229 Let V be a complex vector space equipped with positive hermitian form and with a $\mathbb{C}$-linear symmetric bilinear form $S(v,v')$. Choose an orthogonal basis $\{\xi_i, 1 \le i \le n\}$ for V: $\langle \xi_i | \xi_j \rangle = \delta_{ij}$,

Given $v, v' \in V$ let $x_i = \langle \xi_i | v \rangle$ so that $v = \sum \xi_i x_i$ and similarly with primes: $v' = \sum \xi_j x'_j$. One has

$$S(v,v') = \sum_{i,j} S(\xi_i x_i, \xi_j x'_j) = \sum_{i,j} s_{ij}\, x_i x'_j = \sum_j \Big(\sum_i s_{ij} x_i \Big) x'_j$$

One has a linear transformation $v \longmapsto s_v$ from V to $\hat{V}$ given by $s_v(v') = S(v,v')$. For each v we can represent s_v by inner product with an elt of V.

$$s_v(v') = S(v,v') = \sum_j \Big(\sum_i s_{ij} x_i \Big) x'_j \quad , \quad x'_j = \langle \xi_j, v' \rangle$$

$$= \sum_j \Big\langle \sum_i \bar{s}_{ij}\, \bar{x}_i \xi_j \Big| v' \Big\rangle \qquad ? \; ?$$

$$s_v(v') = s_v \sum_j \xi_j \overbrace{\langle \xi_j, v' \rangle}^{x'_j}$$

$$s_v(v') = \sum_j \underbrace{s_v(\xi_j)}\, \langle \xi_j, v' \rangle$$

$$S(v, \xi_j) = \sum_i S(\xi_i x_i, \xi_j) = \sum_i x_i s_{ij}$$

$$v = \sum_i \xi_i x_i \qquad x_i = \langle \xi_i | v \rangle$$

$$s_v(v') = S(v, v') = \sum_{i,j} S(\xi_i x_i, \xi_j x'_j)$$

$$= \sum_{i,j} s_{ij} x_i x'_j = \sum_{i,j} s_{ij} x_i \langle \xi_j | v' \rangle = \Big\langle \sum_{ij} \bar{s}_{ij} \bar{x}_i \xi_j \Big| v' \Big\rangle$$

You have this ~~map~~ linear fnl s_v which you have represented by scalar product with the vector $\sum_{ij} \bar{s}_{ij} \bar{x}_i \xi_j$

$$\bar{x}_i = \overline{\langle \xi_i | v \rangle} = \langle v | \xi_i \rangle$$

What is $\sum_{ij} \xi_j \bar{s}_{ij} \langle v | \xi_i \rangle = \sum_{ij} \xi_j \bar{s}_{ji} \langle v | \xi_i \rangle$?

This is an anti linear operator from V to V

Repeat V $\langle\ |\ \rangle$ + $S(v, v')$ $\mathbb{C}$ bilinear symmetric

ξ_i orth basis $v = \sum \xi_i \langle \xi_i | v \rangle$ sum with primes

Define $s_v \in \hat{V}$ by $s_v(v') = S(v, v')$. One has

$$S(v, v') = S\Big(v, \sum_j \xi_j \langle \xi_j, v' \rangle\Big) = \sum_{i,j} S\big(\xi_i \langle \xi_i | v \rangle, \xi_j \langle \xi_j | v' \rangle\big)$$

$$= \sum_{i,j} s_{ij} \langle \xi_i | v \rangle \langle \xi_j | v' \rangle = \sum_j$$

231 Repeat V with $\langle v|v'\rangle$ and $S(v,v')$

ξ_i orth basis. $v = \sum \xi_i \langle \xi_i, v\rangle$. Let $S_v \in \hat{V}$ be $S_v(v') = S(v, v')$. Find $\sigma_v \in V$ s.t. $\langle \sigma_v| = S_v$

$$S_v(v') = S(v, \sum_j \xi_j \langle \xi_j | v'\rangle) = \sum_{i,j} S(\xi_i, \xi_j) \langle \xi_i | v\rangle \langle \xi_j | v'\rangle$$

$$= \sum_{i,j} \Big\langle \xi_j \overline{S(\xi_i,\xi_j)}\, \overline{\langle \xi_i | v\rangle} \Big| v' \Big\rangle$$

$$\therefore\ S_v = \Big\langle \sum_{j,i} \xi_j \bar{s}_{ji} \overline{\langle \xi_i | v\rangle} \Big|$$

S_v is rep by $\sum_{j,i} \xi_j \bar{s}_{ji} \overline{\langle \xi_i | v\rangle}$

So you get ~~[illegible]~~ anti linear map from V to V.

$\boxed{x \longmapsto \bar{s}\bar{x}}$ $\longmapsto \bar{s}(\overline{\bar{s}\bar{x}}) = (\bar{s}s)x$

$\bar{s}s = s^*s$ $\qquad$ $\bar{s}\sigma\bar{s}\sigma = \bar{s}s$

So now arises the question of what the case $\bar{s}s = \lambda \geq 0$ looks like. ~~You ought to be able to~~ Ask about polar decomposition

~~[illegible]~~

So far you have done $V \xrightarrow{S} \hat{V} \longrightarrow V$

$x \longmapsto x^t S \longmapsto$ $\quad f \longmapsto \sum f(\xi_i)\langle \xi_i| = \sum_i \overline{f(\xi_i)}\xi_i$

$[x_i] \longmapsto [\sum_i x_i s_{ij}]$

232 $\quad V$ complex vector space orth basis $\xi_i = |i\rangle$

$S : V \longrightarrow \hat{V}$ bilinear form. Maybe you need dual bases for V and $\hat{V}$?

V ~~has~~ orth basis $\xi_i \quad 1 \leq i \leq n$.

$$S(v, v') = \cancel{\ } S\Big(\sum_i \xi_i x_i, \sum_j \xi_j x'_j\Big)$$

$$= \sum_{i,j} \cancel{x}_i s_{ij} x'_j = \Big\langle \sum_{i,j} \overline{x_i}\, \overline{s_{ij}}\, \xi_j \Big| v' \Big\rangle$$

$$S_v = \sum_{i,j} \overline{x_i}\, \overline{s_{ij}}\, \xi_j$$

Start again. V Hilb with orth basis $\xi_i \quad 1 \leq i \leq n$.

or V with $h(v, v')$ anti-linear in v, linear in v'

V with $s(v, v')$ $\mathbb{C}$ bilinear symmetric.

$$V \xrightarrow{S} \hat{V} \xleftarrow[\sim]{H} \overline{V}$$

$$h(v, -) \longleftarrow\!\shortmid v$$

$$v \longmapsto s(v, -)$$

Define $T : V \longrightarrow \overline{V}$ by $H^{-1}S$, ~~[illegible]~~

~~[illegible] the unique~~ $\quad T = H^{-1}S$ or ~~[illegible]~~

$HT = S$ i.e. $\boxed{Tv = !\text{ elt of } V \text{ such that } h(Tv, -) = s(v, -).}$

$$h(T(\lambda v), -) = s(\lambda v, -) = \lambda s(v, -)$$
$$= \lambda h(Tv, -) = h(\bar{\lambda} Tv, -) \Rightarrow T\lambda = \bar{\lambda} T$$

(233) So you verify $T(\lambda v) = \bar\lambda Tv$. Next ~~?~~ $T\lambda = \bar\lambda T \implies T^2\lambda = \lambda T^2$. But from

$$h(Tv, -) = s(v, -), \quad h(Tv, v') = s(v, v'),$$

$$h(v', Tv) = \overline{h(Tv, v')} = \overline{s(v, v')}$$

$$= \overline{s(v', v)}$$ becomes too hard.

Go back to orth basis $\xi_1, \dots, \xi_n$ for V, together with $S(v, v')$ symm $\mathbb{C}$ bil. Need H and H^{-1}

$Hv = h(v, -)$ means Hv is the elt of $\hat V$ given by the $\mathbb{C}$ linear functional $v' \mapsto h(v, v')$. So

v fixed $\quad v = \sum_i \xi_i c_i, \quad h(\xi_j, v) = h(\xi_j, \sum_i^n \xi_i c_i)$

$$= \sum_{i=1}^n h(\xi_j, \xi_i) c_i = c_j.$$ What are you doing?

$Hv = h(v, -)$ ~~Given $f \in \hat V$ want to expand~~

~~$f(-) = \sum_k f(\xi_k) h(v, \xi_k)$~~ ~~you want to construct v so that $h(v, -) = f(-)$~~

Given $f \in \hat V$ want to have $f(v') = (v, v')$ $\forall v'$. $\quad f(\xi_k) = h(v, \xi_k) = \overline{h(\xi_k, v)}$?

---

maybe focus upon orthogonality + completeness. $f \in \hat V$

If $f(v') = h(v, v')$ $\forall v'$, then use expansion $v = \sum_i \xi_i h(\xi_i, v)$

(234) ~~[illegible]~~ Start again. Suppose given $f \in \hat{V}$, you want to construct v so that $h(v,-) = f(-)$

$$f(\xi_k) = h(v, \xi_k) = \overline{h(\xi_k, v)}, \quad h(\xi_k, v) = \overline{f(\xi_k)}$$

$$v = \sum_k \xi_k\, h(\xi_k, v) = \sum_k \xi_k\, \overline{f(\xi_k)}$$

If $\quad v = \sum_k \xi_k \overline{f(\xi_k)}$, then $h(v, v') = f(v') \quad \forall v'$

$\forall j \quad h(\xi_j, v) = \overline{f(\xi_j)} \Leftarrow \quad h(v, \xi_j) = f(\xi_j) \quad \forall j$

$\Rightarrow \quad \sum \xi_k \overline{f(\xi_k)}$

$$V \xrightarrow{S} \hat{V} \xrightarrow{*} V \qquad\qquad *S*S = S^*S = \bar{S}S$$

$$x \longmapsto x^t S \longrightarrow S^* \bar{x} = \bar{S}\bar{x}$$

~~[illegible]~~ So the composite is $T: x \longmapsto \bar{S}\bar{x}$ (antilinear)

$$Tx = \bar{S}\bar{x}, \quad T(Tx) = \bar{S}\,\overline{Tx} = \bar{S}\,\overline{\bar{S}\bar{x}} = \bar{S}Sx$$

~~so~~ $T^2 = \bar{S}S$ which is S^*S ~~[illegible]~~ hermitian self adjoint ≥ 0

~~[illegible]~~ Aim for polar decomposition, which should arise from $(S^*S)^{1/2}$. Use spectral decomp. of $T^2 = \bar{S}S$

Look at $\bar{S}S$: $\quad \overline{\bar{S}S} = S\bar{S}, \quad (\bar{S}S)^t = S\bar{S}, \quad \bar{S} = S^*$

Focus upon the $\lambda = +1$ eigenspace of $\bar{S}S = S^*S$ which should be stable under T. So you now have

$\bar{S}S = 1 \quad$ so $\quad \bar{S} = S^* = S^{-1}$

235 You have anti linear operator T

$T: x \longmapsto \bar{S}\bar{x}$ on V such that $T^2 = 1$.

$\therefore$ T is a real structure on V.

Similarly in the anti-symmetric case:

$$V \xrightarrow{A} \hat{V} \xrightarrow{*} V \qquad\qquad A^* = \overline{A^t} = \overline{-A}$$

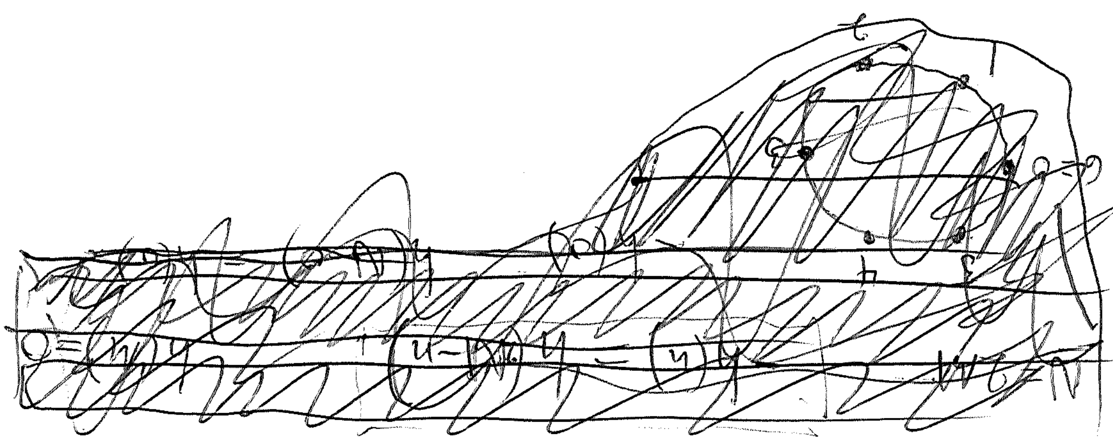
$$x \longmapsto x^t A \longmapsto A^*\bar{x} = -\bar{A}\bar{x}$$

$$Tx = -\bar{A}\bar{x} \qquad T(Tx) = -\bar{A}\left(\overline{Tx}\right) = -\bar{A}\left(\overline{-\bar{A}\bar{x}}\right) = (\bar{A}A)x$$

$A^*A = -\bar{A}A$ has spectrum ≥ 0. Restrict to $+1$ eigenspace for A^*A.

Then $-\bar{A}A = A^*A = 1$ so T is antilinear s.t. $T^2 x = (\bar{A}A)x = -x$. So T is a $\mathbb{H}$ structure on V.

Aim: to link C.T., polar decomposition, and symmetric space inversion. Begin with $Sp(2) = SU(2)$. You believe that the dihedral group formalism in the case of a Grassmannian should carry over to the symplectic symmetric space $Sp(2n)/U(n)$ provided complex structures: $-J = J^* = J^{-1}$ are substituted for involutions: $F = F^* = F^{-1}$.



236 Look for complex structures J in $\mathbb{H}$.

Recall $\mathbb{H} = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : a, b \in \mathbb{C} \right\}$ which contains

$$SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : |a|^2 + |b|^2 = 1 \right\} \quad \text{and}$$

$$\mathcal{L}SU(2) = \left\{ \quad\text{---}\quad : a + \bar{a} = 0 \right\}$$

$$= \left\{ x \underbrace{\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}}_{\hat{i}} + y \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\hat{j}} + z \underbrace{\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}}_{\hat{k}} ; x, y, z \in \mathbb{R} \right\}$$

~~Let $J = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \in \mathbb{H}$. $-J = \begin{bmatrix} -a & -b \\ \bar{b} & -\bar{a} \end{bmatrix} = J^* = \begin{bmatrix} \bar{a} & -b \\ \bar{b} & a \end{bmatrix}$ means that $a = i\alpha$ with $\alpha \in \mathbb{R}$, that is, $J \in \mathcal{L}SU(2)$. Then $J = \begin{bmatrix} i\alpha & b \\ -\bar{b} & -i\alpha \end{bmatrix}$ so $J^2 = \begin{bmatrix} i\alpha & b \\ -\bar{b} & -i\alpha \end{bmatrix}\begin{bmatrix} i\alpha & b \\ -\bar{b} & -i\alpha \end{bmatrix} = \begin{bmatrix} -\alpha^2 - |b|^2 & 0 \\ 0 & -\alpha^2 - |b|^2 \end{bmatrix}$ ... $J^{-1} = \ldots$~~

Let $J = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \in \mathbb{H}$. Then $J^* = \begin{bmatrix} \bar{a} & -b \\ \bar{b} & a \end{bmatrix}$,

$J^{-1} = \dfrac{1}{|a|^2 + |b|^2} \begin{bmatrix} \bar{a} & -b \\ \bar{b} & a \end{bmatrix}$ So $-J = J^* \iff \bar{a} = -a$ i.e. $J \in \mathcal{L}SU(2)$

and $J^* = J^{-1}$ iff $|a|^2 + |b|^2 = 1$, i.e. $J \in SU(2)$.

(Also $J^2 = \begin{bmatrix} a^2 - |b|^2 & (a+\bar{a})b \\ -\bar{b}(a+\bar{a}) & \bar{a}^2 - |b|^2 \end{bmatrix} = -1 \Rightarrow a + \bar{a} = 0$ or $b = 0$.

If $a + \bar{a} = 0$, then $J \in \mathcal{L}SU(2)$, $J^2 = -1$. Note: $\mathbb{R} + \mathbb{R}J$ is a subfield of $\mathbb{H}$ isom. to $\mathbb{C}$.

If $b = 0$, then $J = \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}$ with $a = \pm i$.)

Other comments about the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathcal{L}SU(2)$ being the space of field embeddings $\mathbb{C} \longrightarrow \mathbb{H}$.

237 [redacted] To link C.T., polar decomposition, and the inversion on a symmetric space.

Consider the symmetric space $Sp(2n)/U(n)$ where $U(n) \hookrightarrow Sp(2n)$ is $u \longmapsto \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}$. [redacted]

Lie alg level $\quad \mathcal{L}\,Sp(2n) = \left\{ \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} : a^* = -a \right\} \oplus \left\{ \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} : b^t = b \right\}$

$$\mathfrak{g} \quad = \quad [redacted]\ \mathcal{L}U(n) \quad \oplus \quad \mathfrak{p}$$

This even-odd grading is given by conjugation by $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = i\varepsilon$ [redacted].

Let's now look at $-J = J^* = J^{-1}$ in $\mathfrak{p}$. If $J = \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}$, then [redacted]

$J^* = \begin{bmatrix} 0 & -b^t \\ b^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & -b \\ \bar{b} & 0 \end{bmatrix} = -J$ holds $\forall\, b = b^t$

and $-J^2 = -\begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}\begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} = \begin{bmatrix} b\bar{b} & 0 \\ 0 & \bar{b}b \end{bmatrix}$ so

$J^2 = -1$ iff $b\bar{b} = \bar{b}b = 1$. Since $b^t = b \Rightarrow b^* = \bar{b}$

this means that b is [redacted] a symmetric unitary matrix. 2×2 example: $\begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$

Next return to the spectral theory for complex symmetric matrices b.

$$\begin{array}{ccccc} V & \xrightarrow{\ b\ } & \hat{V} & \xrightarrow{\ *\ } & V \\ x & \longmapsto & x^t b & \longmapsto & \bar{b}\bar{x} \end{array}$$

$V = \mathbb{C}^n$ column
$\hat{V} = \mathbb{C}^n$ row

anti-linear transf $T(x) = \bar{b}\bar{x}$ is s.t. $T^2(x) = \bar{b}\,\overline{T(x)} = \bar{b}\,\overline{\bar{b}\bar{x}} = (\bar{b}b)x$. [redacted]

[redacted] $\therefore T^2 =$ mult by the matrix $\bar{b}b \geq 0$.

238 $T(x) = \overline{bx}$ $T(Tx) = T(\overline{bx}) = \overline{b\,\overline{bx}} = (\bar{b}b)x$

$T^2x = (\bar{b}b)x$ and $\bar{b}b = b^*b$ is hermitian ≥ 0.

Use spectral decomposition of b^*b, the eigenvalues of b^*b are also known as characteristic values of b. You should get a splitting

$$V = \bigoplus_{\lambda \geq 0} V_\lambda \quad \text{where } T^2 = \bar{b}b = \lambda \text{ on } V_\lambda .$$

Since T commutes with T^2, T respects the spectral decomposition of T^2.

Consider T on V_λ where $\lambda > 0$. One has $T^2 = \lambda$, hence $\sigma = \lambda^{-1/2}T$ is an antilinear operator on V_λ whose square is the identity. You therefore should get a reduction of V_λ from a complex vector space with herm. inner product to a real Euclidean space $(V_\lambda)^\sigma$.

Hopefully there would also be a symmetric real bilinear form on this Euclidean space arising from b.

Idea: $Tx = \overline{bx}$ $T^2x = (b\bar{b})x$ On V_λ for $\lambda > 0$ you can rescale: $\lambda^{-1/2}b$ to get $b\bar{b} = 1$.

You've defined an anti linear transf. $Tx = \bar{b}\bar{x}$ on V s.t. $T^2x = (\bar{b}b)x$. $\bar{b}b = b^*b$ is hermitian and ≥ 0, ~~so it has a unique~~ so it has a unique hermitian ≥ 0 square root we denote by $|T| = (\bar{b}b)^{1/2}$.

Repeat: On V you have an anti linear operator $Tx = \bar{b}\bar{x}$ whose square $T^2 x = (\bar{b}b)x$ is the $\mathbb{C}$-linear positive hermitian operator $\bar{b}b = b^*b$. Let $|T| = (\bar{b}b)^{1/2}$ be the pos herm. sqrt of T^2. Let's first handle the case where b is nonsingular. Then $T, T^2, |T|$ are invertible. As T commutes with T^2, T commutes with $|T|$ so $\sigma = |T|^{-1}T = T|T|^{-1}$ is an invertible anti linear operator on V of square 1. So you get a real structure on V which is the Euclidean space of $x \in V$ fixed by σ.

Points bothering you: $b^*b \neq bb^*$.

Grass situation. $X = \begin{bmatrix} 0 & -T^* \\ T & 0 \end{bmatrix}$

$$g = \frac{1+X}{1-X} = F\varepsilon \qquad F = +1 \text{ on } \begin{bmatrix} 1 \\ T \end{bmatrix}, \quad -1 \text{ on } \begin{bmatrix} -T^* \\ 1 \end{bmatrix}$$

$$F(1+X) = (1+X)\varepsilon = \varepsilon(1-X)$$

$$1+X = g(1-X)$$

nonzero eigenvalues should be clear.

How do you make progress? It's clear that $\begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}$ is similar to $X = \begin{bmatrix} 0 & -T^* \\ T & 0 \end{bmatrix}$ so the C.T. should be obvious for $Sp(2n)/U(n)$. So it's a matter of details. The new point is that $b^t = b$ and the isotropy group is $\begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}$ $u \in U(n)$

240 ~~[illegible]~~ You need to clear things up, things being C.T., polar decomp, symmetric space inversion.

First point: describe $J = \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}$: $-J = J^* = J^{-1}$

condition is $b\bar{b} = 1 = \bar{b}b$, i.e. b symmetric and unitary

Is this the same as a real structure. Clearly because $Tx = \overline{bx}$ and $T^2 x = (\bar{b}b)x$

$\bar{b}b = 1 \Rightarrow b$ 1-1 $\Rightarrow b$ onto $\Rightarrow \bar{b} = b^{-1}$

Conclusion is that $J \in \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}$: $-J = J^* = J^{-1}$

$\Leftrightarrow b$ symm. & unitary.

C.T. Here you have general $b^t = b$, which gives $X = \begin{bmatrix} 0 & -\bar{b} \\ b & 0 \end{bmatrix}$ skewherm., so C.T. is defined

$g = \frac{1+X}{1-X}$ $\quad g^{1/2} = \frac{1+X}{(1-X^2)^{1/2}}$

You need clear thinking to get from the C.T. in the Grass case to the C.T. in case of $Sp(2n)/U(n)$

$U(n)$ should be the centralizer of $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, this is clear on $\mathcal{L}$ level. Can you embed $Sp(2n)/U(n)$ into ~~[illegible]~~ $U(2n)/U(n) \times U(n)$. This should be induced by the map $Sp(2n) \hookrightarrow U(2n)$.

$$\mathcal{L}\,Sp(2n) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \begin{array}{l} a^* = -a \\ b^t = b \end{array} \right\}$$

Key idea $\frac{1+tX}{(1-t^2X^2)^{1/2}} \longrightarrow \frac{X}{|X|}$ at $t \to \infty$

241 What to do?. $X = \begin{bmatrix} 0 & -\bar b \\ b & 0 \end{bmatrix}$, $-X^2 = \begin{bmatrix} \bar b b & 0 \\ 0 & b\bar b \end{bmatrix}$

Assume 0 not an eigenvalue of b, so that $|X| = (-X^2)^{1/2}$ is defined + invertible.

$b^t = b$ complex symm. + invertible $X = \left[\begin{array}{c|c} 0 & -\bar b \\ \hline b & 0 \end{array}\right]$

$$-X^2 = \left[\begin{array}{c|c} \bar b b & \\ \hline & b\bar b \end{array}\right], \quad (-X^2)^{1/2} = \begin{bmatrix} (\bar b b)^{1/2} & 0 \\ 0 & (b\bar b)^{1/2} \end{bmatrix} = |X|$$

$$\frac{X}{|X|} = \begin{bmatrix} 0 & -\bar b (b\bar b)^{-1/2} \\ b(\bar b b)^{-1/2} & 0 \end{bmatrix}$$

Describe T antilinear such that $T^2 = 1$

$$\hat V \xrightarrow{*} V \xrightarrow{T} V \qquad T : V \longrightarrow V$$

anti-linear such that $T^2 = 1$.

$$V \xrightarrow{T} V \xrightarrow{*} \hat V$$

Let $T : V \longrightarrow V$ be anti-linear and invertible

$$\begin{tikzcd} & \hat V \arrow[dl, "*"'] \arrow[dr] & \\ V \arrow[rr, "T"] & & V \end{tikzcd}$$

$$\boxed{\begin{bmatrix} \alpha^t & \beta^t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} =}$$

Vague idea that what you are doing is similar to embedding into a hyperbolic space

$$\begin{tikzcd} W \arrow[r, "\begin{bmatrix} \alpha \\ \beta \end{bmatrix}"] \arrow[d, "\alpha^t\beta + \beta^t\alpha"'] & \begin{bmatrix} V \\ \hat V \end{bmatrix} \arrow[d, "\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}"] \\ \hat W & \begin{bmatrix} \hat V \\ V \end{bmatrix} \arrow[l, "\begin{bmatrix} \alpha^t & \beta^t \end{bmatrix}"] \end{tikzcd}$$

(242) Let V be equipped with pos herm form. Let's look at invertible ~~anti-linear~~ $T : V \longrightarrow V$. Obvious thing to do is to compose with $V \xrightarrow{*} \hat{V}$ to get $(*)T : V \longrightarrow \hat{V}$, a bilinear form B on V.

Let $\begin{cases} V = \mathbb{C}^n & \text{col vectors.} \\ \hat{V} = \mathbb{C}^n & \text{row vectors.} \end{cases}$ $B : V \longrightarrow \hat{V}$ is given by a matrix b via $B(x) = x^t b$. Then

$$T = (*)(*)T = (*)B, \quad T(x) = *(B(x)) = b^* \bar{x}$$

and $T^2(x) = T(b^*\bar{x}) = b^* \overline{b^* \bar{x}} = (b^* b^t)x$

T $\mathbb{C}$-anti linear, T^2 $\mathbb{C}$-linear. Assume $T^2 = 1$, i.e.

$$b^* b^t = 1 \overset{(t)}{\Longleftrightarrow} b\bar{b} = 1 \overset{(-)}{\Longleftrightarrow} \bar{b} b = 1$$

$$(*) \Updownarrow$$

$$\bar{b} b^* = 1 \Longrightarrow$$

Still want progress on linking CT, polar decomp, symm space inversion. Considering the cases $Sp(2n)/U(n) \hookrightarrow U(2n)/U(n) \times U(n)$

Puzzle. $U(n) \hookrightarrow Sp(2n)$ $\quad u \longmapsto \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}$

Let $i\varepsilon = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ $\quad -i\varepsilon = (i\varepsilon)^* = (i\varepsilon)^{-1}$. Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sp(2n)$ centralize ~~commute~~ $i\varepsilon$, then $b = c = 0$

g unitary $\Rightarrow a, d$ also. Certainly $d \overset{g \text{ symp} \Rightarrow}{=} (a^t)^{-1} = \bar{a}$.

~~[illegible]~~

243 ~~[scribble]~~ Start again with $Sp(2n)$ acting by conjugation on $J_0 = i\varepsilon$ with isotropy group $U(n) \hookrightarrow Sp(2n)$, $u \mapsto \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}$. This ~~[scribble]~~ gives an embedding of $Sp(2n)/U(n)$ into $\{ J \in M_{2n}\mathbb{C} \mid -J = J^* = J^{-1} \}$.

Consider ~~[scribble]~~ such a J when $n = 1$. Then ~~[scribble]~~ you've seen that for $J = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ one has

$$\begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$

$$\det(J) = \det(-J) = \det(J^*) = \overline{\det(J)}$$

$$\det(J^*) = \det(J^{-1}) = \frac{1}{\det(J)}$$

so $\det(J) = \pm 1$. $\quad J^{-1} = \frac{1}{\det(J)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $\quad -\mathrm{tr}(J) = \mathrm{tr}(J^*) = \overline{\mathrm{tr}(J)}$

Conditions $\quad \bar{a} = -a \quad \bar{d} = -d \quad\quad \overline{a+d} = -(a+d)$

$\quad c = -\bar{b} \quad b = -\bar{c} \quad\quad$ doesn't help.

$$\begin{bmatrix} a & b \\ -\bar{b} & d \end{bmatrix}\begin{bmatrix} a & b \\ -\bar{b} & d \end{bmatrix} = \begin{bmatrix} a^2 - |b|^2 & (a+d)b \\ -\bar{b}(a+d) & d^2 - |b|^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

~~[scribble]~~ If $b = 0$, then $a^2 = d^2 = -1$.
so it's not true that $d = -a$

If $b \neq 0$, then $a + d = 0$. $\quad d = -a = \bar{a}$ etc.

Again you've found problems when you try to compute inside $Sp(2n)$.

So you need to ~~[scribble]~~ use the Lie level. Basic two example ~~[scribble]~~ $Sp(2n)/U(n) \longleftrightarrow U(2n)/U(n) \times U(n)$.
What do you think is true? A point of this Grassmannian is an F whose ± 1 eigenspaces have dim (n). (Note: Here you see already different components of $\{ F = F^* = F^{-1} \}$.) ~~Both you~~

244 In the Grass ~~situation~~ what does C.T. do?
Mainly to construct ~~an~~ explicit conjugacy: $u \varepsilon u^{-1} = F$.
Idea is to form the multiplicative difference $g = F\varepsilon$
which is "anti-symm": $g \mapsto g^{-1}$ under conjugation by
F and ε, then take sqrt $g^{1/2}$ $\quad \mathring{\varepsilon} \quad \mathring{F}$

So you have to work on the Lie level.

$$\mathcal{L}Sp(2n) = \underbrace{\left\{ \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} : a^* = -a \right\}}_{k \simeq \mathcal{L}U(n)} \oplus \underbrace{\left\{ \begin{bmatrix} 0 & -\bar{b} \\ b & 0 \end{bmatrix} : b^t = b \right\}}_{p}$$

In the Grass case

$$\mathcal{L}U(2n) = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} : \begin{matrix} a^* = -a \\ d^* = -d \end{matrix} \right\} \oplus \left\{ \begin{bmatrix} 0 & -b^* \\ b & 0 \end{bmatrix} : \begin{matrix} b \text{ arb} \\ n \times n \end{matrix} \right\}$$

What you want is to understand the symmetric space inversion.

You have two maps $\mathcal{L}Sp(2n) \longrightarrow Sp(2n)$ namely the exponential map e^X and the C.T. $\dfrac{1+X}{1-X}$. These maps are defined by fnl calc. & since X is diagonalizable nothing

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Repeat: To understand symmetric space inversion in the case of $Sp(2n)/U(n)$ and $U(2n)/U(n) \times U(n)$

One thing you forgotten is $K \backslash G / K$. ~~Clifford Taubes~~

245 Still don't understand C.T., polar decomp, symmetric space inversion for $Sp(2n)/U(n)$. Perhaps you should look at $U(2n)/Sp(2n)$. Not even sure this is defined. In any case something is defined namely

$$\mathcal{L}\,SO(2n) = \left\{ \begin{bmatrix} a & 0 \\ 0 & \bar a \end{bmatrix} : a^* = -a \right\} \oplus \left\{ \begin{bmatrix} 0 & b \\ \bar b & 0 \end{bmatrix} ; b^t = -b \right\}$$

So again

$$\begin{array}{ccccc} V & \xrightarrow{b} & \hat V & \xrightarrow{*} & V \\ x & \longmapsto & x^t b & \longmapsto & b^* \bar x = -\bar b \bar x \end{array}$$

$$Tx = -\bar b \bar x \qquad T(Tx) = -\bar b\left(\overline{-\bar b \bar x}\right) = (\bar b b) x$$

$$\therefore T^2 = \bar b b = -b^* b \qquad b^* = \bar b^t = -\bar b$$

so $T^2 \leq 0$ and polar decomposition yields a phase which is an anti linear automorphism T such that $T^2 = -1$. How to visualize?

$$\begin{tikzcd} V \arrow[r, "b"] & \hat V \arrow[r, "*"] & V \\ x \arrow[r, mapsto] \arrow[rr, bend right, "T"'] & x^t b \arrow[r, mapsto] & b^* \bar x \end{tikzcd}$$

$$Tx = b^* \bar x$$

$$TTx = b^* \overline{Tx}$$

$$= b^* \overline{b^* \bar x} = (b^* b^t) \bar x$$

Now assume $b^t = b$. $T^2 \cancel{x} = (\bar b b) x = (b^* b) x$

where $b^* b > 0$

So you need to understand polar decomp. for

$$Sp(2n)/U(n) \hookrightarrow U(2n)/U(n) \times U(n)$$

What's the situation for the Guess? You have the basepoint ε and the variable pt F. Because $i\varepsilon, iF$ are in the Lie algebra you should be able to ~~conjugate use the conjugacy them~~ conjugate iF into the centralizer of $i\varepsilon$.

to conjugate F into the centralizer of ε. Review the variational method. To minimize $\frac{1}{2}\mathrm{tr}\left(gFg^{-1}-\varepsilon\right)^2 =$

$$\frac{1}{2}\mathrm{tr}\left(F^2+\varepsilon^2\right) - \mathrm{tr}\left(gFg^{-1}\varepsilon\right) \quad \text{for } g \in U(2n)$$

Assume $F_s = g_s F g_s^{-1}$ is a 1st order stationary pt. Then you make a variation $\delta g = (1+X)g_s$ to get

$$0 = -\mathrm{tr}\left([X,F_s]\varepsilon\right) = -\mathrm{tr}\left(X,[F_s,\varepsilon]\right) \quad \forall X$$

It should be possible to see what's going on. Is there a vector field on the Grass. associated to ε? The gradient of $\frac{1}{2}\mathrm{tr}\left(gFg^{-1}-\varepsilon\right)^2 =$ grad of $-\mathrm{tr}\left(gFg^{-1}\varepsilon\right)$

Given the Grassmannian $U(2n)/U(n)\times U(n)$ with basepoint ε. Better might be the projective space $U(n)/U(1)\times U(n-1)$. Morse function on Proj space? assoc. to a line l a rank 1 projection (self adjoint): Choose unit vector $z \in V = \mathbb{C}^n$ then zz^* is the orth projection onto l. Then Morse fn. is $\mathrm{tr}\left(zz^*F\right)$ on the Grass; $\mathrm{tr}(zz^*F) = z^*Fz$. You did something before along these lines with F replaced by a s.a. op A. Get fn. z^*Az on $\mathbb{P}\mathbb{C}^n$. Stationary value subject to variation $z^*\delta z = 0$ $\quad \delta(z^*Az) = (\delta z)^*Az + z^*A\delta z$ $= 2(\delta z)^*Az = 0$, z stationary $\Longleftrightarrow Az = \lambda z$

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You seek a flow, a kind of gradient flow, from any point E of the Grassmannian to the basepoint ε. Maybe it would be better to find a Morse function. How might you proceed? The simplest Grass is the Riemann sphere.

$$SU(2)/U(1) \xrightarrow{\sim} U(2)/U(1)\times U(1)$$

You want to link various things.

- conjugacy theorem in the Lie alg. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$

$\mathfrak{p} = \left\{ \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} \right\}$. max abelian subspace $\mathfrak{a} \subset \mathfrak{p}$.

conj thm says any elt of $\mathfrak{p}$ is conjugate to an elt of $\mathfrak{a}$ via an elt of K.

Conjugacy. $\frac{1}{2}\operatorname{tr}\left(kpk^{-1} - p_0\right)^2$. You should figure out what this means in the $SU(2)/U(1)$ case

$SU(2)/U(1) = S^2$

Morse theory for $\mathbb{CP}^n$ take herm. op A, form $\frac{1}{2}z^*Az$, $\|z\| = 1$. Get function on $\mathbb{CP}^n$.

Critical pt. $\delta z^* z = 0$

$$\delta(z^*z) = (\delta z)^* z + z^*\delta z = 2(\delta z)^* z$$

$$0 = \delta\,\tfrac{1}{2} z^*Az = \tfrac{1}{2}\left(\delta z^* Az + z^*A\delta z\right) = \delta z^* Az \implies Az = \lambda z. \quad \text{some } \lambda.$$

248 Study a Grassmannian using conjugacy then in the Lie algebra.

$G = U(n)$ Identify $LU(n)$ with herm. matrices.
Pick a diag matrix Λ with distinct entries. Given A herm. you ~~minimize~~ dist^2 $\frac{1}{2}\operatorname{tr}(uAu^{-1}-\Lambda)^2$ over G.

$$L = \frac{1}{2}\operatorname{tr}(A^2+\Lambda^2) - \operatorname{tr}(uAu^{-1}\Lambda)$$

~~Stationary pt.~~ $\cancel{\delta \operatorname{tr}(uAu^{-1}\Lambda)}$ $\cancel{u+\delta u = (1+X)u}$

$\cancel{= \operatorname{tr}(XuAu^{-1}\Lambda - uAu^{-1}Xu^{-1}\Lambda)}$ $\cancel{\delta u = Xu}$

$\cancel{\delta(u^{-1}) = -u^{-1}Xu^{-1}}$

$\cancel{= \operatorname{tr}}$

Assume u_0 stationary point, put $A_0 = u_0 A u_0^{-1}$

~~Then~~ consider $u = u_0 + \delta u = (1+X)u_0 + O(X^2)$

$$(u_0+\delta u)^{-1} = u_0^{-1}(1-X) \quad ''$$

$$uAu^{-1} = (1+X)u_0 A u_0^{-1}(1-X) = A_0 + [X, A_0]$$

$$\operatorname{tr}(uAu^{-1}\Lambda) = \operatorname{tr}(A_0\Lambda) + \operatorname{tr}([X,A_0]\Lambda)$$

$$\delta \operatorname{tr}(uAu^{-1}\Lambda) = \operatorname{tr}(X[A_0,\Lambda]) = 0 \quad \forall X$$

$$\therefore [A_0, \Lambda] = 0$$

You want this variational arg. in the case $\Lambda = \varepsilon$. What does this mean?

You have $V = \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$ $\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and ~~the~~

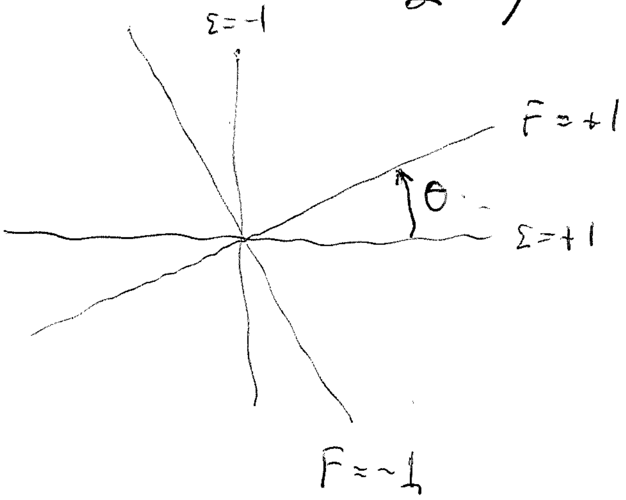
$$F = \begin{cases} +1 & \text{on } W \\ -1 & \text{on } W^\perp \end{cases}$$

(249) Consider the Grass situation $V = \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$, $\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

[redacted] $W \subset V$, $F = \begin{cases} +1 & \text{on } W \\ -1 & \text{on } W^\perp \end{cases}$, F is varying

The function of F used for critical points is

$tr(F\varepsilon) = tr\left(\frac{F\varepsilon + \varepsilon F}{2}\right)$. [redacted] Look at simple case



$$\begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = F \qquad \begin{matrix} tr = 0 \\ det = -1 \end{matrix}$$

$$F\varepsilon = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad tr(F\varepsilon) = 2\cos\theta$$

[redacted] What's the next step?

You have $F_\theta =$ [redacted] $r_{\theta/2}\, \varepsilon\, r_{\theta/2}^{-1} = r_\theta \varepsilon$ $\quad r_\theta =$ rotation thru $\angle\theta$.

this is the ~~fold~~ family of conjugates of ε

Somehow you need ~~[illegible]~~ to understand the Morse theory. Simple case $\mathbb{P}(\mathbb{C}^n) = U(n)/U(1) \times U(n-1)$.

This is the orbit in ~~[illegible]~~ $\mathcal{L}U(n)$ of $e = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right]$. To each $l \subset \mathbb{C}^n$ get projection $e = zz^*$ where $z \in l$, $|z| = 1$.

The Morse fn is $\frac{1}{2} tr(zz^*A)$, where $A = A^*$. Note $tr(zz^*A) = z^*Az$, a well defined fn on $\mathbb{P}(\mathbb{C}^n)$. Critical points? $\Phi = z^*Az + \lambda(1 - z^*z) = z^*(A-\lambda)z + \lambda$

It could be true that the critical points z are ~~[illegible]~~ given by $\forall \delta z \quad \delta z^*(A-\lambda)z = 0$ so

$Az = \lambda z$

$0 = \frac{\partial \Phi}{\partial \bar{z}_i} = \left[(A-\lambda)z\right]_i, \quad \frac{\partial \Phi}{\partial \lambda} = 1 - z^* z$

So what's going on? Let $z_k = x_k + i y_k$

$$\Phi(z, z^*, \lambda) = z^*(A-\lambda)z + \lambda$$
$$= (x - iy)^t (A-\lambda)(x+iy) + \lambda$$

$x - iy$

$$= \sum_{j,k} (x_j - i y_j)\big(\underbrace{a_{jk}}_{a^s_{jk} + i a^{sk}_{jk}} - \lambda \delta_{jk}\big)(x_k + i y_k) + \lambda$$

$$\Phi(z, \lambda) = z^*(A-\lambda)z + \lambda$$

Q: critical points of $z^* B z$ where $B^* = B$?

$$(z + \delta z)^* B (z + \delta z) = z^* B z + \delta z^* B z + z^* B \delta z + \delta z^* B \delta z$$

z such that $v^*(Bz) + (Bz)^* v = 0 \quad \forall v \in \mathbb{C}^n$

$\forall v_1 \in \mathbb{C}:$ $\quad \bar{v}_1 (Bz)_1 + \overline{(Bz)_1}\, v_1 = 0 \quad \therefore (Bz)_1 = 0.$

z critical point $\Leftrightarrow (A-\lambda)z = 0.$

$0 = \partial_\lambda \Phi = 1 - z^* z$

$$\tfrac{1}{2}\operatorname{tr}(F-\varepsilon)^2 = \tfrac{1}{2}\operatorname{tr}(F^2 + \varepsilon^2) - \operatorname{tr}\left(\frac{F\varepsilon + \varepsilon F}{2}\right)$$
$$= \operatorname{tr}\left(1 - \frac{g + g^{-1}}{2}\right) \geq 0$$
$$= \operatorname{tr}\left(1 - \frac{F\varepsilon + \varepsilon F}{2}\right) \geq 0$$
$$= \operatorname{tr}(1 - F\varepsilon) = \operatorname{tr}(1-g) \geq 0$$

> 0 if $\|F - \varepsilon\| < 1$

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$$\begin{bmatrix} b^*c - c^*b & 0 \\ 0 & bc^* - cb^* \end{bmatrix} = \left[\begin{bmatrix} 0 & -c^* \\ c & 0 \end{bmatrix}, \begin{bmatrix} 0 & -b^* \\ b & 0 \end{bmatrix} \right]$$

Take b to be simple, ask what c's give 0. e.g.

let $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\}_{n-1}$ $\quad c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

$$b^*c = c_1 \qquad c^*b = \overline{c_1}$$

$$bc^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} [\overline{c_1} \cdots \overline{c_n}] \qquad cb^* = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} [1\ 0 \cdots 0]$$

$$\begin{bmatrix} \overline{c_1} \cdots \overline{c_n} \\ 0 \end{bmatrix} = \begin{bmatrix} \begin{matrix} c_1 \\ \vdots \\ c_n \end{matrix} & 0 \end{bmatrix} \Rightarrow \begin{matrix} \overline{c_1} = c_1 \\ c_2 \cdots c_n = 0 \end{matrix}$$

You want a better way. You have two vectors [redacted] b, c which yield a herm. pairing $\langle b, c \rangle = b^*c$ and a rank 1 operator cb^*

$$\begin{bmatrix} b^*c & \\ & cb^* \end{bmatrix} \qquad \begin{bmatrix} b^*c & b^* \\ c & cb^* \end{bmatrix}$$

What you really want is a proof that

$$\left. \begin{matrix} b^*c - c^*b = 0 \\ bc^* - cb^* = 0 \end{matrix} \right\} \Longrightarrow c, b \text{ are } \mathbb{R}\text{-[redacted] dep.}$$

$$b^*c = \mathrm{Re}(b^*c) + i\,\mathrm{Im}(b^*c)$$

252 Go back to the Morse function on

$\mathbb{C}P^{n+1} = U(n+1)/U(1)\times U(n)$ symm. space

$$\varepsilon = \left[\begin{array}{c|c} 1 & \\ \hline & -1 \end{array}\right]$$

Let's try once more

Go back to $bc^* - cb^*$.
This operator is skew-adjoint of rank ≤ 2.
Does it have a meaning for $\mathbb{C}P^n$. Recall
that $b, c \in \mathbb{C}^n$ the big open cell.

Review: $$\mathcal{L}U(n+1) = \underbrace{\left\{\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \in \begin{bmatrix} \mathcal{L}U(1) & 0 \\ 0 & \mathcal{L}U(n) \end{bmatrix}\right\}}_{k} \oplus \underbrace{\left\{\begin{bmatrix} 0 & -\bar{b} \\ b & 0 \end{bmatrix}\right\}}_{p}$$

What is your aim? The symmetric
space $\mathbb{C}P^n$ = Grassmannian $U(n+1)/U(1)\times U(n)$.
What happens? Have conj by $\varepsilon = \left[\begin{array}{c|c} 1 & \\ \hline & -1 \end{array}\right]$
Check commutation

$$\begin{bmatrix} 0 & -b^* \\ b & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -c^* \\ c & 0 \end{bmatrix} = \begin{bmatrix} -b^*c & 0 \\ 0 & -bc^* \end{bmatrix}$$

$$-\begin{bmatrix} 0 & -c^* \\ c & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -b^* \\ b & 0 \end{bmatrix} = -\begin{bmatrix} -c^*b & 0 \\ 0 & -cb^* \end{bmatrix}$$

$$\begin{bmatrix} c^*b - b^*c & 0 \\ 0 & cb^* - bc^* \end{bmatrix}$$

Here b, c are column vectors. Probably there's a
curvature interpretation

253 Go to Morse theory proof of periodicity

$U(2n)/U(n)\times U(n) \longrightarrow \Omega(SU(2n);\, 1, -1)$

Consider the group $SU(2n)$. Use Morse theory for the loop space, energy function – critical points are geodesics.

What is a geodesic joining 1 to -1? It's given by an elt X of $\mathcal{L}SU(2n)$ such that $\exp(\pi X) = -1$. ?? A geodesic starting from 1 is a 1-parameter subgroup $\exp(tX)$, $X \in \mathcal{L}SU(2n)$. You can conjugate X into the Cartan subalg of diag

In general if you want geodesics from 1 to a point $g \in SU(2n)$, you diagonalize g

---

Clifford algebras. $\mathrm{Cliff}(\mathbb{R}^n)$ anti commuting generators $s_i^2 = -1$, $1 \le i \le n$.

$\underset{0}{\mathbb{R}},\ \underset{1}{\mathbb{C}},\ \underset{2}{\mathbb{H}},\ \underset{3}{}$ $\qquad \mathbb{H} +$

i, j, k, s

$\mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k \qquad \mathbb{H} \oplus \mathbb{H}s$ can you find a central element., extra special 2 groups, extensions of an elementary 2 group, classification via Quadratic fn. Clifford

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See how much you understand of complex periodicity

$\Omega(SU(2n);\, 1, -1) \qquad \exp(tX) = 1$ at $t = \pi$

$X = \mathrm{diag}(i\lambda_j) \qquad e^{\pi i \lambda_j} = -1 \qquad j = 1, \ldots, 2n$

$\mathrm{diag}\ e^{it\lambda_j} \qquad \lambda_j = \pm 1.$

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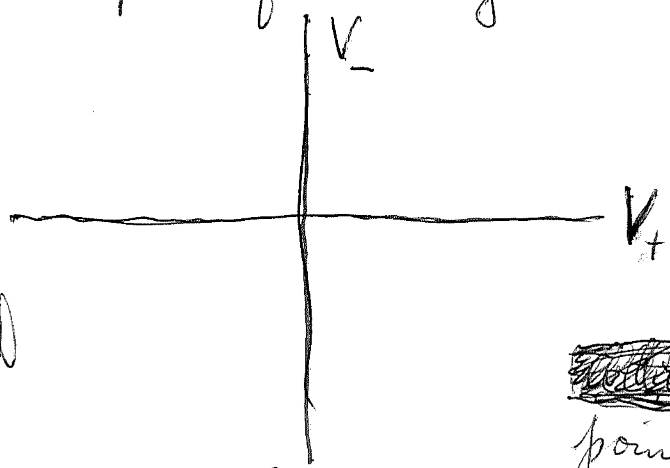
nondeg crit manifold of min. geodesics e^{tX} X has eigenvalues $\pm i$ geodesic e^{tX} $0 \leq t \leq \pi$

$$\det\left(e^{tX}\right) = e^{t\,\mathrm{tr}(X)} = 1$$

Space of min. geodesics is $\underbrace{SU(2n)/U(n)\times U(n)}_{Gr(n,n)}$



You want space of paths in Grass from V_+ to V_-. Critical points are geodesics. How are these described? Probably $\exp(tp)$ where $p \in \mathfrak{p}$ for the symmetric space. $\mathcal{L}U(2n) = \left\{ \begin{bmatrix} a & -b^* \\ b & d \end{bmatrix} ; \begin{matrix} a^* = -a \\ d^* = -d \end{matrix} \right\}$. Put another way: The tangent space to any Grassmannian is $\mathrm{Hom}(V_+, V_-)$. What can you say about geodesics in the Grassmannian? Use ε, F notation: Your Grass has the origin V_+, a tangent vector at the origin is a $b \in \mathrm{Hom}(V_+, V_-)$. To exponentiate this tangent means what? It should involve the symmetric space reflections.

You want to claim that the geodesic in the Grass starting from the origin with tangent vector b (really $X = \begin{bmatrix} 0 & -b^* \\ b & 0 \end{bmatrix}$) is simply related to $\exp(tX)\varepsilon = \exp\left(\frac{t}{2}X\right)\varepsilon \exp\left(\frac{-t}{2}X\right)$.

To understand this better you need the eigenvalue decomposition of X, which is the characteristic value decomposition for b.

$$X = \begin{bmatrix} 0 & -b^* \\ b & 0 \end{bmatrix}, \quad -X^2 = \begin{bmatrix} +b^*b & 0 \\ 0 & +bb^* \end{bmatrix} \text{ and}$$

so $$|X| = \begin{bmatrix} (b^*b)^{1/2} & 0 \\ 0 & (bb^*)^{1/2} \end{bmatrix}, \quad J = \frac{X}{|X|} = \begin{bmatrix} 0 & -b^*(bb^*)^{-1/2} \\ b(b^*b)^{-1/2} & 0 \end{bmatrix}$$

so $J^2 = -1$. You need to assume b invertible

All you've done is the polar decomposition. What's the point? You want $\exp(tX)$ to relate the geodesic in the symm space via $\exp(tX)\varepsilon = \exp(\frac{t}{2}X)\,\varepsilon\,\exp(\frac{-tX}{2})$.

You probably want the eigenvalues of X to be $\pm i$ so that $\exp(\pi X)\varepsilon = -\varepsilon$. The interesting case seems to be where $X^2 = -1$, whence b unitary, and you get $U(n) \hookrightarrow \Omega\,\mathrm{Grass}(\varepsilon, -\varepsilon)$

Try to recall real periodicity. Start with

$SO(2n) = \{ g \in GL(2n, \mathbb{R}) \mid g^t g = 1 \}$

$\mathcal{L}SO(2n) = \{ X \in M_{2n}(\mathbb{R}) \mid X^t + X = 0 \}$, X skew adjoint

Cartan subalg of $\mathcal{L}SO(2)^{\oplus n}$. $\Omega(SO(2n))(1,-1)$?

Instead try the complex picture. $V = \mathbb{C}^n$ $H(V) = \begin{bmatrix} V \\ \hat{V} \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ \xi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ \xi_2 \end{bmatrix} \qquad O(2n, \mathbb{C}) = \{ g \in GL(2n, \mathbb{C}) \mid g^t \overbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}^{S} g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \}$$

$$\mathcal{L}O(2n, \mathbb{C}) = \{ X \in gl(2n, \mathbb{C}) \mid X^t S + SX = 0 \}$$

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad SX^tS = S \begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} S = \begin{bmatrix} d^t & b^t \\ c^t & a^t \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$

So $d = -a^t$, $b^t = -b$, $c^t = -c$ $\quad \left\{ X = \begin{bmatrix} a & b \\ c & -a^t \end{bmatrix} : \begin{matrix} b^t = -b \\ c^t = -c \end{matrix} \right\}$

pos herm. $\begin{bmatrix} x_1 \\ \xi_1 \end{bmatrix}^* \begin{bmatrix} x_2 \\ \xi_2 \end{bmatrix}$ $\qquad U(2n) = \{ g \in Gl(2n, \mathbb{C}) : g^* g = 1 \}$.

$$\mathcal{L}U(2n) = \{ X^* + X = 0 \}.$$

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$$\mathcal{L}O(2n) = \mathcal{L}O(2n,\mathbb{C}) \cap \mathcal{L}U(2n)$$

$$\begin{bmatrix} a & b \\ c & -a^t \end{bmatrix}$$

$$O(2n,\mathbb{C}) = \{ g \in GL(2n,\mathbb{C}) : g^t S g = S \} \qquad S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$n=1$: $\quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(g)^2 = 1 \qquad \det(g) = \pm 1$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}(\pm 1)$$

$$= \begin{bmatrix} a & -c \\ -b & d \end{bmatrix}(\pm 1)$$

If -1 get $a = d = 0$

so $g = \begin{bmatrix} 0 & b \\ b^{-1} & 0 \end{bmatrix}$. If $\det = +1$, get $b = c = 0$ $\quad g = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$

$$\mathcal{L}O(2n,\mathbb{C}) = \{ X \in \mathfrak{gl}(2n,\mathbb{C}) ; X^t S + S X = 0 \}$$

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$

$\left\{ X = \begin{bmatrix} a & b \\ c & -a^t \end{bmatrix} : \begin{array}{l} b^t = -b \\ c^t = -c \end{array} \right\} \qquad \begin{bmatrix} c^t & a^t \\ d^t & b^t \end{bmatrix} + \begin{bmatrix} c & d \\ a & b \end{bmatrix} = 0$

$$U(2n) = \{ g \in GL(2n,\mathbb{C}) : g^* g = 1 \}.$$

$$\mathcal{L}U(2n) = \{ X \in M_{2n}\mathbb{C} : X^* + X = 0 \}$$

$$= \left\{ X = \begin{bmatrix} a & b \\ -b^* & d \end{bmatrix} : \begin{array}{l} a^* = -a \\ d^* = -d \end{array} \right\}. \qquad \begin{array}{l} c = -b^* = \overline{(-b)^t} \\ = \bar{b} \end{array}$$

$$\mathcal{L}SO(2n) = \mathcal{L}SO(2n,\mathbb{C}) \cap \mathcal{L}U(2n)$$

$$= \left\{ X = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} : \begin{array}{l} a^* = -a \\ b^t = -b \end{array} \right\}$$

$$-d = +a^t = +d^*$$
$$a = \bar{d}$$

$$\frac{2n(2n-1)}{2} = n^2 + n^2 - n$$

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3 conditions $X^* + X = 0$, $X^t S + SX = 0$, $SX = \bar{X} S$

Any two $\Rightarrow$ third.

$$\begin{bmatrix} \bar a & \bar b \\ \bar c & \bar d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$$

$$\left\{ X = \begin{bmatrix} a & b \\ \bar b & \bar a \end{bmatrix} : \right.$$

Combine with $X^* = -X$

$a^* = -a \Leftrightarrow a^t = -\bar a$, $\bar b = -b^* \Leftrightarrow b = -b^t$

Now you want to do Morse theory on $SO(2n)$. The question, problem concerns this picture of $SO(2n)$. namely $\left\{ \begin{bmatrix} a & b \\ \bar b & \bar a \end{bmatrix} : \begin{matrix} a^* = -a \\ b^t = -b \end{matrix} \right\}$. Because $SO(2n)$ is a group you can work in the Cartan subalg $a = (\omega_j)$, $1 \le j \le n$. Should be able to assume $b = 0$.

Main case is $X = i\varepsilon \ni X^* + X = 0$, $X^2 = -1$. You get the geodesic $e^{\theta X}$, $0 \le \theta \le \pi$. Centralizer of geodesic = centralizer of X, should be $\left\{ \begin{bmatrix} a & 0 \\ 0 & \bar a \end{bmatrix} : a^* = -a \right\}$

Good viewpt: $SO(2n)/U(n)$

Start again. You're essentially handling the groups

$\Omega(SO(2n); 1, -1) \longleftrightarrow SO(2n)/U(n)$.

Next. $Sp(2n)$, $H(\mathbb{C}^n)$, $\mathcal{L}Sp(2n) = \mathcal{L}U(2n) \cap \mathcal{L}Sp(2n,\mathbb{C})$.

$X \in M_{2n}(\mathbb{C})$: $\quad X + X^* = 0$, $\quad X^t J + JX = 0 \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\bar X + X^t = 0 \qquad -\bar X J + JX = 0$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \overset{?}{=} \begin{bmatrix} \bar a & \bar b \\ \bar c & \bar d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c & d \\ -a & -b \end{bmatrix} = \begin{bmatrix} -\bar b & \bar a \\ -\bar d & \bar c \end{bmatrix} \quad \therefore X = \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix}$$

$a^* = -a$

$-\bar b = -b^*$

$b = b^t$

258 $\quad Sp(2n) = \left\{ X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* = -a \\ b^t = b \end{matrix} \right\}$

geodesics from 1 to -1. Cartan subalg

$\mathfrak{a} = \mathrm{diag}(i\lambda_1, \dots, i\lambda_n)$, again $X = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ $\quad e^{\pi X} = -1$

centralizer of X is $\left\{ \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix} : u \in U(2n) \right\}$. $\quad \therefore \Omega Sp(2n) \sim \frac{Sp(2n)}{U(n)}$

$\mathcal{L}SO(2n) = \mathcal{L}U(2n) \cap \mathcal{L}O(2n, \mathbb{C}) \qquad S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\boxed{X^* + X = 0, \quad X^t S + SX = 0, \quad SX = \bar{X} S}$$

$\bar{X} + X^t = 0 \qquad \bar{X} + X^t = 0 \Rightarrow -\bar{X} S + SX = 0$

$\begin{bmatrix} d & c \\ b & a \end{bmatrix} = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix} \quad \therefore \quad X = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \iff SX = \bar{X} S$

$X^* + X = 0 \iff \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \qquad \begin{matrix} a^* = -a \\ c = -b^* \quad c = \bar{b} \\ \Rightarrow -b^t = b \end{matrix}$

$\mathcal{L}SO(2n) = \left\{ X = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* = -a \\ b^t = -b \end{matrix} \right\}$

$\mathcal{L}Sp(2n) = \mathcal{L}U(2n) \cap \mathcal{L}Sp(2n, \mathbb{C})$

$$\underbrace{\boxed{X^* + X = 0, \quad X^t J + JX = 0, \quad JX = \bar{X} J}}_{X^t = -\bar{X}}$$

$JXJ^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} c & d \\ -a & -b \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix}$

$\mathcal{L}Sp(2n) = \left\{ X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* = -a \\ b^t = b \end{matrix} \right\}$

$b^* = \bar{b}$? $\quad b^t = b$

You now want to understand why

$SX = \bar{X}S$ is a real structure condition

$JX = \bar{X}J$ —— $\mathbb{H}$ ——.

This you did at some point, and you expect it is related to your anti-linear symm + skew-sym maps.

~~Recall~~ $T: V \xrightarrow{b} \hat{V} \xrightarrow{*} V$, $x \mapsto x^t b \mapsto b^*\bar{x}$

~~with square~~ $T(Tx) = T(\overline{bx}) = (\bar{b}b)x$

~~Recall~~ [redacted] $b^* = \overline{b^t} = -\bar{b}$ SO case

$b^* = \overline{b^t} = \bar{b}$ Sp case

$T(Tx) = T(b^*\bar{x}) = b^*(b^*\bar{x})^*$

so $T^2 = \begin{cases} \bar{b}b = -b^*b < 0 & SO \\ \bar{b}b = b^*b > 0 & Sp \end{cases}$ ~~Because~~ $[T, T^2] = 0$

~~T respects the eigenspaces of~~ T^2; ~~eigenvalues of~~ T^2 ~~are~~ > 0 ~~assuming b invertible. On the~~ λ ~~eigenspace of~~ T^2 ~~one~~

$$T: V \xrightarrow{b} \hat{V} \xrightarrow{*} V \qquad T(Tx) = b^*\overline{Tx}$$

$$x \longmapsto x^t b \longmapsto b^*\bar{x} \qquad = b^*\overline{b^*\bar{x}} = (b^*b^t)x$$

$$\therefore T^2 = b^*b^t = \overline{b^t}\,b^t = \bar{b}b \quad \text{in both SO, Sp cases.}$$

$$= \begin{cases} b^*b > 0 & Sp \\ -b^*b < 0 & SO \end{cases} \quad \text{assuming } b^{-1} \exists.$$

Polar decomposition of T. Because $[T, T^2] = 0$, T respects the eigenspaces of T^2, eigenvalues are $\begin{cases} > 0 & Sp \\ < 0 & SO \end{cases}$.

Consider an eigenspace V_λ of T^2 in Sp case (so that $\lambda > 0$). On V_λ you have $b^*b = \lambda$ so $\lambda^{-1/2}b$ is a symmetric unitary operator. On V_λ you have the polar decomp of T: $x \longmapsto \lambda^{-1/2}b^*\bar{x} = \frac{T}{|T|}x$ whose square is ??

260 On V_λ $Tx = b^*\bar{x} = \lambda^{1/2}\lambda^{-1/2}b^*\bar{x} =$
$|T|\cdot|T|^{-1}T(x)$.

Program. To understand the stuff about $\mathbb{R}$ and $\mathbb{H}$ structures. You have

$\mathcal{L}SO(2n) = \{ X \in M_{2n}\mathbb{C}$
$X^* + X = 0, \quad X^tS + SX = 0, \quad SX = \bar{X}S$

$\mathcal{L}Sp(2n) = \{ X \in M_{2n}\mathbb{C} : X^* + X = 0, \ X^tJ + JX = 0, \ JX = \bar{X}J \}$

Meaning of $SX = \bar{X}S$? $X \in End(\mathbb{C}^{2n}) = End\begin{bmatrix} V \\ \hat{V} \end{bmatrix}$, you should see a real structure arising from S on $\begin{bmatrix} V \\ \hat{V} \end{bmatrix}$. A real structure is an anti linear operator with square $= 1$. Ex

$$T\begin{bmatrix} x \\ y \end{bmatrix} = S\sigma\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \bar{y} \\ \bar{x} \end{bmatrix}$$

general case

$$V \longrightarrow \hat{V} \xrightarrow{*} V$$
$$x \longmapsto x^tb \qquad b^*\bar{x}$$

$T(Tx) = b^*\overline{b^*\bar{x}} = \underbrace{b^*b^t}_{\bar{b}^t b^t} x$

half dim $\qquad \bar{b}^t b^t = \bar{b}b$

T anti linear square ± 1

$T\sigma$ linear $\quad T\sigma T\sigma$

Classify real structures. ~~What have~~ Ti:

You need a basepoint. Check for $H(\mathbb{C}^n)$ that the basepoint $\mathbb{R}$ structure is $H(\mathbb{R}^n)$

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[redacted] $H(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} H(\mathbb{C}^n)$, what's the corresponding σ (anti linear $\sigma^2 = 1$). ~~Where~~ where to start?

$$\begin{bmatrix} x_1 \\ \xi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1^t y_2 + \xi_1^t x_2$$

~~explains everything. This splits~~ This formula describes both $H(\mathbb{R}^n)$ and $H(\mathbb{C}^n)$, so the corresponding σ should be $\sigma \begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \bar{\xi} \end{bmatrix}$. You want to twist this somehow.

Consider T anti linear on $H(\mathbb{C}^n)$ preserving the symm form. Then $T\sigma$ is linear and preserves the symplectic form. Q: Is $T\sigma$ a orth gp elt or skew symm operator? it might be both.

Review: $H(\mathbb{C}^n) = \begin{bmatrix} \mathbb{C}^n \\ \mathbb{C}^n \end{bmatrix}$ $\quad \begin{bmatrix} x_1 \\ \xi_1 \end{bmatrix}^t \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{S} \begin{bmatrix} x_2 \\ \xi_2 \end{bmatrix} = x_1^t \xi_2 + \xi_1^t x_2$

$$\begin{bmatrix} x_1 \\ \xi_1 \end{bmatrix}^{\overset{\text{adjoint}}{*}} \begin{bmatrix} x_2 \\ \xi_2 \end{bmatrix} = x_1^* x_2 + \xi_1^* \xi_2 .$$

$$\mathcal{L}SO(2n) = \left\{ X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{array}{l} \text{i) } X^t S + SX = 0 \\ \text{ii) } X^* + X = 0 \\ \text{iii) } SX = \bar{X} S \end{array} \right\}$$

i) says $\begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} = \begin{bmatrix} -d & -b \\ -c & -a \end{bmatrix} \Leftrightarrow c = -b^t,\ d = -a^t$

$X = \begin{bmatrix} a & b \\ -b^t & -a^t \end{bmatrix}$ $\quad$ ii) $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ -b^* & d \end{bmatrix}$ $\quad a^* = -a, \ d^* = -d$

$\left\{ X = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} : \begin{array}{l} a^* = a \\ b^t = -b \end{array} \right\}$ check dims.

262 Review. Aim? To understand the real (resp. $\mathbb{H}$) structure on the basic rep $H(\mathbb{C}^n)$ of $\mathcal{L}SO(2n)$ (resp $Sp(2n)$).

This means that the basic rep of $\mathcal{L}SO(2n)$ (resp $\mathcal{L}Sp(2n)$) commutes with an antilinear operator of square $+1$ (resp -1). So what you need to do is exhibit this operator.

Begin with $\mathcal{L}SO(2n) = \left\{ X = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* = -a \\ b^t = -b \end{matrix} \right\}$

$$X \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ \bar{b}x + \bar{a}y \end{bmatrix} \quad ? \; ?$$

Start with $SXS^{-1} = \bar{X} = \sigma X \sigma^{-1}$

$X = S\sigma X \sigma S = S\sigma X (S\sigma)^{-1}$

Therefore $T = S\sigma$ on $H(\mathbb{C}^n)$ is antilinear $sq = 1$.

Next. $JX = \bar{X}J = \sigma X \sigma J$

$X = J^{-1}\sigma X \sigma J$

So $T = \sigma J$ on $H(\mathbb{C}^n)$ is antilinear $sq = -1$.

$T^2 = \sigma J \sigma J = J^2 = -1$. Things become clearer.

Start with $H(\mathbb{C}^n) = \begin{bmatrix} \mathbb{C}^n \\ \mathbb{C}^n \end{bmatrix}$ $\quad \begin{bmatrix} x_1 \\ \xi_1 \end{bmatrix}^t \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{S} \begin{bmatrix} x_2 \\ \xi_2 \end{bmatrix} = x_1^t \xi_2 + \xi_1^t x_2$

$\sigma \begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \bar{\xi} \end{bmatrix}$ $\quad S = \sigma S \sigma \quad T = S\sigma$ on $H(\mathbb{C}^n)$

is $S\sigma \begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} \bar{\xi} \\ \bar{x} \end{bmatrix}$ $\quad T$ anti-linear $T^2 = S\sigma S\sigma = S^2\sigma^2 = 1$

(263) Real subspace of $H(\mathbb{C}^n) = \left\{ \begin{bmatrix} x \\ \bar{x} \end{bmatrix} \right\}$

~~Next~~. Centralizer of $T = S\sigma$? Confusing

Let's look at symp. case where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ replaces S. Then J, σ commute so get $T = J\sigma = \sigma J$ is an anti-linear transformation of square -1. Thus $H(\mathbb{C}^n)$ becomes a vector space over $\mathbb{H}$ with $\mathbb{C}$ acting usually and T being right mult. by j (?). What is $T = J\sigma$

$$J\sigma \begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{\xi} \end{bmatrix} = \begin{bmatrix} \bar{\xi} \\ -\bar{x} \end{bmatrix}$$

ve

Review $H(\mathbb{C}^n) = H(\mathbb{C})^{\oplus n}$. $H(\mathbb{C}) = \begin{bmatrix} \mathbb{C} \\ \mathbb{C} \end{bmatrix}$

equipped with pos herm $\begin{bmatrix} x_1 \\ \xi_1 \end{bmatrix}^* \begin{bmatrix} x_2 \\ \xi_2 \end{bmatrix}$

skew-symm. $\mathbb{C}$-bil $\begin{bmatrix} x_1 \\ \xi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ \xi_2 \end{bmatrix}$ $\quad \sigma \begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \bar{\xi} \end{bmatrix}$

Maybe more primitive structure is $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

IDEA: $\mathrm{Hom}_{\mathbb{R}}(V, W) = \mathrm{Hom}_{\mathbb{C}}(V, W) \oplus \mathrm{Hom}_{\mathbb{C}}(\bar{V}, W)$

Start again with $H(V) = \begin{bmatrix} V \\ \hat{V} \end{bmatrix}$. At some point you will have to proceed invariantly. For now just focus upon $\mathbb{R}$, $\mathbb{H}$ structures. There are two lines to link:

- $\mathbb{R}$-structure on basic rep of $SO(2n)$
  (resp $\mathbb{H}$- ——— $Sp(2n)$.)
- phase arising in the polar decomposition for the symmetric space $SO(2n)/U(n)$ (resp. $Sp(2n)/U(n)$).

Continue with the review.

Track A: $\mathbb{R}$ (resp $\mathbb{H}$) - structure on basic repn

IDEA: "similarity" between the basic rep of the compact group $SO(2n)$ (resp. $Sp(2n)$), and Tanaaka duality theory for compact Lie gp reps.

Lets start with Track A the $\mathbb{R}$ (resp $\mathbb{H}$) structure on the basic repn. $H(\mathbb{C}^n) = \begin{bmatrix} \mathbb{C}^n \\ \mathbb{C}^n \end{bmatrix}$ equipped with symm $\mathbb{C}$-bilinear form $w_1^t S w_2$, $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and with herm. pos form $w_1^* w_2$, and with complex conjugation $\sigma(w) = \overline{w}$. Here $w_j = \begin{bmatrix} x_j \\ \xi_j \end{bmatrix}$ $j = 1, 2$.

Track B starts with the basic repn of $\mathfrak{L}SO(2n) = \left\{ X = \begin{bmatrix} a & b \\ \overline{b} & \overline{a} \end{bmatrix} ; \begin{matrix} a^* = -a \\ b^t = -b \end{matrix} \right\}$.

Eventually you restrict to the linear space of $\begin{bmatrix} 0 & b \\ \overline{b} & 0 \end{bmatrix}$; $b^t = -b$ with the $U(n)$ action $\begin{bmatrix} u & 0 \\ 0 & \overline{u}^* \end{bmatrix} \begin{bmatrix} 0 & b \\ \overline{b} & 0 \end{bmatrix} \begin{bmatrix} u^* & 0 \\ 0 & \overline{u}^* \end{bmatrix}$

$$= \begin{bmatrix} 0 & ub \\ \overline{u}\overline{b} & 0 \end{bmatrix} \begin{bmatrix} u^* & 0 \\ 0 & \overline{u}^* \end{bmatrix} = \begin{bmatrix} 0 & ub\overline{u}^* \\ \overline{u}\overline{b}u^* & 0 \end{bmatrix}$$

note $\overline{u}^* = u^t$

The action is $\boxed{u * b = ubu^t}$. Question: Did you have a good way to understand this action? NO. You want the picture

$$\begin{bmatrix} u & 0 \\ 0 & \overline{u} \end{bmatrix} \begin{bmatrix} 0 & b \\ \overline{b} & 0 \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & \overline{u} \end{bmatrix}^* = \begin{bmatrix} 0 & ubu^t \\ \overline{u}\overline{b}u^* & 0 \end{bmatrix}$$

because you want to study the polar decomp.

267 The main point is the polar decomp of $\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix}$, maybe also the exp. map applied to this elt of $\mathfrak{p}$. These should depend on the characteristic values of the operator b, that is the eigenvalues of $(b^*b)^{1/2}$ and $(bb^*)^{1/2}$. Assuming b nonsing the char values should be equiv. to polar decomp. The point is that $\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix}$ is skew adjoint because $b^* = \bar{b}^t = -\bar{b}$. So polar decomp is

$$\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ -b^* & 0 \end{bmatrix} \quad \text{and} \quad X = \overbrace{\frac{X}{(-X^2)^{1/2}}}^{J_X} \overbrace{(-X^2)^{1/2}}^{|X|}$$

$$\begin{bmatrix} 0 & b \\ -b^* & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & b \\ -b^* & 0 \end{bmatrix} \begin{bmatrix} (bb^*)^{-1/2} & 0 \\ 0 & (b^*b)^{-1/2} \end{bmatrix}}_{J_X} \cdot \underbrace{\begin{bmatrix} (bb^*)^{1/2} & 0 \\ 0 & (b^*b)^{1/2} \end{bmatrix}}_{|X|}$$

$$\boxed{J_X = \begin{bmatrix} 0 & b(b^*b)^{-1/2} \\ -b^*(bb^*)^{-1/2} & 0 \end{bmatrix}, \quad |X| = \begin{bmatrix} (bb^*)^{1/2} & 0 \\ 0 & (b^*b)^{1/2} \end{bmatrix}}$$

Now you have another approach. Given b : $b^t = -b$ you have with $V = \mathbb{C}^n$

$$\begin{array}{ccccc} V & \xrightarrow{b} & \hat{V} & \xrightarrow{*} & V \\ x & \longmapsto & x^t b & \longmapsto & b^* \bar{x} = \bar{b}^t \bar{x} = -\bar{b}\bar{x} \end{array}$$

so you have $T(x) = -\overline{bx}$ anti linear from V to V

$T(Tx) = -\bar{b}(\overline{-bx}) = +(\bar{b}b)\,x$ | Note $b^*b = \bar{b}^t b = -\bar{b}b$

(26B) $\therefore\ (\bar b b) < 0$ (when $b^{-1}\ \exists$)

So $T^2 = -b^*b$,

and $|T| = (b^*b)^{1/2}$
so you should have

$J = b|T|^{-1}$ satisf. J ??

$$T(x) = (x^t b)^* = b^*\bar x = \bar b^t \bar x = -\bar b x$$

$$T(Tx) = -\bar b\left(\overline{-\bar b x}\right) = (\bar b b)x \qquad \therefore\ T^2 = \bar b b$$

$$b^*b = \bar b^t b = -\bar b b \qquad \therefore\ T^2 = \bar b b = -b^*b$$

$-T^2 = b^*b$, define $|T| = (-T^2)^{1/2}$. So now ?

$$T|T|^{-1} = |T|^{-1}T \overset{\text{def}}{=} j$$ j is anti-linear, $j^2 = -1$?

$$j^2 = |T|^{-1}T^2|T|^{-1} = \frac{1}{|T|^2}T^2 = \frac{1}{-T^2}T^2 = -1$$

Today you propose to straighten out the polar decomposition, $\mathbb{R}$ and $\mathbb{H}$ structures

Review: $\mathcal{L}\,SO(2n) = \left\{X = \begin{bmatrix} a & b \\ \bar b & \bar a \end{bmatrix} : \begin{matrix} a^* = -a \\ b^t = -b \end{matrix}\right\}$. Viewpoint should be via the basic representation $\begin{bmatrix} x \\ \xi \end{bmatrix} \in \begin{bmatrix} \mathbb{C}^n \\ \mathbb{C}^n \end{bmatrix}$. Thus you list structure of the hyperbolic orthogonal space

$$\begin{bmatrix} x_1 \\ \xi_1 \end{bmatrix}^* \begin{bmatrix} x_2 \\ \xi_2 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ \xi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ \xi_2 \end{bmatrix}, \quad S\sigma \begin{bmatrix} x \\ \xi \end{bmatrix} = S\begin{bmatrix} \bar x \\ \bar \xi \end{bmatrix} = \begin{bmatrix} \bar \xi \\ \bar x \end{bmatrix}$$

$$w_1^* w_2 \qquad w_1^t S\, w_2 \qquad \sigma w = \bar w$$

$$X^* + X = 0 \qquad X^t S + SX = 0 \qquad \bar X S = -X^t S = SX$$

267 need better [redacted] understanding of the anti linear operator, tentatively denoted T.

T should be $S\sigma = \sigma S$? Possible choices for T since S, σ have order 2 + commute are $T = \sigma$, $\sigma S = S\sigma$. You have $\bar{X} = \sigma X \sigma$

oo $\quad TXT = S\sigma X \sigma S = S\bar{X}S = X$.

You want any $X = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}$ to commute with

$T = \sigma S$

Now $\mathcal{L}Sp(2n) = \left\{ X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} ;\ \begin{matrix} a^* = -a \\ b^t = b \end{matrix} \right\}$.

$w_1^* w_2$, $\quad w_1^t J w_2$, $\quad \sigma(w) = \bar{w}$ $\qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$X^* + X = 0$, $\quad X^t J + JX = 0$,

$-JX = X^t J = (-\bar{X})J \Rightarrow \boxed{JX = \bar{X}J}$

You want any $X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$ to commute with $T = \sigma J$

$\sigma J \sigma = -J$

$TX = \sigma J X = \sigma \bar{X} J = X \sigma J = XT$

$T = \sigma S \qquad TX = \sigma S X = \sigma \bar{X} S = X \sigma S = XT$

268 Review: $H(\mathbb{C}^n)=\begin{bmatrix}\mathbb{C}^n\\ \mathbb{C}^n\end{bmatrix}$ with

$$\begin{bmatrix}x_1\\ \xi_1\end{bmatrix}^*\begin{bmatrix}x_2\\ \xi_2\end{bmatrix},\quad \begin{bmatrix}x_1\\ \xi_1\end{bmatrix}^t\underbrace{\begin{bmatrix}0&1\\ 1&0\end{bmatrix}}_{S}\begin{bmatrix}x_2\\ \xi_2\end{bmatrix},\quad \sigma\begin{bmatrix}x\\ \xi\end{bmatrix}=\begin{bmatrix}\bar{x}\\ \bar{\xi}\end{bmatrix}$$

$X^*+X=0$, $X^tS+SX=0$, $SX=\bar{X}S$

$T=\sigma S:\begin{bmatrix}x\\ \xi\end{bmatrix}\longmapsto\begin{bmatrix}\bar{\xi}\\ \bar{x}\end{bmatrix}$ anti-linear of square 1.

Claim $SX=\bar{X}S \iff TX=XT$

$$TX=(\sigma S)X=\sigma\bar{X}S=X(\sigma S)=XT$$

Assume $TX=XT$ i.e. $\sigma SX=X\sigma S$

$\sigma SX=X\sigma S \implies SX=\sigma X\sigma S=\bar{X}S.$

Next. $H(\mathbb{C}^n)=\begin{bmatrix}\mathbb{C}^n\\ \mathbb{C}^n\end{bmatrix}$

$$\begin{bmatrix}x_1\\ \xi_1\end{bmatrix}^*\begin{bmatrix}x_2\\ \xi_2\end{bmatrix},\quad \begin{bmatrix}x_1\\ \xi_1\end{bmatrix}^t\underbrace{\begin{bmatrix}0&1\\ -1&0\end{bmatrix}}_{J}\begin{bmatrix}x_2\\ \xi_2\end{bmatrix},\quad \sigma\begin{bmatrix}x\\ \xi\end{bmatrix}=\begin{bmatrix}\bar{x}\\ \bar{\xi}\end{bmatrix}$$

$\sigma J=-J\sigma$, $J^2=-1$

$X^*+X=0$, $X^tJ+JX=0$, $JX=\bar{X}J$

$JX=-X^tJ=\bar{X}J$ ✓. $TX=\sigma JX=\sigma\bar{X}J=XT$

269 So now look at the symmetric spaces $SO(2n)/U(n)$, $Sp(2n)/U(n)$ on the Lie algebra level.
$$\mathcal{L}SO(2n) = \left\{ \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} : \begin{array}{l} a^* = -a \\ b^t = -b \end{array} \right\}.$$

You have $U(n) \hookrightarrow SO(2n)$, $u \longmapsto \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}$ acts by conjugation on $\mathfrak{p} = \left\{ \begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix} : b^t = -b \right\}$.

$$\begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix} \begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix} \begin{bmatrix} u^* & 0 \\ 0 & u^t \end{bmatrix} = \begin{bmatrix} 0 & ubu^t \\ \bar{u}\bar{b}u^* & 0 \end{bmatrix}$$

You want to understand polar decomposition for an elt $\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix} \in \mathfrak{p}$. It's related to (special case of) polar decomposition for an $\overbrace{\text{invertible}}$ skew hermitian operator X: $X = |X|\,\Phi$ where
$$|X| = (X^*X)^{1/2} = (-X^2)^{1/2} \quad \text{and} \quad \Phi = \frac{X}{|X|}.$$

Polar decomp yields phase operators

Φ	anti linear	square -1	$SO(2n)/U(n)$
		$+1$	$Sp(2n)/U(n)$

You are hoping to construct the symmetric space using such phase operators.

270 Back to $\begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ -b^* & 0 \end{bmatrix} \in SO(2n)$

because $b^* = \bar{b}^t = -\bar{b}$. Find $\exp \begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix}$. This should be related to the polar decomposition.

$$X = \begin{bmatrix} 0 & b \\ -b^* & 0 \end{bmatrix} \qquad X^2 = \begin{bmatrix} -bb^* & 0 \\ 0 & -b^*b \end{bmatrix}$$

Try $$\frac{1+X}{(1-X^2)^{1/2}} = \begin{bmatrix} 1 & b \\ -b^* & 1 \end{bmatrix} \begin{bmatrix} (1+bb^*)^{-1/2} & 0 \\ 0 & (1+b^*b)^{-1/2} \end{bmatrix}$$

$\begin{bmatrix} 0 & b \\ -b^* & 0 \end{bmatrix}$ should have a simple ~~eigenvalue~~ spectral decomposition, purely imaginary

$$e^X = 1 + X + \frac{X^2}{2!} + \cdots = \cosh(X) + \frac{\sinh(X)}{X} X$$

$$\frac{1+x}{(1-x^2)^{1/2}} = \underbrace{(1+x)\left(1 + \left(-\tfrac{1}{2}\right)x^2\right.}_{1+x-\frac{1}{2}x^2-\frac{1}{2}x^3} \left. + \left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\frac{x^4}{2!}\right) \quad ?$$

What's your philosophy about the C.T.? You have in the graded case $V = \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$ an $F = F^* = F^{-1}$, and in the ungraded case you have just a unitary g. In the graded case you have a unitary $g = F\varepsilon$ s.t. $\varepsilon g \varepsilon = g^{-1}$. These are compact data, which you want to convert to a time evolution operator X, which should be skew adjoint.

271 To do this perhaps you might use the inverse C.T. $g = \frac{1+X}{1-X}$, or better might be $\frac{s+X}{s-X} = g$, s Laplace Transform variable.

Start where? Aim: To organize the stuff involving C.T., polar decomp., eigenvalues, maybe time evolution for oscillators.

Review some of yesterday's stuff: symmetric spaces $SO(2n)/U(n)$, $Sp(2n)/U(n)$. These should be related respectively to complex structures on the Euclidean space $\mathbb{R}^{2n}$ and to something involving reduction of an $\mathbb{H}$-module to a $\mathbb{C}$-module. Perhaps you take the basic repn $H(\mathbb{C}^n)$ which is an $\mathbb{H}$-vector space and you try to write it $\mathbb{H} \otimes_{\mathbb{C}} V$. In other words you have the hyperbolic functor

$$V \longmapsto \begin{bmatrix} V \\ \hat{V} \end{bmatrix} = \begin{bmatrix} V \\ \bar{V} \end{bmatrix}$$

and you ask for all reductions of the basic repn. of $Sp(2n)$ to $\mathbb{H} \otimes_{\mathbb{C}}$(basic rep of $U(n)$).

So now you have a program.

Begin again studying $SO(2n)/U(n)$ and $Sp(2n)/U(n)$. These are symmetric spaces. How do you understand them? Guess that $SO(2n)$ and $Sp(2n)$ act naturally on the basic representations $\pm H(\mathbb{C}^n)$ and that these symmetric spaces are orbits of some natural operator

272 So you need to understand these symm. spaces. Let's start with $SO(2n)$ acting on the basic repn $\begin{bmatrix} \mathbb{C}^n \\ \mathbb{C}^n \end{bmatrix}$ preserving the three structures

$$\begin{bmatrix} x_1 \\ \xi_1 \end{bmatrix}^* \begin{bmatrix} x_2 \\ \xi_2 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ \xi_1 \end{bmatrix}^t \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{S} \begin{bmatrix} x_2 \\ \xi_2 \end{bmatrix}, \quad \sigma S \begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} \bar{\xi} \\ \bar{x} \end{bmatrix}$$

You are puzzled again by the Lie group $SO(2n)$ as opposed to the Lie algebra: $\mathcal{L}SO(2n) = \left\{ \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* = -a \\ b^t = -b \end{matrix} \right\}$.

Try defining $SO(2n) = \{ g \in GL(2n, \mathbb{C})$ such that

$$g^* g = 1, \quad g^t S g = S, \quad g^{-1} T g = T$$

where T is the antilinear operator $T = \sigma S$ of square $1.\}$

Can you deduce the third condition from the first two?

$$g^{-1} T g = g^* \sigma S g = \sigma g^t S g = \sigma S = T$$

alt. $g^t S g = S \implies \underbrace{\sigma g^t S g}_{g^* \sigma S g \,=\, g^{-1} T g} = \sigma S = T$ YES.

Now $T = \sigma S$ is an anti-linear op on $\mathbb{C}^{2n}$ of square $= 1$, so you have a real structure on $\mathbb{C}^n$ given by $\left\{ \begin{bmatrix} x \\ \bar{x} \end{bmatrix} : x \in \mathbb{C}^n \right\}$.

Next you want an involution on $SO(2n)$ with centralizer $U(n)$. On the Lie alg level

$$\mathcal{L}SO(2n) = \left\{ \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* = -a \\ b^t = -b \end{matrix} \right\}$$

you want the involution given by $\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \longmapsto \begin{bmatrix} a & -b \\ -\bar{b} & \bar{a} \end{bmatrix}$ e.g. conjugation by $\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Better is $i\varepsilon = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ which satisfies

$(i\varepsilon)^* = -i\varepsilon = (i\varepsilon)^{-1}$ $\quad J^* = -J = J^{-1} \Rightarrow -J^2 = 1$

Question. Is $i\varepsilon \in SO(2n)$? $\quad (i\varepsilon)^*(i\varepsilon) = 1 \quad S \,\shortparallel$

$$(i\varepsilon)^t S(i\varepsilon) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

YES!

What does this mean? You have some kind of analog of F, ε in the Grass case. But there is this conjugation, or anti-linear operator that you must handle.

Review what you learned yesterday about the symmetric space $SO(2n)/U(n)$. You begin with $\mathcal{L}SO(2n)$ IDEA: the orbits of the adjoint repn of a compact Lie group G are the flag manifolds associated to G. They have the form ~~[illegible]~~ G/K where K is the centralizer of a torus. In the case of $SO(2n)$, $Sp(2n)$ you get the ~~orthogonal and symplectic~~ varieties of Lagrangian subspaces, where "Lagrangian" means "maximal isotropic".

It seems that you have a description of the symmetric space $SO(2n)/U(n)$ as the space of "polarizations" of the basic representation W. Meaning: a splitting of W into ~~complementary~~ orthogonal Lagrangian subspaces, orthogonal wrt inner product.

~~[illegible]~~ Past IDEA: Recall starting with a $\mathbb{C}$-linear symplectic space W, then choosing a pos. herm form on W, ~~[illegible]~~ an inner product on W. There's ~~Better: Recall starting~~ a compatibility problem it seems.

274 Given W a $\mathbb{C}$-linear symplectic space you can split it into symplectic planes by choosing a nonzero $w_1 \in W$, forming the hyperplane w_1° = ann of w_1 for symp form, next choosing $w_2 \notin w_1^\circ$, in fact can arrange $A(w_1, w_2) = 1$.

$\mathbb{C}w_1 + \mathbb{C}w_2$ whence A is nondeg on and $W = \underbrace{(\mathbb{C}w_1 + \mathbb{C}w_2)}_{P} \oplus P^\circ$

This inductive construction gives a standard form for the symplectic space.

Next arises the compatibility of this symplectic basis with the inner product. You had some idea of using the conjugacy theorem in the negative space $\mathfrak{p} = \left\{ \begin{bmatrix} 0 & b \\ \pm b & 0 \end{bmatrix} \right\}$ in the Lie alg. Recall for $Sp(2n)$ case you want to conjugate b to $b_0 = \begin{bmatrix} 1 & 0 \\ 0 & * \end{bmatrix}$ and for $SO(2n)$ case you want to conjugate b to $b_0 = \begin{bmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & 0 \\ 0 & * \end{bmatrix}$

This approach described seems to establish the "diagonal" maximal abelian subspace of $\mathfrak{p}$.

The key idea should be "polarization", that is, a splitting of the basic repn space $H(\mathbb{C}^n)$ into complementary Lagrangian subspaces which are orthogonal for the inner product.

It should be easy to describe a first order variations of a polarization, say the obvious one $\begin{bmatrix} \mathbb{C}^n \\ \mathbb{C}^n \end{bmatrix} = \begin{bmatrix} V \\ \hat{V} \end{bmatrix}$

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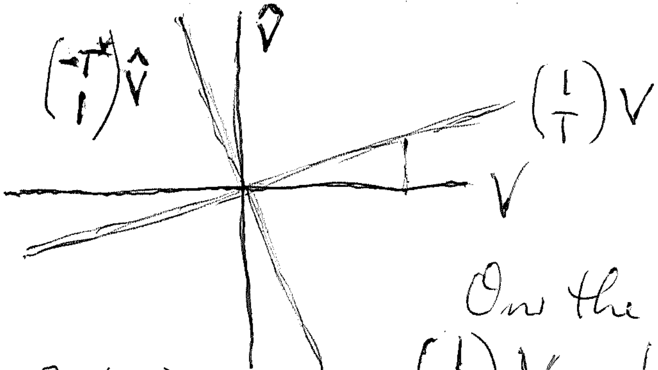

$$X = \begin{bmatrix} 0 & -T^* \\ T & 0 \end{bmatrix}$$

On the other hand you want $\begin{pmatrix}1\\T\end{pmatrix}V$ to be Lagrangian.

In the $SO(2n)$ case this means

$$\left(\begin{pmatrix}1\\T\end{pmatrix}V\right)^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix}1\\T\end{pmatrix}V = 0$$

which should be $\begin{bmatrix} 1 & T^t \end{bmatrix} \begin{bmatrix} T \\ 1 \end{bmatrix} = T + T^t = 0.$

So it seems that [redacted] an infinitesimal variation [redacted] of a polarization $\begin{bmatrix} V \\ \hat V \end{bmatrix}$ [redacted] can be identified with a skew-symmetric $b : V \to \hat V$. It would be better to say that a 1st order variation of a polarization $\begin{bmatrix} V \\ \hat V \end{bmatrix}$ has the form

$$X = \begin{bmatrix} 0 & -b^* \\ b & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} \quad \text{where } b^t = -b.$$

It seems now that you ought to be able to carry over the C.T. theory for the ~~[illegible]~~ symmetric space $Grass(2n, n) = U(2n)/U(n) \times U(n)$ to the symm. spaces $SO(2n)/U(n)$, $Sp(2n)/U(n)$.

How to proceed?

276 Start with the space of creation and annihilation operators, call it W, it has basis $\{a_i, a_j^*, 1 \leq i,j \leq n\}$, an anti-linear auto $*$ of square $+1$ given by $(a_i)^* = a_i^*$, $(a_i^*)^* = a_i$. In the fermion case one has the CAR: $\{a_i, a_j\} = \{a_i^*, a_j^*\} = 0$ ($\forall i,j$) and $\{a_i, a_j^*\} = \delta_{ij}$, so W can be ident. with the hyperbolic orthogonal space $H(\mathbb{C}^n) = \begin{bmatrix}\mathbb{C}^n \\ \mathbb{C}^n\end{bmatrix}$, $\begin{bmatrix}x_1 \\ \xi_1\end{bmatrix}^t \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix} \begin{bmatrix}x_2 \\ \xi_2\end{bmatrix}$

Next? Review yesterday. You need an inner product.

Review: Your approach started with the two forms

$$\begin{bmatrix}x_1 \\ \xi_1\end{bmatrix}^* \begin{bmatrix}x_2 \\ \xi_2\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}x_1 \\ \xi_1\end{bmatrix}^t \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix} \begin{bmatrix}x_2 \\ \xi_2\end{bmatrix}$$

and defined $SO(2n)$ as the set of $g \in GL(2n, \mathbb{C})$ respecting these two forms:

$$g^* g = 1, \qquad g^t S g = S \qquad \text{where } S = \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}$$

Then you get the third condition $Tg = gT$ where $T = \sigma S$ and $\sigma \begin{bmatrix}x \\ \xi\end{bmatrix} = \begin{bmatrix}\bar{x} \\ \bar{\xi}\end{bmatrix}$. Proof: $Tg = \sigma S g$

$Sg = (g^t)^{-1} S = (g^{-1})^t S = (g^*)^t S = \bar{g} S$. $Tg = \sigma S g = \sigma \bar{g} S$ $= g \sigma S = gT$. But $\sigma S \begin{bmatrix}x \\ \xi\end{bmatrix} = \sigma \begin{bmatrix}\xi \\ x\end{bmatrix} = \begin{bmatrix}\bar{\xi} \\ \bar{x}\end{bmatrix}$, which should mean that T on the $\{a_i, a_j^*\}$ is given by the adjoint transformation.

Next: Can you prove that any polarization of the hyperbolic space $H(\mathbb{C}^n)$ is obtained from the basepoint polarization via an elt of $SO(2n)$?

List ideas: polarization is the same as as an element in Lie $SO(2n)$; apply conjugacy thm.

(277) Mimic what happens for $U(2n)/U(n)\times U(n)$.

Actually a polarization is a special type flag, flag varieties are compact, they have cell decomposition, Bruhat decomp. It should be easier to understand polarizations than group elements.

Return to $SO(2n)/U(n) \hookrightarrow U(2n)/U(n)\times U(n)$

A polarization of $H(\mathbb{C}^n)$ should be the same as a Lagrangian subspace, because you get the opposite Lagrangian subspace by applying $*$. Note that the operator picture tells you that applying $*$ preserves isotropic subspaces.

Let's consider a 2n-dim complex v.s. W equipped with nondegenerate $\mathbb{C}$-linear symmetric bilinear form. Show it's hyperbolic. Take $n=1$, pick basis for W, let $x, y \in \mathbb{C}$ be coordinates on W, symm. bil form is determined by the quadratic form $ax^2+2bxy+cy^2$ where $b^2 \neq ac$. You put $y=\lambda x$ to get

$$ax^2+2b\lambda x^2+c\lambda^2x^2 = (a+2b\lambda+c\lambda^2)x^2$$

maybe $x=\lambda y$ $\quad (a\lambda^2+2b\lambda+c)y^2$. So the two distinct roots give two isotropic lines.

This takes care of $a\neq 0$ $\quad b^2\neq ac$

Go thru cases. matrix is $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

assumed invertible $\Rightarrow$ $ac \neq b^2$. If $a\neq 0$ get 2 dist. roots. If $a=0$, then get $\dfrac{-b\pm\sqrt{b^2-0}}{2}$?

Get 2 roots $\lambda = 0$ and $\lambda = b$?.

278 quadratic form $ax^2+2bxy+cy^2$ on $\mathbb{C}^2$

assume non degenerate: $\begin{vmatrix} a & b \\ b & c \end{vmatrix} = ac-b^2 \neq 0.$

If $a \neq 0$, then $\dfrac{x}{y} = \dfrac{-b \pm \sqrt{b^2-ac}}{2a}$ yield 2 distinct isotropic lines

If $a=0$, the quad form is $2bxy+cy^2$ and this vanishes on $y=0$ and on $2bx+cy=0$

$y = -\dfrac{2bx}{c}$, $\dfrac{y}{x} = -\dfrac{2b}{c}$

$\dfrac{y}{x} = -\dfrac{2b}{0}$ means $x=0$. clear.

W 2n-dim$/\mathbb{C}$ + $\mathbb{C}$-bilinear non-deg symm. form. 1st step is to construct an isotropic line

Pick a nonzero vector w_1, by non degeneracy $\exists\ w_2$ such that $S(w_1, w_2) = 1$. This means S is non degenerate on the 2-plane $\mathbb{C}w_1 + \mathbb{C}w_2$

not clear. Start again.

You have W equipped with (non degenerate) symmetric $\mathbb{C}$ bilinear form $S(w, w')$.

The first step is to construct an isotropic vector $w_1 \neq 0$.

279 W complex v.s. with nondeg symm. bilinear form S

Claim there exists an orthogonal splitting of W into nondegenerate lines and hyperbolic 2 planes.

Use induction on $\dim(W)$. ~~Let $w_1 \in W$, $w_1 \neq 0$.~~

~~[illegible]~~ Let $w_1 \in W$ be $\neq 0$.

If $S(w_1, w_1) \neq 0$, then S restricted to $l = \mathbb{C}w_1$ is nondegenerate, so one has an orthogonal decomp: $W = l \oplus l^0$ into nondeg subspaces wrt S. apply induction

If $S(w_1, w_1) = 0$, then $S(w_1, -)$ is a nonzero linear functional by nondeg. of S. Pick w_2 so that $S(w_1, w_2) = 1$. ~~[illegible]~~ $w_2 \notin \mathbb{C}w_1$

S restricted to the 2 plane $\{xw_1 + yw_2 : x, y \in \mathbb{C}\}$ has the form

~~$ax^2 + 2bxy + cy^2$~~

$a = S(w_1, w_1) = 0$
$b = S(w_1, w_2) = 1$
$c = S(w_2, w_2)$.

$$\begin{bmatrix} x \\ y \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$ so you should be able to change variables to make $c = 0$.

$(2x + cy)y$ $\qquad \tilde{w}_2 = w_2 + \lambda w_1$

$$S(\tilde{w}_2, \tilde{w}_2) = S(w_2 + \lambda w_1, w_2 + \lambda w_1)$$
$$= c + 2\lambda + 0 \qquad \therefore \lambda = -\frac{c}{2}$$

$\therefore \tilde{w}_2$ is isotropic.

(280) You would like to link $\mathbb{C}$-bilinear symmetric (resp. skew-symm) forms on a complex inner product space to?

Now what you need is a good review. What's the problem? You've made the step from $\mathcal{L}SO(2n)$ to $SO(2n)$. $SO(2n) = \{ g \in U(2n) : g^* g = 1 ; \ g^t S g = S ;$ any two $\Rightarrow$ third $(\sigma S) g = g (\sigma S)$, i.e. $S g = \bar{g} S \}$

IDEA: Do C.T. $\mathcal{L}SO(2n) \hookrightarrow SO(2n)$. Is there some analog of $F = F^* = F^{-1}$?

Maybe the best approach is to review all the formulas pertaining to the objects you're studying. objects + results. e.g. Anti-linear isos. Remember you want to obtain on any f.d. Hilbert space a spectral decomposition for symmetric (resp skew-symm) $\mathbb{C}$-bilinear forms. This result should follow from the conjugacy theorem in the appropriate symmetric space $\mathcal{L}(SO(2n)/U(n))$, $\mathcal{L}(Sp(2n)/U(n))$.

Let's start somewhere different. real picture of $SO(2n)$. This means changing S - you want to keep the condition $g^* g = 1$, and you want $\sigma g = g \sigma$ equiv. $\bar{g} = g$. From $g^* g = 1$, $\bar{g} = g$ you get $g^t g = 1$ i.e. g is a real orthogonal matrix. On $\mathcal{L}SO(2n)$ level you get $X^* + X = 0$, $\bar{X} = X \Rightarrow X^t + X = 0$ i.e. X is real skew-symmetric.

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Puzzle here. You have lots of choices for the symmetric form S. In the real picture you have $S = I_{2n}$. In the hyperbolic quadratic space $S = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$. Look at $S = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$, $p+q=2n$. Take some simple cases: $n=1$. consider $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Look at $O(2) = \left\{ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in gl_2(\mathbb{R}) : g^t g = 1 \right\}$.

$$g^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = g^{-1} = \frac{1}{\det(g)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \begin{matrix} \det(g)^2 = 1 \\ \det(g) = \pm 1 \end{matrix}$$

~~$\det(g) = 1 \Rightarrow a = d, d = a$~~

$$\det(g) = -1 \implies \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} -d & b \\ c & -a \end{bmatrix} \qquad \begin{matrix} b = c \\ a = -d \end{matrix}$$

and $-1 = ad - bc$, $-1 = -a^2 - b^2$

$$\det(g) = +1 \implies \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad a = d,\ b = -c$$

and $1 = ad - bc$
$= a^2 + b^2$

Confused: You define $O(2)$ via $g^* g = 1$, $g^t g = 1$, $\bar{g} = g$

There's no problem: ~~for $\det(g) = 1$ ...~~

For $\det(g) = 1$ you get $\left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : \begin{matrix} a^2 + b^2 = 1 \\ a, b \in \mathbb{R} \end{matrix} \right\}$.

which is exactly $SO(2)$.

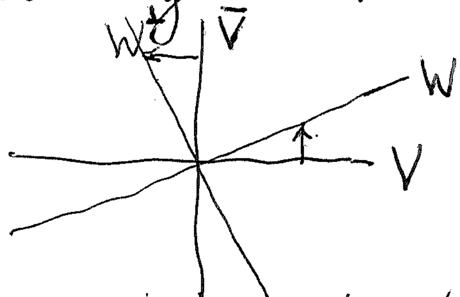
For $\det(g) = -1$ you get $\left\{ \begin{bmatrix} a & b \\ b & -a \end{bmatrix} : \begin{matrix} +a^2 + b^2 = 1 \\ a, b \in \mathbb{R} \end{matrix} \right\}$

which is the space of nontrivial involutions F, orthogonal reflections through any line. Get dihedral gp $\mathbb{Z}/2 \ltimes SO(2)$.

282 Note: $SO(2n)/U(n)$ is a flag manifold, = the space of polarizations of $H(\mathbb{C}^n)$. = the space of Lagrangian subspaces of $H(\mathbb{C}^n)$. General theory says any flag manifold is G/K, K = centralizer of a torus, also G/K = orbit of adj repn of G.

You want, need a description of the minimal flag manifold which agrees with the embedding (?) $SO(2n)/U(n) \hookrightarrow U(2n)/U(n)\times U(n)$.

One way to proceed is to use the linear space theory involved with the C.T. This means that your look for analogs of F, ε. So start with $H(\mathbb{C}^n)$, write it $\begin{bmatrix} V \\ \hat{V} \end{bmatrix} = \begin{bmatrix} V \\ \bar{V} \end{bmatrix}$ and consider another polarization transversal to the basepoint polarization:



You should be able to understand this easily
W = graph of a linear $b : V \longrightarrow \bar{V}$.

Better might be $V \xrightarrow{b} \hat{V} \underset{\sim}{\xrightarrow{*}} V$. Now might be the chance to clarify the nature of these maps. Find a clean statement.

Adopt the canon isom. $\hat{V} \simeq \bar{V}$ arising from the (pos herm) inner product on V. This canonical isom equivalent to an anti linear isomorphism between V and $\hat{V}$. Q: How does an anti-linear isom $V \xrightarrow{\sim} \hat{V}$ arise?

283 Your problems now seem to be notational. You begin with the f.d. cx. Hilb space V. The pos. herm. inner product on V yields an anti-linear isom. $V \longrightarrow \hat{V}$, ~~better~~ an invertible anti-linear transformation $T: V \longrightarrow \hat{V}$.

$$\bar{V} \xrightarrow{\sim} \hat{V} \implies V \xrightarrow{\sim} \hat{\bar{V}}$$

$$V \xrightarrow{b} \hat{V} \xrightarrow{*} V$$

$$x \longmapsto x^t b \longmapsto b^* \bar{x}$$

$$\hat{V} \xrightarrow{*} V$$

$$V \xleftarrow{(*)^t} \hat{V}$$

$$[y_1 \cdots y_n] \xmapsto{*} \begin{bmatrix} \bar{y}_1 \\ \bar{y}_n \end{bmatrix}$$

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_n \end{bmatrix} \xleftarrow{*} [x_1 \cdots\cdots x_n]$$

$$\sum_j y_j \bar{x}_j \qquad \sum_j x_j \bar{y}_j$$

$$\begin{tikzcd} \hat{V} \arrow[r, "*"] & V \arrow[d, "*"] \\ V \arrow[u, "*"] & \hat{V} \arrow[l, "*"] \arrow[ul, phantom, "="] \end{tikzcd}$$

$$V \xrightarrow{b} \hat{V} \xrightarrow{*} V$$

$$x \longmapsto x^t b \longmapsto b^* \bar{x}$$

Something else was

$$\hat{V} \xrightarrow{c} V$$

$$[x_1 \cdots x_n] \longmapsto \sum \qquad \sum_i x_i c_{ij}$$

maybe you should be using tensor notation

[redacted] V v.s. with pos herm inner product

$\exists$ canonical invertible anti-linear transf. $V \xrightarrow{T} \hat{V}$

$$x \longmapsto (x, -)$$

T should be "hermitian symmetric"

$$V \xrightarrow{T^t} \hat{V}$$

IDEA

Tannaka duality

creation & annihilation operator formalism, you have a hyperbolic linear space with basis consisting of these operators: $a_1, \dots, a_n$ and $a_1^*, \dots, a_n^*$, so it's a $\mathbb{C}$ vector space of dim $2n$ with conjugation operator $*$ and hyperbolic symmetric bilinear form given by $\{a_i, a_j^*\} = \delta_{ij}$, $\{a_i, a_j\} = \{a_i^*, a_j^*\} = 0$.

So you have this basic space $H(\mathbb{C}^n)$ of 1st order operators together with the hyperbolic quadratic form. 1st order operator refers to Clifford algebra.

Next comes "polarization" of $H(\mathbb{C}^n)$, equivalently Lagrangian subspace.

Idea from the past: A polarization is an n-diml isotropic subspace $W \subset V \oplus \bar{V}$. In operator terms W isotropic means $\{W, W\} = 0$. In the bosonic situation this becomes $[W, W] = 0$. There's also a positivity condition $[w, w^*] > 0$ for $w \neq 0$.

Always the difficulty seems to be the identification $V \longrightarrow \hat{V}$. How to set things up. ~~Start~~ with V equipped with pos herm form x^*y anti-linear in x, linear in y. Have antilinear map $V \longrightarrow \hat{V}$, $x \longmapsto (y \mapsto x^*y)$ which is invertible. Then form $H(V) = \begin{bmatrix} V \\ \bar{V} \end{bmatrix}$

$$\begin{bmatrix} x \\ \xi^* \end{bmatrix}^* \begin{bmatrix} y \\ \eta^* \end{bmatrix} = x^*y + \eta^*\xi$$

$$\begin{bmatrix} x \\ \xi^* \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ \eta^* \end{bmatrix} = x^t\eta^* + \xi^* y$$

(285) Maybe you should avoid row + column vectors. Start w. V, ~~and~~ $\hat{V}$, and a pos hermitian form (x,y) | anti in x, linear in y. Then you have canonical invertible anti-linear map $V \xrightarrow{\tau} \hat{V}$, $x \longmapsto (y \longmapsto (x,y))$. Use $x \longmapsto x^\dagger$ for this map maybe. Ask about

$$V = (\hat{V})^\wedge \xrightarrow{\tau^t} \hat{V} \qquad \text{can hope that } \tau^t = \tau$$

Start again with V, (x,y) antilinear in x, linear in y

$$\tau : V \longrightarrow \hat{V} \qquad \tau \text{ is antilinear}$$
$$x \longmapsto \{y \longmapsto (x,y)\}$$

Take transpose of τ

$$\underset{\text{canon}}{V = (\hat{V})^\wedge} \xrightarrow{\tau^t} \hat{V} \qquad \tau^t \text{ is antilinear}$$

As (x,y) is herm symm, it should be true that $\tau^t = \tau$.

Perhaps easier to work with Euclidean spaces E and complex structures? A complex structure is an operator J on E s.t. $-J = J^t = J^{-1}$. How can you use this?

Let's go back to the study of a symmetric bilinear form on V, say nondegenerate, where V is equipped with an inner product.

Better would be to study Lagrangian subspaces of $H(V) = \begin{bmatrix} V \\ \hat{V} \end{bmatrix}$ equipped with the hyperbolic symm. form $\begin{bmatrix} x_1 \\ \xi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ \xi_2 \end{bmatrix} = x_1^t \xi_2 + \xi_1^t x_2$

(286) What does this notation mean? $x_1 \in V$ so x_1 can be identified with the map $\mathbb{C} \longrightarrow V$, $\lambda \mapsto x\lambda$

$\xi_1 \in \hat{V}$ so ξ_1 cbiw the map $\mathbb{C} \xrightarrow{\xi_1} \hat{V}$, which ~~[illegible]~~ in turn cbiw ——— $V = (\hat{V})^\wedge \xrightarrow{\xi_1^t} \hat{\mathbb{C}} = \mathbb{C}$

$x \in V$, $\mathbb{C} \xrightarrow{\cdot x} V$, $\hat{V} \xrightarrow{x^t} \hat{\mathbb{C}} = \mathbb{C}$??

$\lambda \longmapsto x\lambda$

$x \in V$ same as the map $\mathbb{C} \xrightarrow{x} V$, $\lambda \mapsto x\lambda$

$\hat{V} \xrightarrow{x^t} \mathbb{C}$, $\xi \longmapsto \xi x$

How to make sense of this??

$$H(V) = \begin{bmatrix} V \\ \hat{V} \end{bmatrix} \qquad S\left(\begin{bmatrix} x_1 \\ \xi_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ \xi_2 \end{bmatrix}\right) = (x_1, \xi_2) + (\xi_1, x_2)$$

$$\begin{bmatrix} x_1 \\ \xi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ \xi_2 \end{bmatrix} = x_1^t \xi_2 + \xi_1^t x_2$$

$H(V) = \begin{bmatrix} V \\ \hat{V} \end{bmatrix}$ equipped with $\begin{bmatrix} x \\ \xi \end{bmatrix}^* \begin{bmatrix} y \\ \eta \end{bmatrix} = x^* y + \xi^* \eta$

and $\begin{bmatrix} x \\ \xi \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ \eta \end{bmatrix} = x^t \eta + \xi^t y = (x, \eta) + (\xi, y)$

Next you want R-structure on $H(V)$, which is a consequence of ~~[illegible]~~ the forms $\omega_1^* \omega_2$, $\omega_1^t S \omega_2$

~~[illegible]~~ The difference between ~~of~~ $\omega_1^* \omega_2$, $\omega_1^t S \omega_2$ should be σS

~~2625~~ ~~11750~~ ~~date of completion~~ ~~insurance~~ ~~[illegible] signature~~

(287) Consider a bilinear form

$$\tilde{T}(v, w), \quad v \in V,\ w \in W$$

Then $\tilde{T}$ is equivalent to the linear operator $T : V \longrightarrow W^t$ defd by $\tilde{T}(v, w) = \langle Tv, w\rangle$ where $\langle \omega, w\rangle$ denotes the canonical pairing between W^t and W.

Associated to T is the transpose operator

$$(W^t)^t \xrightarrow{\ T^t\ } V^t \qquad T^t(\mu) = \mu \cdot T \quad \forall \mu \in (W^t)^t.$$

Assuming our vector spaces fin. diml, we have a canon. isom. $W \xrightarrow{\sim} (W^t)^t$ given by

$$\underset{\substack{\cap \\ W}}{w} \longmapsto \underbrace{\Big\{ \underset{\substack{\cap \\ W^t}}{\omega} \longmapsto \langle \omega, w\rangle \Big\}}_{\in (W^t)^t}$$

Now apply T^t to $\{\omega \mapsto \langle \omega, w\rangle\}$, which means applying this linear fnl to $\omega = Tv$. This yields the map

$$W \xrightarrow{\ T^t\ } V^t, \quad w \longmapsto \langle Tv, w\rangle$$

~~This seems strange — both T and T^t yield the same bilinear form $\langle Tv, w\rangle$ on $V \times W$~~

The corresponding bilinear form is

$$\widetilde{T^t}(w, v) = \langle T^t w, v\rangle$$

$$\boxed{\therefore\ \langle Tv, w\rangle = \langle T^t w, v\rangle}$$

Let's go back to Cayley Transform.

What is the rough idea? There are objects which seem to be related.

Let's start with V equipped with positive hermitian form $\langle x|y\rangle$. and with a symmetric bilinear form $S(x,y)$. There should be a spectral theory for S.

V equipped with a pos. herm. form $\langle x|y\rangle = x^* y$ and a symm. bilinear form

$$\tilde{S}(x,y) = (Sx)^t y = x^t S y$$

Can you find a variational problem yielding the spectral theory desired for S?

First case: Go back to harmonic oscillator situation. Newton: $m\ddot{x} = -kx$, results from $L(x,\dot{x}) = \frac{1}{2}\dot{x}^t m \dot{x} - \frac{1}{2}x^t k x$ + Lagrange DE

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0 \quad \text{Lagrange}$$

$$\frac{\partial L}{\partial x} = -kx \qquad\qquad \dot{p} + kx = 0$$

Hamilton: $H = \frac{1}{2}p^t m^{-1} p + \frac{1}{2}x^t k x$ Hamiltonian

$$\dot{x} = \frac{\partial H}{\partial p} = m^{-1}p.$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -kx$$

Idea: Similarity between $H(V) = \begin{bmatrix} V \\ V^t \end{bmatrix}$ and phase space.

(289) [redacted] Next is [redacted] to solve the equations of motion, i.e. to find the flow on configuration (resp phase) spaces. Take phase space:

$$\begin{bmatrix}\dot{x}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0 & m^{-1}\\ -k & 0\end{bmatrix}\begin{bmatrix}x\\ p\end{bmatrix}$$

But ~~the~~ Hamilton's theory yields an interpretation of this flow, namely $AX = H$ where A is the symplectic form on phase space

$$\begin{bmatrix}x_1\\ p_1\end{bmatrix}^t A \begin{bmatrix}x_2\\ p_2\end{bmatrix} = \begin{bmatrix}x_1\\ p_1\end{bmatrix}^t \begin{bmatrix}0 & -1\\ +1 & 0\end{bmatrix}\begin{bmatrix}x_2\\ p_2\end{bmatrix} = x_1^t p_2 - p_1^t x_2$$

and H is the positive symmetric form

$$\begin{bmatrix}x_1\\ p_1\end{bmatrix}^t H \begin{bmatrix}x_2\\ p_2\end{bmatrix} = \begin{bmatrix}x_1\\ p_1\end{bmatrix}^t \begin{bmatrix}k & 0\\ 0 & m^{-1}\end{bmatrix}\begin{bmatrix}x_2\\ p_2\end{bmatrix} = x_1^t k x_2 + p_1^t m^{-1} p_2$$

$$X = A^{-1}H = \begin{bmatrix}0 & +1\\ -1 & 0\end{bmatrix}\begin{bmatrix}k & 0\\ 0 & m^{-1}\end{bmatrix} = \begin{bmatrix}0 & +m^{-1}\\ -k & 0\end{bmatrix}$$

Recall why the infinitesimal time translation X preserves A and H: $X^tA + AX = (-AX)^t + AX = -H^t + H = 0$. $0 = X^tAX + AXX = X^tH + HX$.

Recall that [redacted] the point of interest at the moment is [redacted] taking the difference (multiplicatively) $A^{-1}H$. You have other examples you want to understand better. You forgot to mention that A, H are bilinear forms on phase space so that assuming nondegeneracy, the difference

$$X = A^{-1}H$$ is an operator on phase space.

List possible things to do. Look at Hamilton's phase space picture for a harmonic oscillator. So you have W a $2n$-diml $\mathbb{R}$ vector space equipped with symplectic form A, and you have a positive symmetric form H. The time evolution (dynamics) is given by $X = A^{-1}H$. You know that X is skew-symmetric: $X^t H + HX = 0$ and nonsingular. There's a spectral theory for skew-symmetric operators X on a Euclidean space which splits X into orthogonal 2 diml rotations. The usual way to view this intrinsically is via the polar decomp $X = |X| J$, where

$$|X| = (X^* X)^{1/2} = (-X^2)^{1/2} \quad \text{and} \quad J = \frac{X}{|X|}.$$

The pos. self adjoint operator $|X|$ has for its eigenvalues the frequencies $\omega > 0$ for the oscillator. J is a complex structure on phase space: $J^* = -J = J^{-1}$. You make phase space into a complex ~~vector~~ Hilbert space by defining mult by i to be given by J. Then $|X|$ is a positive hermitian operator on phase space giving the multiplicities for the different frequencies.

(291) Let (W, A) be a real symplectic vector space, and let H be a symmetric bilinear form on W. You know that H can be identified with an "infinitesimal symplectic transformation X on W. Is $X = A^{-1}H$?

This should be true — a proof using a polarization of (W, A) i.e. isom $W \simeq \begin{bmatrix} V \\ V^t \end{bmatrix}$, making W a phase space, should be easy.

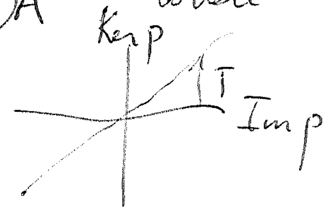
In the preceding you reviewed the real phase space + symm. form H. You have this very simple picture of a real $2n$-diml v.s. with non deg symplectic form A and positive symmetric form H.

Next you want to understand complex examples, especially those **arising** from $Sp(2n)$ and $SO(2n)$, better might be $Sp(2n)/U(n)$ and $SO(2n)/U(n)$.

Digress: Return to the harmonic oscillator with symplectic phase space of dim $2n$ and H pos definite. All this is real, but the polar decomposition of $X = A^{-1}H$ yields a complex structure J on phase-space. Then it seems that J together with A enable one to define a positive hermitian form on phase-space with imaginary part A.

292 Try today to get the spectral theory of a symmetric bilinear form on a Hilbert space one eigenvalue at a time in increasing order. The idea: for each line l in V you restrict S to l and take the corresponding invariant $\lambda \geq 0$ in the case of rank 1. Then you minimize $\lambda(l)$ for $l \in \mathbb{P}V$.

How does this compare to the case of a hermitian operator A on V? Identify l with a rank 1 hermitian projection p, the functional is $tr(pA)$. $\delta\, tr(pA) = tr\,(\delta p)A$ where δp ~~can be~~ is a tangent to $\mathbb{P}V$ at p.



$$\delta p = \begin{bmatrix} 0 & +T^* \\ T & 0 \end{bmatrix}$$

$$tr \begin{bmatrix} 0 & T^* \\ T & 0 \end{bmatrix} \begin{bmatrix} pAp & pAp' \\ p'Ap & p'Ap' \end{bmatrix} = tr\left(T pAp' + T^* p'Ap \right)$$

$= 0 \quad \forall T \iff \quad p'Ap = 0$ which means $Al \subset l$

Try this for the symmetric bilinear form cases. Things are different because S is not an operator, so you expect to square S in some way. This should occur in the form $\begin{bmatrix} 0 & S \\ \bar{S} & 0 \end{bmatrix}$ roughly.

[redacted] You want to restrict $S : V \longrightarrow V^t$ to $l \subset V$, which means the composition $l \hookrightarrow V \xrightarrow{S} \bar{V} \longrightarrow \bar{l}$. [redacted]

Let's review how on a complex Hilbert space V a symmetric bilinear form S yields an antilinear transformation. Choose an orthonormal basis for V and identify $V = \mathbb{C}^n$ the space of column vectors. Then V^t can be identified with $\mathbb{C}^n$, the space of row vectors, and the canonical pairing of $x \in V$, $\xi \in V^t$ is $x^t \xi = \xi^t x$, matrix multiplication. The symmetric bilinear form S is given by $S(x,y) = (Sx)^t y$, which is equal to $x^t S^t y = x^t S y$ by symmetry of S.

So far you have the map

$$\begin{aligned} V &\xrightarrow{S} V^t \\ x &\longmapsto (Sx)^t = x^t S \end{aligned}$$

Next you bring in the canonical anti-linear invertible transformation $V^t \overset{*}{\underset{\sim}{\longleftrightarrow}} \overline{V}$ which expresses elements of the dual V^t using the pos. herm. inner product: A linear functional on V is uniquely represented as $x \longmapsto y^* x$, so that $y \longmapsto y^*$ is an invertible transformation from V to V^t, which is anti-linear, resulting in $\mathbb{C}$-linear isomorphisms $\overline{V} \xrightarrow{\sim} V^t$ or $V \xrightarrow{\sim} V^*$, $V^* =$ conjugate dual.

Consider now the composition

$$\begin{aligned} T: \quad V &\xrightarrow{S} V^t \xrightarrow{*} \overline{V} \\ x &\longmapsto (Sx)^t \longmapsto \left((Sx)^{\cancel{t}}\right)^* = \overline{Sx} \end{aligned}$$

T is antilinear, $T(Tx) = T(\overline{Sx}) = \overline{S\,\overline{Sx}} = (\overline{S}S)x$

294

Review. You have the combination of $S : V \to V^t$ and $* : V^t \xrightarrow{\sim} V$,

$$x \mapsto (Sx)^t \longmapsto ((Sx)^t)^* = \overline{Sx} \qquad Tx = \overline{Sx}$$

$$T(Tx) = \overline{S(Tx)} = \overline{S(\overline{Sx})} = (\bar{S}S)x.$$

Next you want to take a line $l \subset V$ and restrict T to this line. Meaning:

$$l \subset V \xrightarrow{S} V^t \longrightarrow l^t \xrightarrow{*} \bar{l}$$

Let $x, y \in l$

$$x \mapsto x \longmapsto (Sx)^t \longmapsto (Sx)^t \longmapsto \overline{Sx}.$$

This seems right but there a lot to check.

Basically you have $V \xrightarrow{S} V^t \xrightarrow{*} V$, whose composition is an anti-linear operator on V. If $L \subset V$ then you have the restriction of [redacted] the bilinear form S to the subspace L.

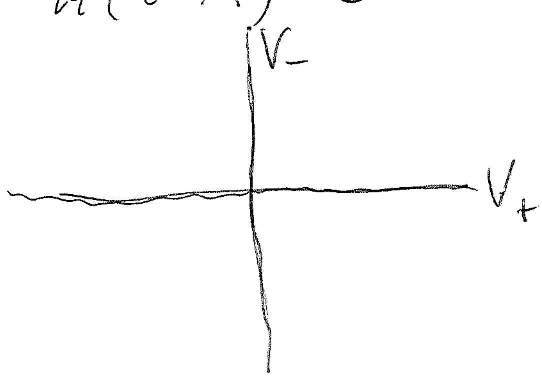
$$\begin{tikzcd} V \arrow[r, "S"] & V^t \arrow[r, "*"] \arrow[d, "i^t"'] & V \arrow[d, "i^*"] \\ L \arrow[u, "i"] \arrow[r, "S_L"] & L^t \arrow[r, "*"] & L \end{tikzcd}$$

The projection (orthogonal) operator with image L is ii^*.

Thus the restriction T_L of T should be $T_L = i^*(*S)i = i^*Ti$. Next square to get i^*TpTi.

This is too confused, but it seems that the functional you want is something like $tr(T_{\bar{p}}T_p)$.

V pos herm. A hermitian op. Function on the full Grass: $tr(FA)$. What are its critical points $tr(\delta F A) = 0$? When is $\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ critical.



$$F^2 = 1 \implies \delta F F + F \delta F = 0$$

so δF can be any hermitian operator on V anti-commuting with F

$$\delta F = \begin{bmatrix} 0 & T^* \\ T & 0 \end{bmatrix} \qquad A = \begin{bmatrix} p_+ \\ p_- \end{bmatrix} A \begin{bmatrix} p_+ & p_- \end{bmatrix}$$

295 [redacted] $\delta \underset{\wedge}{\overset{A}{F}} = \begin{bmatrix} 0 & T^* \\ T & 0 \end{bmatrix} \begin{bmatrix} p_+Ap_+ & p_+Ap_- \\ p_-Ap_+ & p_-Ap_- \end{bmatrix}$

$= \begin{bmatrix} T^*p_-Ap_+ & T^*p_-Ap_- \\ Tp_+Ap_+ & Tp_+Ap_- \end{bmatrix}$ $\quad tr(\delta F A) = tr(T^*p_-Ap_+ + Tp_+Ap_-)$

zero $\forall T$ means $p_-Ap_+ = p_+Ap_- = 0$

$\Rightarrow AV_+ \subset V_+$, $AV_- \subset V_-$. Clear

[redacted] Next you want to handle [redacted] a symmetric bilinear form: $T : V \xrightarrow{S} V^t \xrightarrow[\sim]{*} V$, $Tx = \bar{S}\bar{x}$, $T^2x = (\bar{S}S)x$

You want ~~[illegible]~~ a spectral theory for the anti-linear transf. T. It's sort of clear that you have to square T somehow.

What you want is to unify various ideas.

- Spectral theory for $\mathcal{L}\{Sp(2n)/U(n)\}$ (*)
- [redacted] Symmetric bilinear forms / unitary equiv.
- Conjugacy theorem for (*)
- Flag [redacted] manifolds

Start [redacted] with an analog of $\overset{(F\mapsto)}{tr(FA)}$, where F [redacted] varies over Grass(V). Instead of A? There seem to be two possibilities:

$$\begin{tikzcd} V \arrow[r, "T"] & V \arrow[r, "T"] \arrow[d, shift right, "\iota^*"'] & V \arrow[d, "\iota^*"] \\ L \arrow[u, "\iota"] & L \arrow[u, shift right, "\iota"'] & L \end{tikzcd}$$

either $\iota^* T^2 \iota$ or $(\iota^* T \iota)^2$.

$tr(pT^2)$ $\qquad$ $tr(Tp)^2$

296 You want to extend the critical point argument for $\mathrm{tr}(pA)$ on a Grassmannian to the cases where A is replaced by [redacted] a symmetric bilinear form S. Recall you have the antilinear operator $T = {*S} : V \longrightarrow V$ s.t. $Tx = \bar{S}\bar{x}$, $T^2x = (\bar{S}S)x$. Let L be a subspace of V [redacted] and let p be the corresp orthogonal projector. $\therefore$ $p = jj^*$ where $j : L \longrightarrow V$ is the inclusion. [redacted] Consider:

$$\begin{tikzcd}
V \arrow[r, "T"] & V \arrow[r, "T"] \arrow[d, shift right, "j^*"'] & V \arrow[d, "j^*"] \\
L \arrow[u, "j"] & L \arrow[u, shift right, "j"'] & L
\end{tikzcd}$$

You can restrict T to get the anti-linear operator j^*Tj. [redacted] There doesn't seem to be an interesting trace for an anti linear operator. If you take the [redacted] trace over $\mathbb{R}$, then because T and J = mult. by i anti-commute, you get $\mathrm{tr}_{\mathbb{R}} T = \mathrm{tr}_{\mathbb{R}} JTJ^{-1} = -\mathrm{tr}_{\mathbb{R}} T$. So you need to square T and restrict to L, or restrict to L and then square to get an interesting trace. This gives two possibilities

$$\mathrm{tr}(j^*T^2j) = \mathrm{tr}(T^2p) \quad , \quad \mathrm{tr}(j^*Tj)^2 = \mathrm{tr}(TpTp)$$

The former is $\mathrm{tr}(\bar{S}S)p$ where $\bar{S}S = S^*S$ is non-negative [redacted] hermitian, so its critical points are the invariant subspaces for $T^2 = \bar{S}S$.

Next let's study the difference $\mathrm{tr}(T[T,p]p)$ $= \mathrm{tr}(T^2p^2 - TpTp)$.

297 $tr(j^*T^2j) = tr(T^2p) = tr(pT^2p)$

$tr(j^*Tj)^2 = tr(j^*TpTj) = tr(T_pT_p) = tr(pT_pT_p)$

Let $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ relative to $V = \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$

$$pTTp = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} a^2+bc & 0 \\ 0 & 0 \end{bmatrix}$$

$$pT_pT_p = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ 0 & 0 \end{bmatrix}$$

So $tr(pT^2p) = tr(a^2+bc)$ of course $pTp = a$

$tr(pT_pT_p) = tr(a^2)$

~~Or~~ primordial elt. W subspace of ~~[illegible]~~ F^n

w is primodial when

Go thru the argument. You have a Galois field extension $F \subset E$ and a subspace W of E^n. Make ~~[illegible]~~ $Gal(E/F)$ act on E^n componentwise, Assume $\forall\sigma$ $\sigma(W) \subset W$. You want to show that W is spanned by the F vector space W^G

Idea. Take $w \in W^G$ ~~Enough to~~

~~write w — Assume primordial, which means that~~

Look at the ~~nonzero~~ support of w, i.e. if $w = (c_1, \dots, c_n) \in E^n$, $Supp(w) = \{j \mid c_j \neq 0\} \subset \{1, \dots, n\}$

This support ~~do~~ is unchanged by the Galois action.

Suppose one coeff $= 1$. Then $w - \sigma w$ has smaller support which implies $w = \sigma w$

298 $W \subset E^n$ any $w = (c_1, \ldots, c_n)$ has a support $\mathrm{Supp}(w) = \{ j : 1 \leq j \leq n \text{ and } c_j \neq 0 \}$.
w is primordial when

$$\forall w' \in W \quad \mathrm{Supp}(w') < \mathrm{Supp}(w) \Rightarrow w' = 0.$$

You want to show that W is spanned by its primodial elements. Let $0 \neq w \in W$. If w primordial, done. If not $\exists\, 0 \neq w' \in W$ such that $\mathrm{Supp}(w') < \mathrm{Supp}(w)$. Then $\exists c$ such that $\mathrm{Supp}(w - cw') < \mathrm{Supp}(w)$, so w is the sum $w = w' + (w - cw')$, so you should be able to proceed by induction on card Supp

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So what comes next? Let T be an anti-linear operator from V to itself. Is there some sort of matrix you can attach to T? NO.

Better question: If X, Y are anti linear transfs

$$V \xrightarrow{X} W \xrightarrow{Y} V \xrightarrow{X} W \quad \text{is} \quad tr_V(YX) = tr_W(XY)?$$

Form

$$V \xrightarrow{X} \bar{W} \xrightarrow{Y} V \xrightarrow{X} \bar{W} \Rightarrow tr_V(YX) = tr_{\bar{W}}(XY)$$

so the question becomes whether for $S : V \longrightarrow V$ $\mathbb{C}$-linear you have $tr_V(S) = tr_{\bar{V}} S$. Choose orth basis for V. S

(299) How to organize all this anti linear operator ~~stuff~~? First describe anti-linear maps $\mathbb{C}^m \longrightarrow \mathbb{C}^n$. Probably such a map should be viewed as a bilinear form (?).

$$\cancel{X} : \mathbb{C}^m \longrightarrow \mathbb{C}^n \qquad \cancel{X}(e_k) = \sum_j e_j \cancel{x}_{jk}$$

$$\cancel{X}\Big(\sum_k e_k x_k\Big) = \sum_k \Big(\sum_j e_j \cancel{x}_{jk}\Big)\bar{x}_k$$

Start again: An anti-linear $X : \mathbb{C}^m \longrightarrow \mathbb{C}^n$

Review the problem. Example. Spectral theory for a hermitian operator A on a Hilb space V. Introduce a function on the Grass of subspaces $F \mapsto tr(FA)$. Smooth real valued fn on compact space so critical points $\exists$. ~~Suppose F critical~~ Actual Grass described by $F^2 = 1$, smooth submanifold of all hermitian operators on V. Tangent space at F is space of hermitian ops anti-comm with F. Let F be a critical point $V = \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$ $F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$A = \begin{bmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{bmatrix} \qquad \delta F = \begin{bmatrix} 0 & X^* \\ X & 0 \end{bmatrix}$$

$$\cancel{tr}\ \delta F A = \begin{bmatrix} 0 & X^* \\ X & 0 \end{bmatrix}\begin{bmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{bmatrix} = \begin{bmatrix} X^*A_{-+} & X^*A_{--} \\ XA_{++} & XA_{+-} \end{bmatrix}$$

$$tr(\delta F A) = tr\left(X^*A_{-+} + XA_{+-}\right) = 0 \qquad \forall X$$

~~take~~ $X = A_{-+}$ use $X^* = A_{+-}$

$$tr\Big((A_{-+})^*A_{-+} + A_{-+}\underbrace{A_{+-}}_{(A_{-+})^*}\Big)$$

Simpler $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$tr(\delta F A) =$
$tr(X^*c + Xb)$
take $b = X^* \Rightarrow c = X$
$tr(X^*X + XX^*)$.

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So F critical $\Rightarrow$ b and $c = 0$ i.e. $[F,A] = 0$. Better calc. [redacted] $\mathrm{tr}(\delta F A) =$
$\mathrm{tr}(\delta F A F^2)$ $= \mathrm{tr}(F(\delta F) A F)$
$= -\mathrm{tr}((\delta F)(FA)F) = -\mathrm{tr}((F\delta F)(FA))$
$\mathrm{tr}((F\delta F)(AF - FA))$. [redacted] Not as clear

Next extend this method to symplectic, orthog cases. The analog of the Grass is the variety of Lagrangian subspaces $Sp(2n)/U(n) \hookrightarrow U(2n)/U(n)\times U(n)$

The infinitesimal picture is needed for the critical point analyses: $\mathcal{L}(Sp(2n)/U(n)) = \left\{ \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} : b^t = b \right\}$, and where

$u \in U(n)$ acts via $$\begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}\begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}\begin{bmatrix} u^* & 0 \\ 0 & u^t \end{bmatrix} = \begin{bmatrix} 0 & u b u^t \\ -\bar{u}\bar{b}u^* & 0 \end{bmatrix}$$

This is just the spectral theory of a symmetric bilinear form on a Hilbert space of dim n.

What's the next step? You need the appropriate trace functional on the symplectic Grassmannian.

You are trying to handle the situation by means of the basic representation $H(V)$. However, there is a conjugacy thm. for $\mathcal{L}(Sp(2n)/U(n))$ which should do the job, but you had difficulties with calculating the centralizer of a general diagonal elt, b. You would like to [redacted] calculate in the basic repn rather than the adjoint repn if this is possible.

So what ideas are available?

$$H(V) = \begin{bmatrix} V \\ V \end{bmatrix} \qquad V \text{ pos herm. space}$$

On $\underset{W}{\underset{\shortparallel}{H(V)}}$ you have pos herm form

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}^* \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_0^* & y_0^* \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_0^* x_2 + y_0^* y_2$$

and symplectic form

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1^t y_2 - y_1^t x_2$$

Assoc. to the two forms $w_0^* w_2$, $w_1^t J w_2$ are maps

$$W \xrightarrow{J} W^t \xleftrightarrow{*} W$$

$$w_1 \longmapsto w_1^t J, \quad w_0^* \longleftrightarrow w_0$$

J is $\mathbb{C}$-linear, $*$ is anti-linear. So the "$\overset{\text{quotient}}{\text{difference}}$" of the two forms is an invertible anti-linear transformation.

$$w_1 \longmapsto *(w_1^t J) = J^*(w_1^t)^* = -J\bar{w}_1 \quad \text{call this } Tw_1.$$

$$\begin{bmatrix} x_1^t & y_1^t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} x_0^* & y_0^* \end{bmatrix}$$

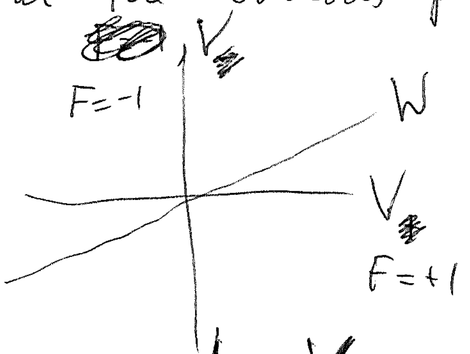
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \bar{x}_0 \\ \bar{y}_0 \end{bmatrix},$$

Apply $\bar{\ }$ to get

$$\therefore \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{y}_1 \end{bmatrix}$$

Polarizations of $H(V)$. Splitting of $H(V)$ into orthogonal Lagrangian subspaces. So the space of polarizations is a kind of Grassmannian, which should be the space of Lagrangian subspaces.

For the moment assume that the ~~orthogonal~~ $\perp$ subspace of a Lagrangian subspace L is Lagrangian. This is true locally. For example look at the obvious polarization of $H(V) \simeq \begin{bmatrix} V \\ V \end{bmatrix}$



You know that a Lagrangian subspace W transversal to V_- is the graph of a symmetric map $b: V \longrightarrow V$, $W = \begin{bmatrix} 1 \\ b \end{bmatrix} V$. In order for this to have meaning you need to identify V and V^t, which you can do via $\xi_1^t J \xi_2$. Then $W^\perp = \begin{bmatrix} -b^* \\ 1 \end{bmatrix} V$, but $-b^* = -\bar{b}$ is also symmetric, so $W^\perp$ is Lagrangian.

The local picture near a polarization is given by the C.T. $X = \begin{bmatrix} 0 & -\bar{b} \\ b & 0 \end{bmatrix}$ $\quad 1+X = \begin{bmatrix} 1 & -\bar{b} \\ b & 1 \end{bmatrix}$

$$F(1+X) = (1+X)\varepsilon$$

$$F(1+X)\varepsilon = (1+X) \qquad \therefore \; F\varepsilon = \frac{1+X}{1-X}$$

$$\parallel$$

$$F\varepsilon(1-X)$$

303 You want to relate polarization in the symplectic case to the $\mathbb{H}$ structure. You know that $H(V)$ is an $\mathbb{H}$-vector space. Why? because you have on $H(V)$ an invertible anti-linear operator T of square -1. (invertible follows from $T^2=-1$) Recall T is the "ratio" "quotient" of the forms $\xi_1^*\xi_0$, $\xi_2^t J\xi_0$. If $Z=H(V)$, then

$$Z \xrightarrow{J} Z^t \xrightarrow{*} Z$$
$$\xi_2 \longmapsto \xi_2^t J \longmapsto -J$$

Actually $\xi_2 \longmapsto (J\xi_2)^t = \xi_2^t(-J) \overset{*}{\longmapsto} J\bar{\xi}_2$

So if $\xi_2 = \begin{bmatrix} x \\ y \end{bmatrix}$, then $\underline{J\bar{\xi}_2} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \bar{y} \\ -\bar{x} \end{bmatrix}$.

Next try to understand polarizations

Look at $n=1$. $\begin{bmatrix} x \\ y \end{bmatrix}^*\begin{bmatrix} x \\ y \end{bmatrix} = |x|^2+|y|^2$

$$\begin{bmatrix} x \\ y \end{bmatrix}^t\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = x^t y' - y^t x'$$

Digress: recall 3 conditions satisfied by inf. automs. of structure

$$X^*+X=0, \quad X^tJ+JX=0, \quad JX=\bar{X}J$$

$$T=\sigma J \qquad TX = \sigma JX = \sigma(-X^tJ) = (-X^*)\sigma J = XT$$

Now that you have operators $\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$ $a^*=-a$, $b^t=b$.

you can try to understand polarizations. [redacted]

[redacted] You should first handle polarizations

(304) close to the basepoint polarization:

You want a simple description of a polarization involving the $\mathbb{H}$ structure, i.e. the operator T. Recall [redacted] conditions of an inf. symmetry X of $H(V)$

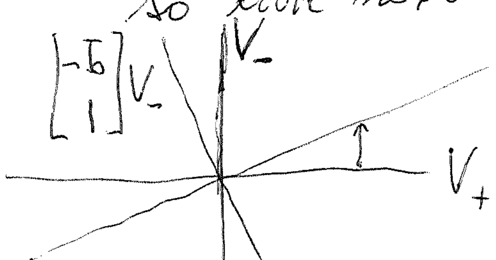
$$X^* + X = 0, \quad X^t J + JX = 0, \quad JX = \bar{X}J, \quad (\sigma J)X = X(\sigma J)$$

A polarization of $H(V)$ is described by an $F = F^* = F^{-1}$; e.g. the basept is $\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ on $\begin{bmatrix} V \\ V \end{bmatrix}$. [redacted]

$T\varepsilon = \sigma J \varepsilon = J\varepsilon\sigma$, $\quad \varepsilon T = \varepsilon\sigma J = \varepsilon J \sigma$. ε and $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ anti commute. $\therefore$ $T\varepsilon = -\varepsilon T$. Use $i\varepsilon$ instead: $\quad i\varepsilon = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, you know this commutes with T.

So a polarization seems to be a complex structure commuting with T. Complex structure should be any $iF = J$. Check: $-J = J^* = J^{-1}$, $-(iF) = (iF)^* = $ [redacted] $(iF)^{-1}$

So look next at C.T.



$\begin{bmatrix} 1 \\ b \end{bmatrix} V_+ \qquad F\varepsilon(1 \tilde{\otimes} X) = (1 + X)\cancel{\varepsilon}$

$$g = \frac{1+X}{1-X} \qquad g^{1/2} = \frac{1+X}{(1-X^2)^{1/2}}$$

You are ~~going~~ over the Grassmannian picture when $V_\pm$ [redacted] are different, but in the symplectic situation these are related by an anti linear isometry.

X has the form $\begin{bmatrix} 0 & -\bar{b} \\ b & 0 \end{bmatrix} \in \mathfrak{p}$. i.e. $b^t = b$.

What next? Maybe clean up $H(V) = \begin{bmatrix} V \\ \bar{V} \end{bmatrix} = \mathbb{H} \otimes_{\mathbb{C}} V$

Aim: Periodicity Real Bott

Questions concerning the open cell of the symplectic Grass. This is an affine space of dim $= \frac{1}{2}(2n)(2n+1) = 2n^2 + n$. What's the boundary like?

 Assume b invertible, i.e. nondegenerate.

Form the C.T. with scaling $\quad g_t = \dfrac{1+tX}{1-tX} = F_t \, \varepsilon$

Actually you want $\quad g_t^{1/2} = \dfrac{1+tX}{(1-t^2X^2)^{1/2}} \longrightarrow \dfrac{X}{|X|}$ as $t \to \infty$

which is the phase in the polar decomposition of X.

If $\quad X = \begin{bmatrix} 0 & -\bar{b} \\ b & 0 \end{bmatrix} \qquad -X^2 = \begin{bmatrix} 0 & \bar{b} \\ -b & 0 \end{bmatrix} \begin{bmatrix} 0 & -\bar{b} \\ b & 0 \end{bmatrix} = \begin{bmatrix} \bar{b}b & 0 \\ 0 & b\bar{b} \end{bmatrix}$

What is interesting here? You want a link to anti linear transf. $*b$

You also want the analog of $\operatorname{tr}(FA)$ in the Grass case. You know what F is to be. Probably you want to impose the condition of some S <u>commuting with</u> $T = \sigma J$

Suppose $b^t = b$. Recall assoc. to b is an anti-linear transf.

$$\begin{array}{ccccc} V & \xrightarrow{\ b\ } & V^t & \xrightarrow{\ *\ } & V \\ x & & x^t b & & b^* \bar{x} = \bar{b}\bar{x} \end{array}$$

So $*b$ is an anti-linear transf on V ~~with~~ square

$$(*b)(*b)x = (*b)(\bar{b}\bar{x}) = \bar{b}\,\overline{\bar{b}\bar{x}} = (\bar{b}b)\,x$$

IDEA Could there exist in infinite-dimensions an interesting index associated to $\begin{bmatrix} \bar{b}b & 0 \\ 0 & b\bar{b} \end{bmatrix}$. Note that these two quadratic expressions are self adjoint (assuming $b^t = b$) and get interchanged under $-$ and transpose.

Next you want the analog of $\overset{F \mapsto}{\operatorname{tr}(FA)}$ on the Grassmannian, but in the case of the symplectic Grassmannian which is the space of complex structures commuting with $T = *J_0$, $\quad J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(306) Let's straighten the notation: $H = \begin{bmatrix} V \\ \bar V \end{bmatrix} = \mathbb{H} \otimes_{\mathbb{C}} V$

$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$, $\mathbb{H} \otimes V = 1 \otimes V \oplus j \otimes \bar V \doteq \begin{bmatrix} V \\ \bar V \end{bmatrix}$?

Review: $X \in \mathcal{L}\, Sp(2n)$ means $J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

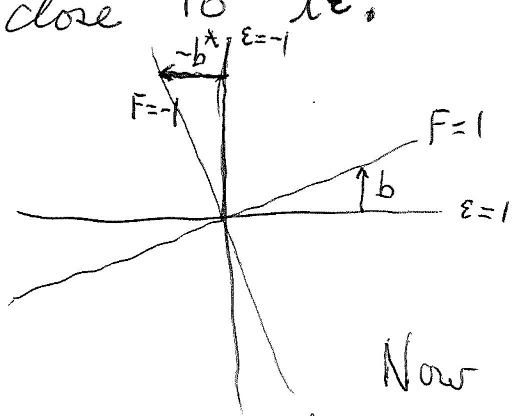
$X^* + X = 0$, $X^t J_0 + J_0 X = 0$, $J_0 X = \bar X J_0$, $(\sigma J_0) X = X(\sigma J_0)$,

$(\sigma J_0)(\sigma J_0) = \sigma^2 J_0^2 = -1$. You want operators on the left, so ?

Let's explore the idea that a polarization in the symplectic is a complex structure iF which commutes with T. True for $i\varepsilon$ because $T\varepsilon = \sigma J \varepsilon$ ~~and~~

$\varepsilon T = \varepsilon \sigma J = \sigma \varepsilon J$ but $\varepsilon J = -J\varepsilon \Rightarrow T\varepsilon = -\varepsilon T$,

then $T(i\varepsilon) \overset{?}{=} -iT\varepsilon = (i\varepsilon)T$. Next look at an iF close to $i\varepsilon$:



$$F \begin{bmatrix} 1 & -b^* \\ b & 1 \end{bmatrix} = F(1+X) = (1+X)\varepsilon$$

$$\Rightarrow \quad F\varepsilon = \frac{1+X}{1-X}, \quad (F\varepsilon)^{1/2} = \frac{1+X}{(1-X^2)^{1/2}}$$

Now $iF(-i\varepsilon)$ commutes with T since iF and $i\varepsilon$ do. Since $X = \dfrac{F\varepsilon - 1}{F\varepsilon + 1}$ it follows that T commutes with X. As $X = \begin{bmatrix} 0 & -b^* \\ b & 0 \end{bmatrix}$ and you know that X commutes with T iff it has the form $X = \begin{bmatrix} a & -\bar b \\ b & \bar a \end{bmatrix}$ you conclude that $\bar b = b^*$, equivalently $b = b^t$.

So you seem to understand polarizations better in the symplectic case at least.

Program: You can identify the space of polarizations with the orbit of iE in the adjoint repn — centralizer of a torus. What should the program be? Polarizations can be identified with iF which commute with T, or with F anti commuting with T.

You want a conjugacy theorem like in the case of the Grassmannian. Recall the functional is $\mathrm{tr}(FA)$ A hermitian.

Perhaps you should review the conjugacy theorem.

Let's begin again with the space of polarizations of $H(V)$. No, first you should emphasize the geometry. You

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Begin with the geometry on $H(V)$, whose symmetry group is $Sp(2n)$. Note that the geometry on V has the symmetry group $U(n)$. What's important is the space of polarizations. This is the smallest of the flag manifolds.

A polarization should have equivalent descriptions (i) Lagrangian subspace, (ii) An iF which commutes with $T = \sigma J_0$; (iii) an F anti commuting with T.

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philosophy. You want to adapt the conjugacy thm. in the Lie alg, make it work in the basic repn. Instead of a functional on the compact group you have a functional on a Grassmannian. This should be very simple ultimately.

308 ~~[illegible]~~ Symplectic Grass consisting of

$\{F = F^* = F^{-1} : TF + FT = 0\}$. ~~[illegible]~~

What does this mean? $H = \begin{bmatrix} W_+ \\ W_- \end{bmatrix}$ $F = \pm 1$ on $W_\pm$

since T anticommutes with F: $T(W_\pm) = W_\mp$

also $T^2 = -1$. $\Longrightarrow$ $H = \mathbb{H} \otimes W_+$

Tangent space to Symp Grass at F. δF is hermitian, anticommutes with F and T.

~~is skew hermitian~~ $F^2 = 1$, $FT + TF = 0$, $T^2 = -1$ ~~Looks like~~

$(iF)^2 = -1$, $T^2 = -1$, iF and T commute

Check things again. for $Sp(2n)$ $H = \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$

$z^* z_1$, $z^t J z_2$ $\bar{z}_1 = J z_2$ or $z_1 = J\bar{z}_2$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_2 \\ \bar{y}_2 \end{bmatrix} = \begin{bmatrix} \bar{y}_2 \\ -\bar{x}_2 \end{bmatrix}$$

Problem: You want to show for any polarization F that the ± 1 eigenspaces are isotropic. What do you know? You have the V_+, V_- splitting [redacted] which is flipped by T. It should be possible to express $z_1^t J_0 z_2$ in terms of $z_1^* z_2$ and T

Start at the basepoint $H = \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$, $\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

[redacted] Let's review $H = \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$. Can you establish a canonical isomorphism of H with $\mathbb{H} \otimes_{\mathbb{C}} V$

So you now ~~[illegible]~~ want to identify a polarization of H with a reduction from $\mathbb{H}$ to $\mathbb{C}$.

309 You want to understand polarizations of $H(V)$, = the basic repn. of $Sp(2n)$ using the inner product and $\mathbb{H}$-module structures. What do you know about $H(V)$? Conclusion you want is a **canonical isom** $H(V) = V \otimes \mathbb{H}$.

How to **start**? What is $H(V)$? V is a complex vector space equipped with pos herm form. Let's try defining $H(V)$ in terms of its inner product and the antilinear operator T. This should amount to something like an $\mathbb{H}$-vector space equipped w. some sort of compatible inner product.

Idea. You are studying polarizations, equiv. max isotropic subspaces for the symp form. So you are looking at the smallest flag manifold for $Sp(2n)$. The largest flag manifold should consist of all isotropic flags, i.e. a composition series in a Lagrangian subspace. Question: Is an isotropic flag in $H(V)$ equivalent to a quaternionic flag? It seems unlikely - and it might be interesting to see why.

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point $n=1$. $\mathcal{L}\, Sp(2) = \left\{ \begin{bmatrix} i|a| & -\bar{b} \\ b & -i|a| \end{bmatrix} \right\}$ smaller than $\mathbb{H} = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \right\}$.

problem about limitations of $\mathbb{H}$ viewpt

Look at $\mathbb{H} \otimes_{\mathbb{C}} V = (\mathbb{C} \oplus \mathbb{C}_j) \otimes_{\mathbb{C}} V$

[redacted] You want to understand $H(V)$ as an $\mathbb{H}$-module equipped with a [redacted] suitably compatible [redacted] inner product.

Better: $H(V)$ should be the $\mathbb{H}$-module $\mathbb{H} \otimes_{\mathbb{C}} V$ equipped with a suitable inner product. Here you have mentioned 2 structures namely the inner product and [redacted] the $\mathbb{H}$-module structure i.e. [redacted] an <u>anti-linear operator T</u> [redacted] with $T^2 = -1$.

Review: What is $H(V)$? You want to understand this using the inner product and the $\mathbb{H}$-structure, that is, the <u>anti-linear operator T such that $T^2 = -1$.</u>

Let's try a different approach. Let V be a complex vector space with pos. herm. inner product, i.e. a f.d. Hilbert space. Define $H(V)$ to be $\mathbb{H} \otimes_{\mathbb{C}} V$ equipped with a suitable inner product. Assuming this can be done OK you then have [redacted] two of the structures you want on $H(V)$, and you can examine the 3rd structure: symplectic form.

[redacted] Let U be a complex v.s. let $T : U \to U$ be an antilinear transform. such that $T^2 = -1$. Then there should be a canon isom $U = \mathbb{H} \otimes V$ where $V = \ ?$

311 You want to consider $\mathbb{H}$ as a right $\mathbb{C}$-module and you want a linear functional $\mathbb{H} \longrightarrow \mathbb{C}$. Work inside $\mathbb{H} = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \right\}$

$\mathbb{H} = \mathbb{C} + \mathbb{C}j$. ?

Define $H(V) = \begin{bmatrix} V \\ \bar{V} \end{bmatrix}$ $\quad c \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ \bar{c} y \end{bmatrix}$?

$U = H(V) = \begin{bmatrix} V \\ V \end{bmatrix}$, $\quad \begin{bmatrix} x \\ y \end{bmatrix}^* \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, $\quad \begin{bmatrix} x \\ y \end{bmatrix}^t \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{J_0} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$$U \xrightarrow{J_0} U^t \xrightarrow{*} U$$

$$z \longmapsto z^t J_0 \longmapsto J_0 \bar{z}$$

$$z = \begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \bar{y} \\ -\bar{x} \end{bmatrix}$$

Still, what is the defn of $H(V)$ for V a f.d. complex Hilbert space. $H(V) = \begin{bmatrix} V \\ V^t \end{bmatrix}$ $\quad \begin{bmatrix} x \\ y \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = x^t y_1 - y^t x_1$

$V \xrightarrow{\sim} V^*$ $\quad x \longmapsto x^*$

$\bar{V} \xrightarrow{\sim} V^t$

$H(V) = \begin{bmatrix} V \\ \bar{V} \end{bmatrix} = \begin{bmatrix} V \\ V^t \end{bmatrix}$ Try something else

You still need to start with a definition

It should be possible to define $H(V)$ as $\mathbb{H} \otimes_{\mathbb{C}} V$ with suitable pos herm. form. TENSOR PRODUCT of inner products on $\mathbb{H}$, V.

You need some sort of pairing

$$(\mathbb{H}\otimes_{\mathbb{C}} V)^*(\mathbb{H}\otimes_{\mathbb{C}} V)$$

$$\begin{bmatrix}1\otimes x_1^* & -j\otimes y_1^*\end{bmatrix}\begin{bmatrix}1\otimes x_2\\ j\otimes y_2\end{bmatrix} = x_1^*x_2 + y_1^*y_2$$

It's clear what you want namely

$$(1\otimes x_1, 1\otimes x_2) = (x_1, x_2)$$

$$(j\otimes y_1, j\otimes y_2) = (y_1, y_2)$$

$1\frac{3}{8} = \frac{11}{8} = 2\times\frac{11}{16}$

You are still working on a definition of $H(V)$ with V pos. herm. It should be easy. What is it? A complex vector space equipped with pos. herm. form, symplectic form, $\mathbb{H}$ action.

$$H(V) = \begin{bmatrix}V\\ V^t\end{bmatrix} = \begin{bmatrix}V\\ \bar{V}\end{bmatrix} = \mathbb{H}\otimes_{\mathbb{C}} V$$

Idea: On $\begin{bmatrix}V\\ V^t\end{bmatrix}$ you have a canonical invertible anti linear transformation with square $= -1$. This defines the $\mathbb{H}$ action. This means that on $H(V) = \begin{bmatrix}V\\ V^t\end{bmatrix}$ you are given the symplectic form and the ~~invertible~~ antilinear operator with square $= -1$. You should then be able to derive the positive hermitian form

$$Z = \begin{bmatrix}V\\ V^t\end{bmatrix} \qquad \begin{bmatrix}x_1\\ \xi_1\end{bmatrix}^t\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x_2\\ \xi_2\end{bmatrix} = x_1^t\xi_2 - \xi_1^t x_2$$

$$z_1^* z \;,\quad z_2^t J z$$

$$Z \xrightarrow{J} Z^t \xrightarrow{*} Z$$

$$z_2 \longmapsto z_2^t J \longmapsto -J\bar{z}_2$$

313 Things are becoming clear. Start with complex structure: $H(V) = \begin{bmatrix} V \\ V^t \end{bmatrix}$ $\quad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Then ~~introduce~~ introduce $\quad * : V \longrightarrow V^t$

$$x \longmapsto x^* = \langle x|$$

$*$ is antilinear of square 1.

$* : V^\sigma \xrightarrow{\sim} V^t$ Maybe $*$ is ~~fishy~~ but there is a canonical antilinear transformation. So you get $\begin{bmatrix} V \\ V^\sigma \end{bmatrix} \xrightarrow{\sim} \begin{bmatrix} V \\ V^t \end{bmatrix}$, you can pull back the symplectic form from $H(V)$ to $\begin{bmatrix} V \\ V^\sigma \end{bmatrix}$. Next you need to identify $\begin{bmatrix} V \\ V^\sigma \end{bmatrix} = \mathbb{H} \otimes_{\mathbb{C}} V$

Start again with V $\mathbb{C}$-v.s. Define

$H(V) = \begin{bmatrix} V \\ V^t \end{bmatrix}$ equipped with $\begin{bmatrix} x_1 \\ \xi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ \xi_2 \end{bmatrix} = x_1^t \xi_2 - \xi_1^t x_2$

This makes a symplectic form on $H(V)$.

Now suppose V equipped with a positive hermitian form. Then get $\quad V \longrightarrow V^t$

$$x \longmapsto \langle x|y\rangle$$

anti linear isom. Also get $\quad V^\sigma \longrightarrow V^t$

$$V \longrightarrow V^* \qquad * = \sigma t$$

so $\quad \begin{bmatrix} V \\ V^\sigma \end{bmatrix} \xrightarrow{\sim} \begin{bmatrix} V \\ V^t \end{bmatrix} = H(V) \qquad y \in V \quad \mathbb{C} \xrightarrow{y} V$

$$\mathbb{C}^* \xleftarrow{y^*} V^*$$

$x \longmapsto x$

$\xi^* \longmapsto \xi$ $\qquad ??\; x_1^t \bar{y}_2 - \bar{y}_1 x_2 = \begin{bmatrix} x_1 \\ y_1^* \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

(314) V complex v.s., V^t its dual, $H(V) = \begin{bmatrix} V \\ V^t \end{bmatrix}$

symplectic form $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1^t y_2 - y_1^t x_2$

Suppose $\langle x_1, x_2 \rangle$ is positive hermitian; maybe you should consider non-degenerate hermitian eventually.

Then you get $V \longrightarrow V^t$

Review the problem. To define $H(V)$ and explain its structure. Begin with V a $\mathbb{C}$-vector space, let V^t be its dual spaces.

Canonical pairing: From $x \in V$, $y \in V^t$ get $y(x)$

can interpret x as the linear map $\mathbb{C} \longrightarrow V$, $c \mapsto cx$.

assoc. to this linear map is its transpose

can identify x, y with linear maps

$$\mathbb{C} \xrightarrow{\hat{x}} V, \quad \mathbb{C} \xrightarrow{\hat{y}} V^t$$

which have transposes:

$$V^t \xrightarrow{\hat{x}^t} \mathbb{C}^t = \mathbb{C}, \quad V = V^{tt} \xrightarrow{\hat{y}^t} \mathbb{C}^t = \mathbb{C}$$

what is $\hat{x}^t(y) = y \circ \hat{x} = y \circ (c \mapsto cx) = (c \mapsto y(cx))$

evaluate at $c = 1$ to get $\hat{x}^t(y) = y(x)$

What is $\hat{y}^t : V^{tt} \longrightarrow \mathbb{C}^t = 0$, $\hat{y}^t(\lambda) = \lambda \circ \hat{y}$

? ?

First explain: $V \xrightarrow{\varphi} (V^t)^t$ $\quad \varphi(x)(\lambda) = \lambda(x)$

Next take transpose of $\mathbb{C} \xrightarrow{\hat{y}} V^t$, $\hat{y}(c) = cy$

$$\mathbb{C} \xleftarrow{\hat{y}^t} V^{tt} \xleftarrow{\varphi} V$$

$$\hat{y}^t(\varphi(x)) = \hat{y}^t(\lambda \mapsto \lambda(x)) = (\lambda \mapsto \lambda(x)) \circ \hat{y} = \hat{y}(x) = y(x)$$

315 You seem to be missing something.

V finite dim'l vector space, $V^t = \mathrm{Hom}(V, \mathbb{C})$

To construct a canonical isom $V \xrightarrow{\varphi} (V^t)^t$

$\varphi(x)(\lambda) = \lambda(x)$. φ linear injective by extn of linear fnls.

then $\dim(V) = \dim(V^t) = \dim(V^{tt}) \Rightarrow \varphi$ onto.

Let $x \in V$, $y \in V^t$, let $\mathbb{C} \xrightarrow{\hat{x}} V$, (resp $\mathbb{C} \xrightarrow{\hat{y}} V^t$)

be $c \mapsto cx$ (resp $c \mapsto cy$).

Next if $T: V \longrightarrow W$ linear, define

$T^t : W^t \longrightarrow V^t$ $\qquad T^t(\mu) = \mu T$

Q: What are $\mathbb{C} = \mathbb{C}^t \xleftarrow{\hat{x}^t} V^t$, $\mathbb{C} = \mathbb{C}^t \xleftarrow{\hat{y}^t} V^{tt}$?

Let $\lambda \in V^t$. $\hat{x}^t(\lambda) = (\lambda \circ \hat{x} : c \mapsto \lambda(cx)) \overset{\text{set } c=1}{=} \lambda(x)$

Let $\mu \in (V^t)^t$. ~~You can assume that $\mu = \varphi(x)$~~

~~$\mu = \hat{y}^t \varphi(x)$~~ i.e. $\mu : V^t \longrightarrow \mathbb{C}$. You know that $\mu : \lambda \longmapsto \lambda(v)$ $\exists ! v$

$$\hat{y}^t(\mu) = \mu \circ (c \overset{\hat{y}}{\longmapsto} cy) = (c \longmapsto c\mu(y)) = \mu(y).$$