The Mathematical Beauty

Of Physics

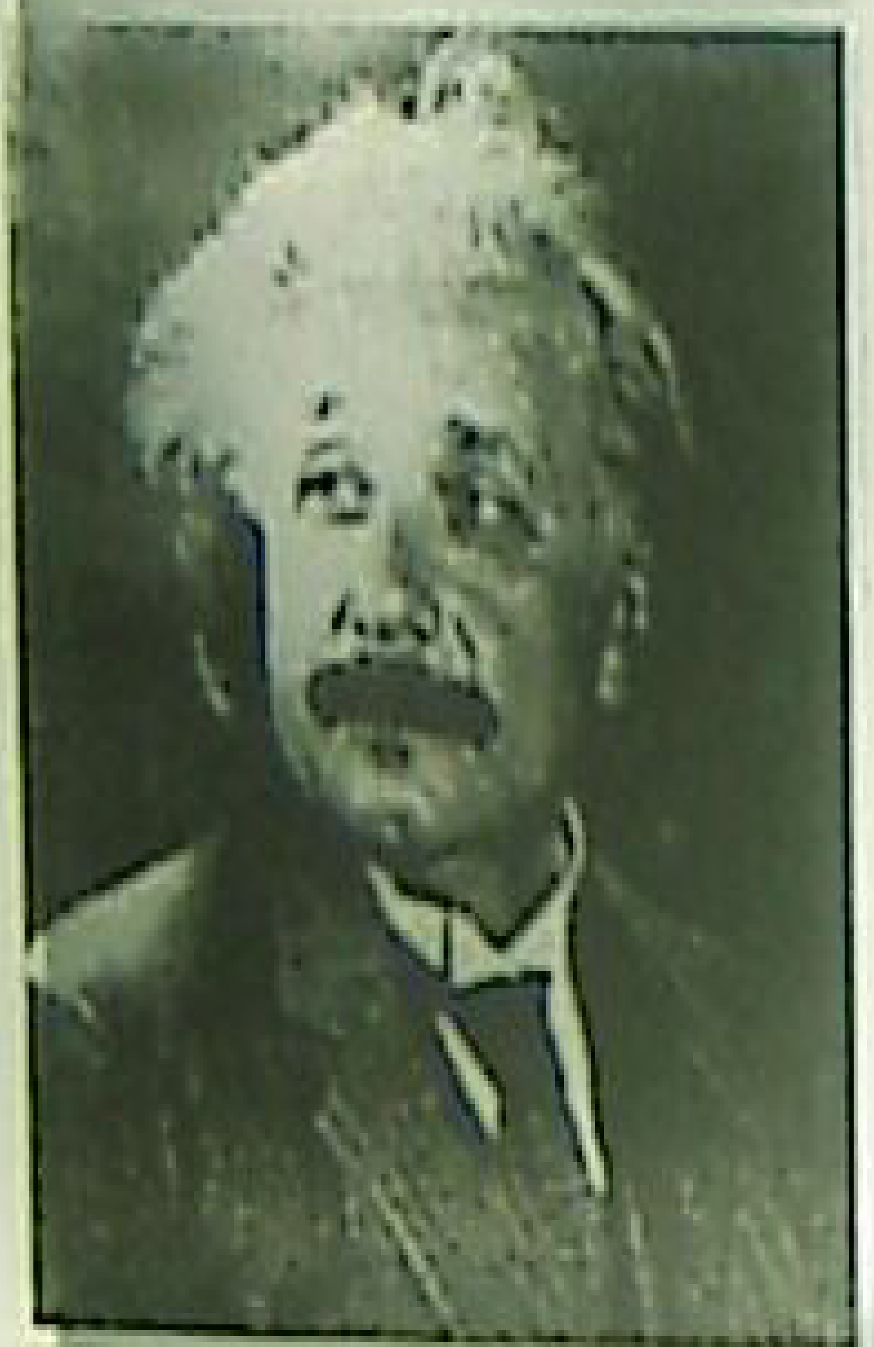
Sergei Gukov

Hilbert's 6th Problem (1900):

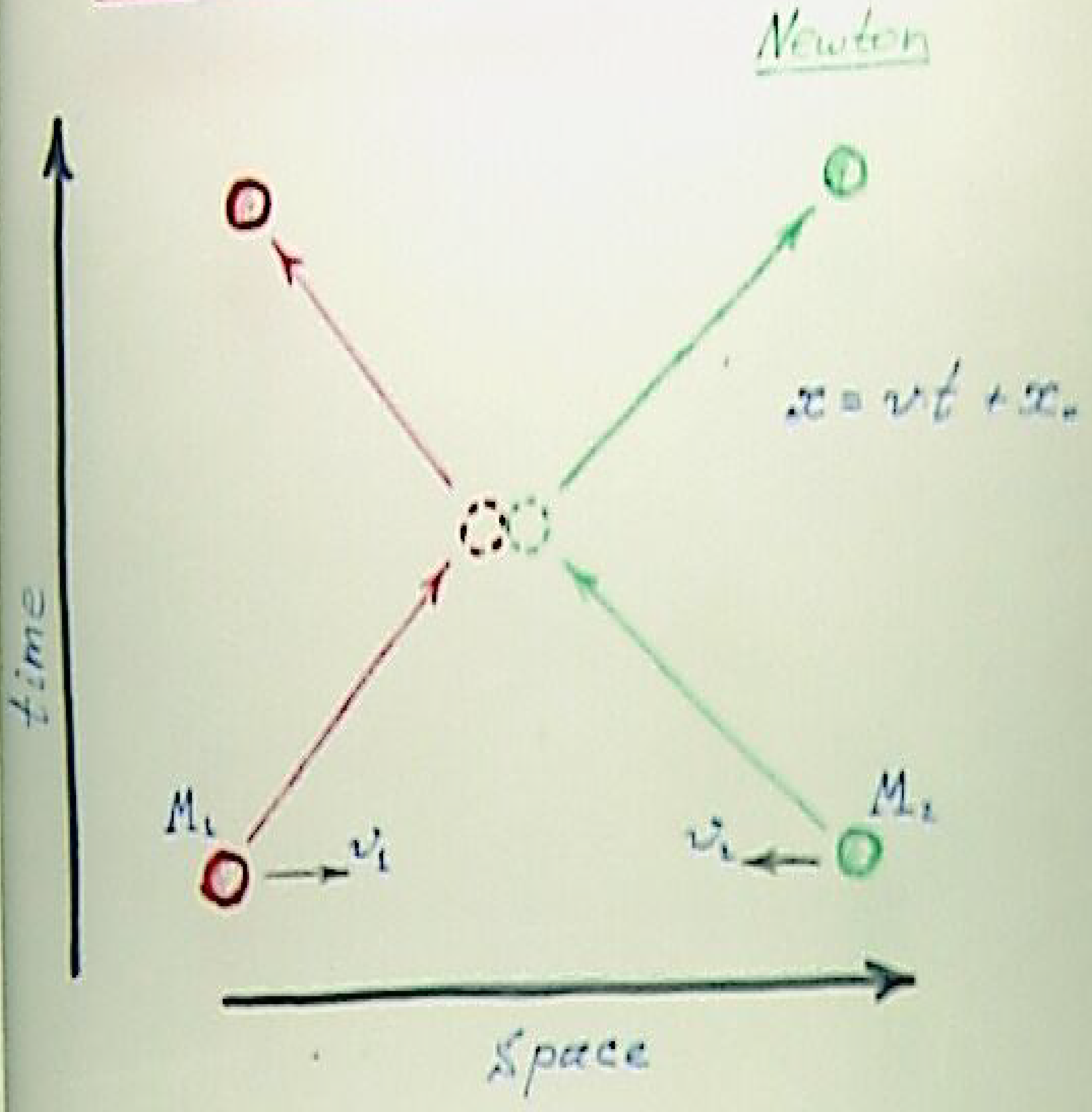
"Mathematical treatment of the axioms of physics"

"If geometry is to serve as a model for the treatment of physical axioms, ... the mathematician will have to take account not only of those theories coming near to reality, but also, as in geometry, of all logically possible theories." –

The history of physics and mathematics is greatly interconnected. For example, Newton's theory of mechanics and the developments of techniques of calculus are a classical example of this phenomenon. Another example is the development of Maxwell's theory of electromagnetism and Einstein's theory of general relativity inspired by the ideas in differential geometry.



Classical Mechanics



- Given the initial state of the system, one describe its evolution at all later moments of time.

$$\boxed{\vec{F} = M \cdot \frac{d^2\vec{x}}{dt^2}}$$

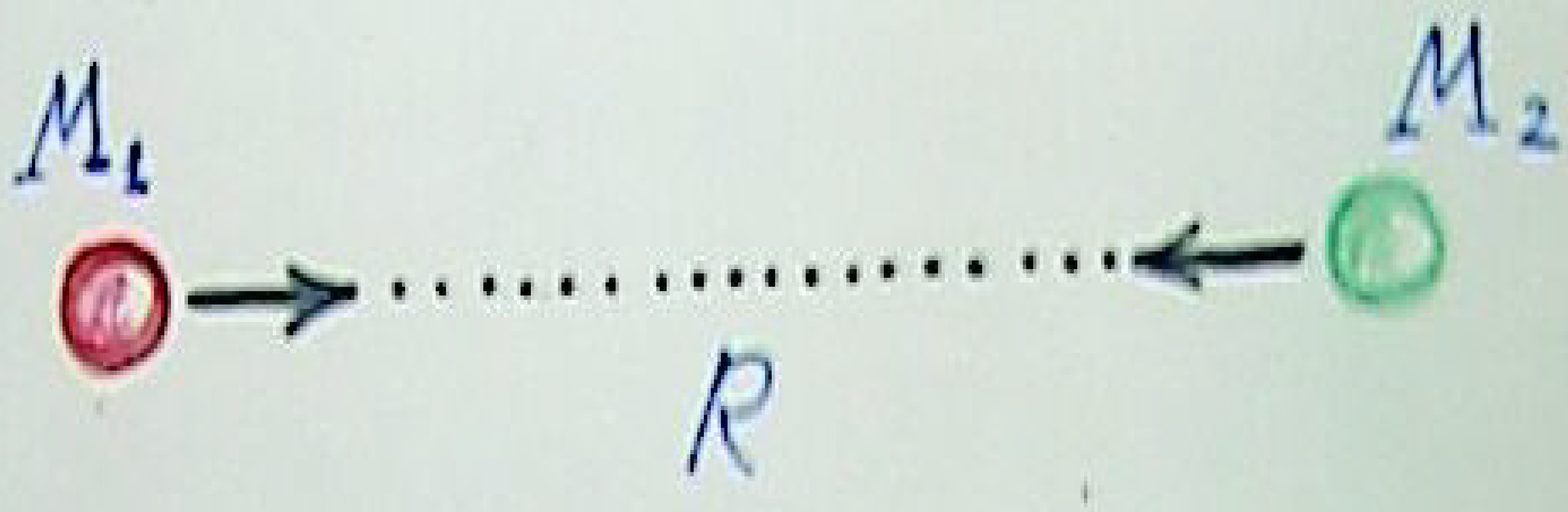
IF NEWTON HAD MISSED THE POINT . . .

Gravity

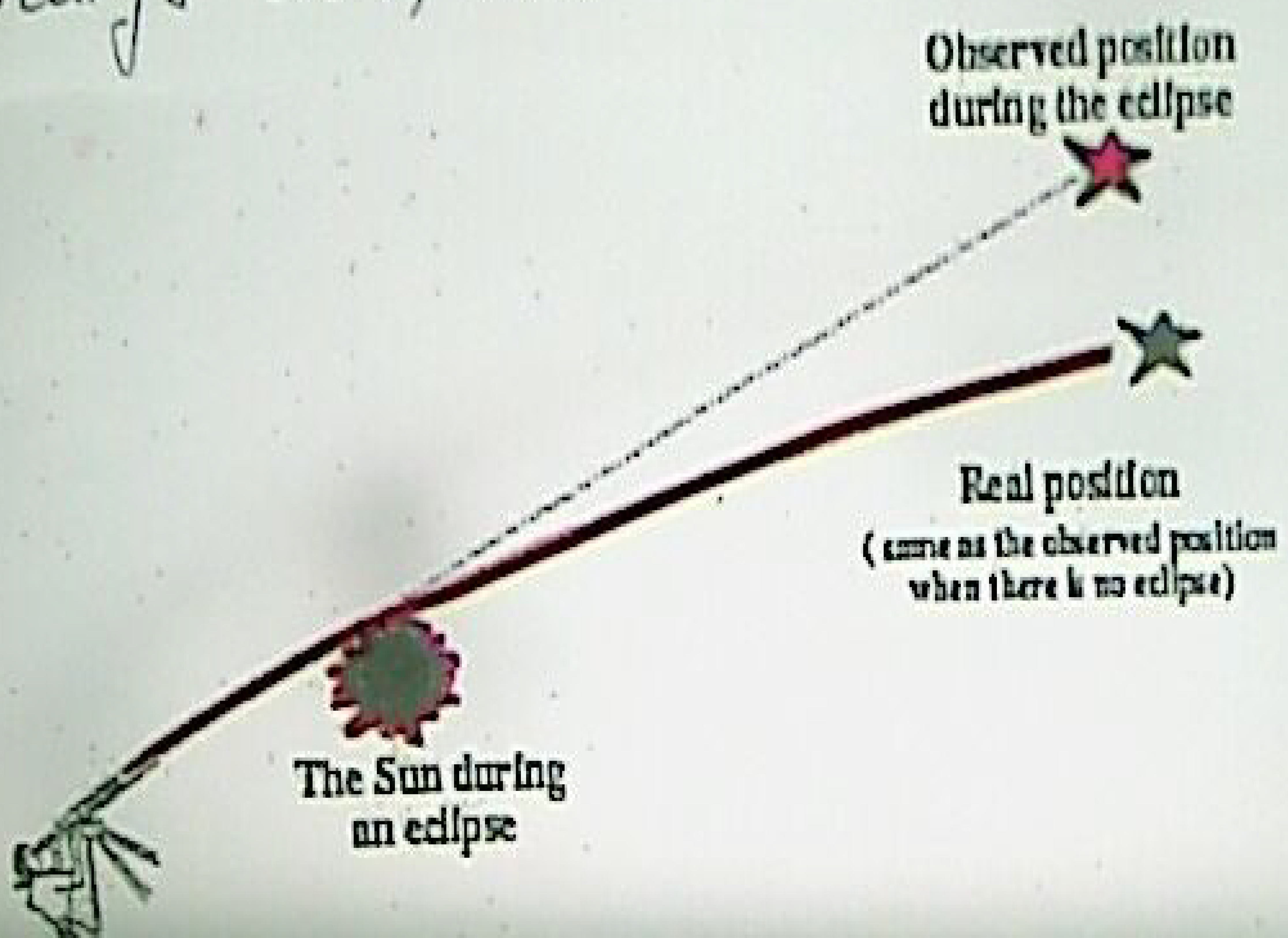
Newton

The force between two masses separated by a distance R is

$$F = G \frac{M_1 \cdot M_2}{R^2}$$



However, Newtonian gravity does not expla the bending of light, the precession of Mercury's orbit, etc.



Special Relativity (1905)

Lorentz
Einstein

- Space-time: t time

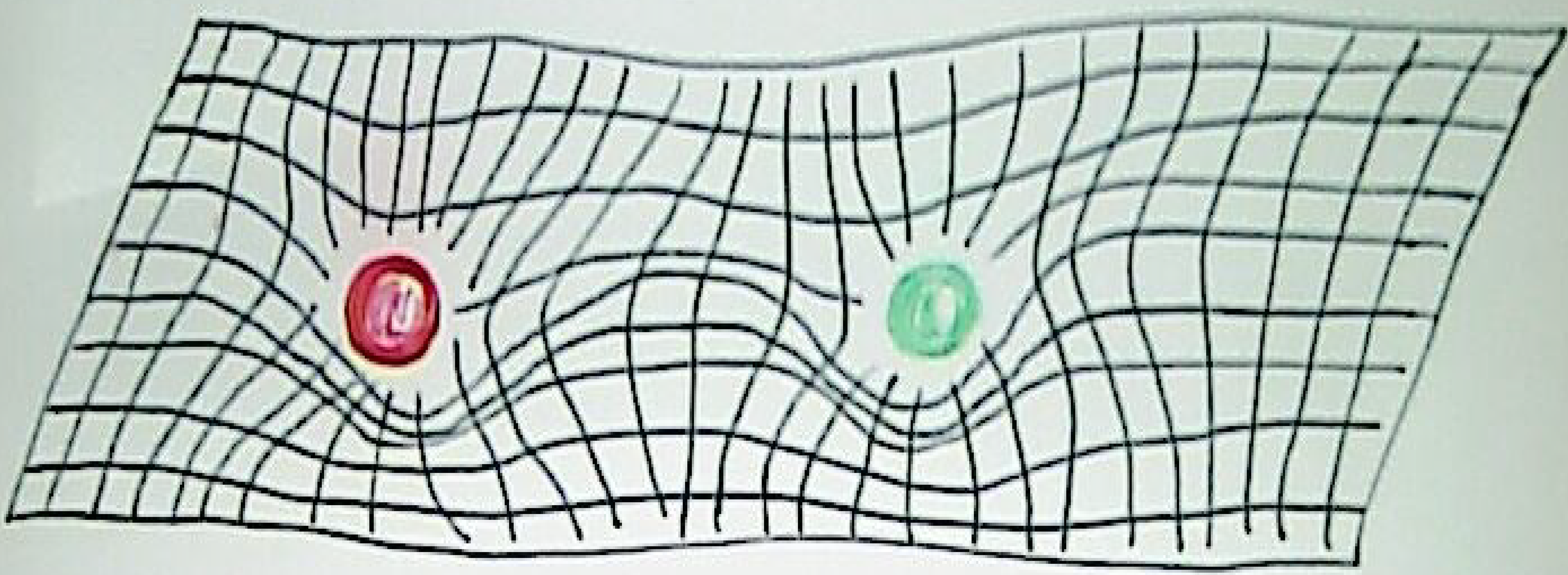
$\left.\begin{array}{l} x \\ y \\ z \end{array}\right\}$ space

- $c = \text{const}$

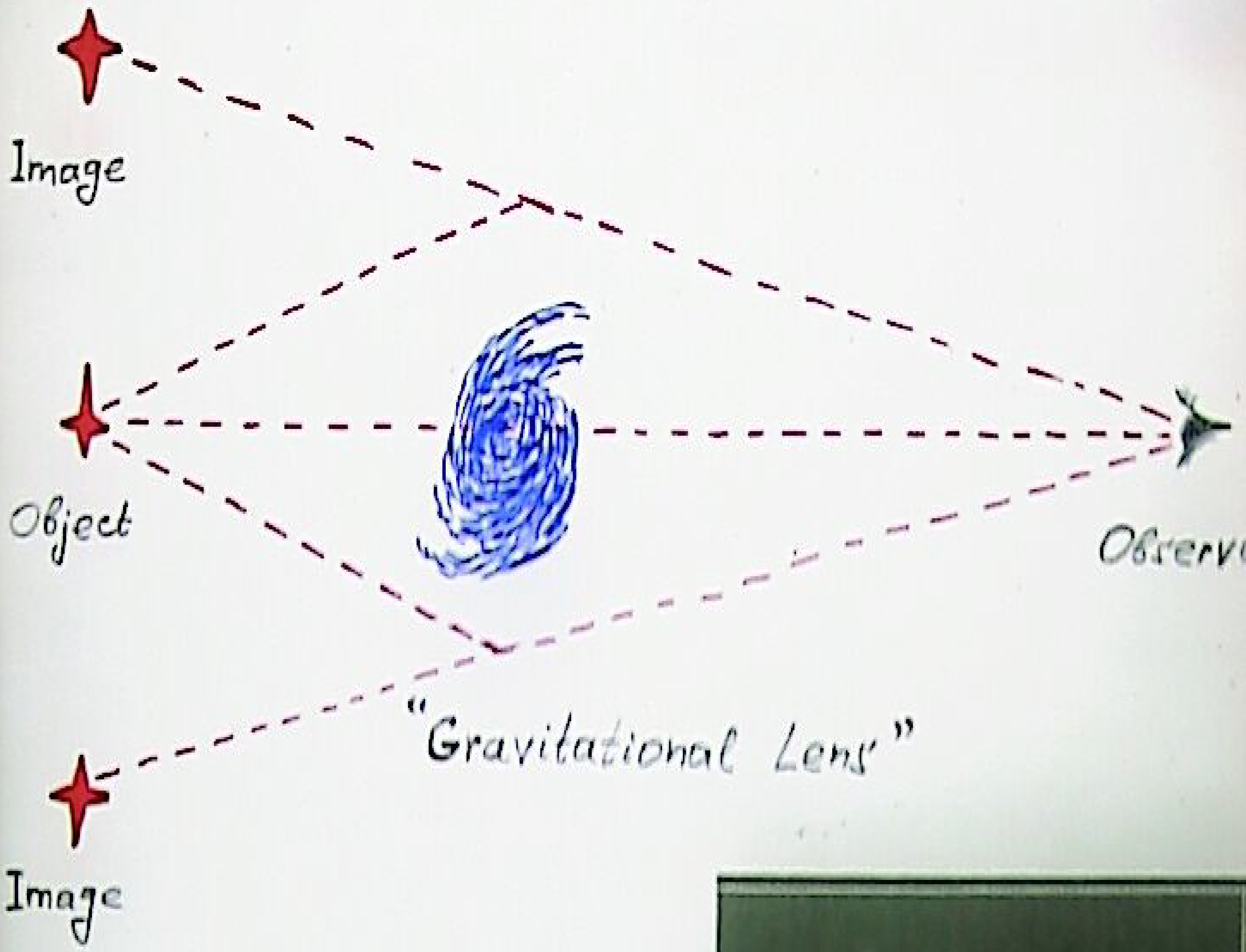
$$\boxed{E = mc^2}$$

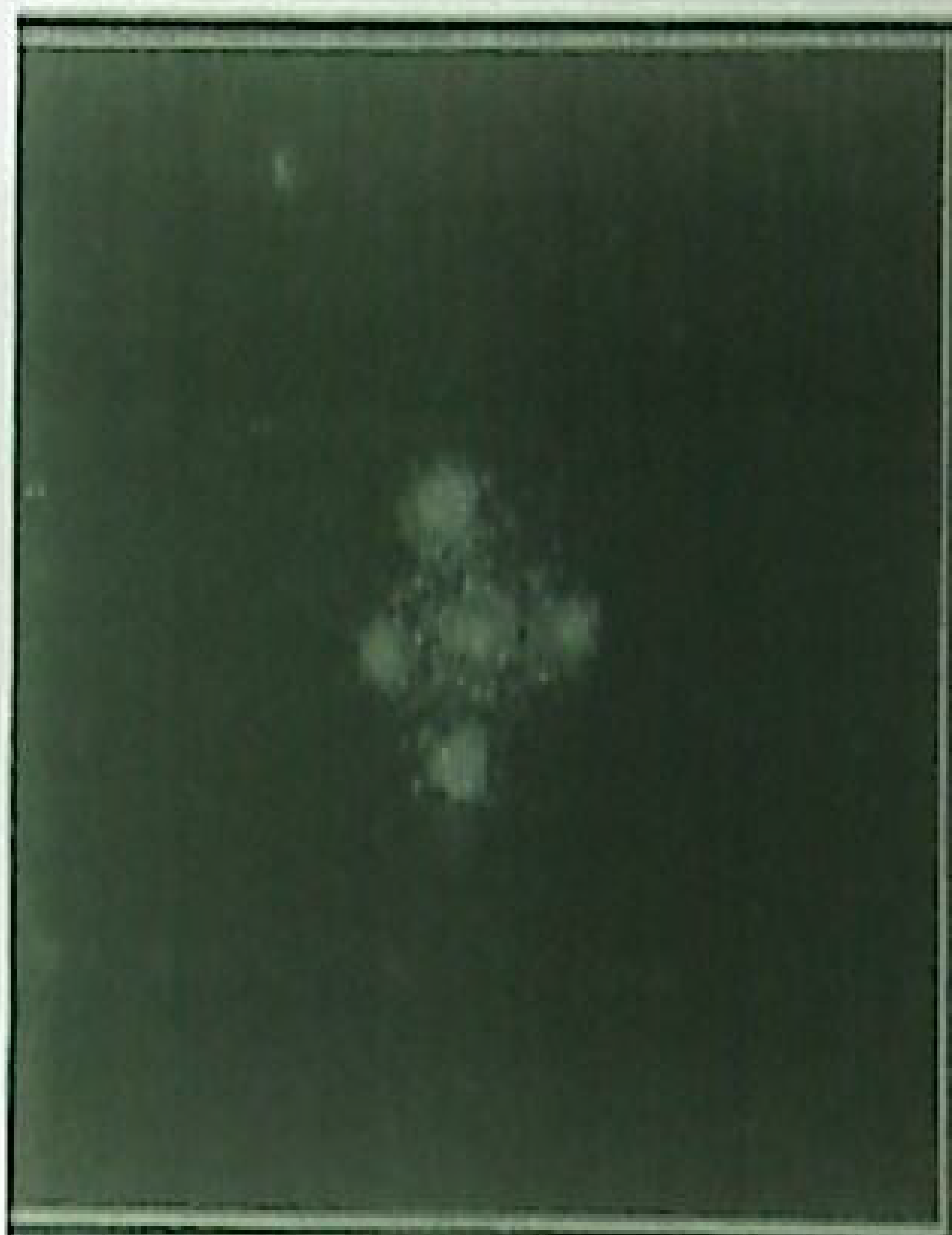
General Relativity (1915)

Einstein

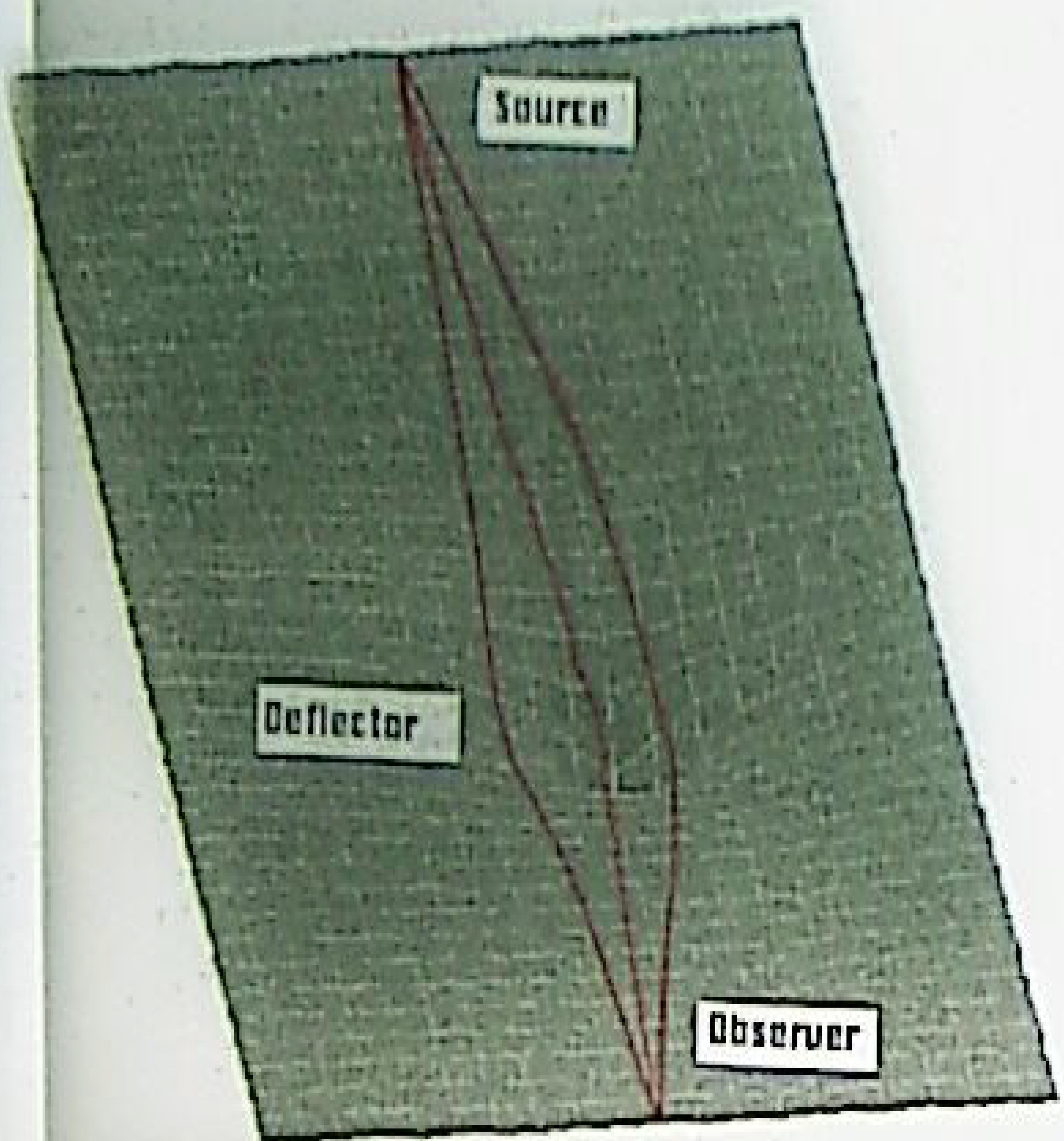


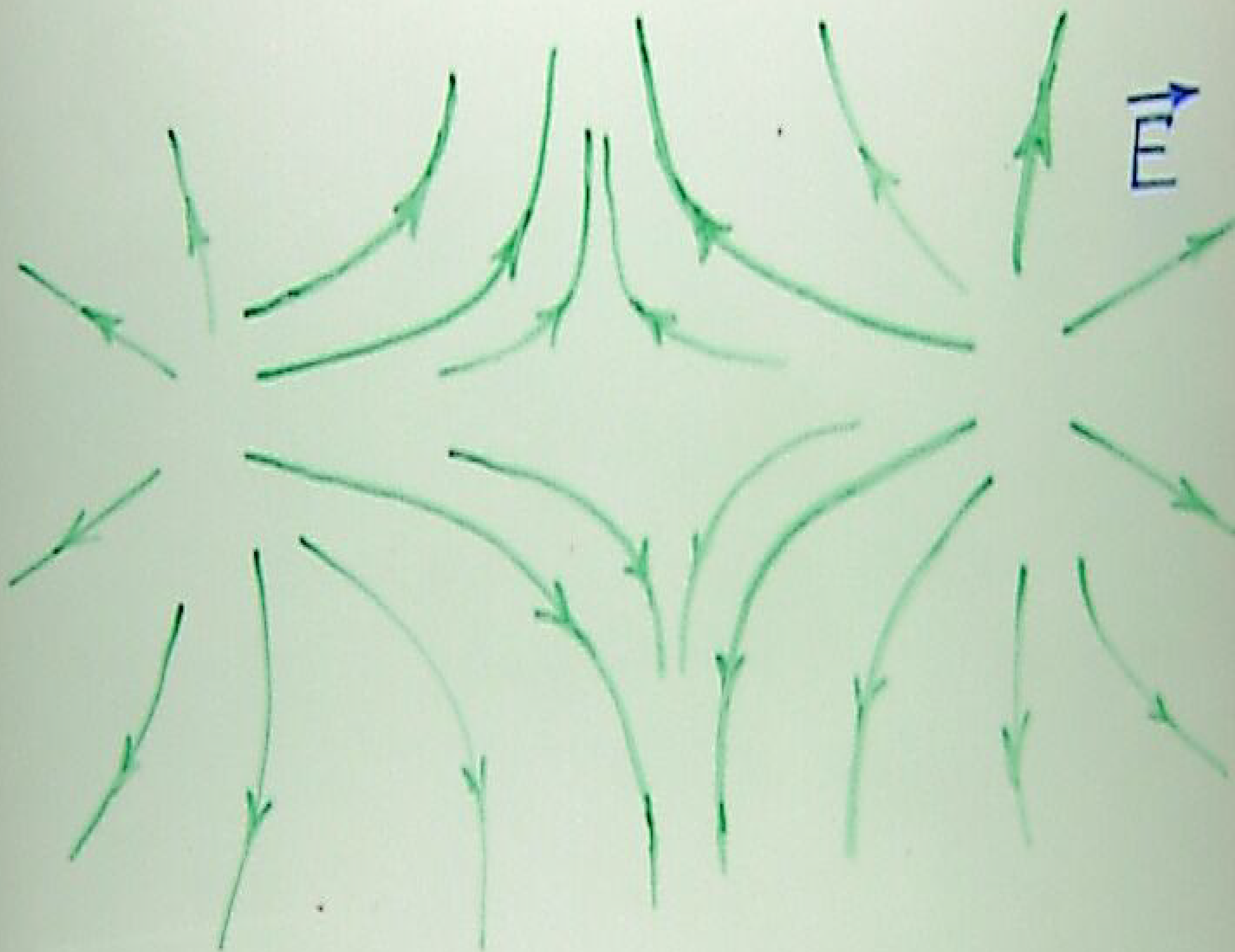
Gravitational Lens





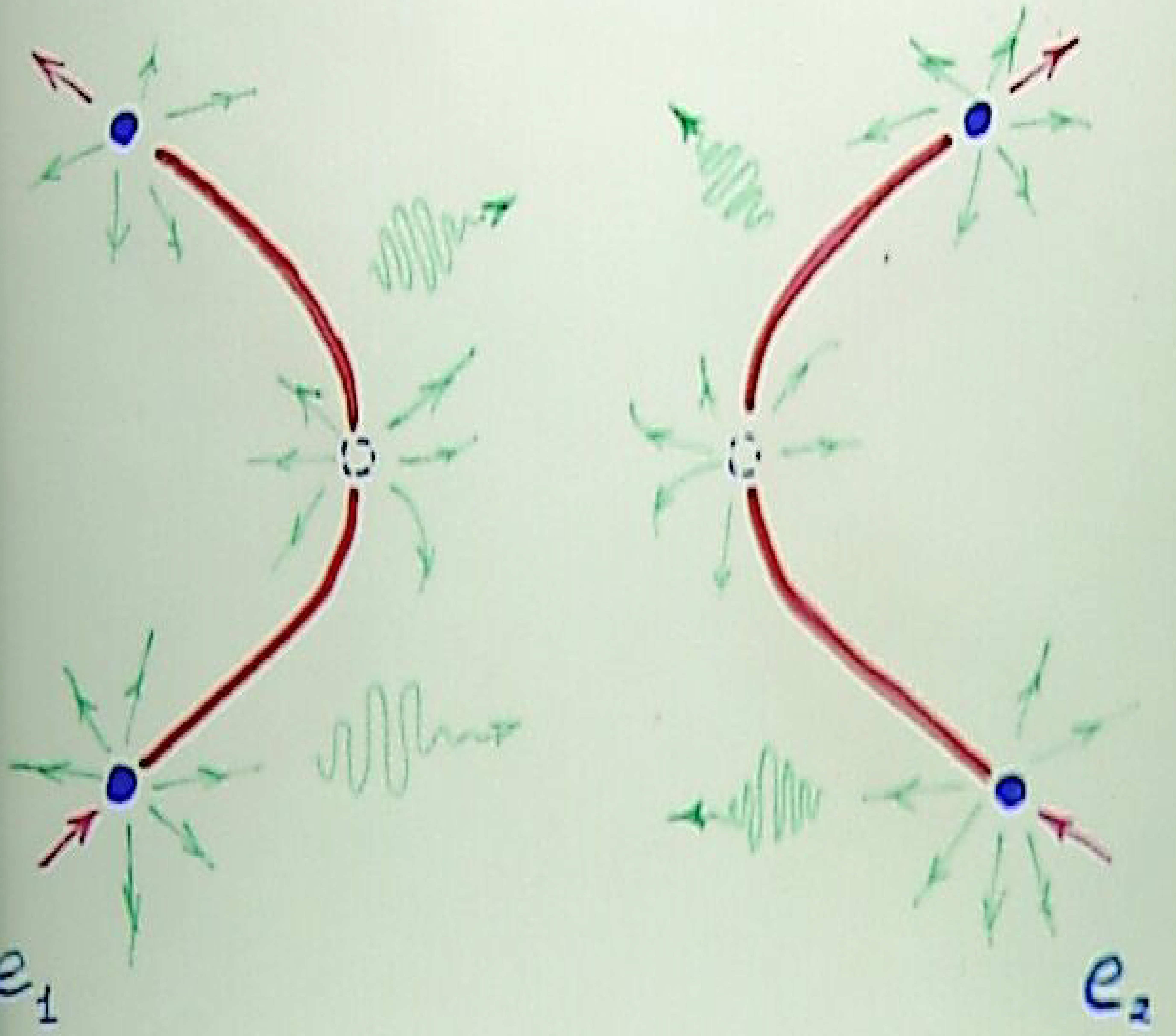
Gravitational Lens G2237+0305





is a result of the interaction between the charges and the electric field they produce.

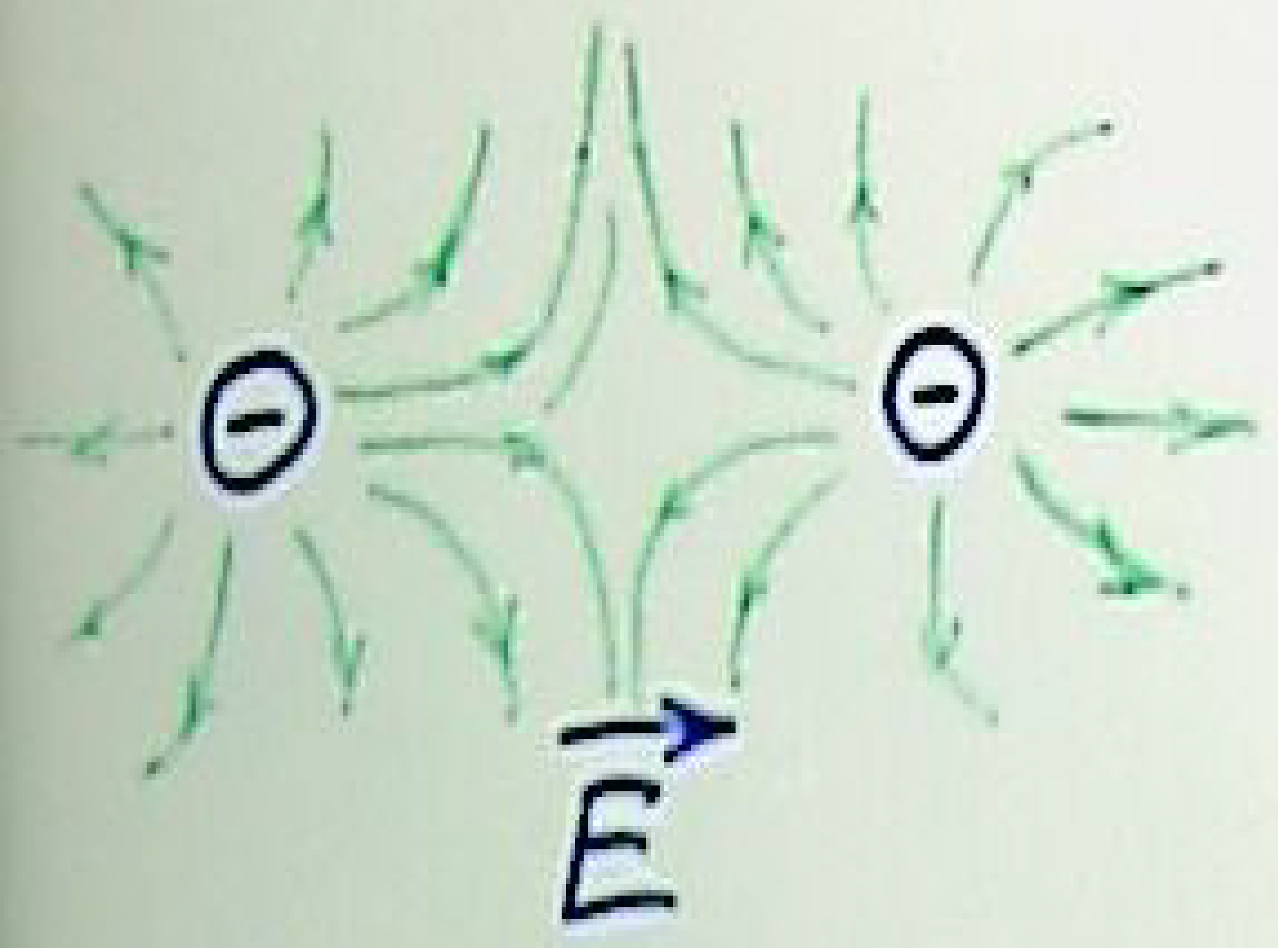
For example, two electrons can scatter without "touching" each other:

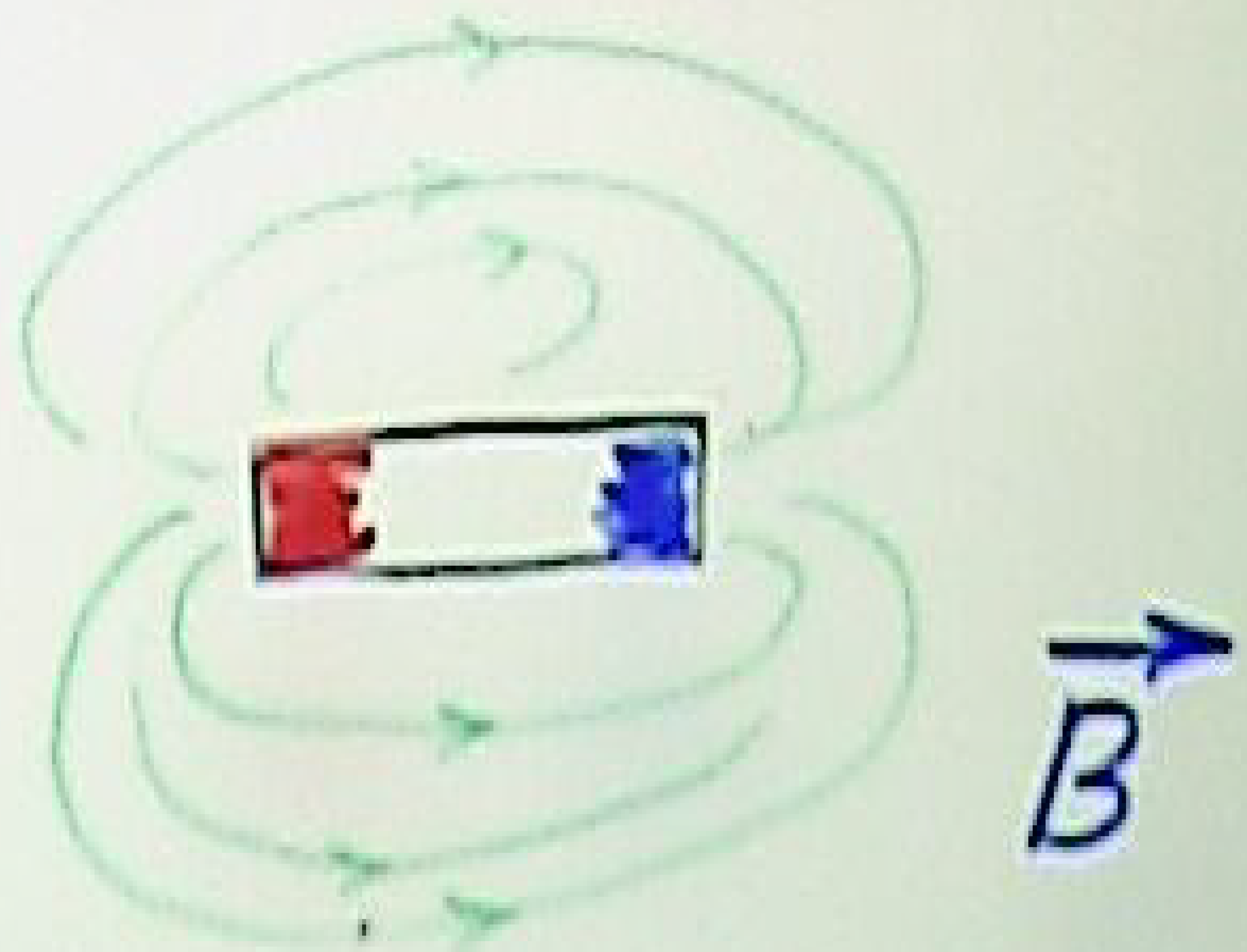


- What is the electric field?

- Does it move "along" with the electrons?

Magnetism





Maxwell's Equations:

$$\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 4\pi \vec{J} \quad , \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad , \quad \vec{\nabla} \cdot \vec{E} = 4\pi \rho$$

- Electricity + Magnetism = Electromagnetis
- In a vacuum, electromagnetic fields travel with the speed of light, $c = 300{,}000{,}000\ m/c$
- Light is a form of electromagnetic waves.

Particles	Fields
electron	gravitational
proton	electromagnetic

In Quantum Theory:

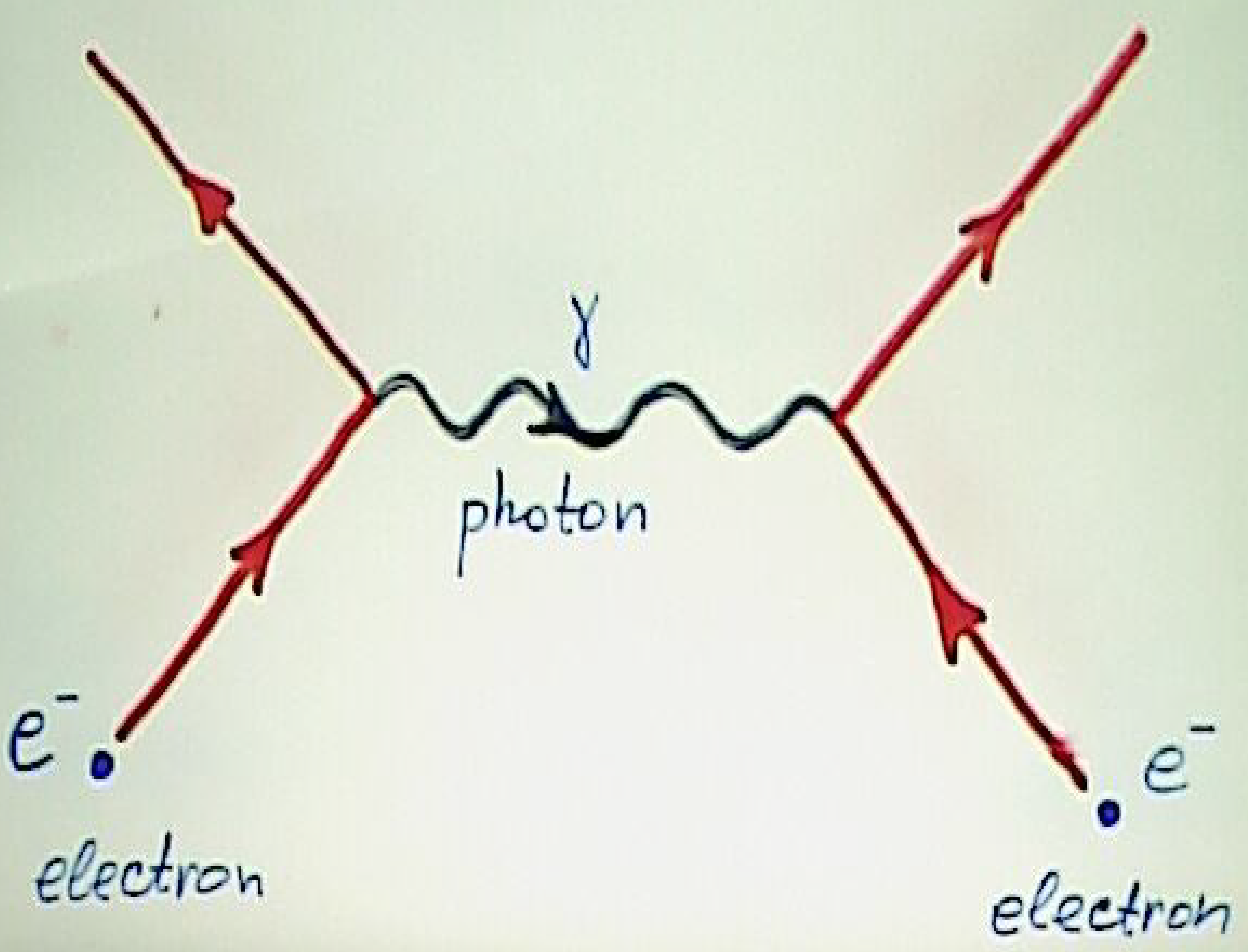
$\Longleftrightarrow$

$\longrightarrow$ spinor fields — P. Dirac

"graviton" $\longleftarrow$

"photon" $\longleftarrow$

Scattering of two electrons can be described by a "Feynman diagram":



Physics and Mathematics in the 20th Century

Quantum — **Abstract**

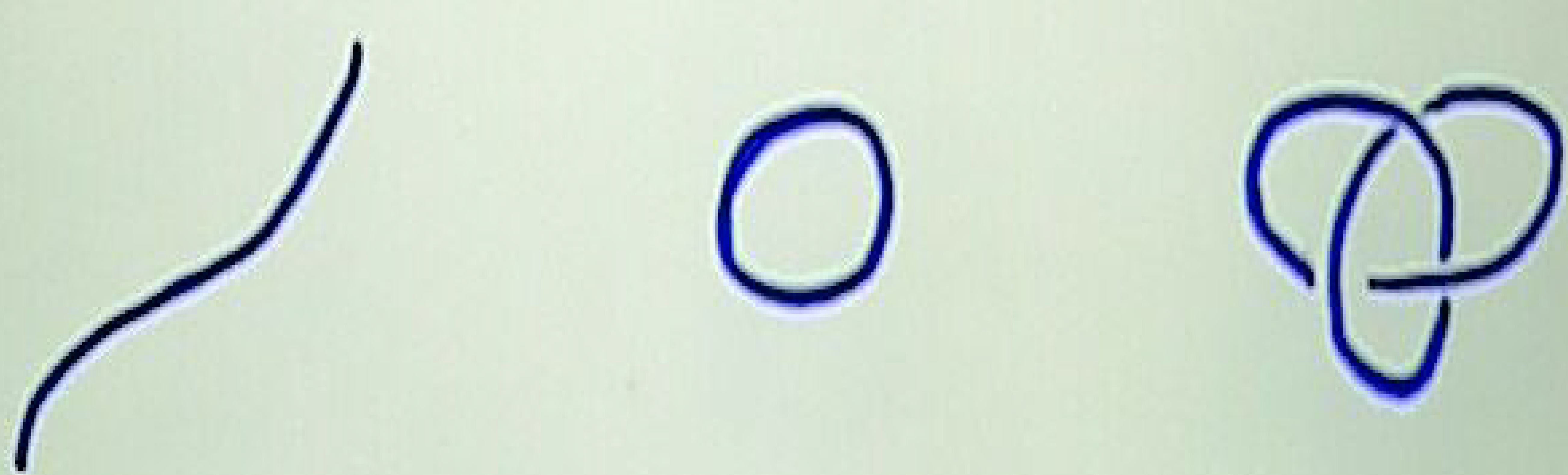
Paul Dirac (1939):

"Quantum mechanics requires the introduction into physical theory a vast new domain of pure mathematics — the whole domain connected with non-commutative multiplication. This, coming on top of the introduction of new geometries by the theory of relativity, indicates a trend which we may expe[ct] to continue. We may expect that in the future further big domains of pure mathematics will hav[e] to be brought in to deal with the advances in fundamental physics."

Examples of Spaces with Various Numbers of Dimensions

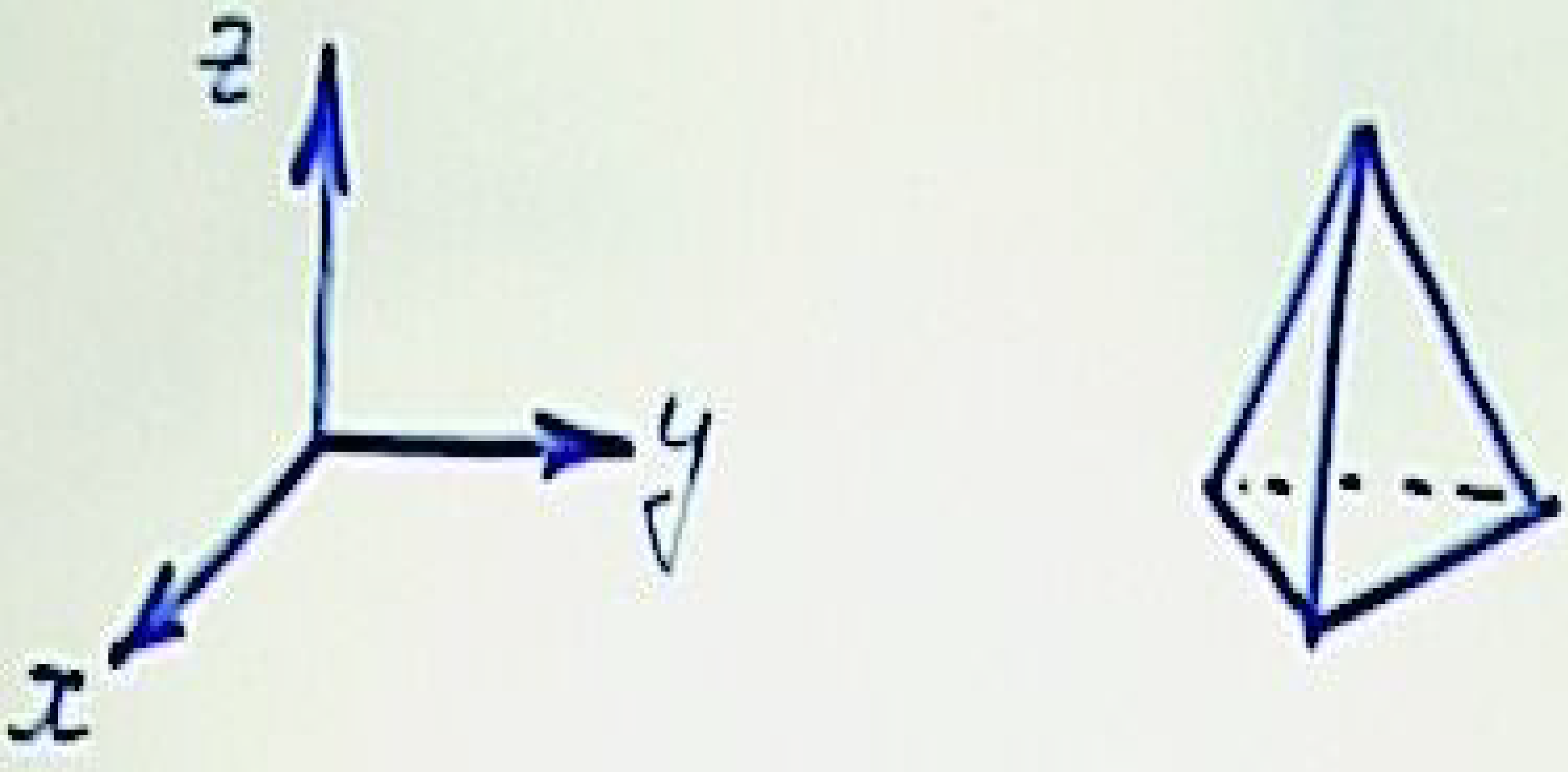
D=0: $\bullet$ a point $\qquad \underbrace{\bullet\,\bullet\,\bullet\,\cdots\,\bullet}_{N \text{ points}}$

D=1:



D=2:

D=3:



D=4:

t time

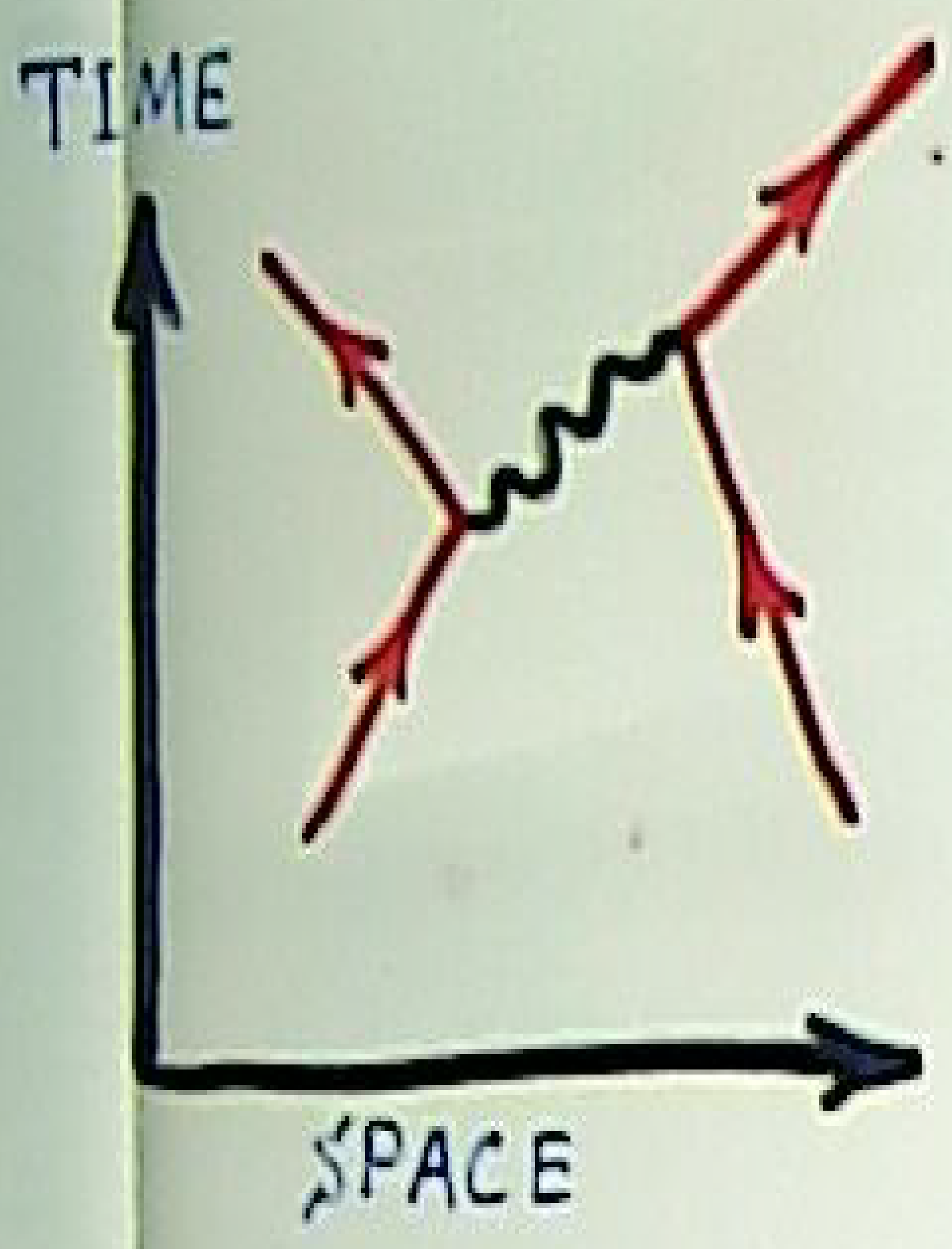
space-time: $\left.\begin{matrix} x \\ y \end{matrix}\right\}$ space

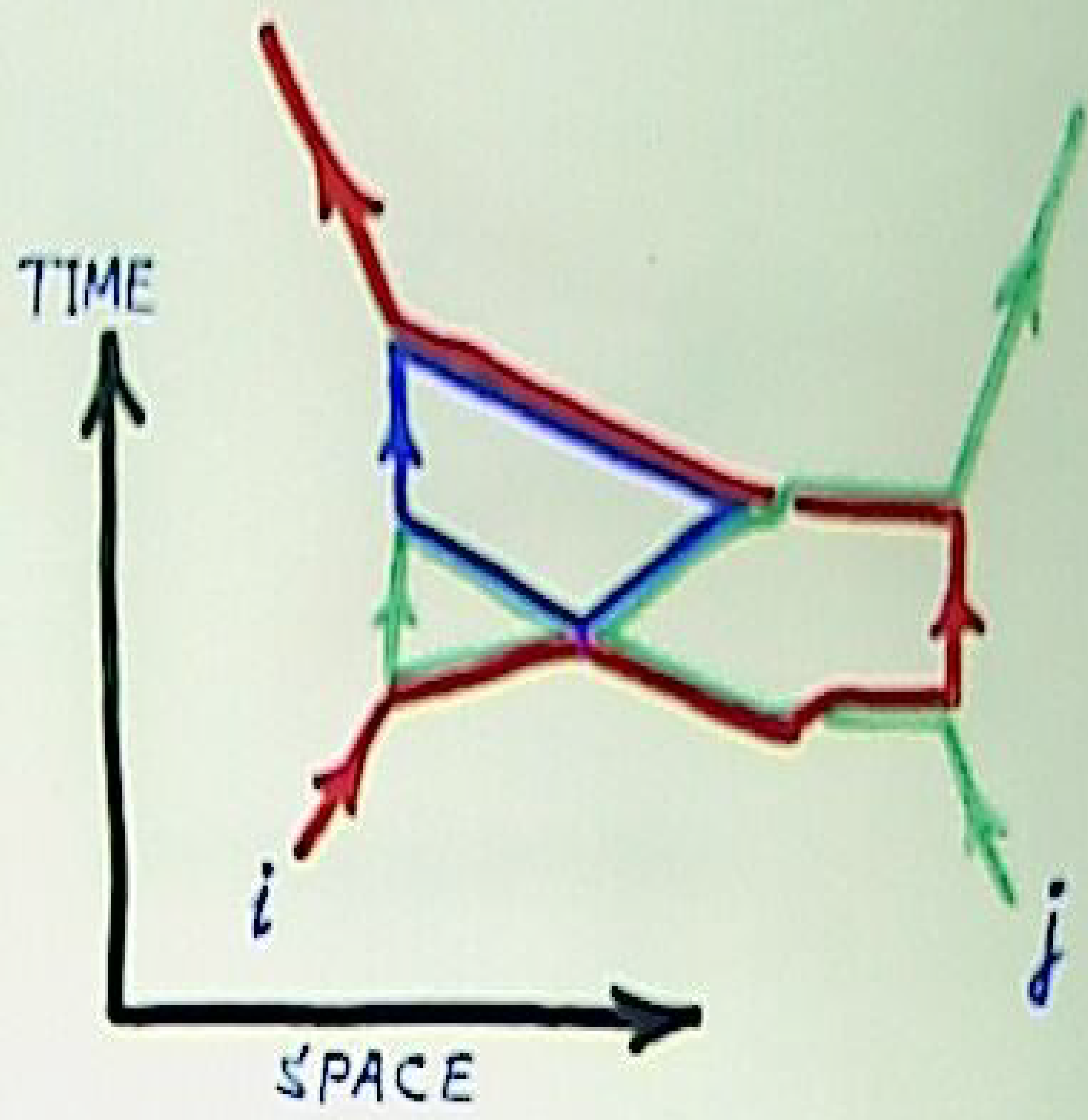
Yang-Mills Theory

1950's

- A matrix version of Maxwell's equations (calle
Yang-Mills equations) forms a basis for present day theory of all particles and their interactions (except gravity!).

Maxwell	Yang-Mills
linear	non-linear



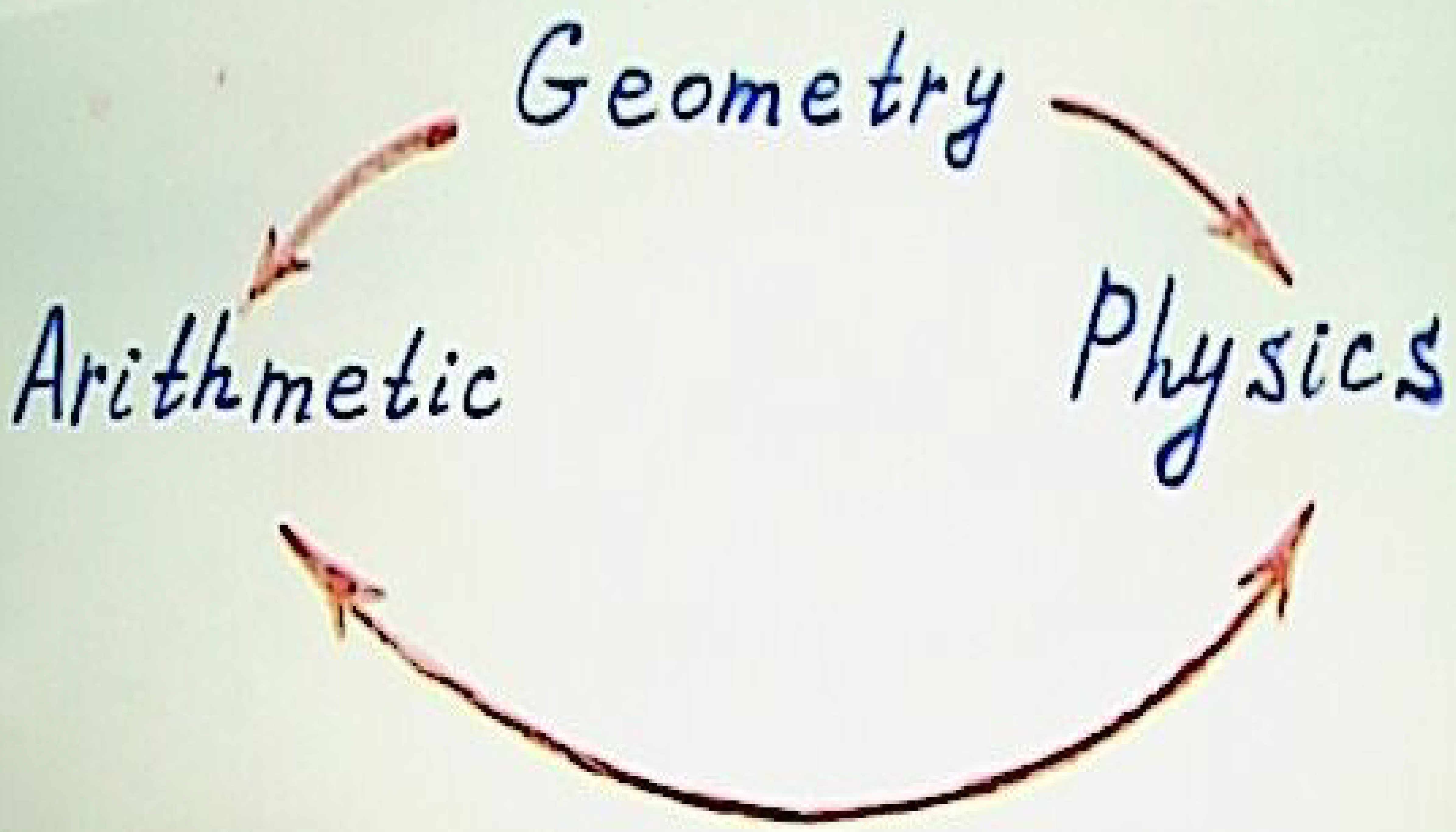


\$ 1,000,000 Prize Problem: Mathematical foundations of Yang-Mills theory? Mass gap?

Michael Atiyah (Bonn, 1984)

Commentary on the Article of V. Manin

"In recent years there has been a remarkable resurgence of the traditional links between mathematics and physics. A number of striking ideas and problems from theoretical physics have penetrated into various branches of mathematics including areas such as algebraic geometry and number theory which are rarely disturbed by such outside influences.... The picture is best described by the following schematic diagram: "



(...-1920) Classical Physics (Newton, Maxwell, Einstein, ...

⋮

↓

(1980-1990) Topological Quantum Field Theory

(Witten, ...

- D=2 Sigma-Model / Enumerative Geometry
- D=3 Chern-Simons Theory / Knot Theory
- D=4 Yang-Mills Theory / Donaldson Theory

(1990-...) String Theory, M-theory

- Special Holonomy Manifolds

  (Calabi-Yau, G_2, ...)

- K-theory and D-branes

⋮

The history of physics and mathematics is greatly interconnected.
For example, Newton's theory of mechanics and the developments
of techniques of calculus are a classical example of this phenomenon.
Another example is the development of Maxwell's theory
of electromagnetism and Einstein's theory of general relativity
ispired by the ideas in differential geometry.

In 1867, Lord Kelvin, also known as William Thomson, put forward
a comprehensive theory of atoms which, through heuristic reasoning,
seemed to explain several of the essential qualities of the chemical elements.
Kelvin's theory conjectured that atoms were knotted tubes of ''ether''.
% The topological stability and the variety of knots were thought to mirror
% the stability of matter and the variety of chemical elements.

==

Throughout the history of physics and mathematics, there are many occasions
when progress in one subject is motivated by the developments in the other.
Starting with a brief review of the history of this phenomenon,
I will focus on some recent developments, which include the study
of special holonomy manifolds (such as Calabi-Yau and G2 manifolds)
and their geometric structures, motivated by the ideas in string theory.
As an example, I will discuss coassociative fibrations and mirror symmetry
for G2 manifolds, studied in a joint work with S.T.Yau and E.Zaslow.

==

Built on work of Weil and Zariski and effected fundamental advances
in algebraic geometry. He introduced the idea
of K-theory (the Grothendieck groups and rings).
Revolutionized homological algebra in his celebrated "Tohoku paper".

The history of physics and mathematics is greatly interconnected.
Sometimes new mathematics gets developed in connection with
understanding physical questions (for example, the development
of calculus was not independent of the questions raised by
classical mechanics). Sometimes new physics gets developed from
known mathematics (for example, general theory of relativity
found its natural setting in the context of Riemannian geometry).

Newton's theory of mechanics and the developments of techniques
of calculus are a classical example of this phenomenon.
Another example is the developments in differential geometry
ispired by Maxwell theory of electromagnetism and Einstein
theory of general relativity.

A recent glorious example is the developments of topological
quantum field theories and their relevance to the study of
geometry and topology of low dimensional manifolds.

Until the late 1800's no one was really interested in the idea of knots.
Lord Kelvin, also known as William Thomson, was an English scientist
who taught at the University of Glasgow.
One of his greatest achievements was the idea of "ether."
His belief was that the universe was filled with an invisible
and frictionless fluid called "ether," and the atoms were
the vortices in this fluid in the shape of knots.
Thus a table of knots would be a table of elements.
This of course was later on disproved when atomic ideas began to emerge.
The Knot Theory soon faded away until the 1950's.

In the nineteenth century physicists were speculating about the underlying
principles of atoms. In 1867, Lord Kelvin put forward a comprehensive theory of

atoms which, through heuristic reasoning, seemed to explain several of the essential qualities of the chemical elements. Kelvin's theory conjectured that atoms were knotted tubes of ether. (To a topologist a knot in 3-space is any closed loop having no self-intersections and a link is any collection of non-intersecting closed loops.) The topological stability and the variety of knots were thought to mirror the stability of matter and the variety of chemical elements.

With this poliferation of new polynomials it was natural to ask whether any of these invariants had a natural extensions to all 3-manifolds. Two facts worked in favor of having such extensions: 1) all 3-manifolds can be describe in terms of knots and links via an operation called Dehn surgery; 2) there exists a set of moves, the Kirby calculus, that allow one to move between differing Dehn surgery descriptions of the same homeomorphic 3-manifold. Using the Kirby calculus as a means to generalizing the polynomial invariants, Edward Witten, a theoretical physicist, proposed new invariants for 3-manifolds. His invariants came out of the theoretical area of physics know as quantum field theory. These new invariants can be realized as certain averages of link polynomials obtained from a given Dehn surgery representation of the manifold.

Starting with the flawed theory of Kelvin's knotted vortex to the work of Thurston, Jones and Witten, knot theory has circled back to its ancestral origins of theoretical physics.

Topological quantum field theories can be used as a powerful tool to probe geometry and topology in low dimensions.

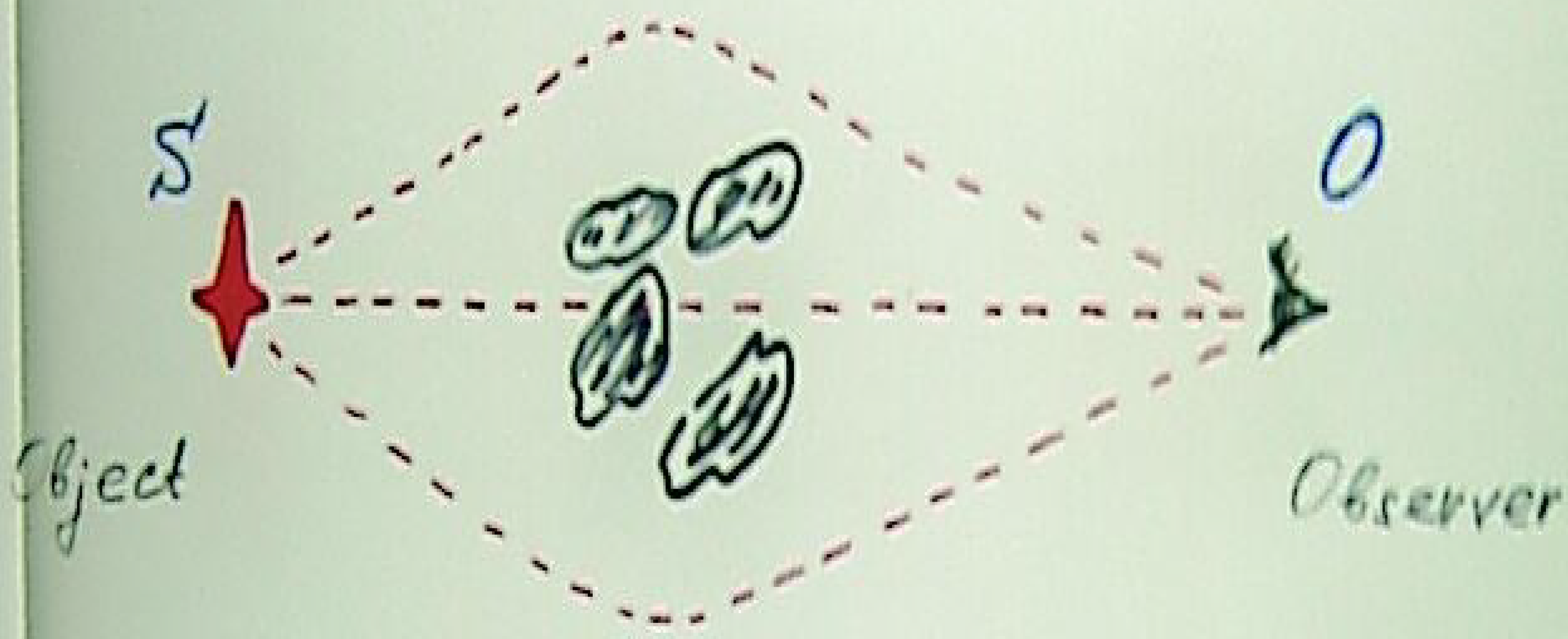
Chern-Simons theories, which are examples of such theories, provide a field theoretic framework for the study of knots and links in three dimensions.

The interplay between quantum field theory and knot theory has paid rich dividends in both directions. Many of the open problems in knot theory have found answers in this process.

Topological Ideas in Physics: Gravitational Lensing

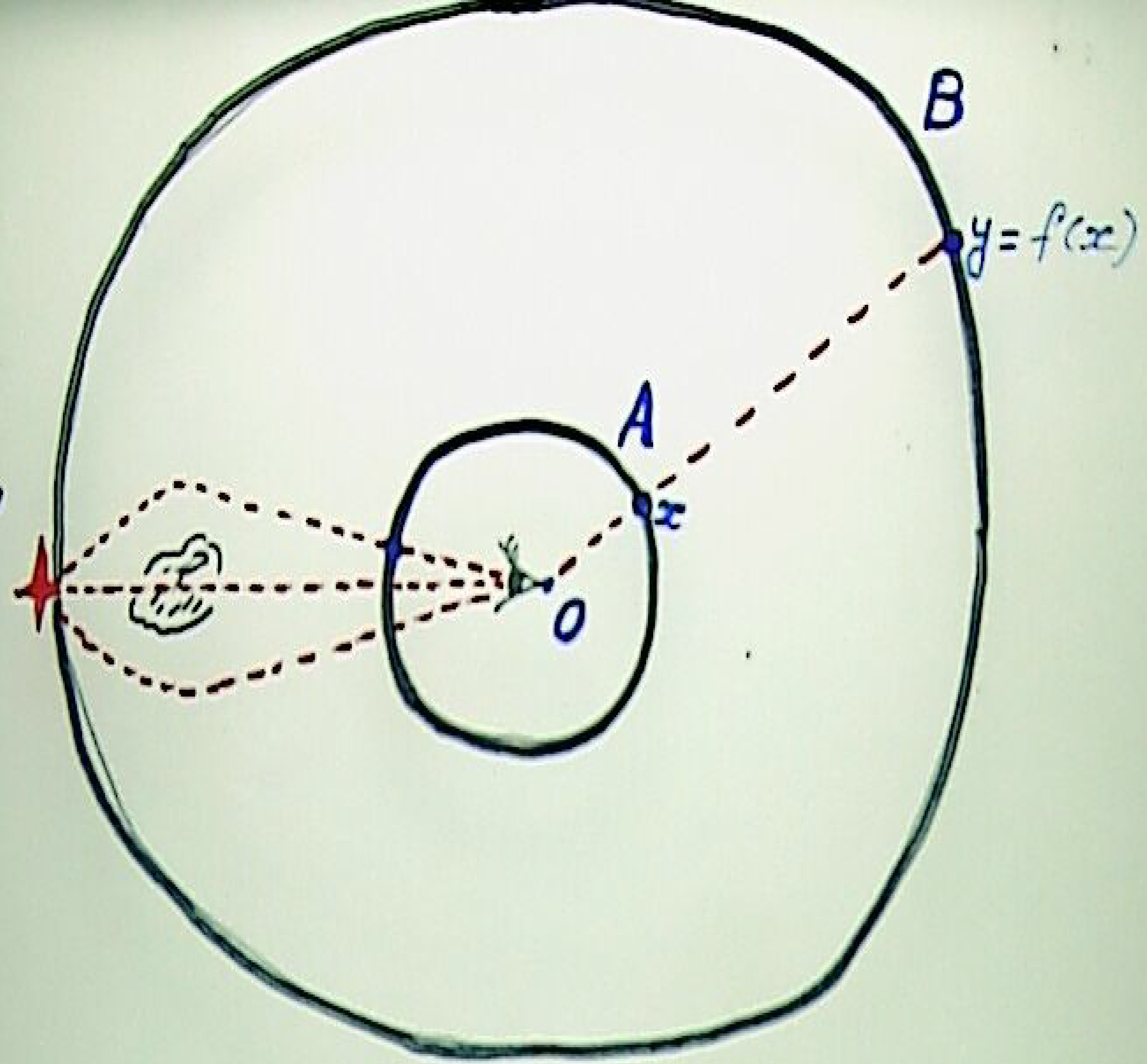
- A gravitational lens must produce an odd number of images.

R. McKenzie '85



- Moreover, $\frac{1}{2}(n-1)$ images have reversed orientation.

➡ To prove this, we first notice that the number of images is equal to the number of light paths between the source S and the observer

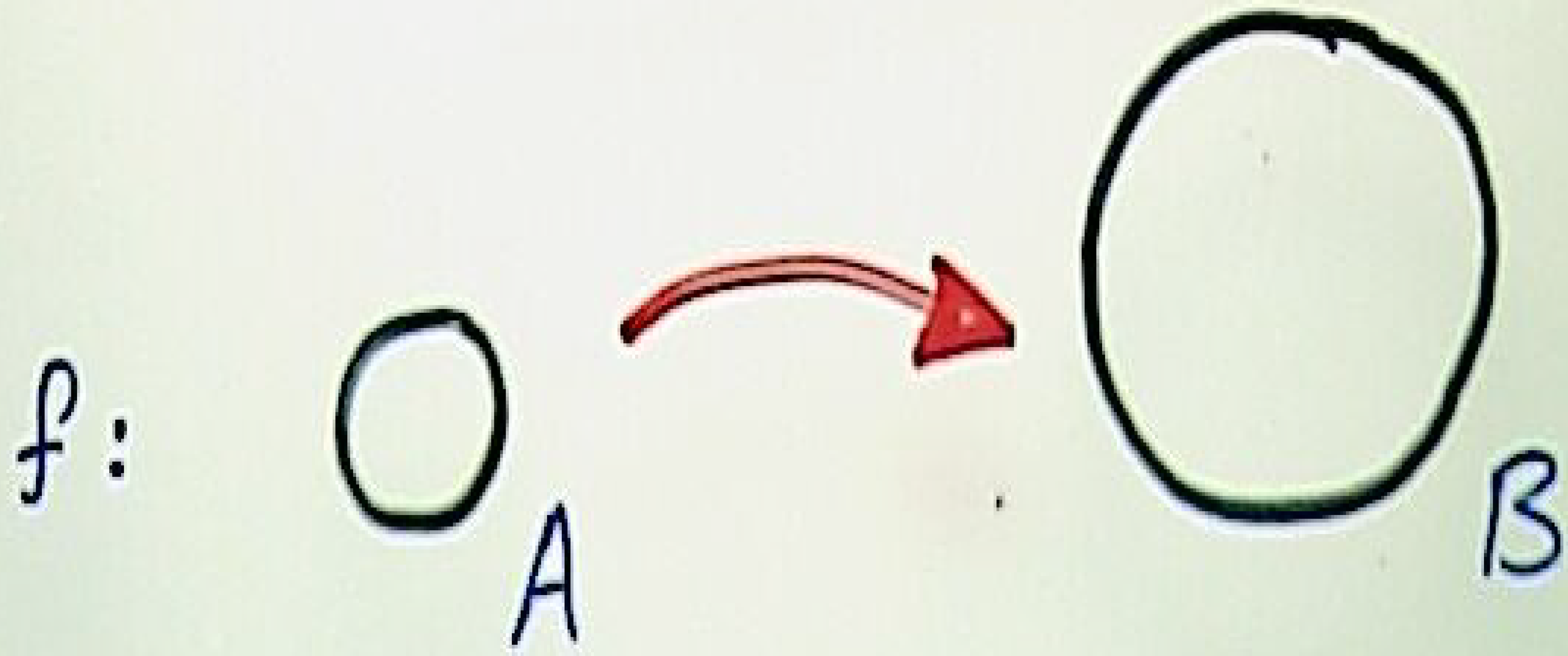


Introduce the map f that maps a point x on the small sphere A to the point y on the sphere B where the light ray through O and x intersects B

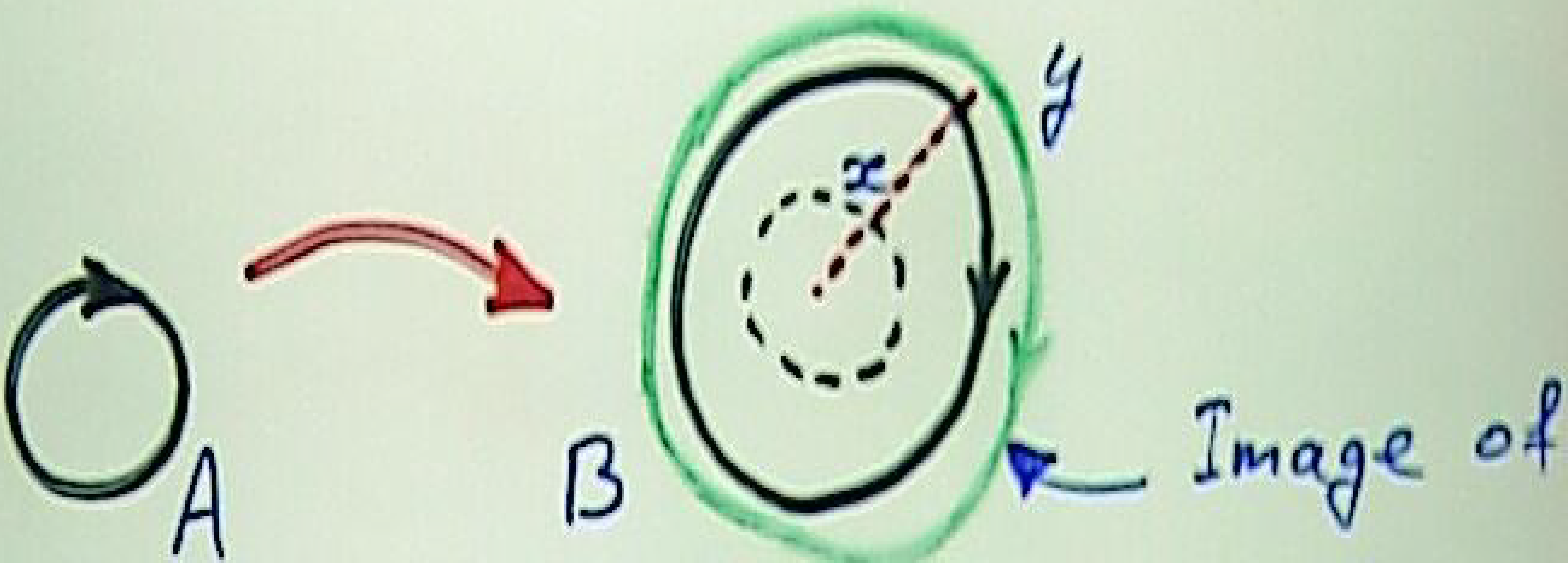
$$f : A \rightarrow B$$

⇒ The number of images of S seen by O is the number of points on A mapped onto S.

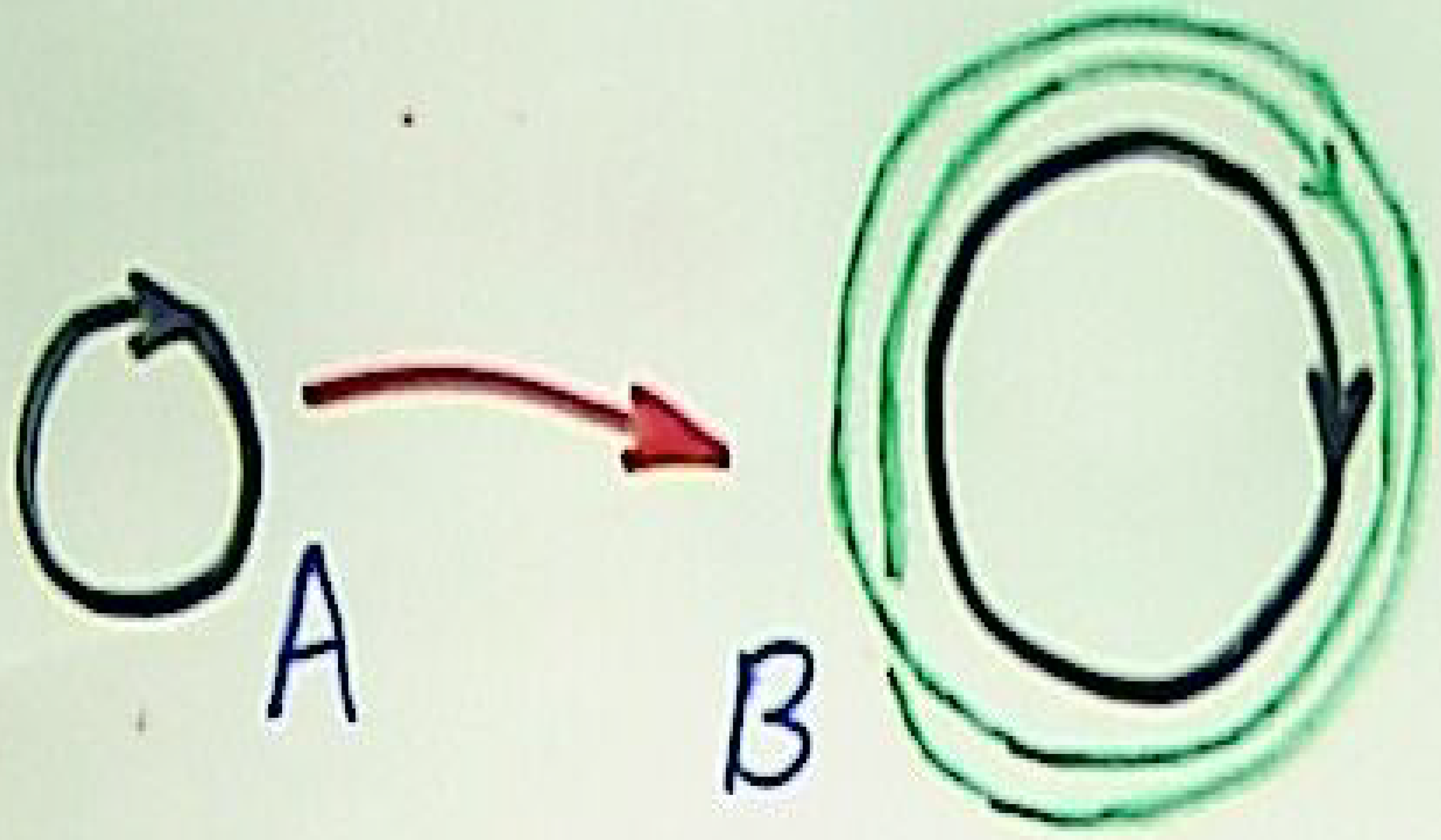
Examples of smooth maps from one circle to another circle



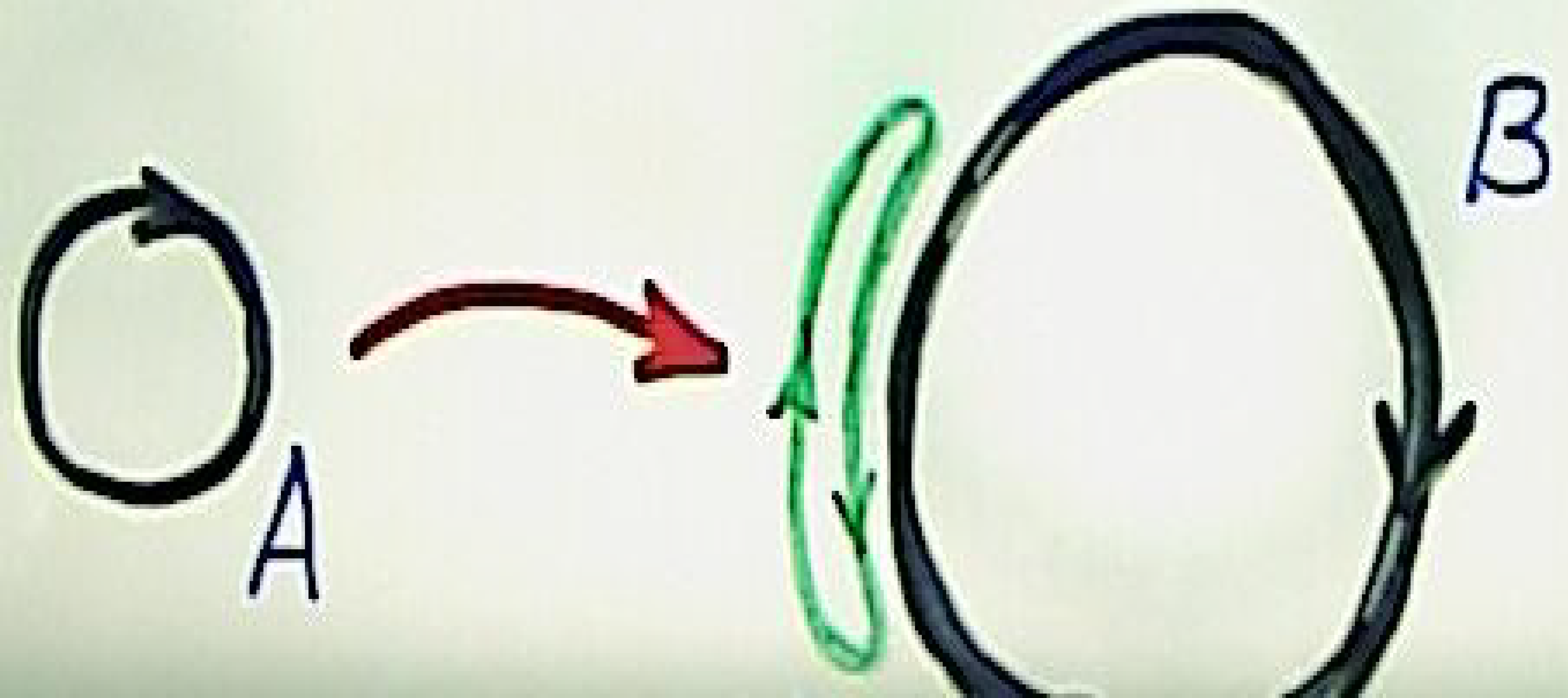
1)



2)



3)



The maps from A to B are characterized by the degree,

<u>$\deg(f)$</u>

- Informally, $\deg(f)$ is the number of times the image of A "winds" around B.

- More accurately, $\deg(f)$ is defined as the number of points on A mapped to a (regular) point $y \in B$, counted with signs:

$$\deg_y(f) = \sum_{x \in f^{-1}(y)} \operatorname{sgn} df_x$$

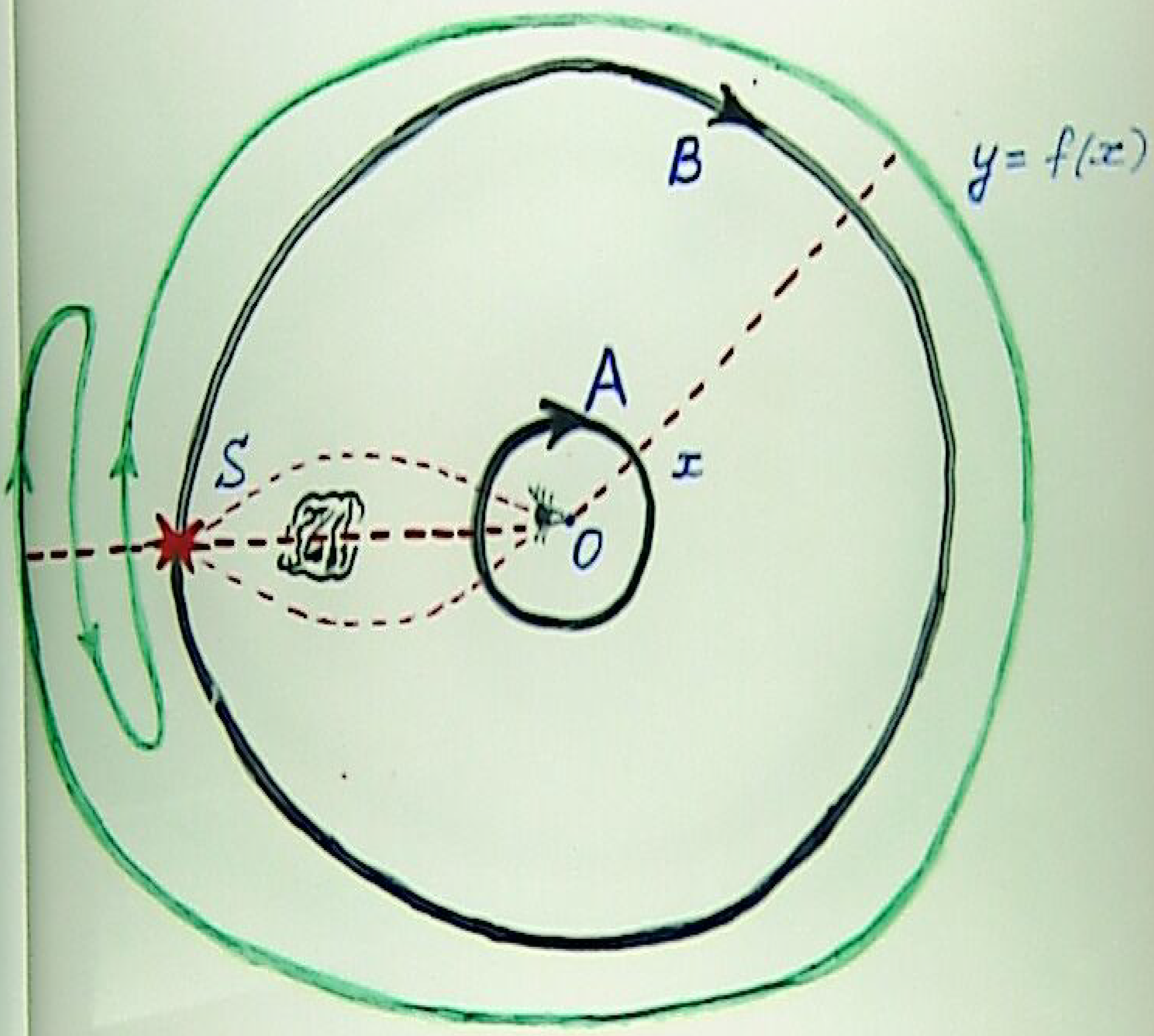
- It turns out that $\deg_y(f)$ is the same for all regular y; it is called the degree of f

$$\deg(f) = \deg_y(f)$$

$\Rightarrow$ The number of images of S seen by O =

= the number of points on A mapped to S =

= $\deg(f)$!

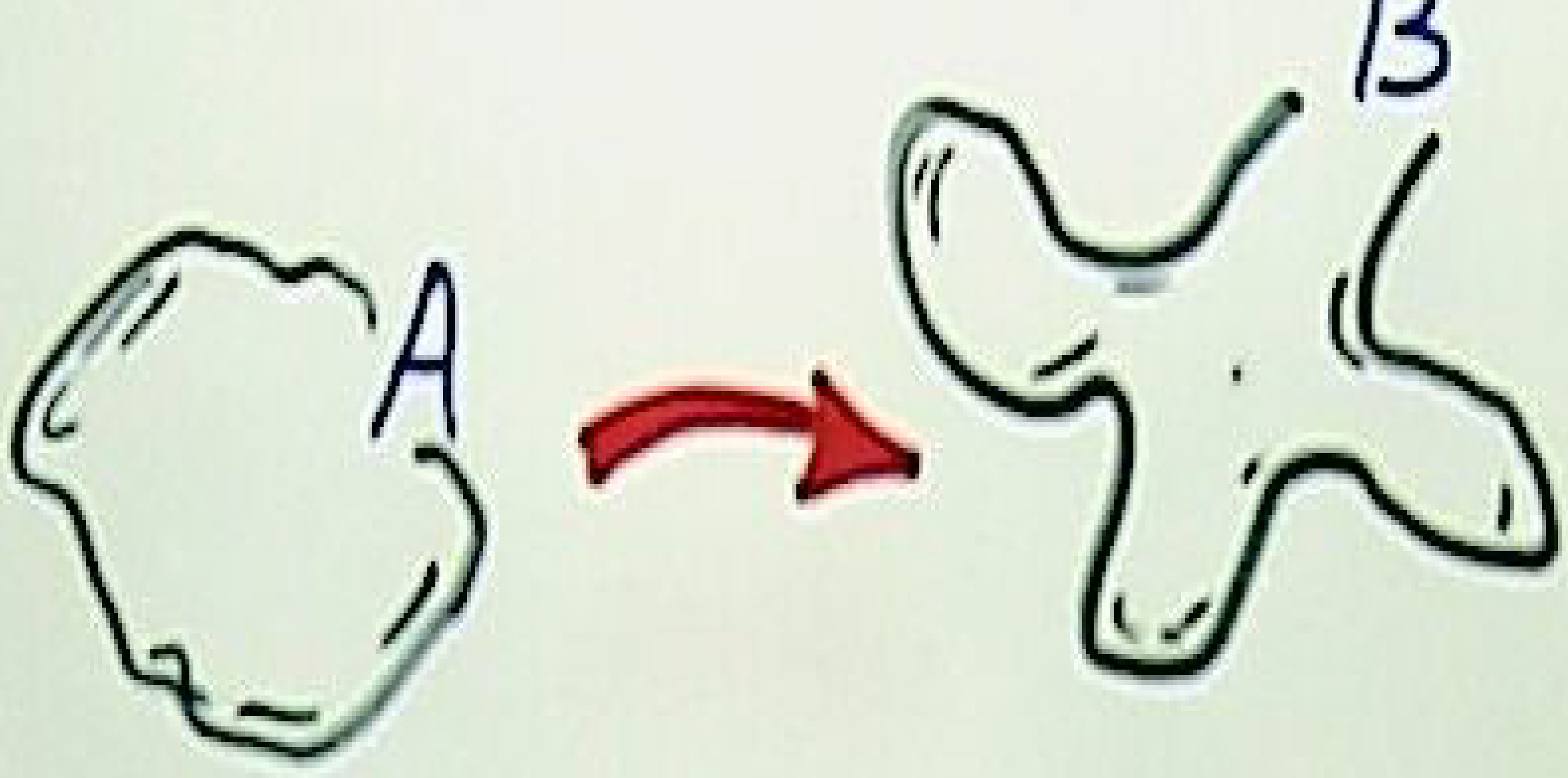


Since

$$\deg(f) = n_+ - n_- = 1$$

a gravitational lens must produce an odd number of images, $\underline{n_+ + n_- = 2n_- + 1}$

- In general, studying maps from A to B one can extract a great deal of subtle information about the geometry of the space B.



- E.g. notice that maps from a point to X reproduce the entire space X

$$\text{Maps}\left\{ \underset{pt}{\bullet} \to X \right\} = X$$

- The next natural step is to study maps from a circle (a loop $\bigcirc$) to X, modulo continuous deformations of the loop:

$$\text{Maps}\left\{ \underset{\text{circle}}{\bigcirc} \to X \right\} \Big/ \text{"equivalen} \atop \text{loops}$$

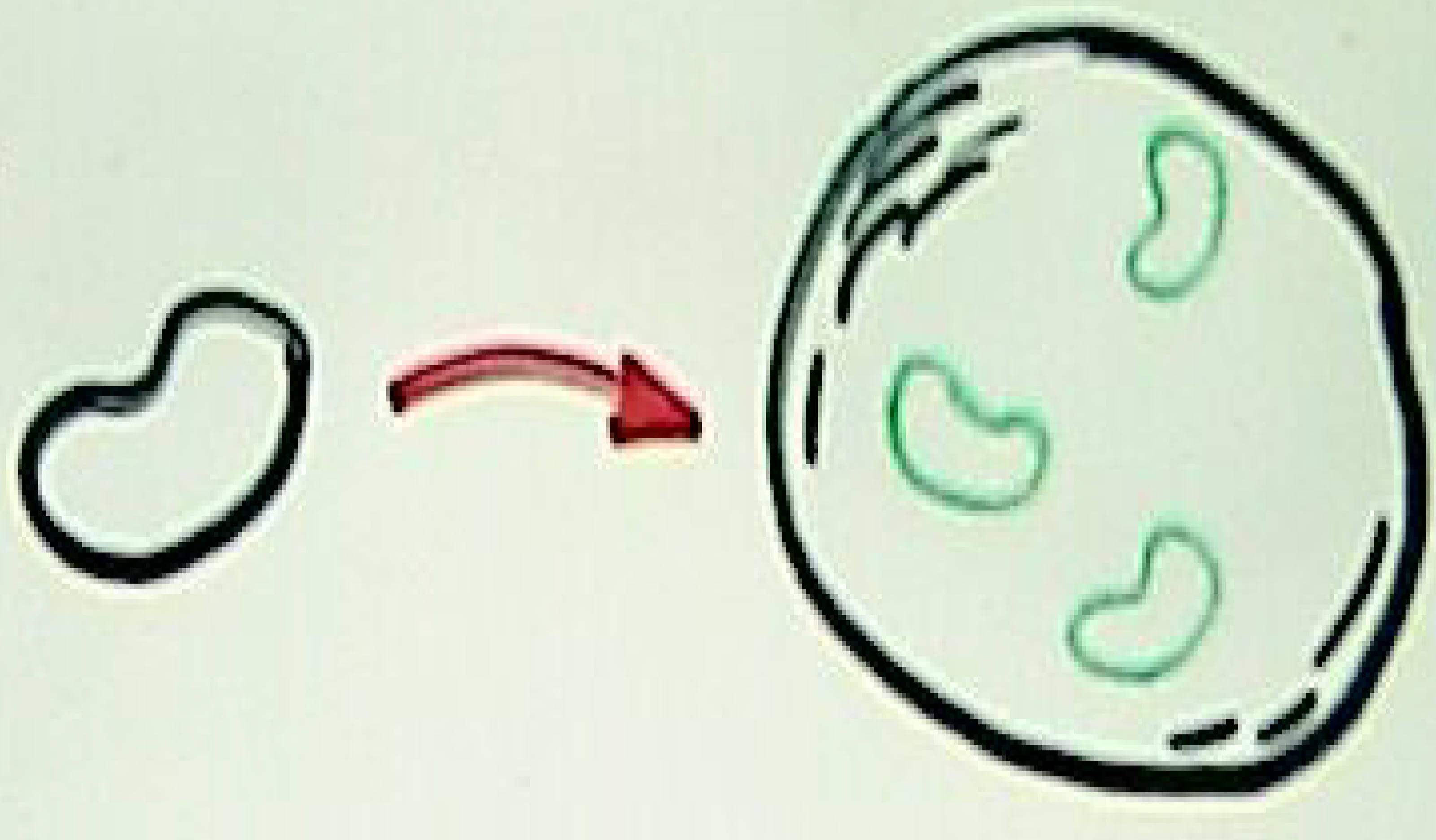
Examples

- If X is also a circle such maps are classified by the degree:

deg = 0   deg = 1   deg = 2

- What if dimension$(X) = 2$?

.... for example, if X is a sphere ....



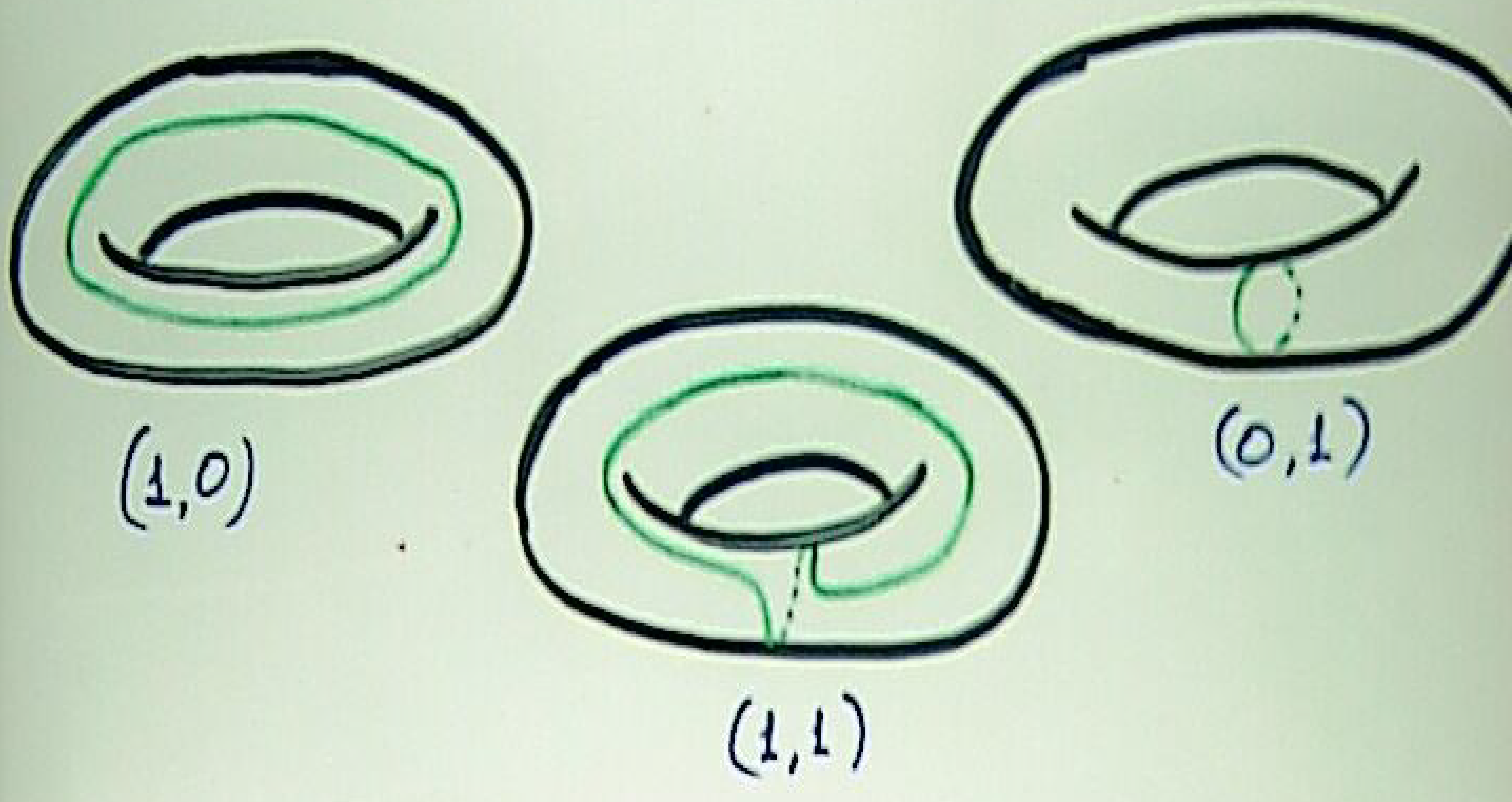
$X = S^2$

all such maps are equivalent (can be continuous deformed into each other) =>

Maps {  loop →  } / equivalent loops = trivia

- It turns out that the sphere is the only closed 2-dimensional surface on which any loop can be shrunk to a point!

- For example, closed loops on a torus (the surface of a doughnut) are classified by two integer numbers, (m,n) = "winding numbers":



What about three-dimensional case?

The Poincaré Conjecture (1904):

If any closed loop in a compact 3-dimensional manifold X can be shrunk (without cutting) to a point, is the manifold X topologically the same as the 3-sphere?

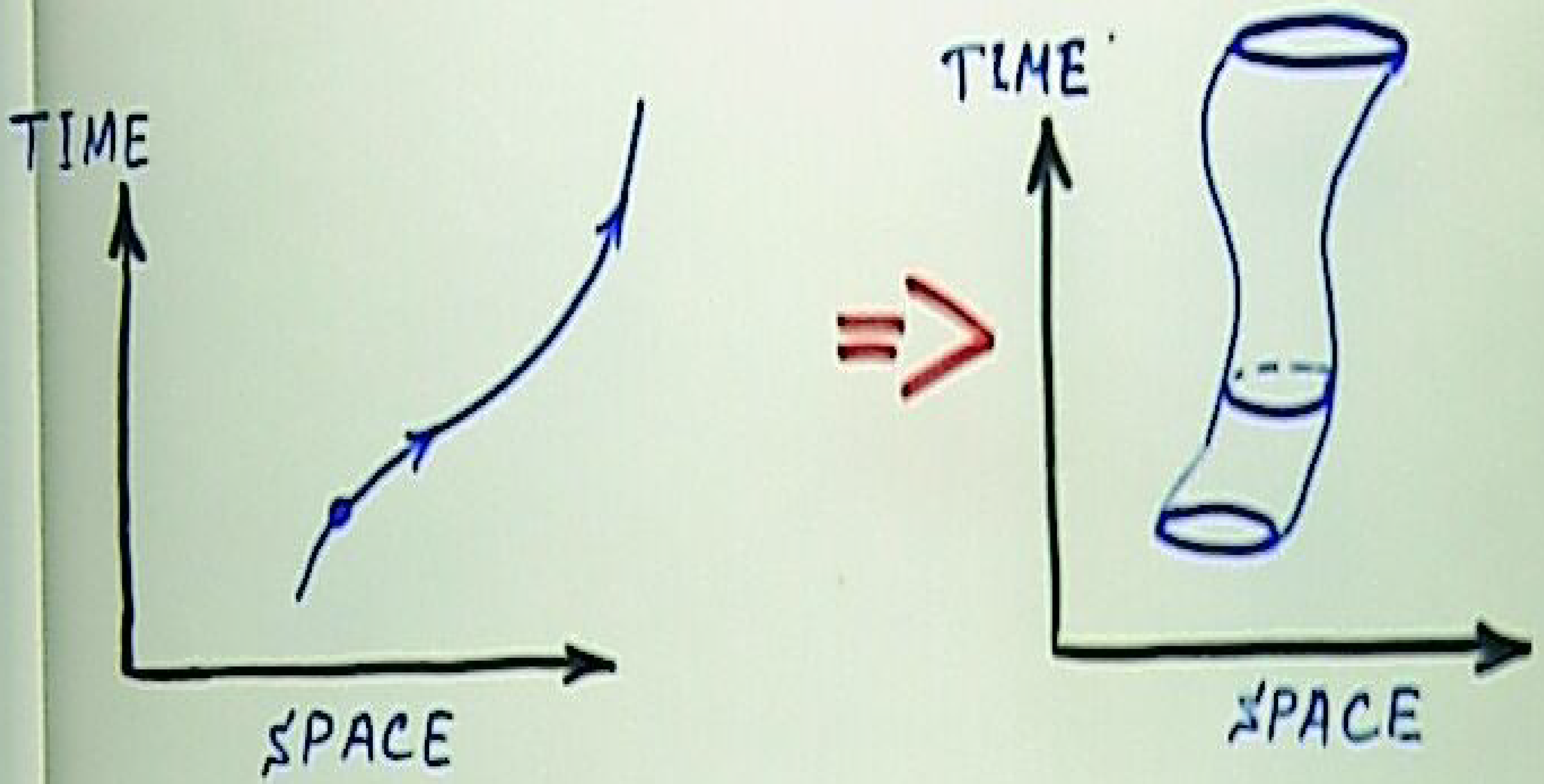
Conjecture: YES

$1,000,000 Prize Problem

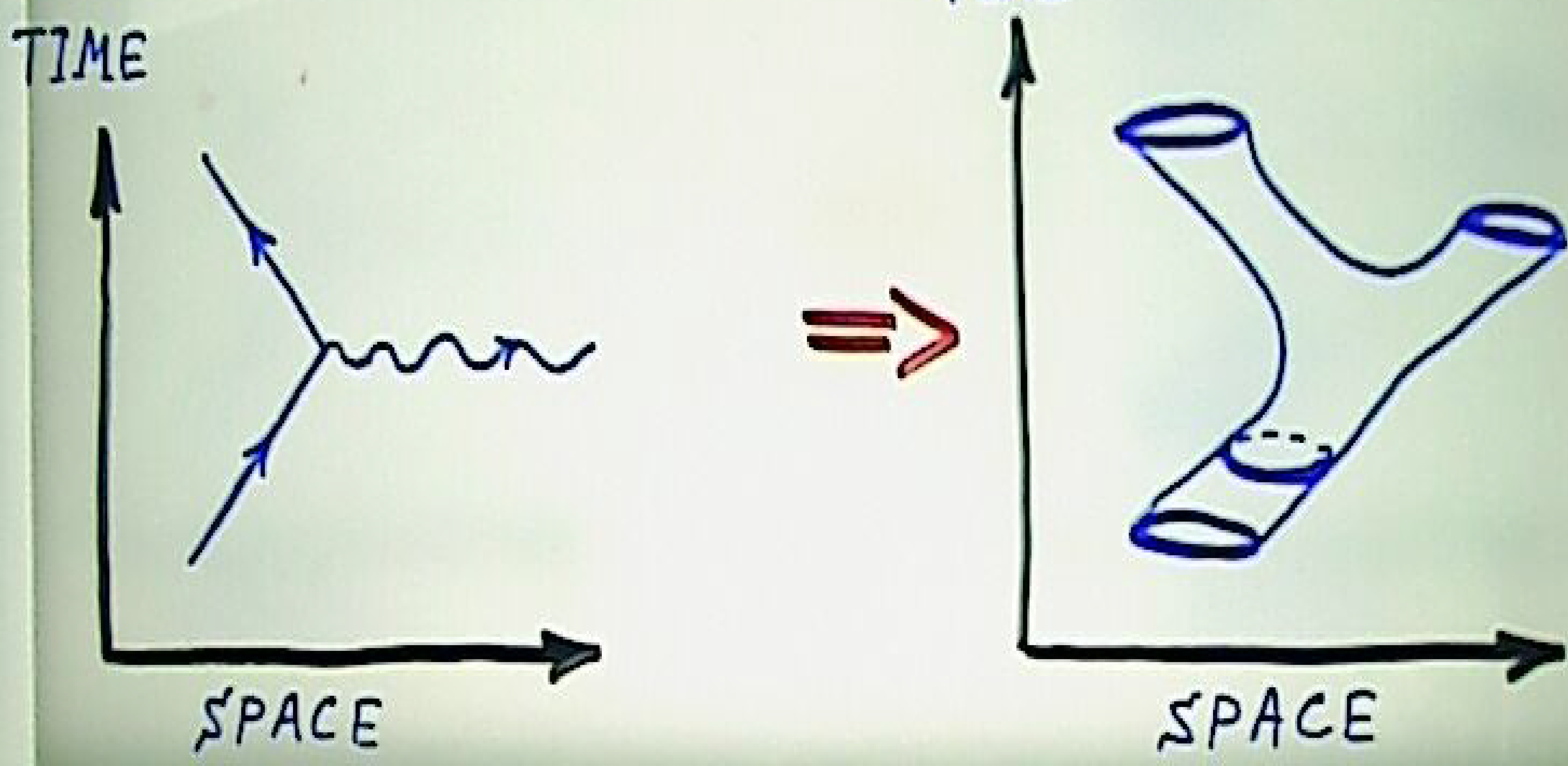

e+
e-
K-
distance, m
10^-35
10^-15
10^-10
10^2
10^8
10^20
10^26
Quantum Mechanics
General Relativity
STRING THEORY
u
u
d

Particle | String

Free Motion:



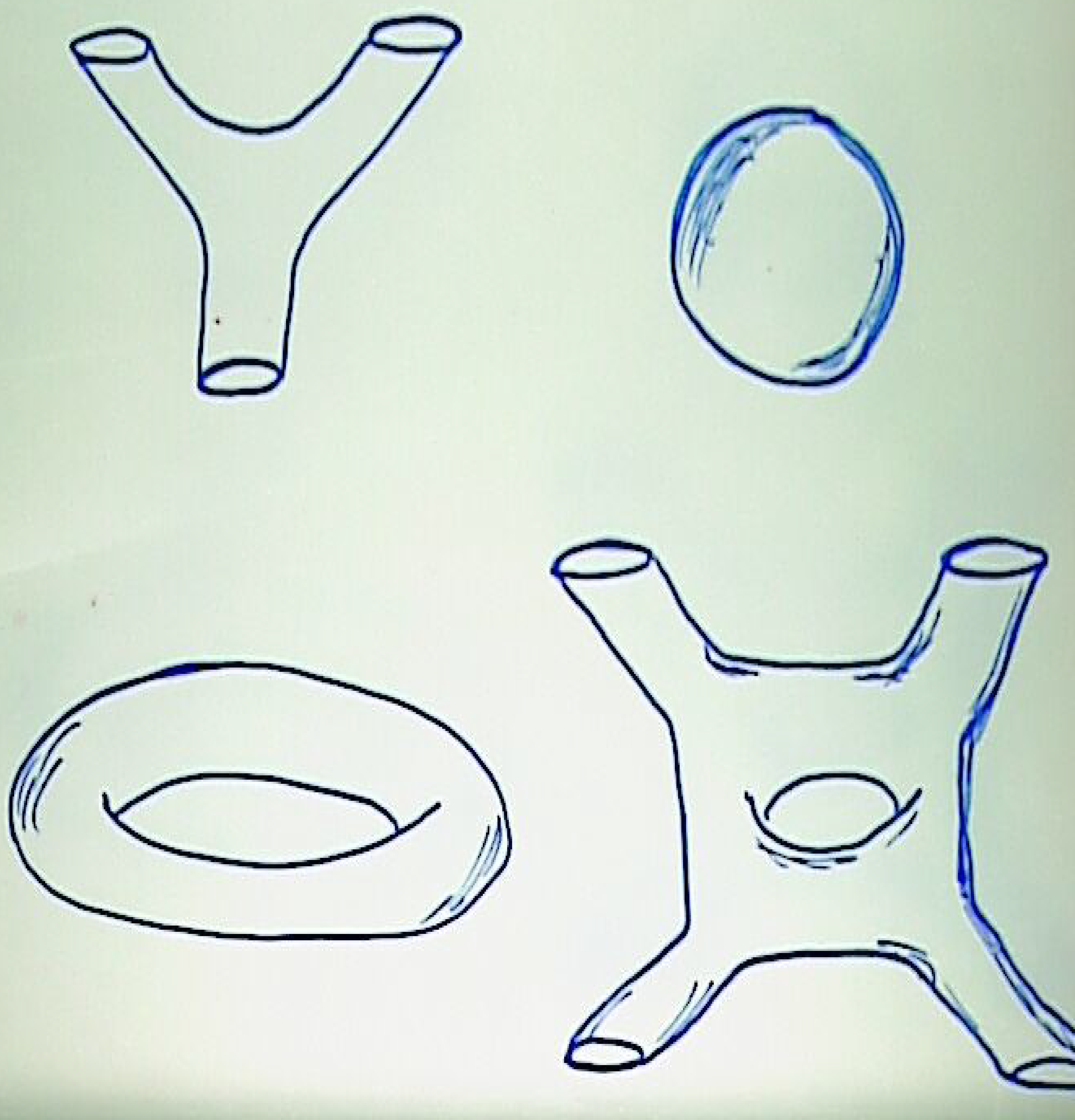
Interaction:

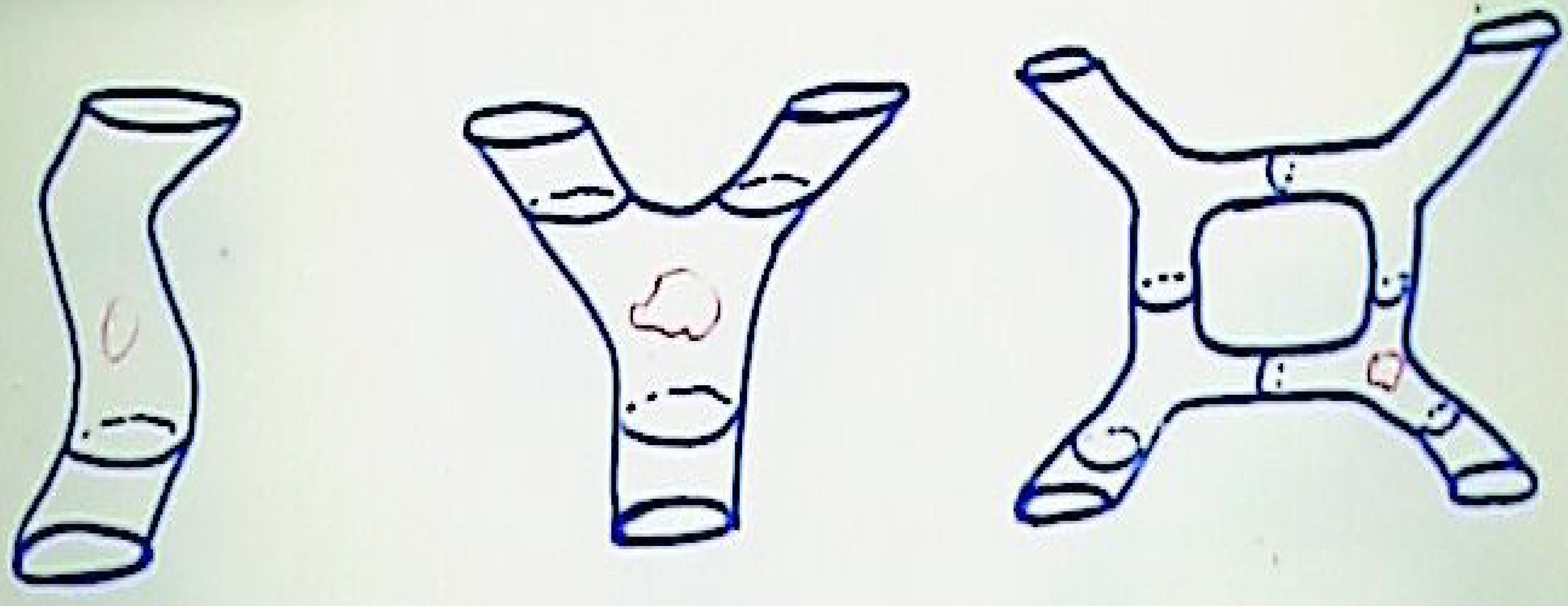


• Since a propagating string sweeps out a two-dimensional surface, one is naturally led to the study of maps from two-dimensional surfaces into the space-time manifold X :

$$\mathrm{Maps}\{\Sigma \to X\}.$$

two-dimensional surface





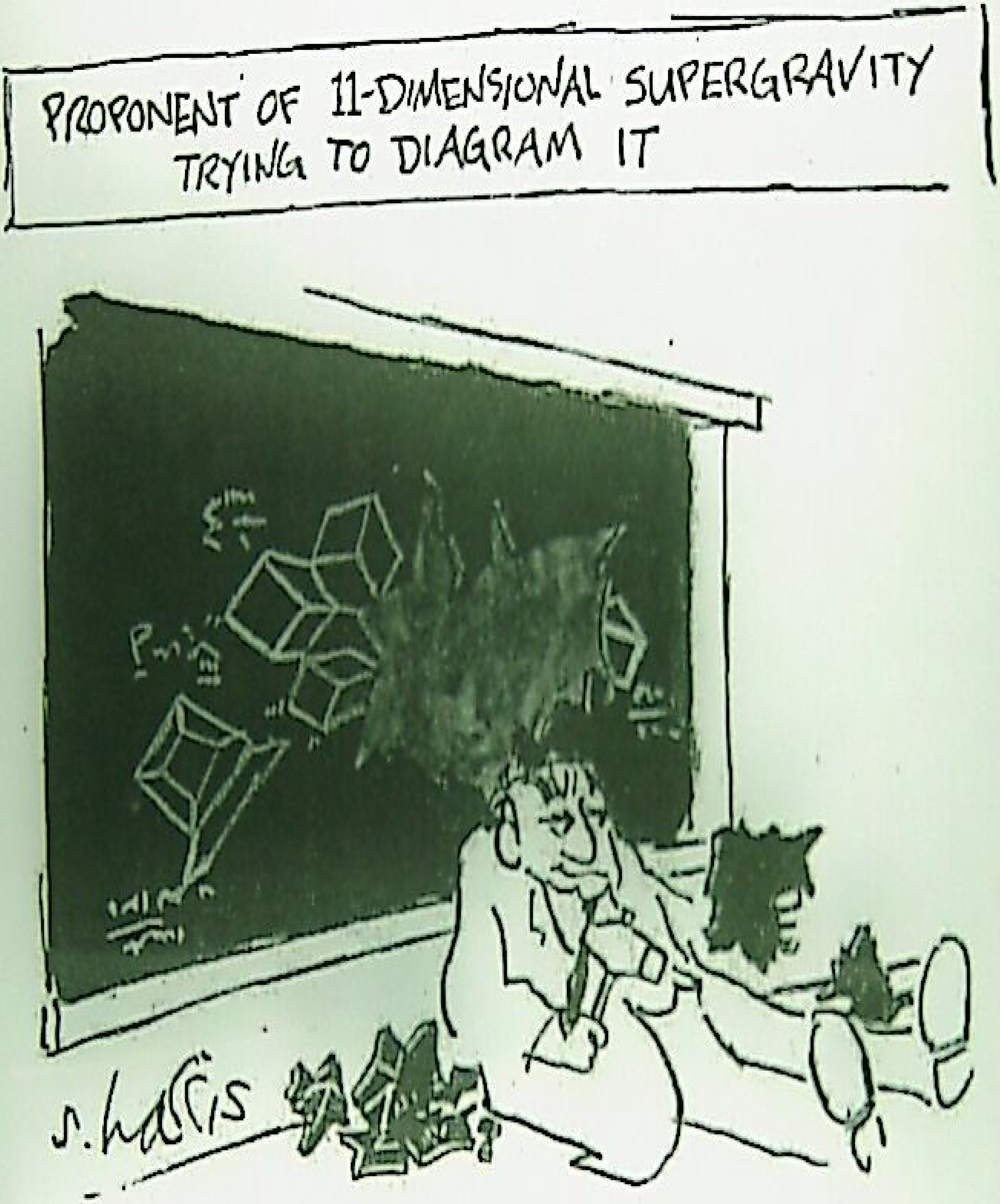
Due to universality of string propagation and interaction, string theory turns out to be very restrictive.

$\boxed{\text{"VERY FEW THEORIES"}}$

Specifically, string theory predicts (requires)

- Gravity
- Yang-Mills theory
- Supersymmetry
- Extra Dimensions

$D = 10$ or 11


PROPONENT OF 11-DIMENSIONAL SUPERGRAVITY TRYING TO DIAGRAM IT
S. Harris

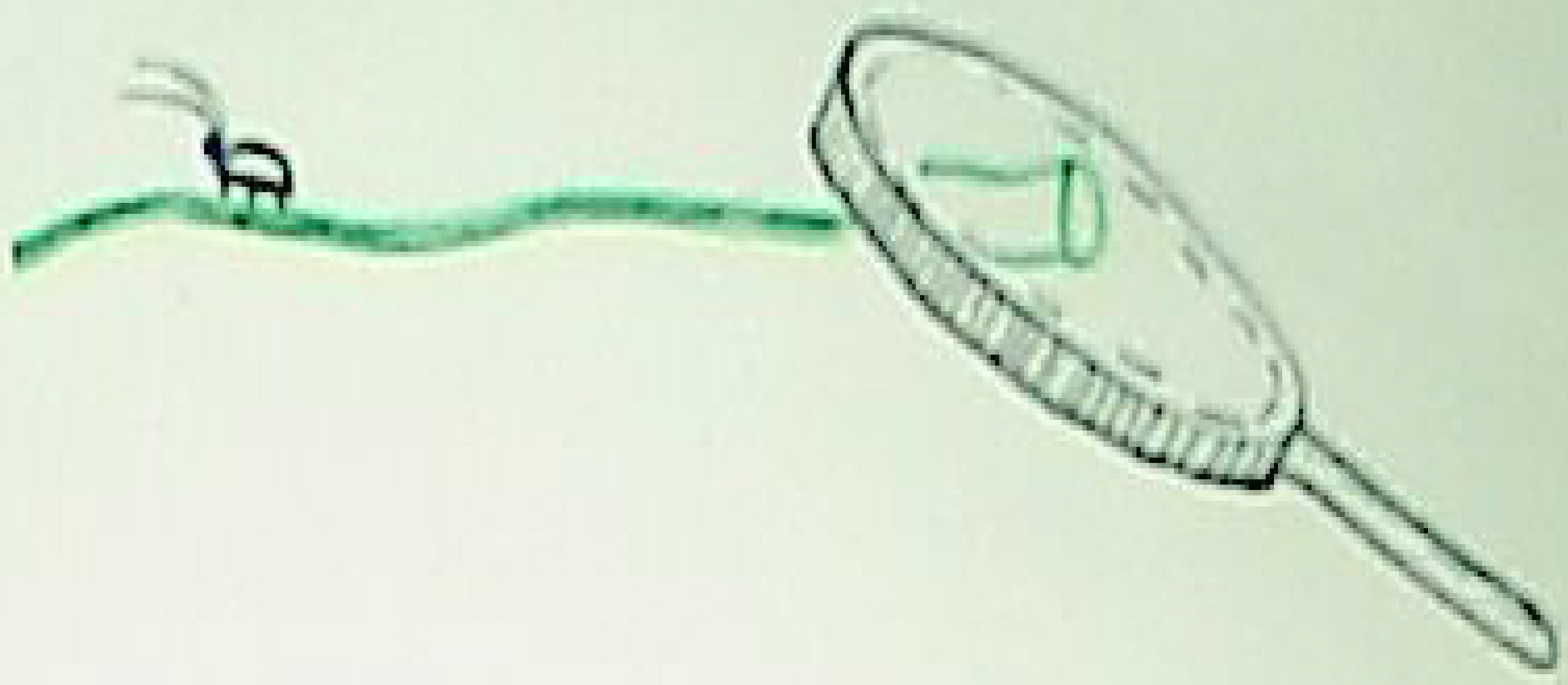
t - time

$$\left.\begin{matrix} x \\ y \\ z \end{matrix}\right\} \text{Space}$$

$$\left.\begin{matrix} w_1 \\ w_2 \\ \vdots \\ w_6 \end{matrix}\right\} \text{extra spatial dimensions}$$

Q: What to do with extra dimensions?

A (Kaluza+Klein): Compactification, dimensiona
reduction on a compact manifold X.



Superstring Compactification

Consider a universe which — apart from three large, extended dimensions — also has additional spatial dimensions that are tightly curled up into a tiny space X — a space so tiny that it is so hard to see even with the most powerful experimental equipment.

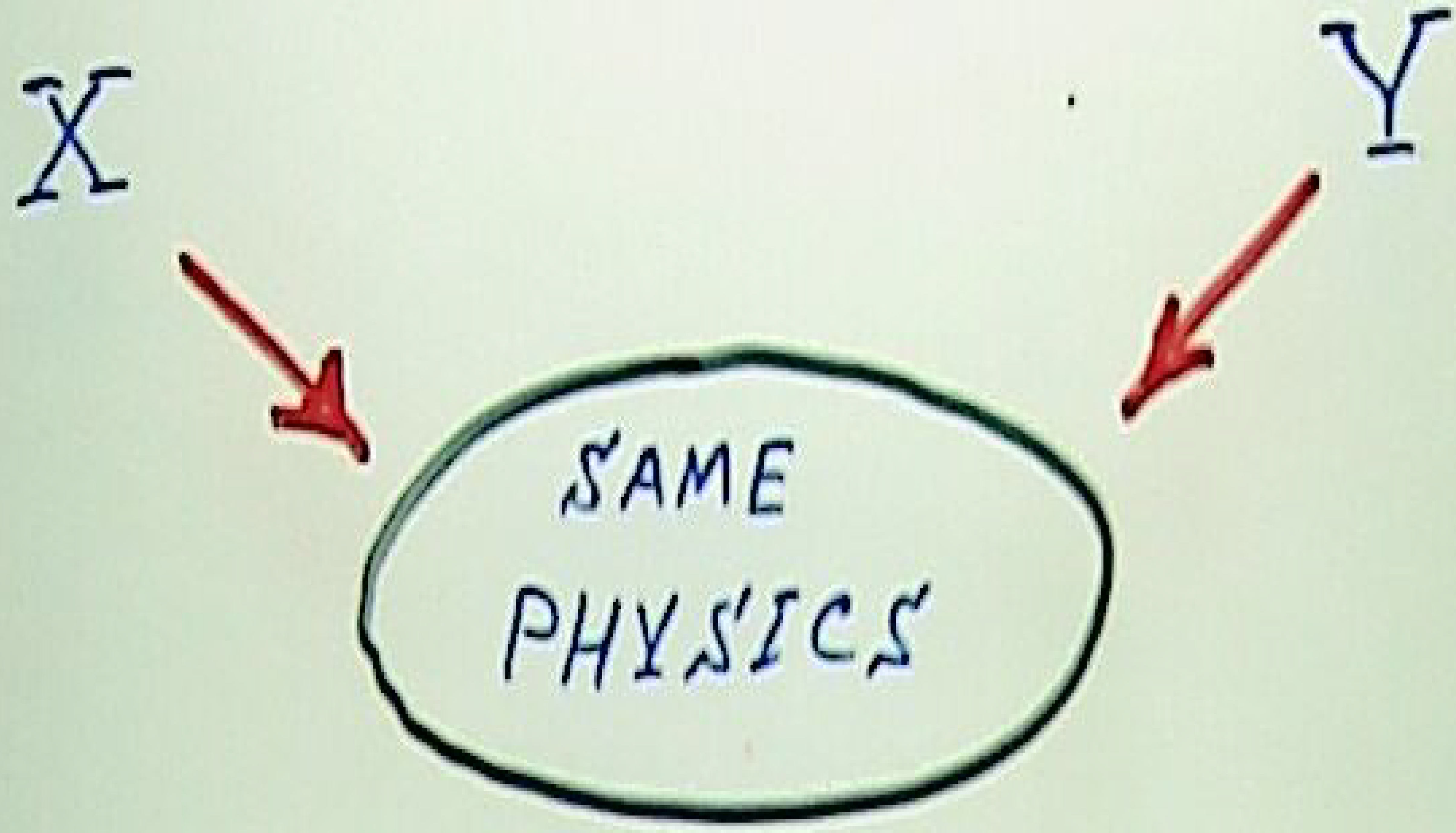
$$M^4 \times X$$

The spectrum of elementary particles and their interactions are encoded in the geometry of the space X.

$$\dim(X) = 10 - (3+1) = 6$$

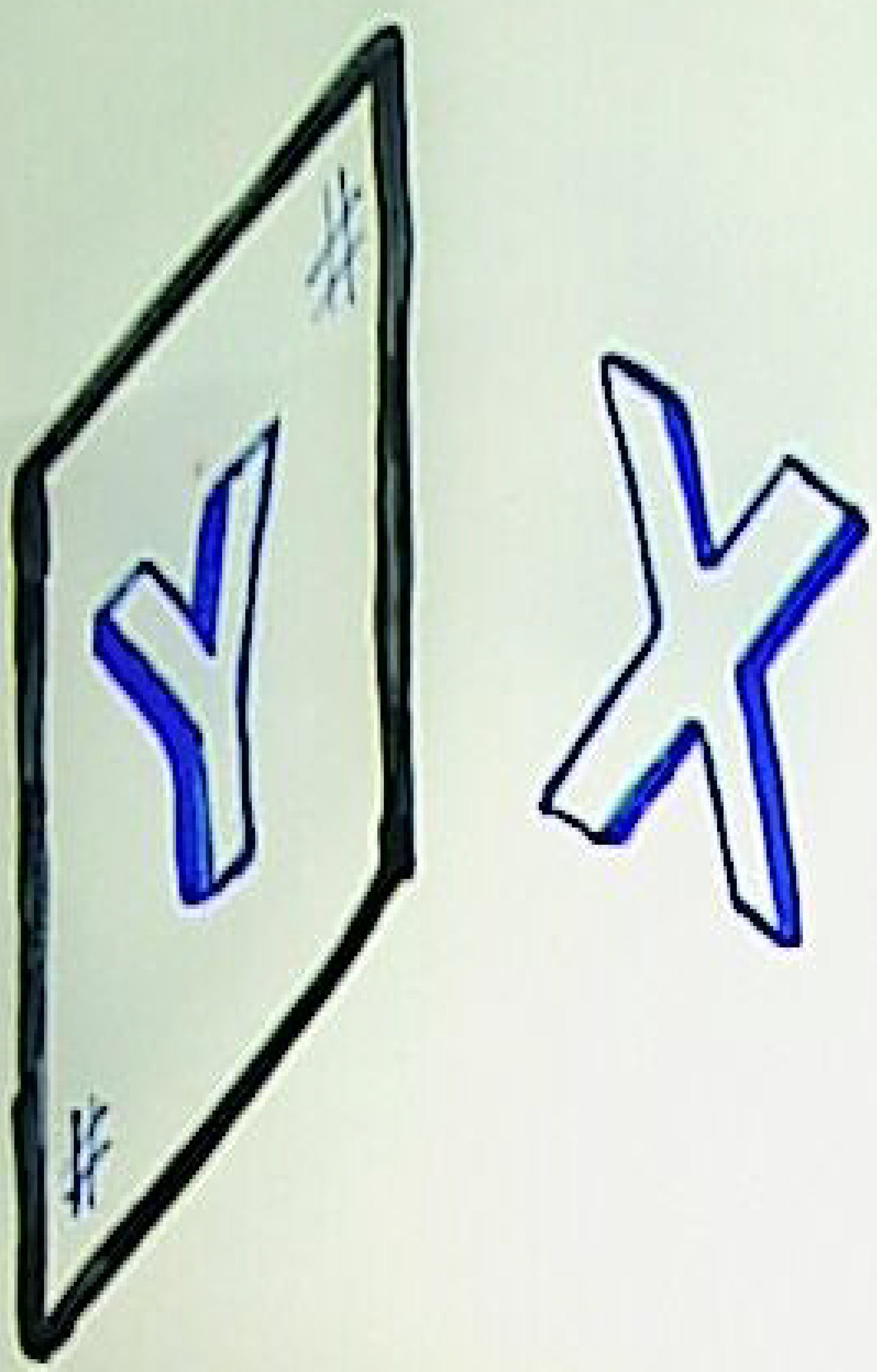
X = "Calabi-Yau manifold"

- However, there may be two different manifolds, X and Y, which lead to the same physical theory:



This phenomenon is called

"Mirror Symmetry"



Mirror Symmetry leads to many fascinating predictions about geometry of Calabi-Yau manifolds!

X		Y
easy	$\longleftrightarrow$	hard
hard	$\longleftrightarrow$	easy

- For example, one can obtain a lot of subtle information about

$$\mathrm{Maps}\{\Sigma \to X\}$$

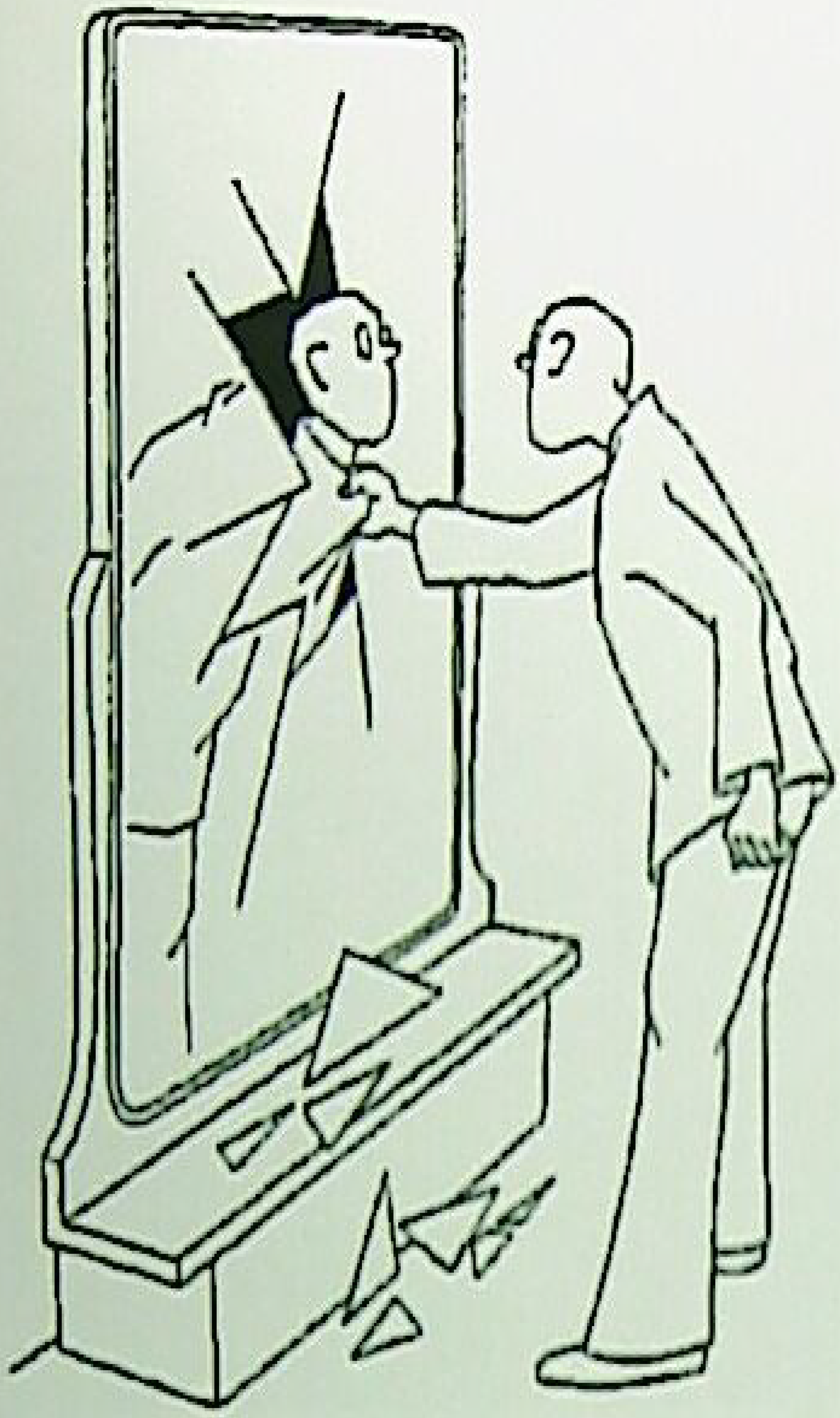
In particular, a generating function of the for

$$F(t) = \sum N_d \cdot e^{-dt}$$

"Number" of maps of degree d

must satisfy a simple differential equation, like

$$\partial^5 F + 5t\partial(5\partial+1)(5\partial+2)(5\partial+3)(5\partial+4)F = 0$$



• String theory dramatically changes our understanding of space and time: