THE MANDELBROT SET, FAREY TREE, AND FIBONACCI SEQUENCE

- HOW TO COUNT
- HOW TO ADD

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ITERATE

$$x^2 + c$$

REAL OR COMPLEX

GIVEN SEED x_0, WHAT IS THE FATE OF ORBIT

$$\left.\begin{aligned} x_1 &= x_0^2 + c \\ x_2 &= x_1^2 + c \\ x_3 &= x_2^2 + c \\ &\vdots \end{aligned}\right\} \text{ORBIT OF } x_0$$

$$x^2 + c$$

FATE OF ORBIT OF 0?

$c = 1 : x^2 + 1$

$0 \to 1 \to 2 \to 5 \to 26$

$\to$ BIG $\to$ BIGGER ....

TO ∞

$c = 0 : x^2$

$0 \to 0 \to 0 \to \ldots$

FIXED

$c = -1$

$0 \to -1 \to 0 \to -1 \ldots$

2 CYCLE

SUMMARY

$c =$	FATE
1	$\to \infty$
0	FIXED PT
−1	2-CYCLE
−1.3	→ 4-CYCLE
−1.38	→ 8-CYCLE
⋮	
−1.75	→ 3-CYCLE
⋮	
−1.9	CHAOS

BIF'N DIAGRAM = SPINE OF THE MANDELBROT SET.

FUNDAMENTAL DICHOTOMY

SOMETIMES ORBIT OF 0 $\to \infty$
($c \notin [-2, .25]$)

SOMETIMES IT DOES NOT

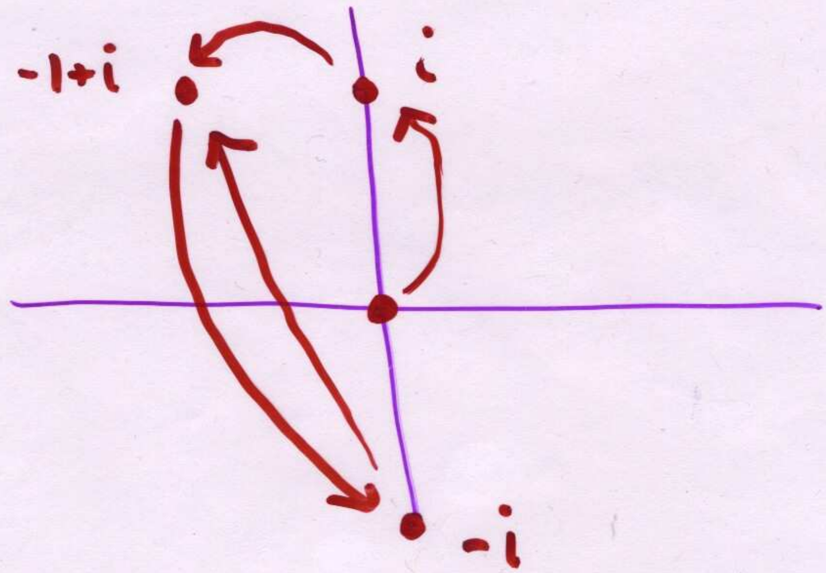
- FIXED PT
- CYCLE
- CHAOTIC

$\vdots$

IF C IS COMPLEX

ORBIT IS A SEQUENCE IN THE PLANE

EX: $x^2 + i$



EVENTUAL 2-CYCLE

EX $x^2 + 2i$

$0 \to 2i \to -4+2i \to 12-14i$
$\to \ldots$

TO ∞

SAME DICHOTOMY

EITHER

- ORBIT OF $0 \to \infty$
- ORBIT OF $0 \not\to \infty$
  - TEND TO FIXED PTS, CYCLES, CHAOTIC, ...
  - POSSIBLE FATES STILL NOT KNOWN*

MANDELBROT SET:

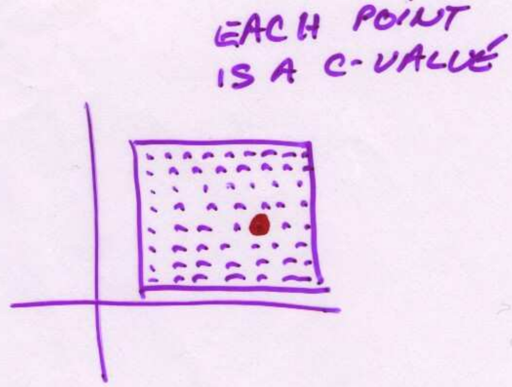
ALL C-VALUES FOR WHICH ORBIT OF 0 DOES <u>NOT</u> GO TO ∞

$i \in$ M-SET

$2i \notin$ M-SET

$[-2, .25] \subset$ M-SET

ALGORITHM



- CHOOSE GRID IN C-PLANE

- GIVEN C IN GRID COMPUTE ORBIT OF 0 UNDER x^2+c

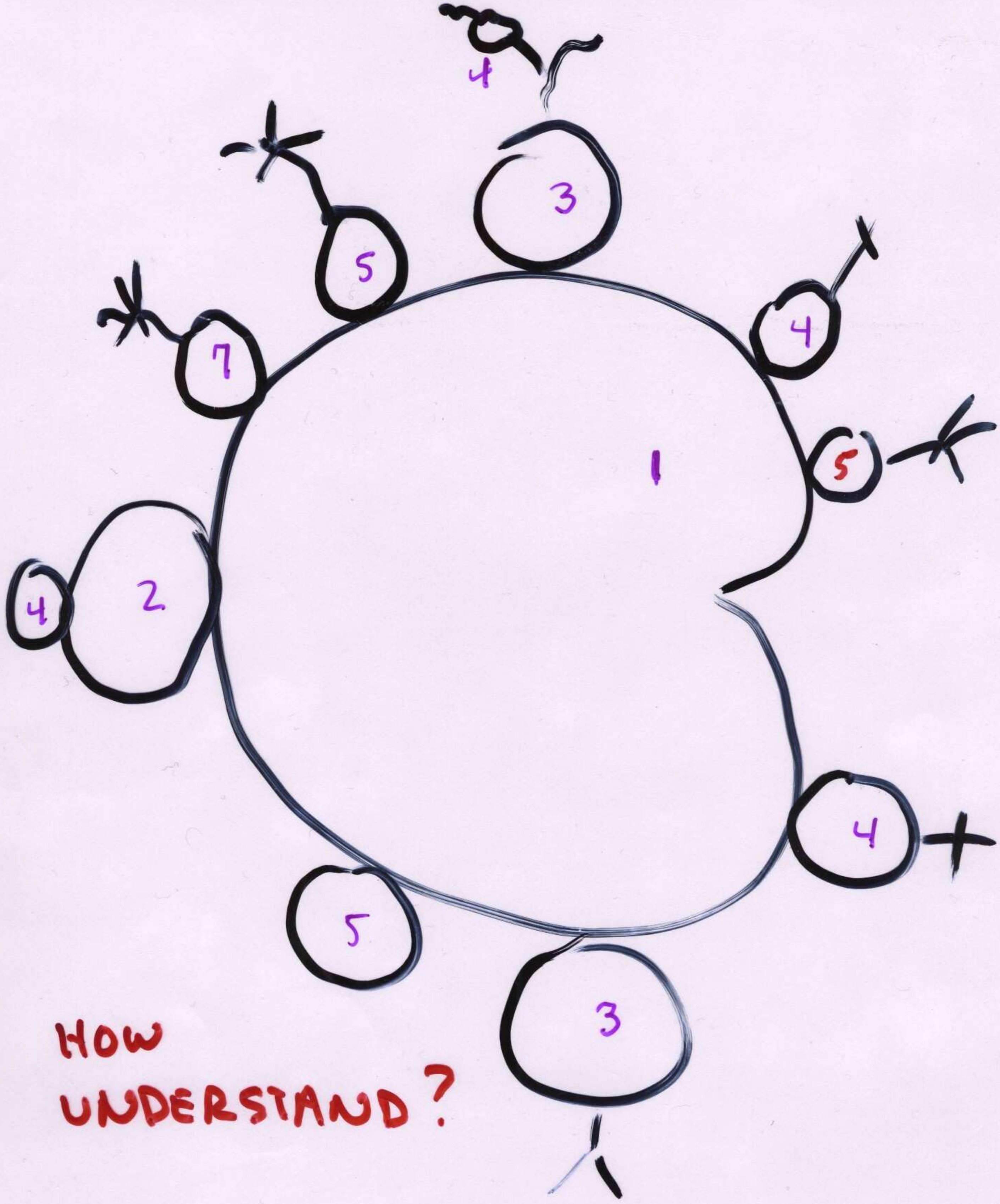
- ORBIT OF 0 ESCAPES THEN COLOR C (NOT IN M-SET)

- ORBIT OF 0 BOUNDED THEN LEAVE C BLACK (IN M-SET)

COLORS INDICATE #
OF ITERATIONS TO
EXCEED BOUND

RED ← FAST ESCAPE
ORANGE
YELLOW
GREEN
BLUE
VIOLET ← SLOW ESCAPE

COMPLICATED MESS



HOW
UNDERSTAND?

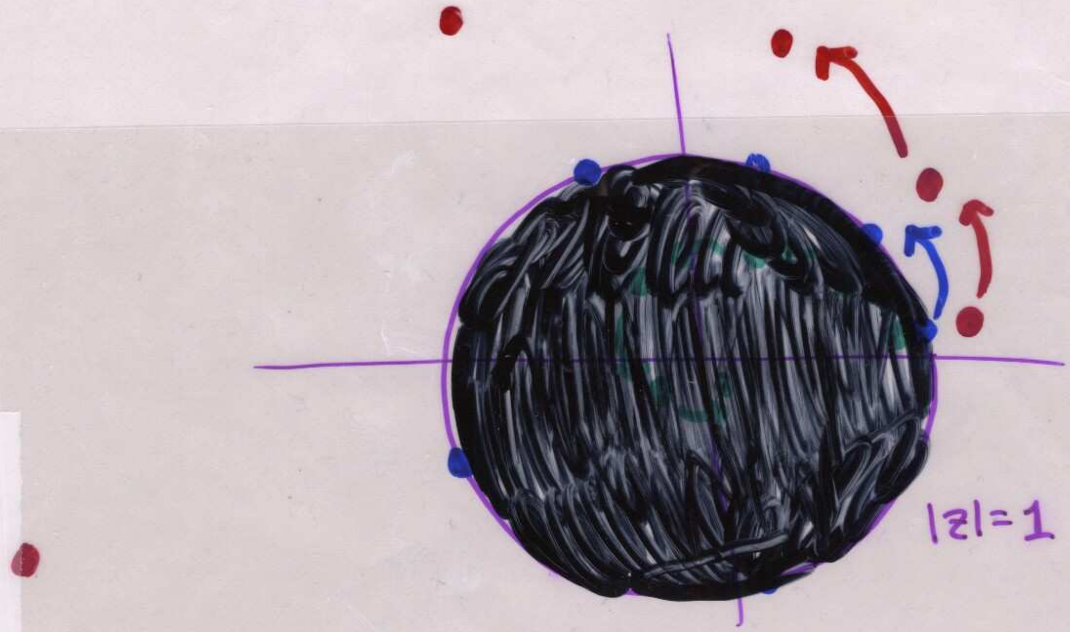
(FILLED) JULIA SET

FIX C.
LET SEED VARY

(FILLED) J-SET = ALL SEEDS WHOSE ORBIT <u>DOES</u> <u>NOT</u> GO TO ∞

DIFFERENT J-SET FOR EACH C-VALUE

EX x^2 $(c=0)$



FILLED J-SET IS ALL SEEDS ON, INSIDE $|z|=1$

IMPORTANT DISTINCTION

M-SET

- PARAMETER PLANE
- VARY C
- COMPUTE ORBIT OF 0

J-SET

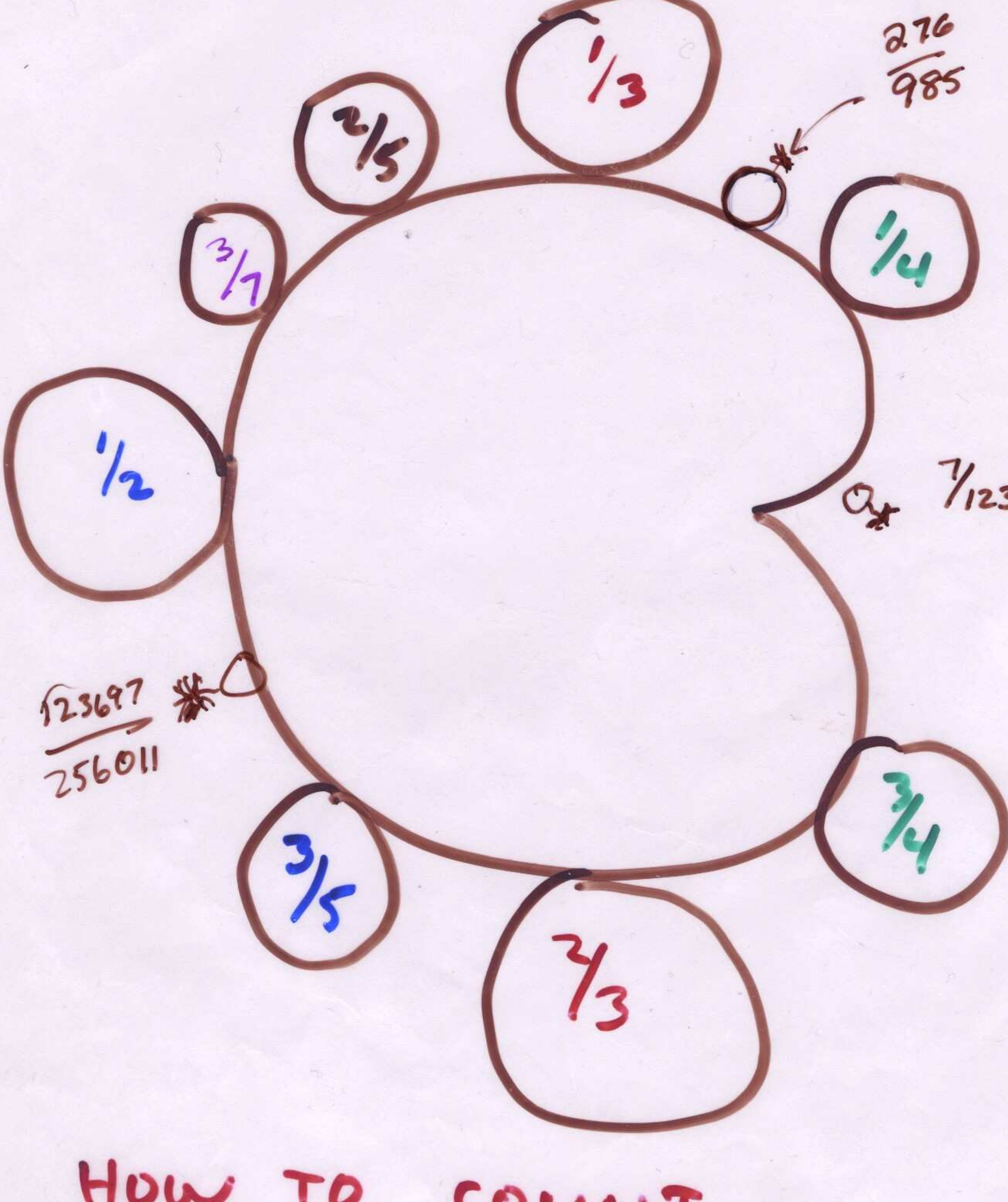
- DYNAMICAL PLANE
- FIX C
- VARY SEEDS

THM

C IN M-SET $\iff$ J-SET IS CONNECTED

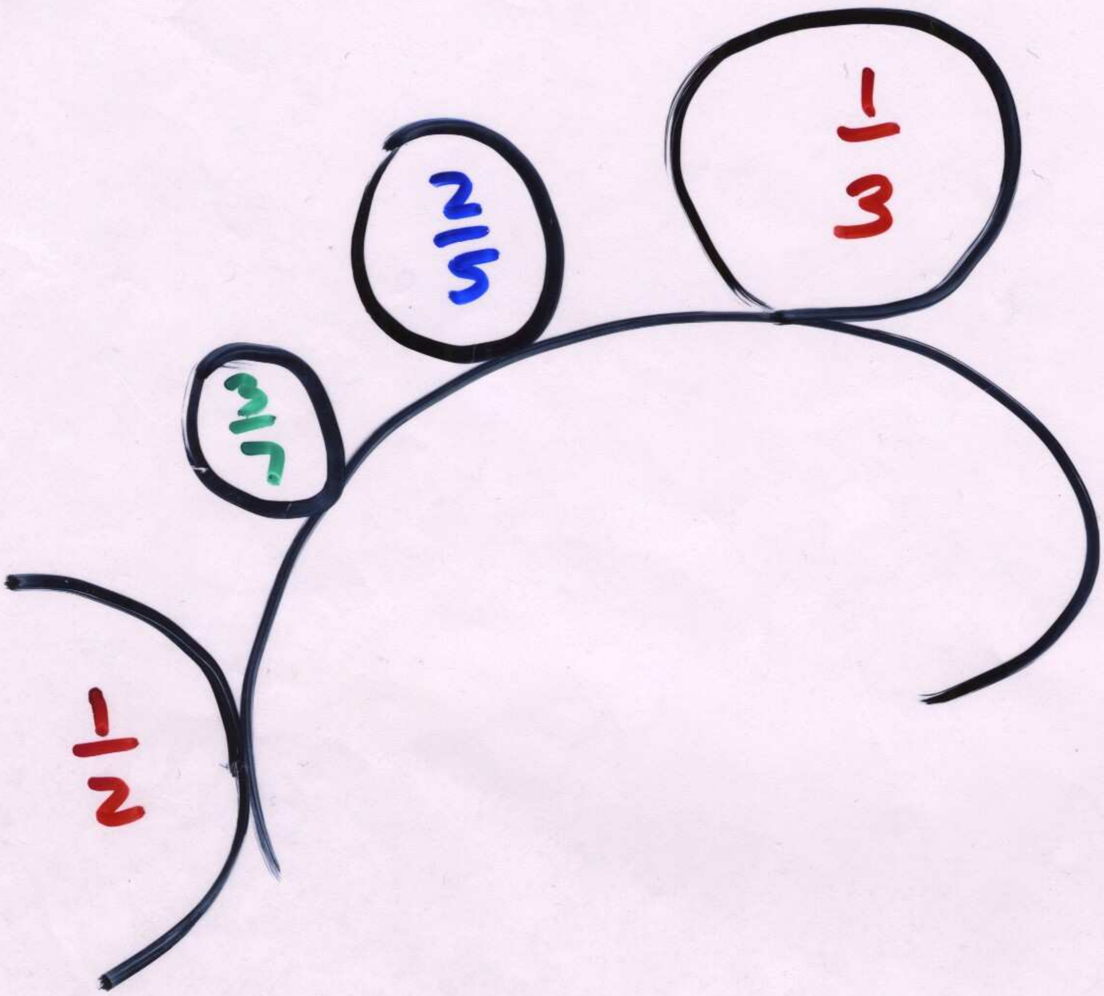
C NOT IN M-SET $\iff$ J-SET IS TOTALLY DISCONNECTED (CANTOR SET)

ROTATION NUMBERS



HOW TO COUNT....

HOW TO ADD



TO GET LARGEST BULB
IN BETWEEN....
ADD NUMERATORS + ADD
DENOMINATORS

$$\frac{1}{2} \oplus \frac{1}{3} = \frac{2}{5}\,!$$

FAREY TREE

$$
\begin{tikzcd}
\frac{0}{1} \arrow[dr] & & \frac{1}{1} \arrow[dl] \\
& \frac{1}{2} &
\end{tikzcd}
$$

FAREY NEIGHBORS

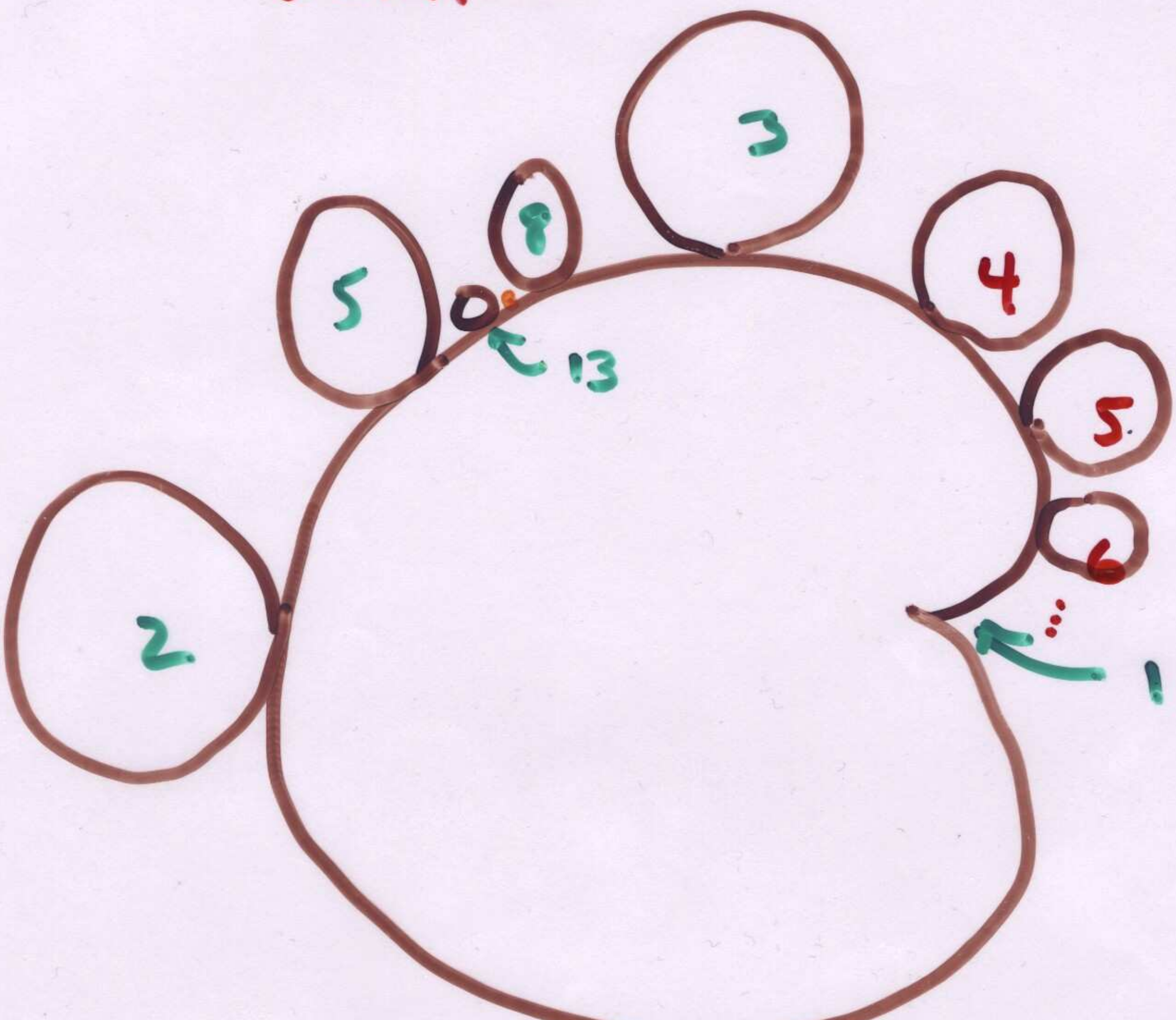
FAREY PARENTS

FAREY CHILD

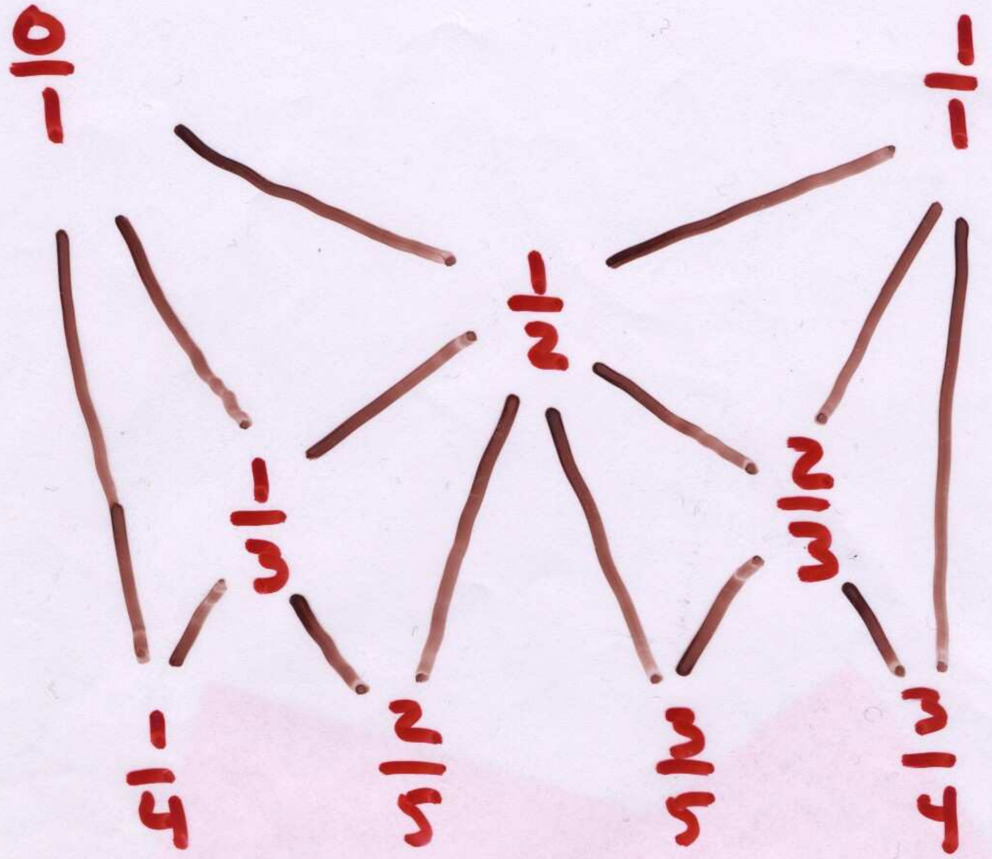
- FRACTION BETWEEN PARENTS WITH SMALLEST DENOM.
- $\frac{1}{2} = \frac{0}{1}$ "+" $\frac{1}{1}$

~~FIBONACCI~~ SEQUENCE

DEVANEY



(DENOMINATORS ONLY)



$\frac{0}{1} \quad \frac{1}{5} \quad \frac{1}{4} \quad \frac{2}{7} \quad \frac{1}{3} \quad \frac{2}{8} \quad \frac{2}{5} \quad \ldots\ldots$

GET ALL FRACTIONS THIS WAY

KNOWN:

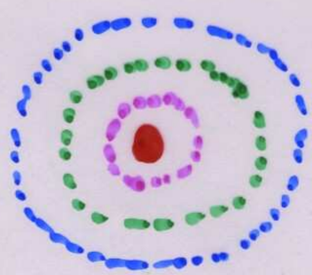
- THE FIBONACCI SEQUENCE CONVERGES TO THE "GOLDEN NUMBER" α (ESSENTIALLY)

  - α IS THE "MOST IRRATIONAL" NUMBER
  - α IS FURTHEST FROM THE RATIONALS
  - $$\left| \alpha - \frac{P}{Q} \right| > \frac{\text{CONST}}{Q^K}$$

    FOR SOME K

- WE UNDERSTAND WHAT IS HAPPENING ON BOUNDARY OF CARDIOID AT
  - HIGHLY IRRATIONAL PTS
  - RATIONAL PTS

HIGHLY IRRATIONAL ⇒ SIEGEL DISK



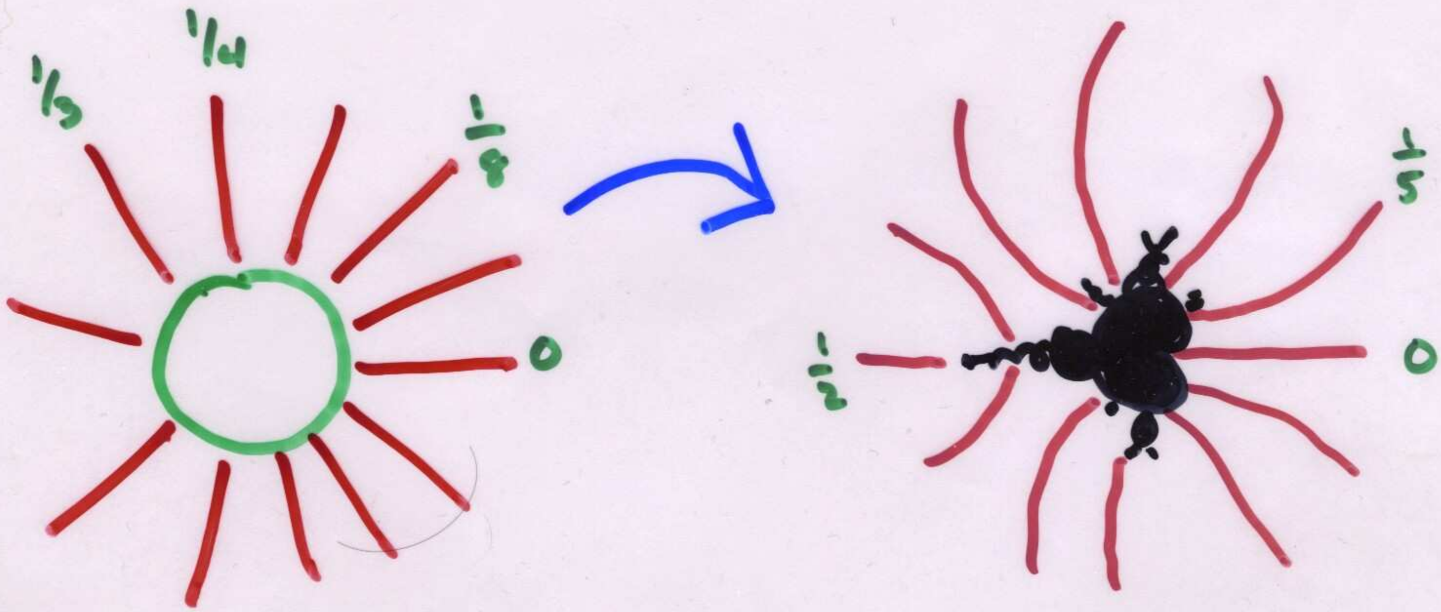
HAVE A FIXED PT — ALL OTHER ORBITS RUN AROUND DENSELY ON NEARBY "CIRCLES" ... IRRATIONAL ROTATIONS

NO IDEA WHAT IS HAPPENING NEAR A FIXED PT WITH A NOT-SO-IRRATIONAL ROTATION

THE "REAL" WAY TO UNDERSTAND M:

(EXTERNAL) RIEMANN MAPPING THM:

⇒ ANALYTIC HOMEO FROM EXTERIOR OF UNIT CIRCLE TO EXT. OF M



EXTERIOR OF $|z|=1$

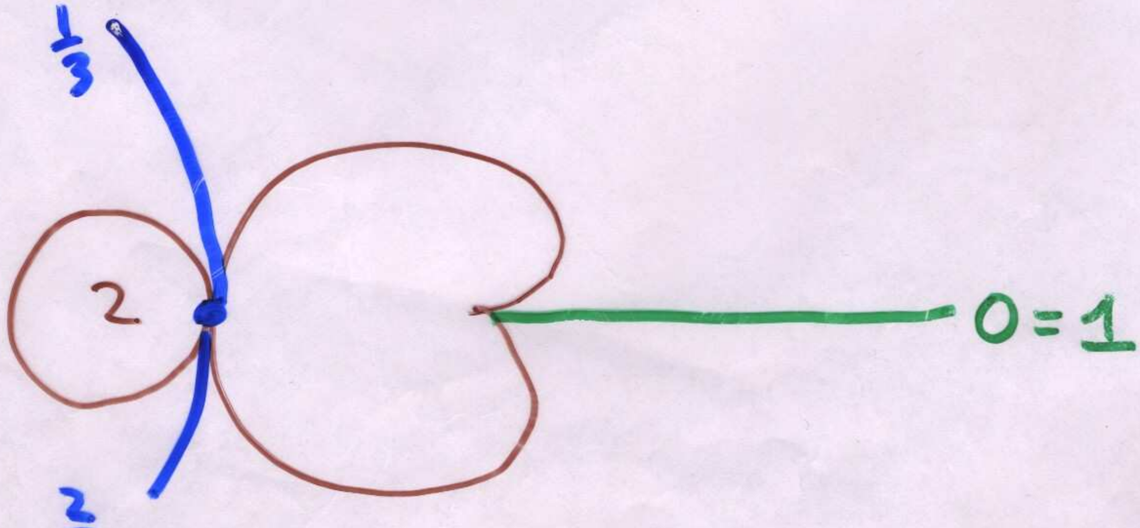
IMAGES ARE "EXTERNAL RAYS"

MEASURE MOD 1

WHERE RAYS LAND DETERMINES M

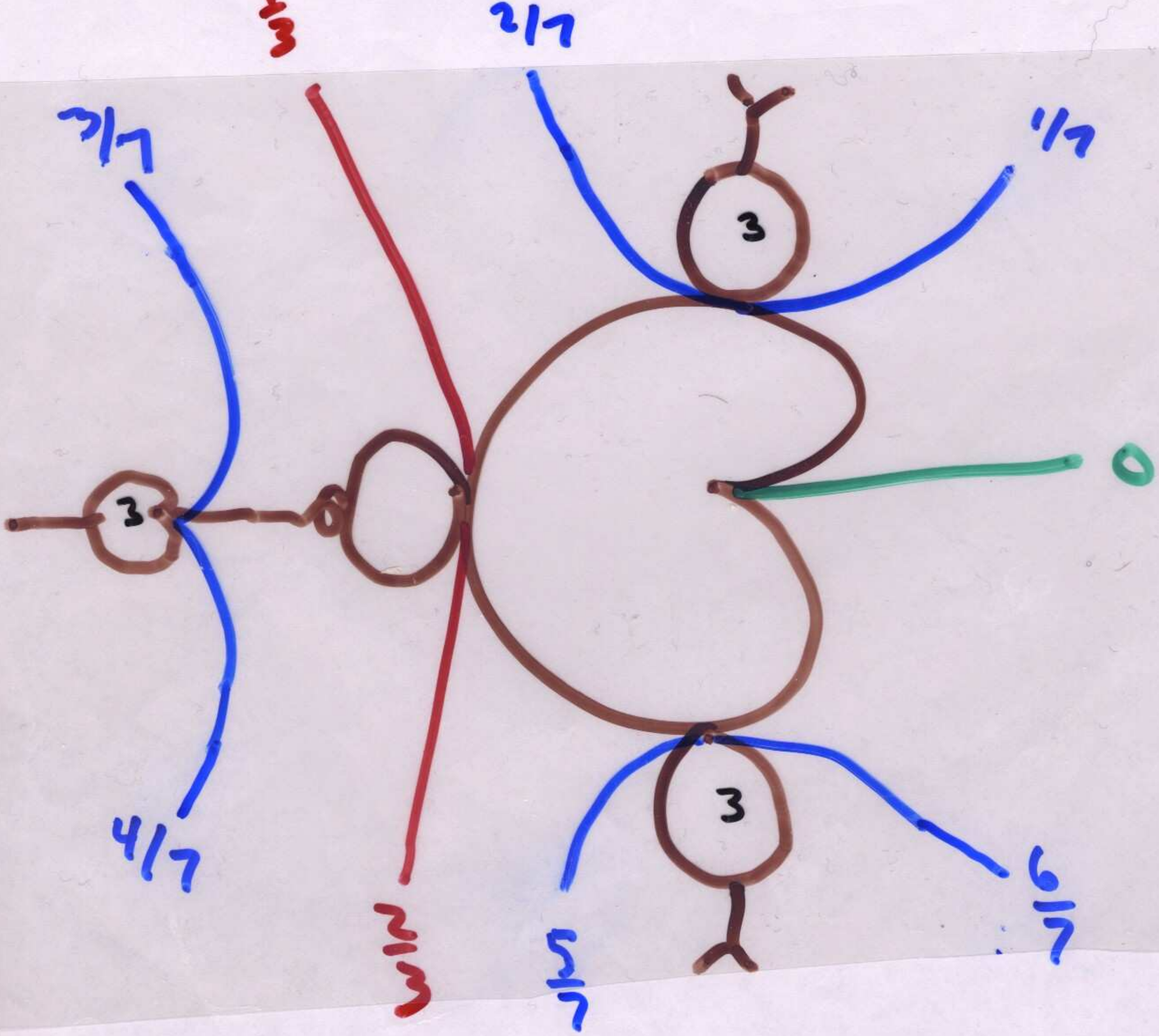
IF P/Q IS PERIODIC OF PERIOD $\underline{\underline{N}}$ UNDER DOUBLING, THEN P/Q RAY "CUTS OFF" PERIOD $\underline{\underline{N}}$-BULB

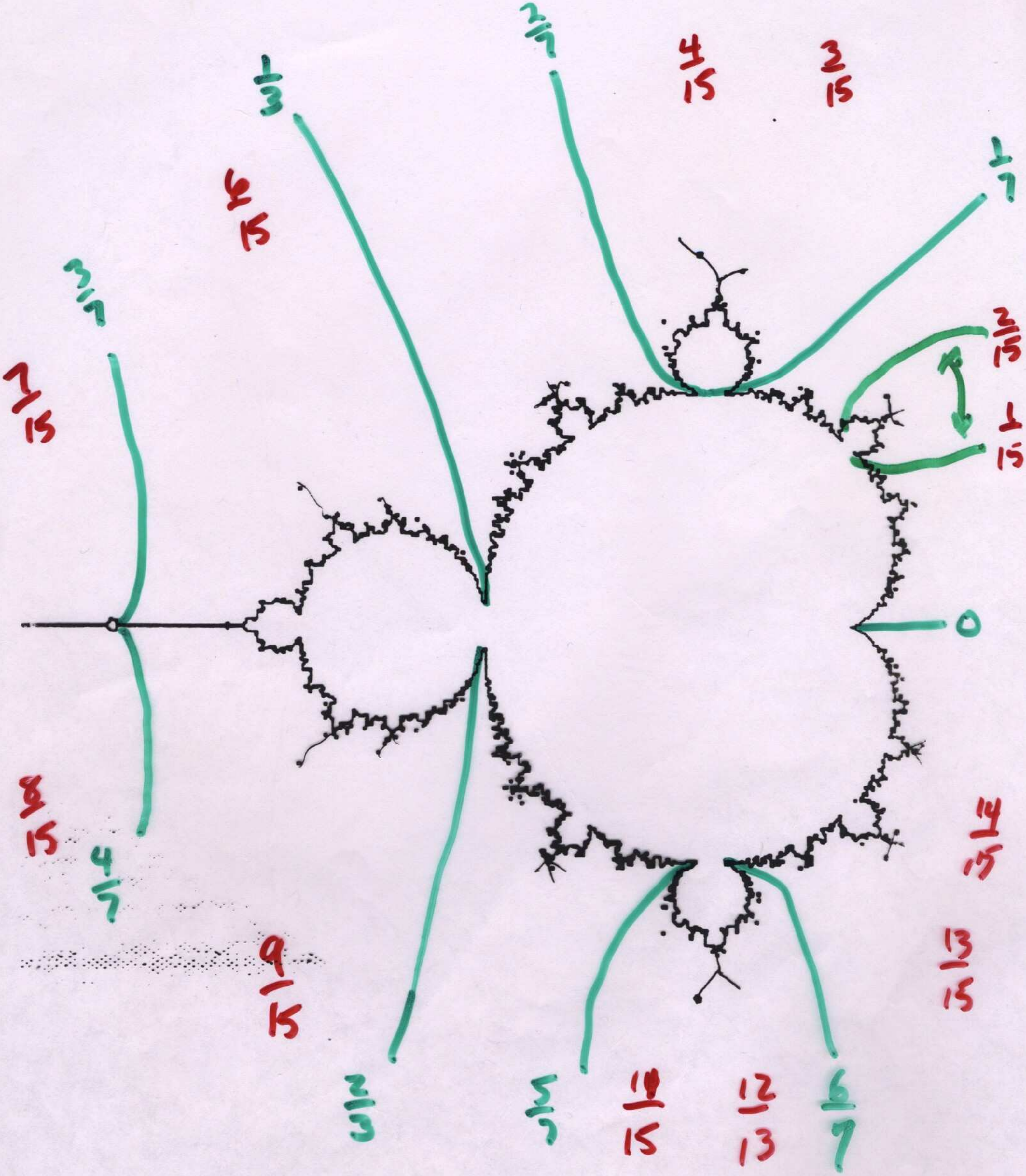
<u>EX</u> $\frac{1}{3} \to \frac{2}{3} \to \frac{1}{3}$ PERIOD 2



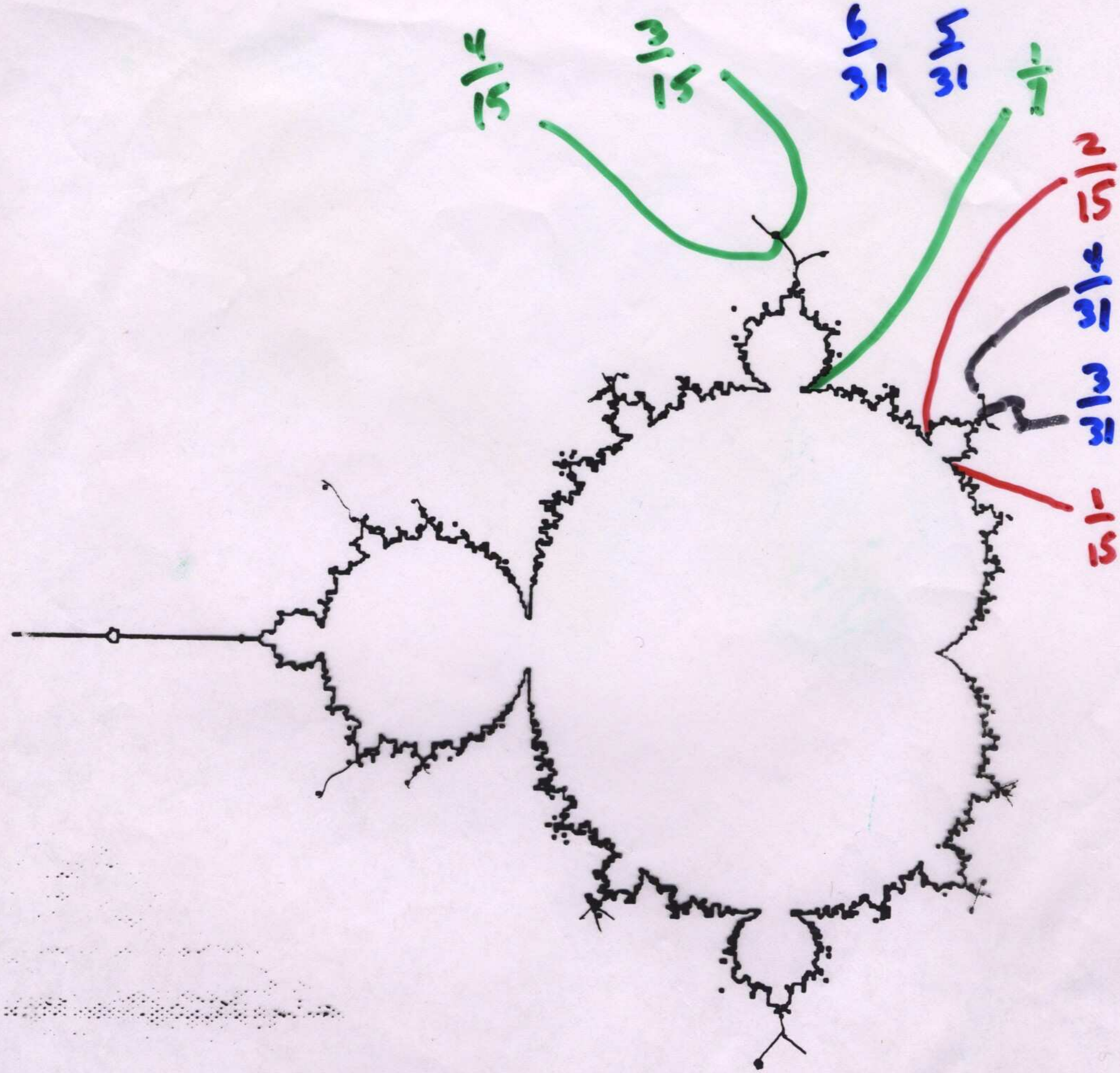
EX $\frac{1}{7} \to \frac{2}{7} \to \frac{4}{7}$ PERIOD 3

$\frac{3}{7} \to \frac{6}{7} \to \frac{5}{7}$





$\frac{1}{15} \to \frac{2}{15} \to \frac{4}{15} \to \frac{8}{15} \to \frac{1}{15} \ldots$ PERIOD 4



$\frac{k}{31}$ HAS PERIOD 5