THE STRONG PERFECT GRAPH THEOREM

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GRAPH $G(V, E)$

VERTICES EDGES

SOME PAIRS OF VERTICES ARE JOINED BY AN EDGE

(ADJACENT VERTICES)

SOME PAIRS OF VERTICES ARE NOT JOINED BY AN EDGE

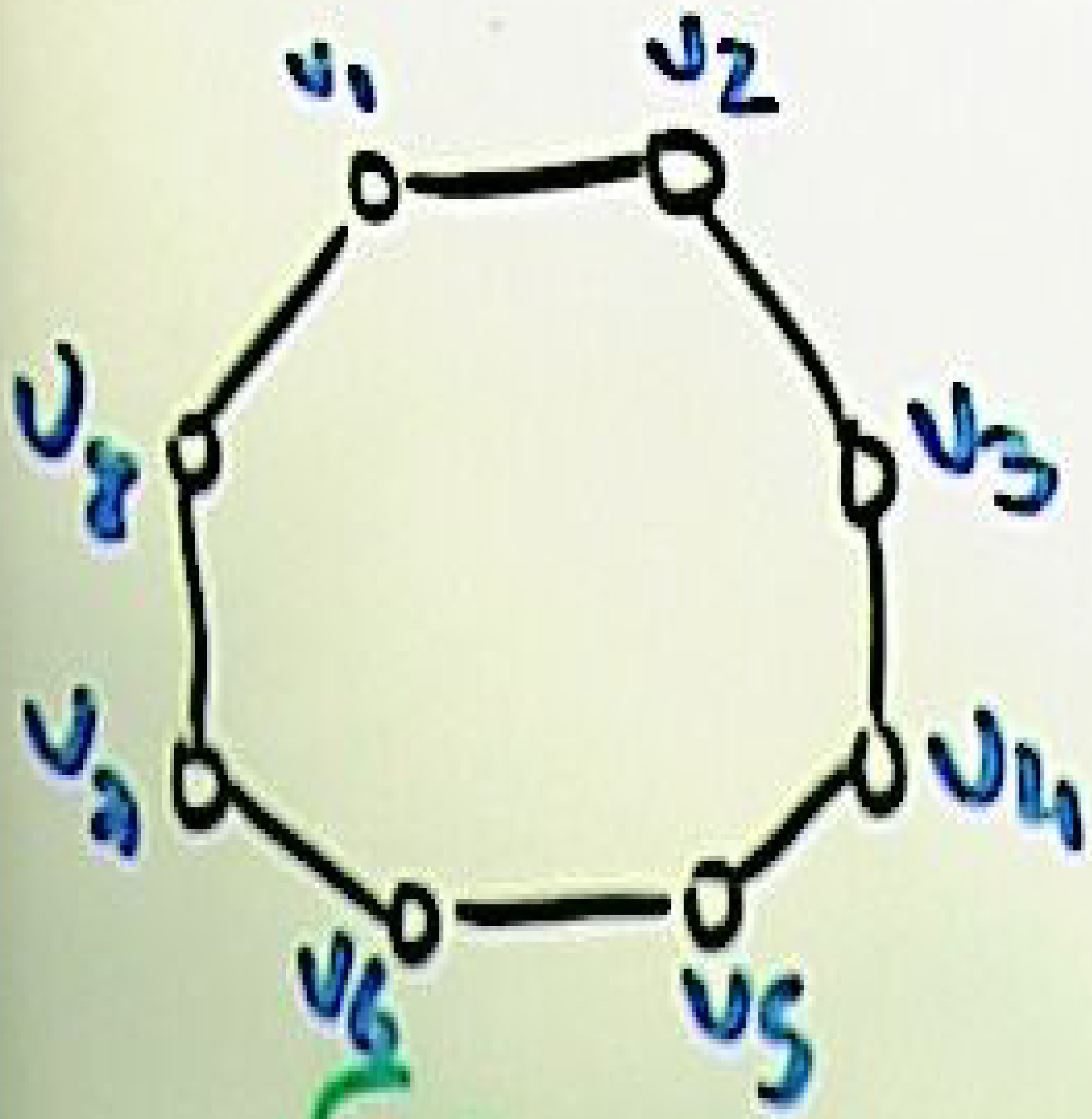
(NON-ADJACENT VERTICES)

V: PEOPLE IN THE ROOM

E: BORN ON SAME DATE

V: COUNTRIES

E: HAVE COMMON BORDER

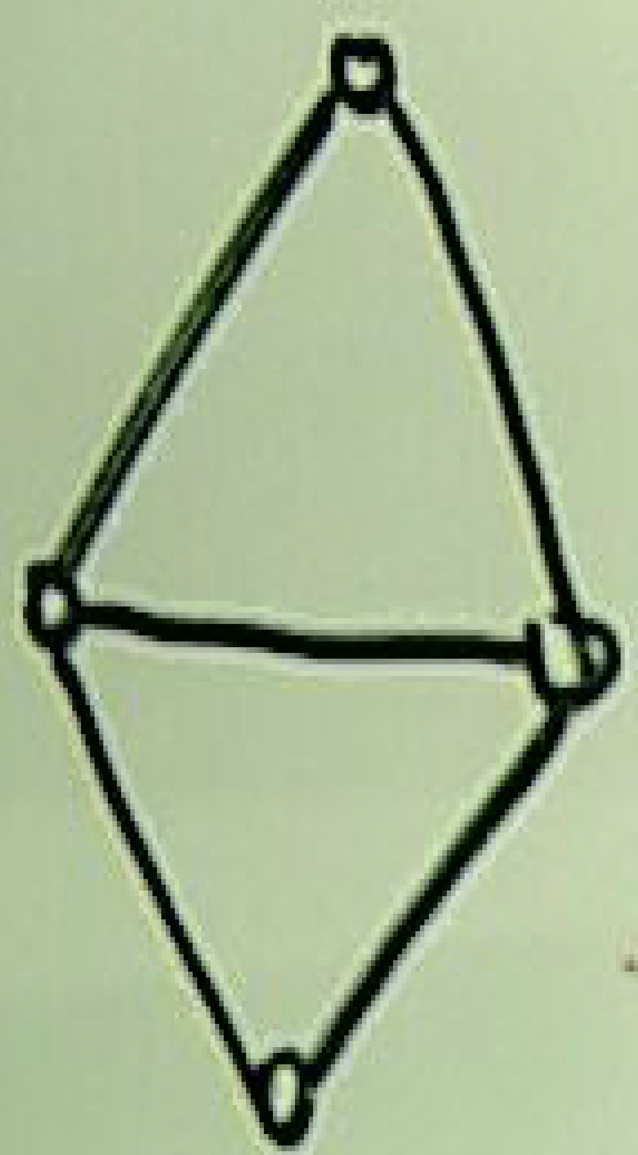


C_8

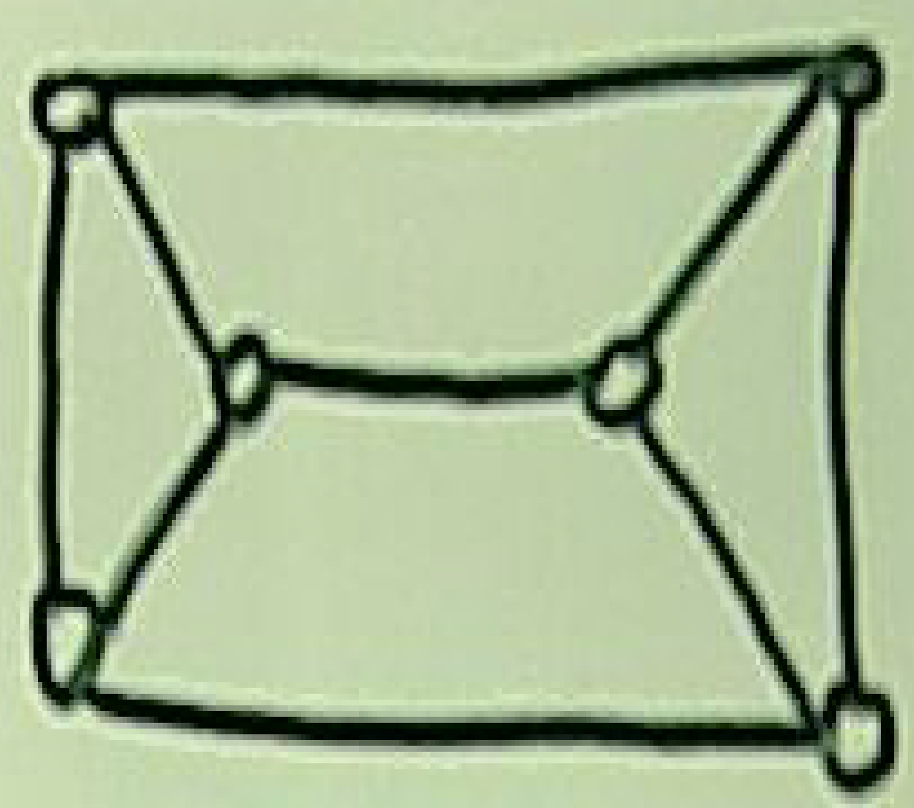
CYCLE OF LENGTH 8

K_4

CLIQUE, COMPLETE GRAPH



DIAMOND



PRISM

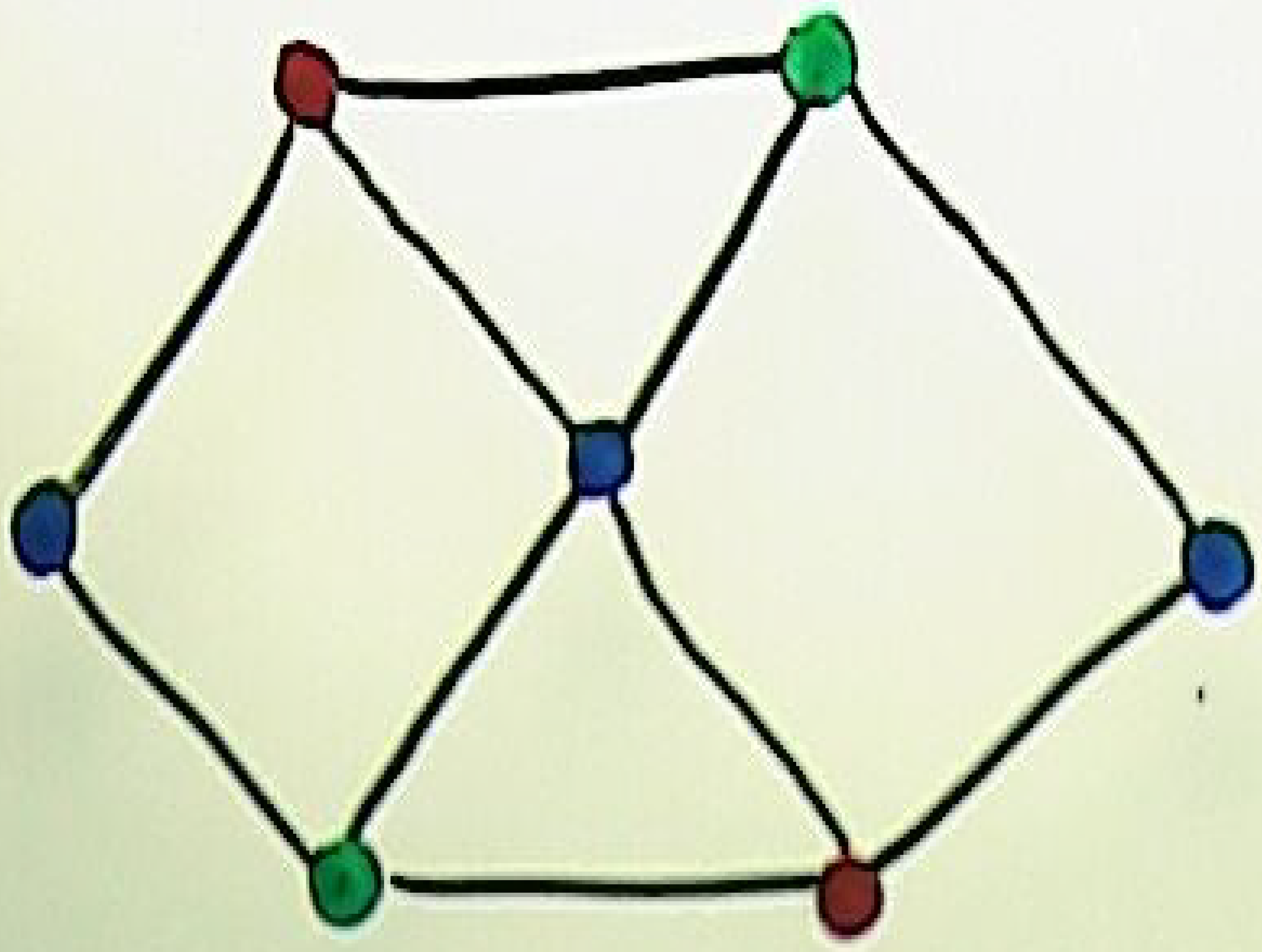
PROPER COLORING

COLORING OF THE VERTICES
S.T. NO TWO ADJACENT VERTICES
RECEIVE SAME COLOR

CHROMATIC NUMBER OF G

MIN NUMBER OF COLORS NEEDED
TO COLOR G PROPERLY

$\chi(G)$



- PROPER COLORING
- USING 3 COLORS

$\chi(G) \leq 3$

- CONTAINS 3 PAIRWISE ADJACENT VERTICES

$\chi(G) \geq 3$

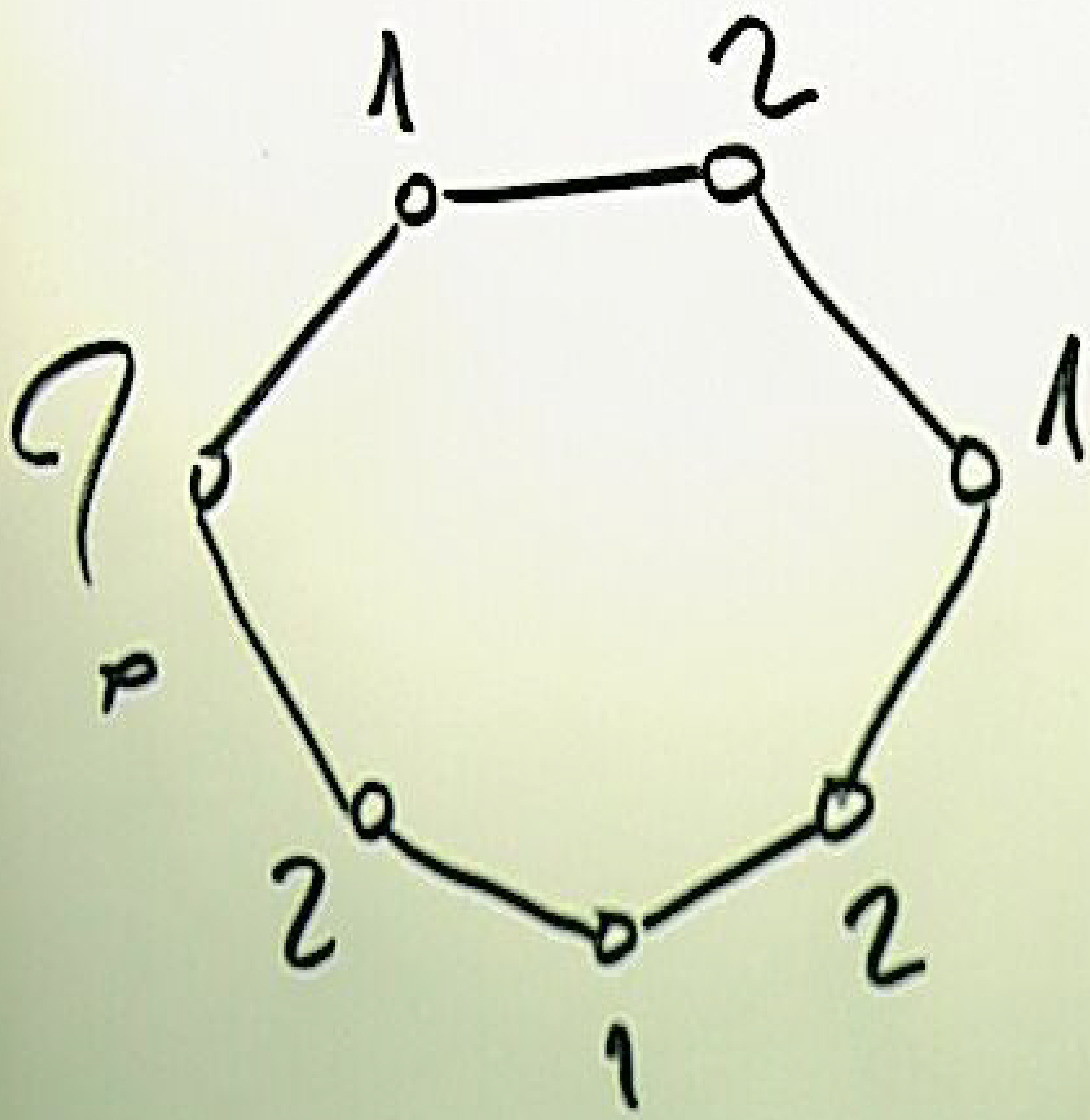
$\Rightarrow \chi(G) = 3$

CLIQUE NUMBER OF A GRAPH

MAX NUMBER OF VERTICES PAIRWISE ADJACENT

$\omega(G)$

$$\boxed{\chi(G) \geq \omega(G)}$$



- CONTAINS 2 ADJACENT VERTICES

$\chi(G) \geq 2$

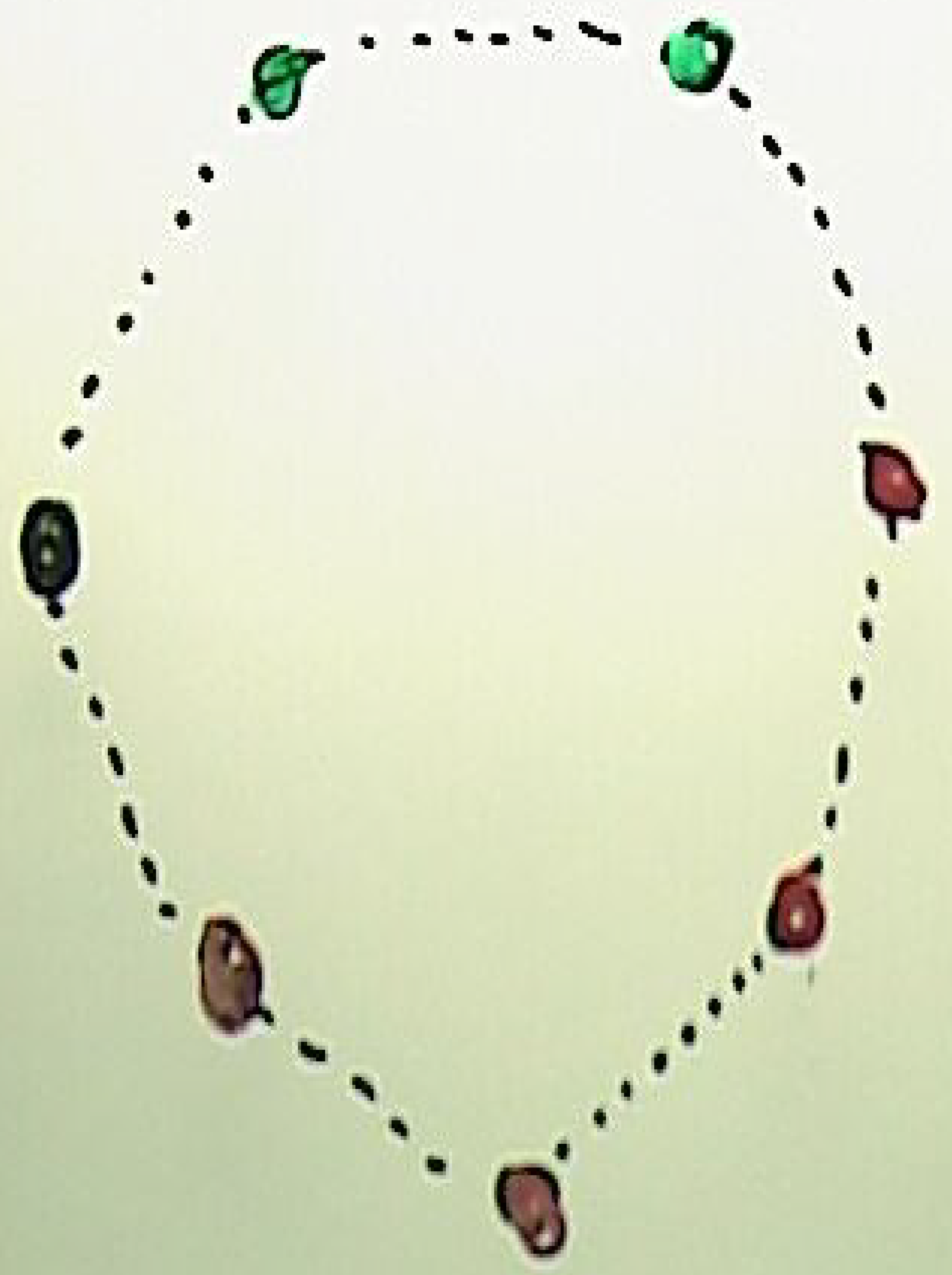
COMPLEMENT OF A GRAPH

G^c

$G(V, E)$

THE VERTICES OF G^c ARE THE VERTICES OF G

TWO VERTICES ARE ADJACENT IN G^c IF AND ONLY IF THEY ARE NON-ADJACENT IN G



C_7^c

$\chi(G) \leq 4$

$\omega(G) = 3$

- AT MOST TWO VERTICES CAN HAVE SAME COLOR

  $\chi(G) \geq 4$

- $\omega(G) = 3$

QUESTION:

FOR WHICH GRAPHS

$\chi(G) = \omega(G)$?

INTERESTING CLASSES OF GRAPHS

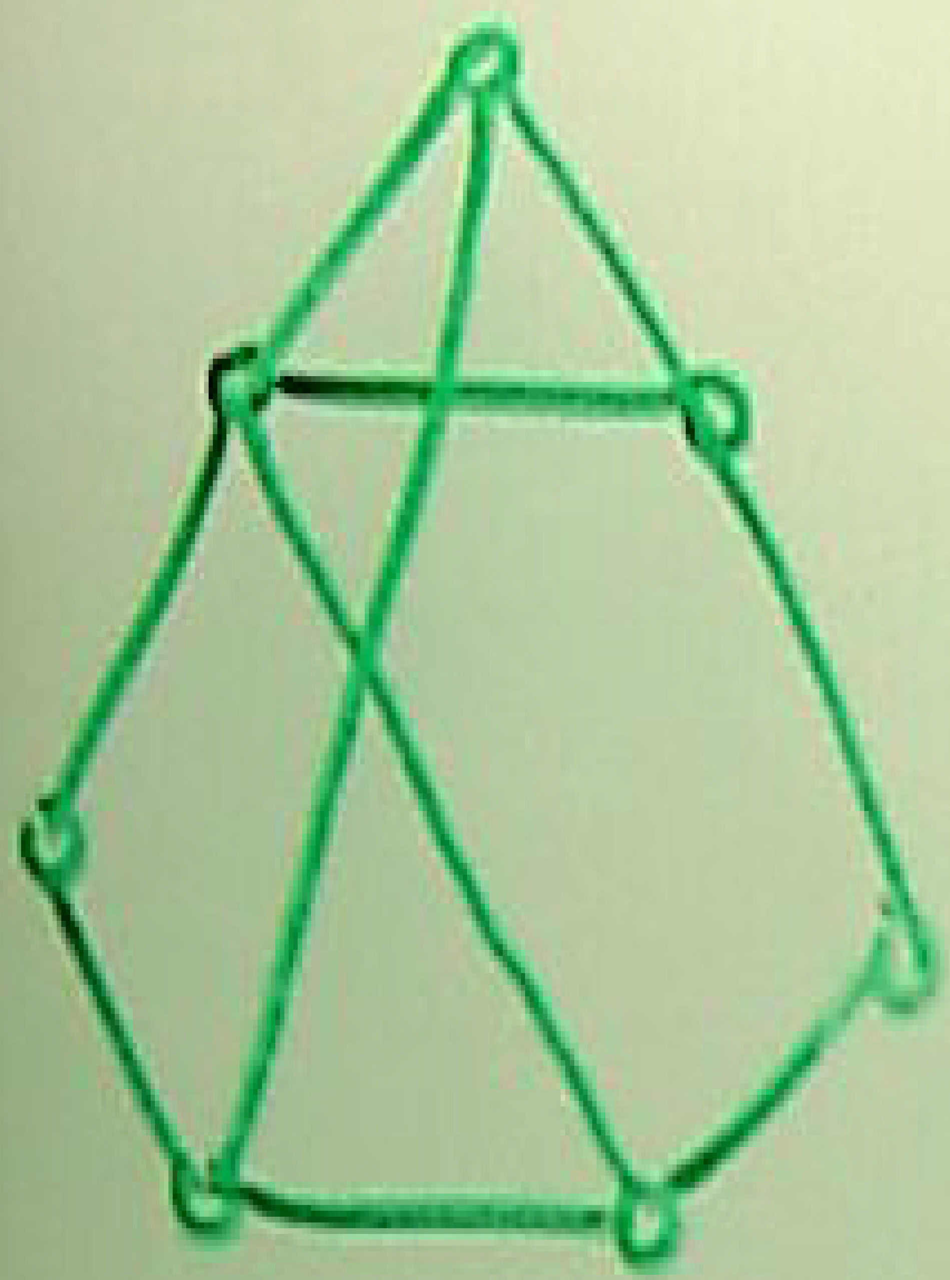
NOT A GOOD QUESTION:

ANY GRAPH ON 99 VERTICES

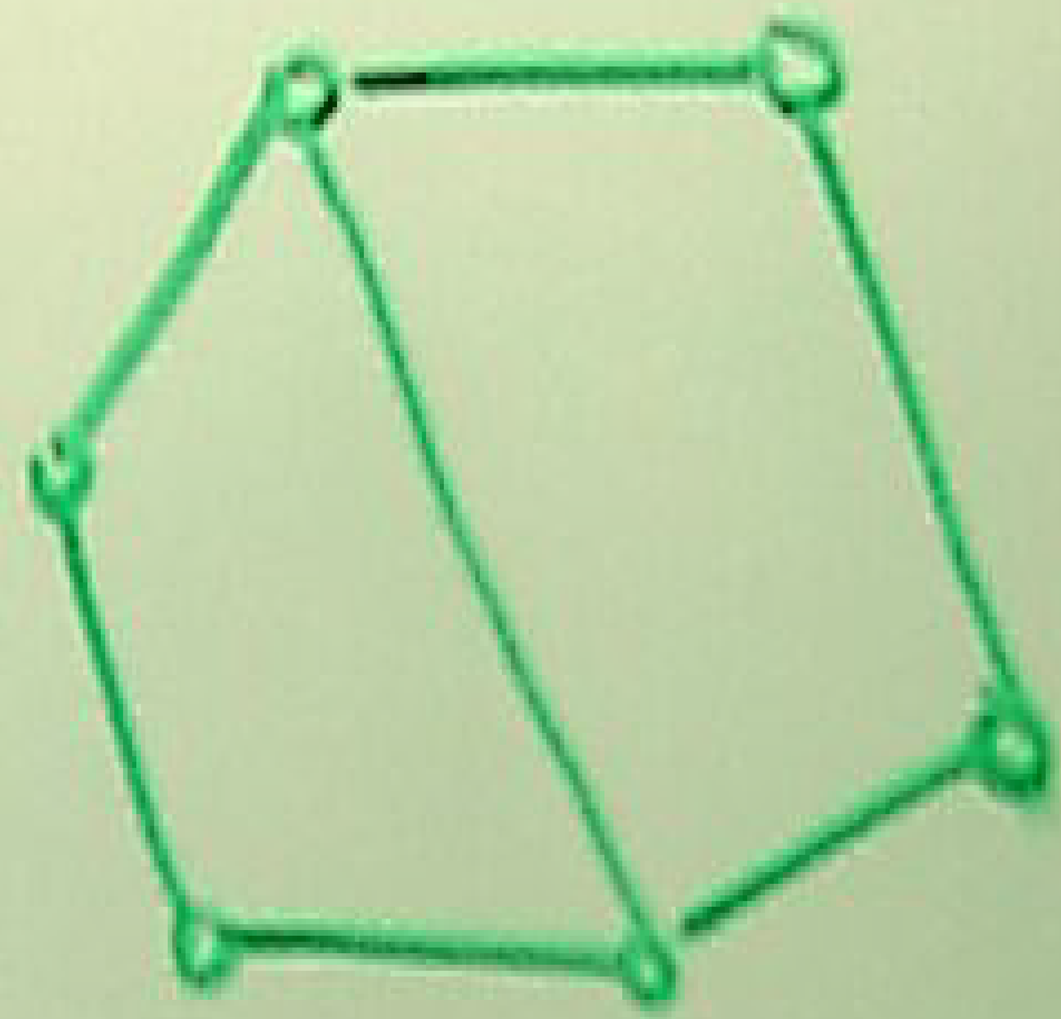
K_{100}

INDUCED SUBGRAPH

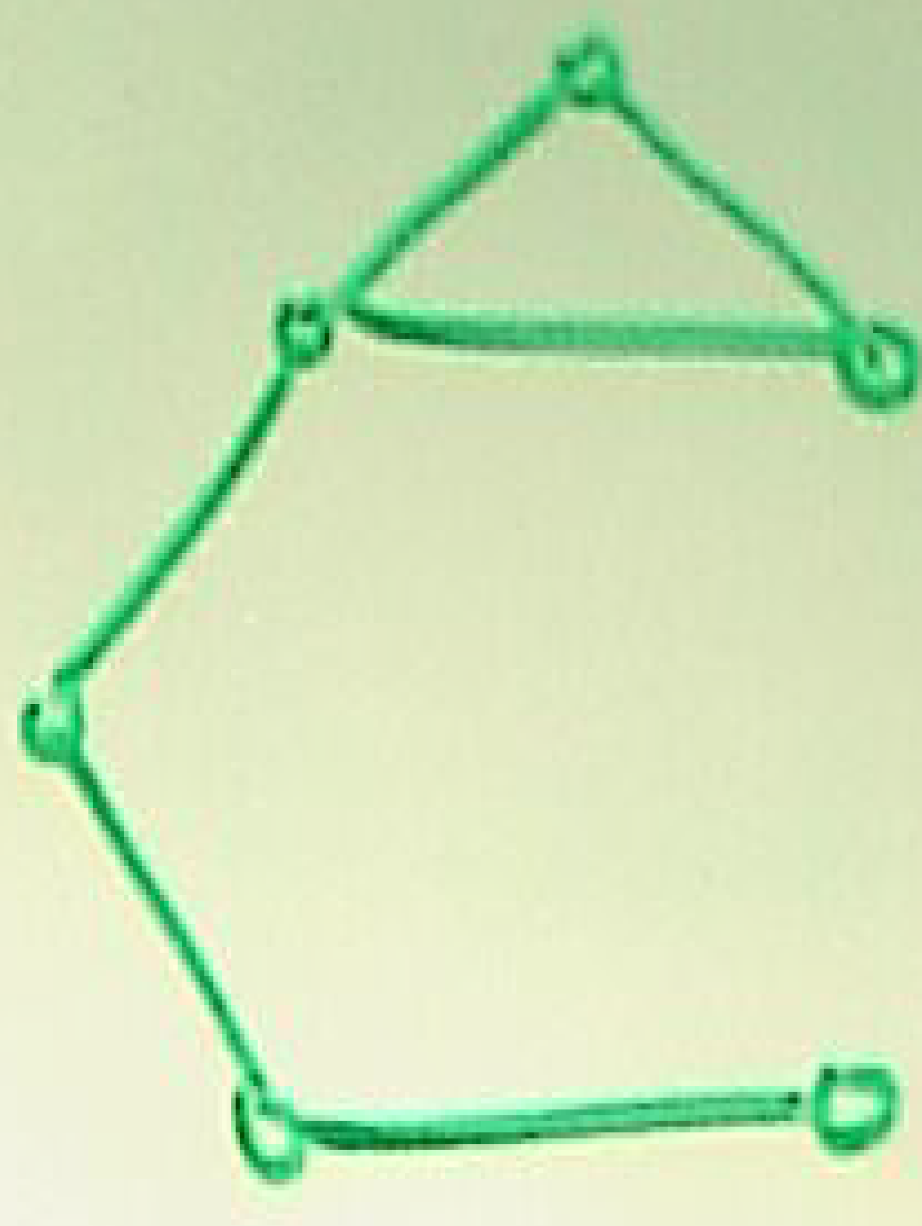
REMOVE SOME VERTICES
KEEP ALL EDGES



G



INDUCED
SUBGRAPH



NOT INDUCED

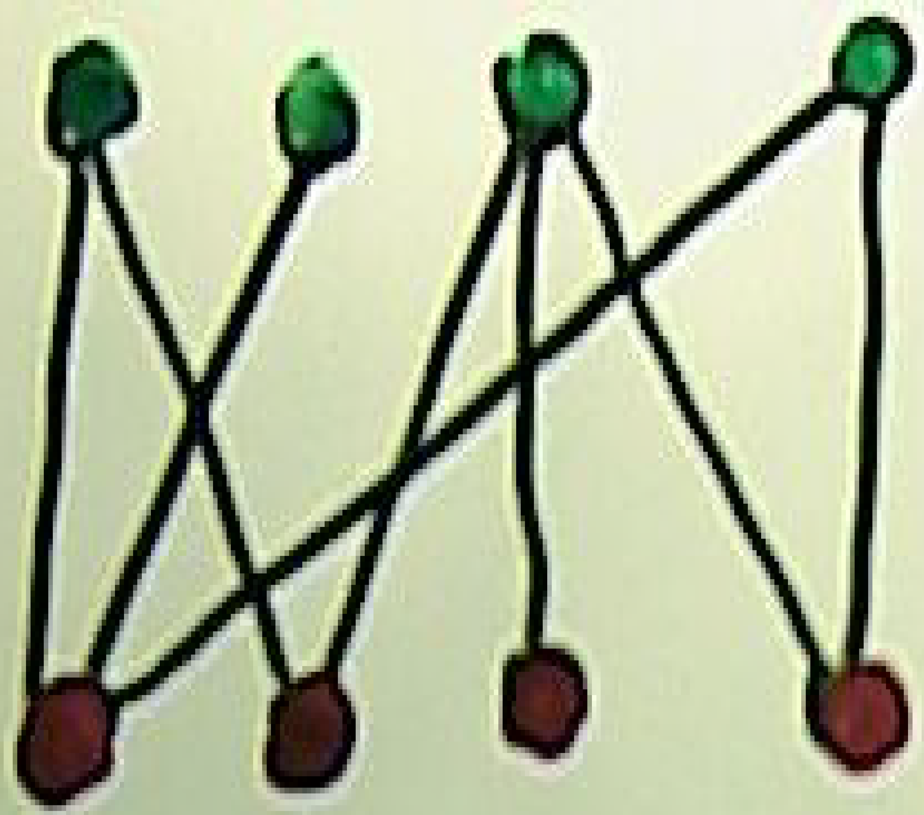
GOOD QUESTION

FOR WHICH GRAPHS $\chi(H) = \omega(H)$
FOR EVERY INDUCED SUBGRAPH
H OF G

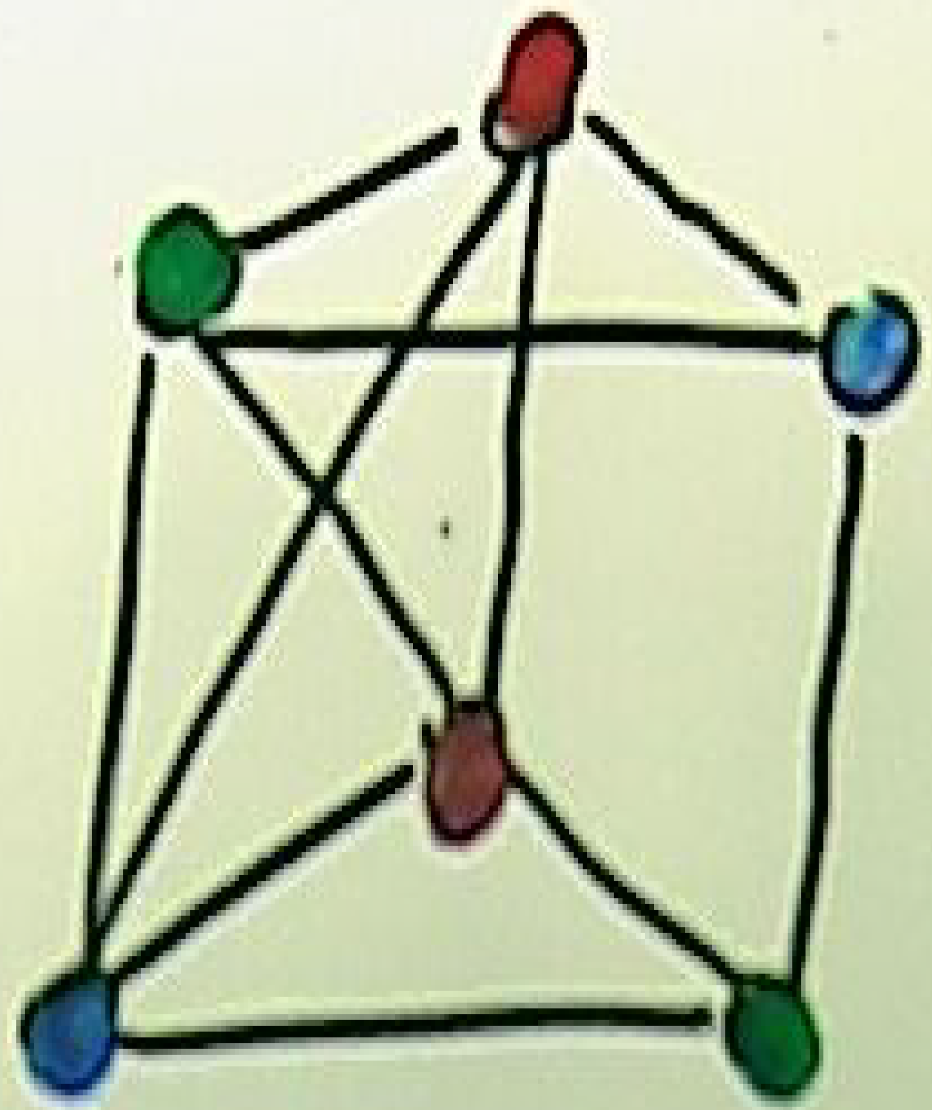
$$\boxed{\text{PERFECT GRAPHS}}$$

CLAUDE BERGE

PERFECT GRAPHS



BIPARTITE



COMPLEMENT OF BIPARTITE (KÖNIG)

LINE GRAPH OF BIPARTITE

COMPLEMENT OF LINEGRAPH OF BIPARTITE

⋮

IMPERFECT GRAPHS

C_7 , C_7^c

C_{2n+1} ODD CYCLE

C_{2n+1}^c COMPLEMENT OF ODD CYCLE

G PERFECT $\Rightarrow$

- NO ODD CYCLE ≥ 5 IS AN INDUCED SUBGRAPH OF G
- NO COMPLEMENT OF AN ODD CYCLE ≥ 5 IS AN INDUCED SUBGRAPH OF G

CONJECTURES ABOUT PERFECT GRAPHS:

(CLAUDE BERGE, 1961)

- G PERFECT $\Leftrightarrow$ G^c PERFECT
- G PERFECT $\Leftrightarrow$

BERGE GRAPHS: NO INDUCED SUBGRAPH OF G IS AN ODD CYCLE (≥ 5) OR THE COMPLEMENT OF AN ODD CYCLE (≥ 5)

WEAK PERFECT GRAPH THEOREM (LOVASZ, 1972)

G IS PERFECT $\Leftrightarrow$

G^c IS PERFECT

STRONG PERFECT GRAPH THEOREM

(CRST, 2002)

G PERFECT $\iff$

NO INDUCED SUBGRAPH OF G IS AN ODD CYCLE OR THE COMPLEMENT OF ONE.

APPROACH:

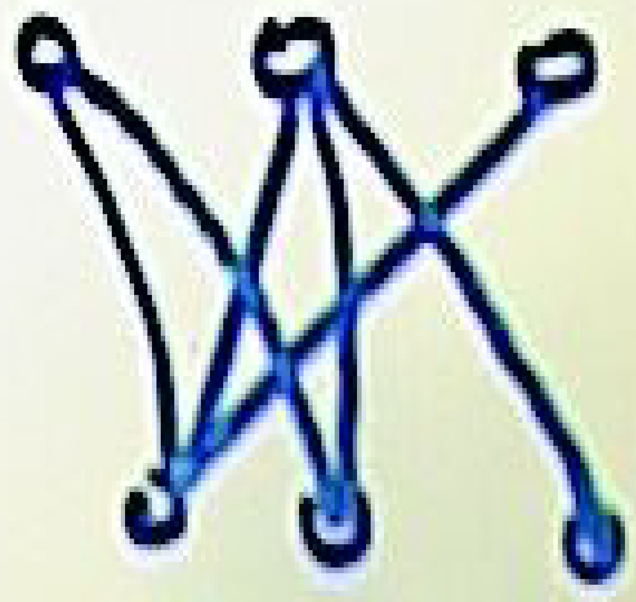
"FIND" ALL BERGE GRAPHS

PROVE THAT:

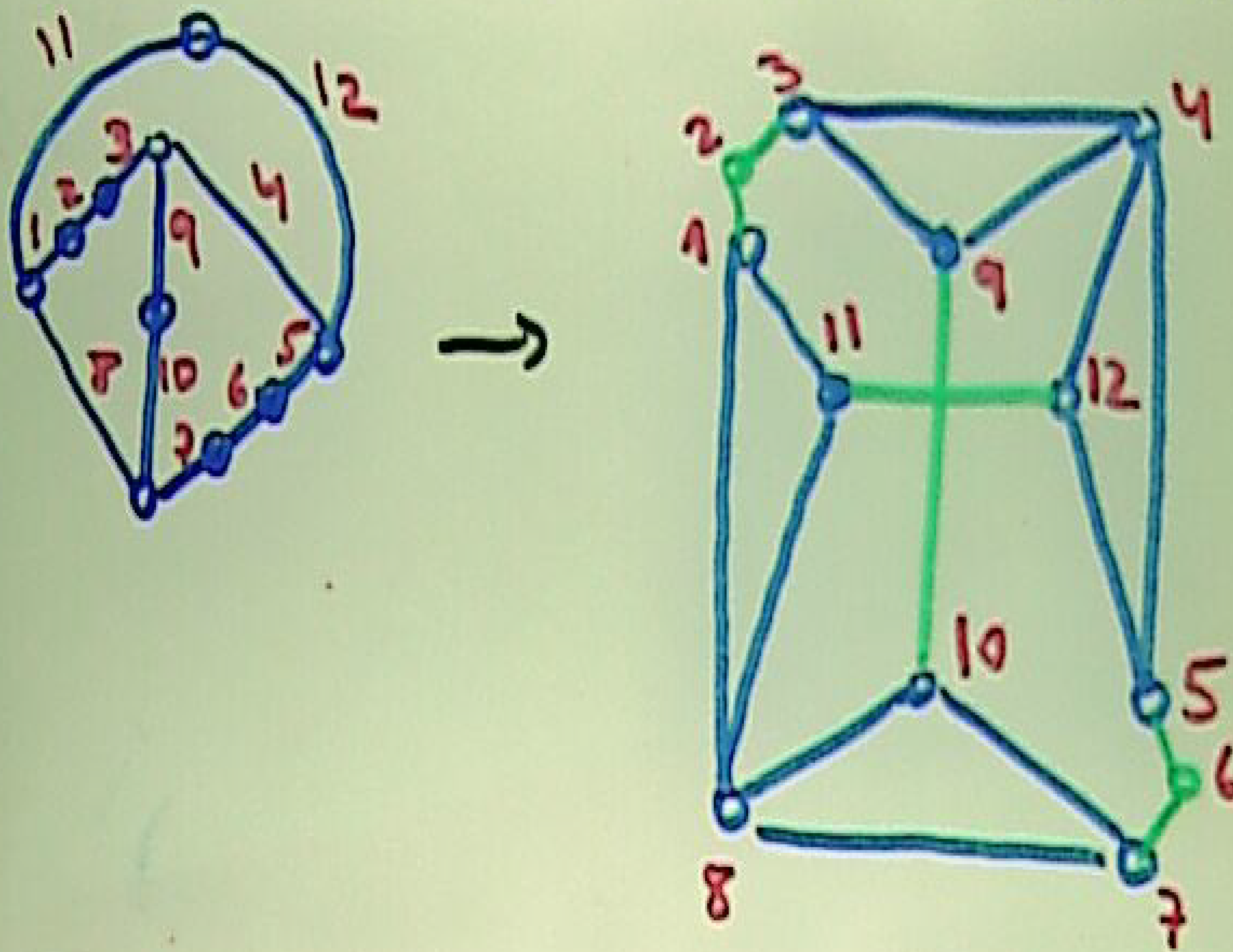
EVERY BERGE GRAPH EITHER BELONGS TO ONE OF A FEW BASIC CLASSES OR HAS A DECOMPOSITION (THAT CANNOT OCCUR IN A MINIMUM COUNTEREXAMPLE TO BERGE'S CONJECTURE)

BASIC CLASSES :

- BIPARTITE GRAPHS



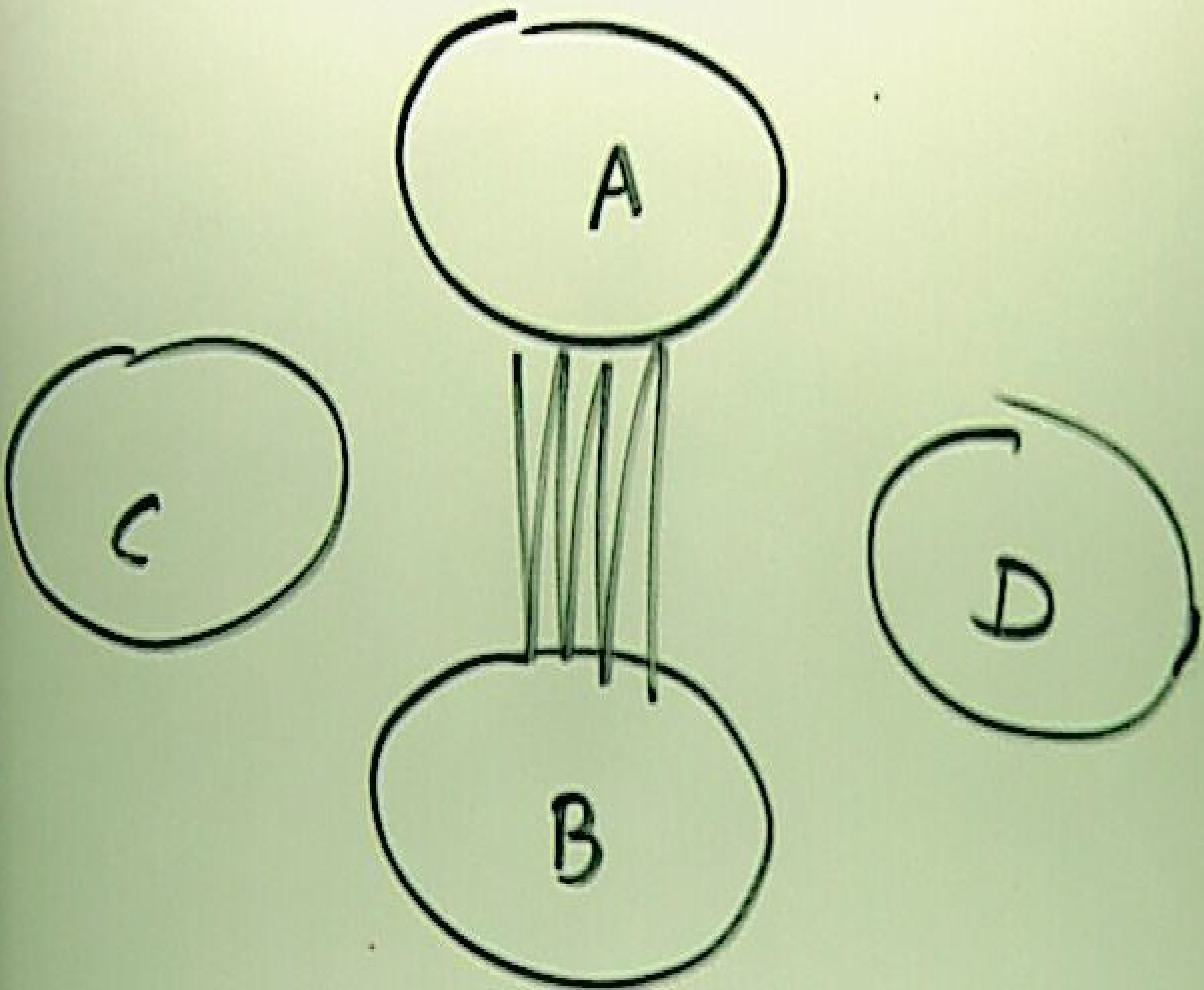
- COMPLEMENTS OF BIP GRAPHS
- LINE GRAPHS OF BIP GRAPHS



- COMPLEMENTS OF LINE GRAPHS OF BIPARTITE GRAPHS
- DOUBLE SPLIT GRAPHS

DECOMPOSITIONS

(BALANCED) SKEW-PARTITION



THM

MIN COUNTEREXAMPLE ADMITS NO BALANCED SKEW PARTITION

IDEA OF PROOF

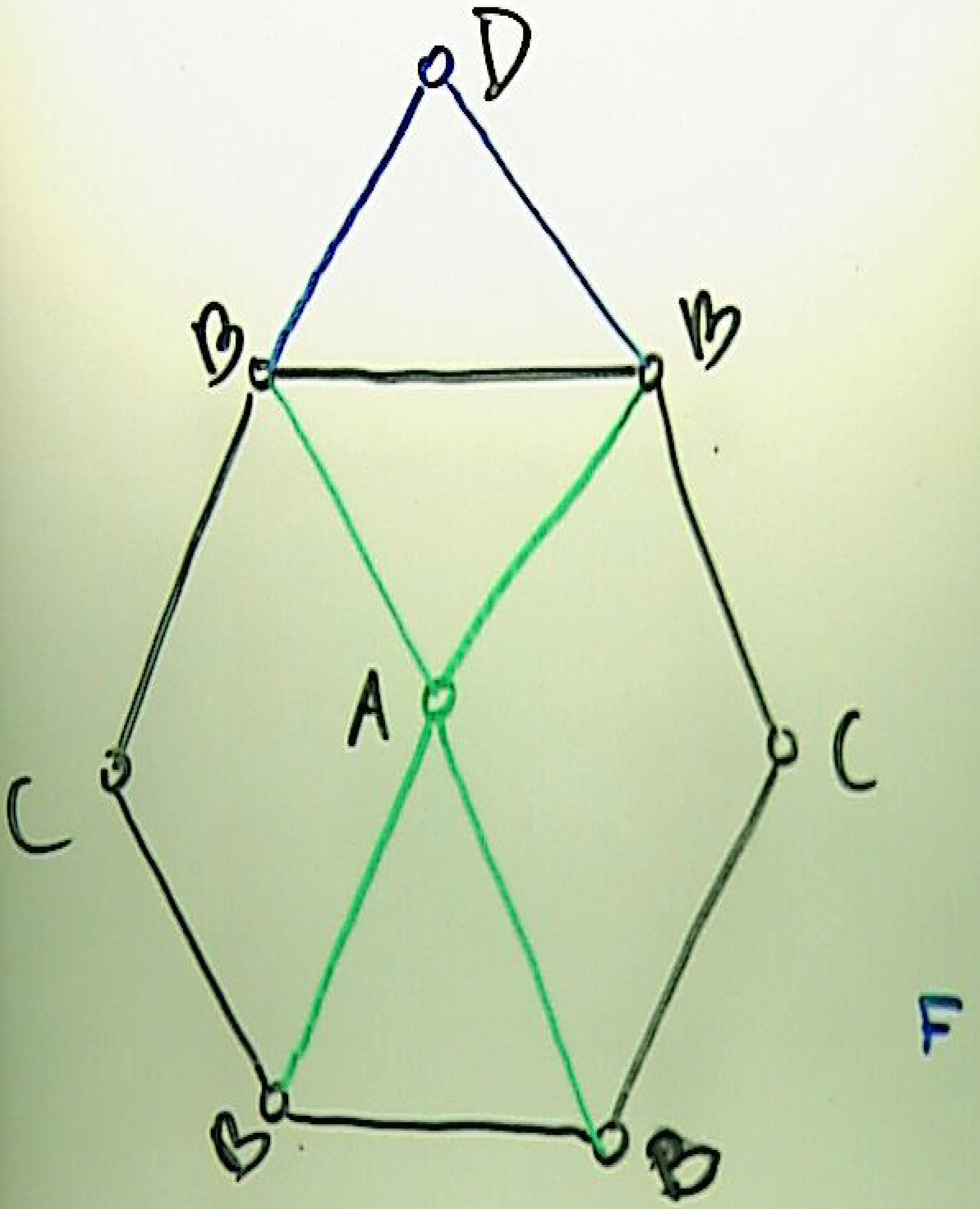
FIND $\left\{\begin{array}{l}\text{SMALL} \\ \text{WELL UNDERSTOOD}\end{array}\right\}$ SUBGRAPHS

THAT DETERMINE THE BEHAVIOR OF THE WHOLE GRAPH

PROVE THEOREMS OF THE FORM:

IF F IS AN INDUCED SUBGRAPH OF G THEN EITHER G IS BASIC OR G ADMITS A DECOMPOSITION

MAKE F SMALLER & SMALLER



THM IF G CONTAINS F
THEN G HAS A SKEW-PARTITION