallest denominator.

\* \[ \frac{1}{2} = \frac{0}{1} + \frac{1}{1} \]
FIBONACCI SEQUENCE

DEVANEY

(DENOMINATORS ONLY)
Get all fractions this way
Known:

- The Fibonacci sequence converges to the "golden number" $\phi$ (essentially)

- $\phi$ is the "most irrational" number

- $\phi$ is furthest from the rationals

- $|\phi - \frac{p}{q}| > \frac{\text{const}}{q^2}$

  for some $q$
We understand what is happening on the boundary of cardioid at:

- Highly irrational pts
- Rational pts

Highly irrational $\implies$ Siegel disk

Have a fixed pt - all other orbits run around densely on nearby "circles"...

Irrational rotations
No idea what is happening near a fixed pt with a not-so-irrational rotation
The "real" way to understand $M$:

(External) Riemann mapping theorem:

$\Rightarrow$ Analytic homeo from exterior of unit circle to ext. of $M$

Exterior of $|z|=1$

Images are "external rays"

Measure mod 1
WHERE RAYS LAND DETERMINES $n$

IF $P/Q$ IS PERIODIC OF PERIOD $n$ UNDER DOUBLING, THEN $P/Q$ RAY "CUTS OFF" PERIOD $n$-BULB

\[ \frac{1}{3} \rightarrow \frac{2}{3} \rightarrow \frac{1}{3} \]  PERIOD 2

0 = 1
Ex: $\frac{1}{7} \rightarrow \frac{2}{7} \rightarrow \frac{3}{7}$

$\frac{3}{7} \rightarrow \frac{6}{7} \rightarrow \frac{5}{7}$

Period 3
\[ \frac{1}{15} \rightarrow \frac{2}{15} \rightarrow \frac{4}{15} \rightarrow \frac{8}{15} \rightarrow \frac{13}{15} \rightarrow \frac{1}{15} \ldots \quad \text{PERIOD 4} \]
$\frac{k}{31}$ has period 5.