57 ~~Also~~ Does σ respect the symmetric form?

$$\left(\sigma\begin{bmatrix} a \\ b^\dagger \end{bmatrix}\right)^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left(\sigma\begin{bmatrix} c \\ d^\dagger \end{bmatrix}\right) = \begin{bmatrix} b \\ a^\dagger \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d \\ c^\dagger \end{bmatrix}$$

$$= \begin{bmatrix} b \\ a^\dagger \end{bmatrix}^t \begin{bmatrix} c^\dagger \\ d \end{bmatrix} = \underset{?}{c^\dagger b} + a^\dagger d \qquad \sum \bar{a}_i d_i$$

not equal, conjugate instead

$$? \sum \bar{d}_i a_i$$

$$\begin{bmatrix} a \\ b^\dagger \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c \\ d^\dagger \end{bmatrix} = \begin{bmatrix} a \\ b^\dagger \end{bmatrix}^t \begin{bmatrix} d^\dagger \\ c \end{bmatrix} = b^\dagger c + d^\dagger a$$

OKAY this looks good. So you take the real subspace of $H(V)$, and you get a real quadratic space consisting of $\begin{bmatrix} a \\ a^\dagger \end{bmatrix}$ with symmetric form

$$\begin{bmatrix} a \\ a^\dagger \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c \\ c^\dagger \end{bmatrix} = \begin{bmatrix} a^t & \bar{a} \end{bmatrix} \begin{bmatrix} c^\dagger \\ c \end{bmatrix}$$

$$\begin{bmatrix} a \\ a^\dagger \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c \\ c^\dagger \end{bmatrix} = \begin{bmatrix} a^t & a^{\dagger t} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c \\ c^\dagger \end{bmatrix}$$

$$= \begin{bmatrix} (a^\dagger)^t & a^t \end{bmatrix} \begin{bmatrix} c \\ c^\dagger \end{bmatrix} = \underline{(a^\dagger)^t c + a^t c^\dagger}$$

$$a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \qquad a^t c'$$

It seems that there is a problem. go back to

$$H(V) = \begin{bmatrix} V \\ V^\wedge \end{bmatrix} \qquad \begin{bmatrix} a \\ \alpha \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b \\ \beta \end{bmatrix} = a^t \beta + \alpha^t b$$

(58) ~~Start~~ again. V $\mathbb{C}$-vector space with pos herm form.

$$H(V) = \begin{bmatrix} V \\ V^\wedge \end{bmatrix}, \quad \begin{bmatrix} a \\ \alpha \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c \\ \gamma \end{bmatrix} = a^t\gamma + \alpha^t c$$

here a, α, c, γ are column vectors. The 1's limit you to the pairings $a^t\gamma$, $\alpha^t c$.

Use the pos. herm form on V to get an isom $\bar{V} \longrightarrow V^\wedge$, $a \longmapsto (b \mapsto a^t b)$

Maybe what's important is $a \longmapsto \bar{a} \longmapsto \bar{a}^t$ $(= a^t)$?

$\langle a | b \rangle = \bar{a}^t b.$

$$\left(\sigma \begin{bmatrix} a \\ \bar{b} \end{bmatrix}\right)^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left(\sigma \begin{bmatrix} c \\ \bar{d} \end{bmatrix}\right) = \begin{bmatrix} b \\ \bar{a} \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d \\ \bar{c} \end{bmatrix}$$

$$= \begin{bmatrix} b^t & \bar{a}^t \end{bmatrix} \begin{bmatrix} \bar{c} \\ d \end{bmatrix} = b^t \bar{c} + \bar{a}^t d$$

conj.

$$\begin{bmatrix} a \\ \bar{b} \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c \\ \bar{d} \end{bmatrix} = \begin{bmatrix} a^t & \bar{b}^t \end{bmatrix} \begin{bmatrix} \bar{d} \\ c \end{bmatrix} = a^t \bar{d} + \bar{b}^t c$$

$$\sigma \begin{bmatrix} a \\ \bar{b} \end{bmatrix} = \begin{bmatrix} b \\ \bar{a} \end{bmatrix} \overset{?}{=} \begin{bmatrix} a \\ \bar{b} \end{bmatrix} \iff a = b$$

real subspace of $H(V)$ is $\left\{ \begin{bmatrix} a \\ \bar{a} \end{bmatrix} \,\middle|\, a \in V \right\}$.

symm bilinear form

$$\begin{bmatrix} a \\ \bar{a} \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b \\ \bar{b} \end{bmatrix} = \begin{bmatrix} a^t & \bar{a}^t \end{bmatrix} \begin{bmatrix} \bar{b} \\ b \end{bmatrix} = a^t \bar{b} + \bar{a}^t b$$

If $a = b$, then $a^t \bar{a} + \bar{a}^t a = \sum_i a_i \bar{a}_i + \sum_i \bar{a}_i a_i = 2\sum |a_i|^2$

(59) Want next the symplectic version, V complex vector space equipped with pos. herm. form $\langle a|b\rangle = \bar{a}^t b$

$$H(V) = \begin{bmatrix} V \\ V^\wedge \end{bmatrix}, \quad \begin{bmatrix} a \\ \alpha \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} b \\ \beta \end{bmatrix} = a^t\beta - \alpha^t b$$

Define a conjugation σ on $H(V)$, NO $\sigma^2 \neq 1$ rather -1.

Try $\sigma \begin{bmatrix} a \\ \bar{b} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \bar{a} \\ b \end{bmatrix} = \begin{bmatrix} b \\ -\bar{a} \end{bmatrix}$. $\sigma^2 \begin{bmatrix} a \\ \bar{b} \end{bmatrix} = \sigma \begin{bmatrix} b \\ -\bar{a} \end{bmatrix} =$ [redacted]

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \bar{b} \\ -a \end{bmatrix} = \begin{bmatrix} -a \\ -\bar{b} \end{bmatrix} = -\begin{bmatrix} a \\ \bar{b} \end{bmatrix}$. Next check the symplectic form is compatible with σ and conjugation.

$$\left(\sigma \begin{bmatrix} a \\ \bar{b} \end{bmatrix}\right)^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \left(\sigma \begin{bmatrix} c \\ \bar{d} \end{bmatrix}\right) = \begin{bmatrix} b \\ -\bar{a} \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} d \\ -\bar{c} \end{bmatrix} = \begin{bmatrix} b \\ -\bar{a} \end{bmatrix}^t \begin{bmatrix} -\bar{c} \\ -d \end{bmatrix}$$

$$= -b^t\bar{c} + \bar{a}^t d, \quad \begin{bmatrix} a \\ \bar{b} \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} c \\ \bar{d} \end{bmatrix} = \begin{bmatrix} a^t & \bar{b}^t \end{bmatrix} \begin{bmatrix} \bar{d} \\ -c \end{bmatrix} = a^t\bar{d} - \bar{b}^t c$$

[redacted] Since $\sigma^2 = -1$ and $\sigma\lambda = \bar{\lambda}\sigma$ for $\lambda \in \mathbb{C}$ $H(V)$ is [redacted] naturally an $\mathbb{H}$-module

Take $V = \mathbb{C}$. $H(\mathbb{C}) = \left\{ \begin{bmatrix} a \\ \bar{b} \end{bmatrix}, a, b \in \mathbb{C} \right\}$.

Look at ~~others~~ the operators you get on $H(\mathbb{C})$

$$\sigma\lambda \begin{bmatrix} a \\ \bar{b} \end{bmatrix} = \sigma \begin{bmatrix} \lambda a \\ \lambda\bar{b} \end{bmatrix} = \begin{bmatrix} \bar{\lambda} b \\ -\bar{\lambda}\bar{a} \end{bmatrix} = \text{[redacted]}\ \bar{\lambda}\sigma \begin{bmatrix} a \\ \bar{b} \end{bmatrix}$$

Review $H(V) = \begin{bmatrix} V \\ V^\wedge \end{bmatrix}$ $\begin{bmatrix} a \\ b \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = a^t d - b^t c$

[redacted] Really $H(V) = \begin{bmatrix} \mathbb{C}^n \\ \mathbb{C}^n \end{bmatrix}$. [redacted] Next define σ on $H(\mathbb{C}^n)$ by

$$\sigma \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix} = \begin{bmatrix} \bar{b} \\ -\bar{a} \end{bmatrix} \longmapsto \sigma \begin{bmatrix} \bar{b} \\ -\bar{a} \end{bmatrix} = \begin{bmatrix} \overline{-\bar{a}} \\ -\bar{\bar{b}} \end{bmatrix} = -\begin{bmatrix} a \\ b \end{bmatrix}$$

$\therefore \sigma^2 = -1$. So $\mathbb{H}$ acts on $H(V)$

(60) You want $|\ |^2$ on $H(\mathbb{C}^n)$. Take $n=1$.

$H(\mathbb{C}) = \begin{bmatrix} \mathbb{C} \\ \mathbb{C} \end{bmatrix}$ $\quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} i \\ 0 \end{bmatrix}, j\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \sigma\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$?

Basis over $\mathbb{H}$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k = \mathbb{C} + \mathbb{C}j$$
$$q = t + xi + yj + zk$$
$$q^* = t - xi - yj - zk$$
$$qq^* = q^*q = t^2 - (xi + yj + zk)^2 = t^2 + x^2 + y^2 + z^2$$

Discuss how to proceed. At the moment you have $H(\mathbb{C}) = \left\{ \begin{bmatrix} a \\ b \end{bmatrix}, a, b \in \mathbb{C} \right\}$. $\sigma\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \bar{b} \\ -\bar{a} \end{bmatrix}$

$$(\lambda + \mu\sigma)\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \lambda a \\ \lambda b \end{bmatrix} + \mu\begin{bmatrix} \bar{b} \\ -\bar{a} \end{bmatrix} = \begin{bmatrix} \lambda a + \mu\bar{b} \\ \lambda b - \mu\bar{a} \end{bmatrix}$$

$\begin{bmatrix} \lambda & \mu \end{bmatrix}\begin{bmatrix} a & \bar{b} \\ b & -\bar{a} \end{bmatrix}$ this is clearly confused.

$\mathbb{H} \otimes_{\mathbb{C}} V$ where V is a complex v.s. with pos herm. scalar product.

Define $\langle \xi_1 \otimes v_1 | \xi_2 \otimes v_2 \rangle = \langle v_1 | \bar{\xi}_1 \xi_2 v_2 \rangle$

$$\mathbb{H} \otimes_{\mathbb{C}} V = \begin{bmatrix} 1 \otimes V \\ j \otimes V \end{bmatrix} = \begin{bmatrix} V \\ \bar{V} \end{bmatrix}$$

How to get started? You ~~???~~ began from the complex hyperbolic symplectic space $\begin{bmatrix} V \\ V^\wedge \end{bmatrix}$ and found σ s.t. $\sigma\lambda = \bar{\lambda}\sigma$, $\sigma^2 = -1$ by choosing pos herm. $\bar{V} \xrightarrow{\sim} V^\wedge$. Another method would be ~~???~~ to form $\mathbb{H} \otimes_{\mathbb{C}} V = \begin{bmatrix} 1 \otimes V \\ j \otimes V \end{bmatrix} = \begin{bmatrix} V \\ \bar{V} \end{bmatrix}$

64 How to represent an elt of $\bar{V}$. Easiest is? Ultimately $\mathbb{H} \otimes_{\mathbb{C}} V$ is a complex vector space with pos herm prod. + dim $= 2n$. It's a $\mathbb{C}$ vs of dim $2n$ arising from the $\mathbb{C}$ vs V of dim n

Perhaps you should put the $\mathbb{H}$ on the other side: $V \otimes_{\mathbb{C}} \mathbb{H} = \begin{bmatrix} V \otimes 1 \\ V \otimes j \end{bmatrix}$. Let $GL(V)$ act by left ~~matrix~~ multiplication. Now $GL(V)$ should also act on $V \otimes j$. Still confused.

Problem. You need to link $H(V) = \begin{bmatrix} V \\ V^{\wedge} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ whose auto gp is $Sp(2n, \mathbb{C})$ to the $\mathbb{H}$-module $\mathbb{H}^n$ whose auto group is $\underbrace{GL(n, \mathbb{H})}$. Note: This is maybe another real form of $Sp(2n, \mathbb{C})$? $\frac{2n(2n+1)}{2} \cdot 2$ for real dim

What's the real dim of $GL(n, \mathbb{H})$? would seem to be $4n^2$??

$n^2 + 2\frac{n(n+1)}{2} = 2n^2 + n$

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do $n = 1$. $\mathbb{H} \xrightarrow{exp} \mathbb{H}^{\times}$

$$Sp(2, \mathbb{C}) = \left\{ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2\mathbb{C},\ g^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

this condition $\Longrightarrow$ $\det(g)^2 = 1$. Suppose $\det(g) = 1$.

$$\overset{g^t}{\begin{bmatrix} a & c \\ b & d \end{bmatrix}} = \overset{J}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \overset{g^{-1}}{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}} \overset{J^{-1}}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -b & -d \\ a & c \end{bmatrix}$$

$$= \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$\therefore Sp(2, \mathbb{C}) = SL(2, \mathbb{C})$

If $\det(g) = -1$ no such g

(62) But $SU(2) = \{ g \in M_2\mathbb{C} \mid \bar{g}^t g = 1 \;\; \det(g) = 1 \}$

If $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\bar{g}^t = \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix} = g^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ if $\det(g)=1$

$\Rightarrow d = \bar{a} \quad c = -\bar{b} \Rightarrow g = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \quad |a|^2 + |b|^2 = 1$

What about $U(2) = \{ g \in M_2\mathbb{C} \mid \bar{g}^t g = 1 \}$. $\quad |\det(g)|^2 = 1$

Careful. $1 = \det(\bar{g}^t g) = \det(\bar{g})\det(g) = |\det(g)|^2$. So $\det(g) = e^{i\theta}$ so $\begin{bmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$ parametrizes $U(2)$

Next look at $\mathbb{H}^\times$. Lie algebra is $\mathbb{H}$. The first thing to understand is ~~whether~~ how $S\mathbb{H} = \{ g \in \mathbb{H} \mid g^* g = 1 \}$. is $SU(2)$.

$\mathbb{H} = \mathbb{R} + \mathbb{R}i + (\mathbb{R} + \mathbb{R}i)j = \mathbb{C} + \mathbb{C}j$. The group of norm 1 quaternions is ???

Question: How do we handle $\mathbb{H}$? A better question might be how to handle $Sp(2)$, and more generally $Sp(2n)$. If possible you want to deal directly with the symplectic space $H(V)$ and its real form, which you have some control over

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begin again. V = complex vector space $\mathbb{C}^n$, $H(V) = \begin{bmatrix} V \\ V^\wedge \end{bmatrix}$ equipped with the symplectic form

$\begin{bmatrix} a \\ b \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = a^t d - b^t c$

$Sp(2n, \mathbb{C})$ = group of autos of $H(V)$ preserving symp form

You want a maximal compact subgroup K of $Sp(2n, \mathbb{C})$. These K should be all conjugate. $\exists$ a pos. herm form on $H(V)$ invariant under K.

(63) But you should be able to construct a pos herm form on $H(V) = \begin{bmatrix} V \\ V^\wedge \end{bmatrix}$ [redacted] starting from a pos. herm form on V. Let's identify $\bar{V} \longrightarrow V^\wedge$, $b \mapsto \bar{b}^t$. Then elements of $H(V)$ have the form $\begin{matrix} a \\ b \end{matrix}$? Go back to $\begin{bmatrix} a \\ b \end{bmatrix} \in \begin{bmatrix} V \\ V^\wedge \end{bmatrix}$ and the pairing is $\begin{bmatrix} a \\ b \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = a^t d - b^t c$

$g \in GL(V)$, then g operates on $\begin{bmatrix} V \\ V^\wedge \end{bmatrix}$ by

$$g \begin{bmatrix} a \\ b \end{bmatrix} = \begin{matrix} ga \\ (g^t)^{-1} b \end{matrix} \qquad \underbrace{g = \begin{bmatrix} g & 0 \\ 0 & (g^t)^{-1} \end{bmatrix}}_{} \qquad \text{suppose } g$$

$(g^t)^{-1} = \bar{g}$

is unitary $\bar{g}^t g = 1$ or [redacted] $\bar{g}^t = g^{-1}$ or $g^t = \bar{g}^{-1}$

So the action of $U(n)$ on $H(V)$ is $g \longmapsto \begin{bmatrix} g & 0 \\ 0 & \bar{g} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$

Consider [redacted] $\mathbb{C}^2$ with volume form acted [redacted] on by $Sp(2, \mathbb{C}) = SL(2, \mathbb{C})$. A Maximal compact subgroup of $SL(2, \mathbb{C})$ is $SU(2)$. Presumably $SU(2) = SL(2, \mathbb{C}) \cap U(2)$. You want a pos. [redacted] herm scalar product on $\mathbb{C}^2$: $\begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix}^t \begin{bmatrix} c \\ d \end{bmatrix}$

[redacted] $Sp(2, \mathbb{C}) = SL(2, \mathbb{C}) \subset GL(2, \mathbb{C})$. Ask about maximal compact subgroups of $SL(2, \mathbb{C})$. There's a symmetric space consisting of pos hermitian forms on $\mathbb{C}^2$ with determinant $= 1$. [redacted] $\det(g) = \pm 1$

Situation: $SL(2, \mathbb{C})$. [redacted]

$$Sp(2, \mathbb{C}) = \left\{ g \in GL(2, \mathbb{C}) \;\middle|\; \overbrace{g^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \right\}$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -b & -d \\ a & c \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

(64) So $Sp(2,\mathbb{C}) = SL(2,\mathbb{C})$ acts in the obvious way on $\mathbb{C}^2$ preserving ω, namely

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{g} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

So SL(2, $\mathbb{C}$) acts on the space of positive hermitian forms on $\mathbb{C}^2$. Such a form is given by a hermitian matrix with positive eigenvalues. You want to pick a nice hermitian form on $\mathbb{C}^2$, and there's an obvious choice, whose stabilizer should be SU(2).

$SL(2,\mathbb{C}) \cap U(2)$ $\qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\qquad a\bar{b} + c\bar{d} = 0$, $\quad \dfrac{ad-bc}{a} = \dfrac{1}{a}$

$$\frac{\bar{b}}{\bar{d}} = -\frac{c}{a}$$

$$d - \frac{bc}{a} = \frac{1}{a}$$

$$d + \frac{|b|^2}{\bar{d}} = \frac{1}{a} \Rightarrow |d|^2 + |b|^2 = \frac{\bar{d}}{a} \qquad \therefore \bar{d} = a$$

$$\bar{b} = -c$$

~~What's happened?~~

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

Review the problem: Given a fin diml $\mathbb{C}$-vector space V, you have its hyperbolic symp space $\begin{bmatrix} V \\ V^\wedge \end{bmatrix}$, $\begin{bmatrix} a \\ b \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$

Take $V = \mathbb{C}^n$, $V^\wedge = \mathbb{C}^n$ duality pairing is $a^t b = b^t a$

Also have positive hermitian symm. pairing $a^t b = \overline{b^t a}$

So you have a $2n$ diml $\mathbb{C}$-vector space with these

Puzzle about meaning of positive

(65) Positivity requires the diagonal. ~~So what can~~

Let's ~~start~~ again with V complex vector space of dim n. Form the hyperbolic symplectic sp $H(V) = \begin{bmatrix} V \\ V^\wedge \end{bmatrix}$. Suppose V equipped with a positive hermitian form $\langle v_1 | v_2 \rangle$, i.e. V is n-diml complex Hilbert space. Describe the structure you have on V: a real $2n$-diml v.s., a pos scalar product, an orthogonal transf J s.t. $J^2 = -1$.

Take $n = 1$. $V = \mathbb{C}$ $\quad J\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ $\quad i(x+iy) = -y + ix$

Let's look again at complex V equipped with a nondegenerate hermitian form, let this be $h(v_1, v_2)$ biadditive $\quad h(z_1 v_1, z_2 v_2) = \bar{z}_1 h(v_1, v_2) z_2$

$f(v_1, v_2)$ is sesquilinear when $f(zv_1, v_2) = \bar{z} f(v_1, v_2)$ and $f(v_1, zv_2) = f(v_1, v_2) z$, $f(v_1 + v_1', v_2) = f(v_1, v_2) + f(v_1', v_2)$ same for v_2.

f same as $\mathbb{C}$-linear map $v_1 \longmapsto f(v_1, -)$ from $\bar{V}$ to $V^\wedge$

$$\operatorname{Hom}_{\mathbb{C}}(\bar{V}, V^\wedge) = \cancel{\{ f : V \to V^\wedge}$$

$$\operatorname{Hom}_{\mathbb{C}}(\bar{V}, V^\wedge) = \operatorname{Hom}_{\mathbb{C}}(\underset{v_1}{\bar{V}}, \operatorname{Hom}_{\mathbb{C}}(\underset{v_2}{V}, \mathbb{C}))$$

$$= \{ f(v_1, v_2) \ \ \mathbb{R} \text{ bilinear} \qquad f(\bar{\lambda} v_1, v_2) = \lambda f(v_1, v_2)$$
$$f(v_1, \lambda v_2) = \lambda f(v_1, v_2)$$

$$= \operatorname{Hom}_{\mathbb{C}}(\bar{V} \otimes_{\mathbb{C}} V, \mathbb{C}) \qquad \overline{V^\wedge} = \operatorname{Hom}_{\mathbb{C}}(\bar{V}, \mathbb{C}) = \operatorname{Hom}_{\mathbb{C}}(V, \bar{\mathbb{C}})$$

sesquilinear same as $T : V \longrightarrow (\bar{V})^\wedge$

66 $\quad V = \mathbb{C}^n, \quad W = \mathbb{C}^m$

$f(v, w)$ antilinear in v
linear in w

$$f\Big(\sum_i v_i \hat{i}, \sum_j w_j \hat{j}\Big) = \sum_{i,j} \bar{v}_i f(\hat{i}, \hat{j}) w_j$$

f is equiv. to a linear map $\quad \bar{V} \longrightarrow W^\wedge$
$\qquad\qquad\qquad\qquad\qquad\qquad v \longmapsto (w \mapsto f(v,w))$

f is also equiv to a linear map $\quad W \longrightarrow \bar{V}^\wedge = \mathrm{Hom}_{anti}(V, \mathbb{C})$
$\qquad\qquad\qquad\qquad\qquad\qquad w \longmapsto (v \mapsto f(v,w))$

maybe you should ~~adopt~~ use $\otimes$

$$V \longrightarrow \bar{W} \iff \bar{V} \longrightarrow W$$
$$V \longrightarrow \overline{W^\wedge} \iff \bar{V} \longrightarrow W^\wedge \iff W^\wedge \otimes \bar{V} \to \mathbb{C}$$

$\bar{V} \otimes W \longrightarrow \mathbb{C}$
$\bar{v} \otimes w \longmapsto f(v, w)$
$\parallel$
$\sum \bar{v}_i \hat{i} \otimes \sum w_j \otimes j \longmapsto \sum_i \bar{v}_i f(\hat{i} \otimes \hat{j}) w_j$

IDEA: basic symbols are matrices, operations: transpose + conjugation and matrix product when row number of first = column number of 2nd factor

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$V = \mathbb{C}^n \qquad V^\wedge = \{ b^t \mid b \in V \}$, basic pairing

$\mathbb{C} \longleftarrow V^\wedge \times V \qquad (b^t a)^t = a^t b$
$b^t a \longleftarrow (b^t, a)$

symplectic form on $\begin{bmatrix} V \\ V^\wedge \end{bmatrix}$: $\begin{bmatrix} a \\ b^t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} c \\ d^t \end{bmatrix} =$

$V \times V^\wedge \longrightarrow \mathbb{C}$
$(a, b^t) \longmapsto a^t b$

$$\begin{bmatrix} a \\ b^t \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} c \\ d^t \end{bmatrix} = [a^t \; b] \begin{bmatrix} d^t \\ -c \end{bmatrix}$$

doesn't work. What does work:

$$\begin{bmatrix} a \\ b \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = [a^t \; b^t] \begin{bmatrix} d \\ -c \end{bmatrix} = a^t d - b^t c$$

(67) $$\begin{bmatrix} c^t & d^t \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c^t & d^t \end{bmatrix}\begin{bmatrix} -b \\ a \end{bmatrix} = -c^t b + d^t a$$

$V = \mathbb{C}^n$, $V^* = \mathbb{C}^n$. NO take $n=1$.

Consider $\begin{bmatrix} \mathbb{C} \\ \mathbb{C} \end{bmatrix} = \left\{ \begin{bmatrix} a \\ \bar{b} \end{bmatrix} \,\middle|\, a, b \in \mathbb{C} \right\}$ sesquilinear

pairing $(b, a) \longmapsto \bar{b} a = \langle b | a \rangle$. Confused again.

~~Go back to~~ $$\begin{bmatrix} a \\ \bar{b} \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} c \\ \bar{d} \end{bmatrix} = a^t \bar{d} - \bar{b}^t c$$
$$= \bar{d}^t a - \bar{b}^t c$$

Start again with $\begin{bmatrix} \mathbb{C} \\ \mathbb{C} \end{bmatrix} = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \,\middle|\, a, b \in \mathbb{C} \right\}$

equipped with $$\begin{bmatrix} a \\ b \end{bmatrix}^\dagger \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = a^\dagger c + b^\dagger d$$

and the skew form $$\begin{bmatrix} a \\ b \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = a^t d - b^t c$$

$$\begin{bmatrix} a \\ b \end{bmatrix}^\dagger \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \bar{a} c + \bar{b} d$$

$$\begin{bmatrix} a \\ b \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = ad - bc$$

$U(2)$ is the group preserving the pos hermitian form

$U(2) = \{ g \in GL(2, \mathbb{C}) \mid g^\dagger g = 1 \}$.

$SL(2,\mathbb{C})$ is the group preserving the symp. form

$$g^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \implies \det(g)^2 = 1 \quad \text{etc.}$$

$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in U(2) \cap SL(2, \mathbb{C})$ means $\det(g) = 1$, $g^\dagger = \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix} = g^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $\quad d = \bar{a}$, $c = -\bar{b}$.

$g = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$, $|a|^2 + |b|^2 = 1$.

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So you learn that the ~~auto~~ symmetries of $\mathbb{C}^2$ preserving the pos. herm form $|a|^2 + |b|^2$ and the $\mathbb{C}$ linear ~~[illegible]~~ symp. form ~~[illegible]~~

$$g\left(\begin{bmatrix} x \\ y \end{bmatrix} \wedge \begin{bmatrix} x' \\ y' \end{bmatrix}\right) = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} \wedge \begin{bmatrix} ax'+by' \\ cx'+dy' \end{bmatrix}$$

$$= (ax+by)(cx'+dy') - (cx+dy)(ax'+by')$$

$$= \begin{matrix} \cancel{acxx'} \\ + adxy' \checkmark \\ + bcx'y \\ + \cancel{bdyy'} \end{matrix} \quad - \left(\begin{matrix} \cancel{caxx'} \\ + dayx' \\ + cbxy' \checkmark \\ + \cancel{dbyy'} \end{matrix} \right.$$

$$= (ad-bc)xy' - (ad-bc)x'y$$

$$= xy' - x'y$$

---

natural question? You have 2 diml complex ~~symplectic~~ space ~~pos herm form~~ equipped with both $\mathbb{C}$-linear symplectic form and pos. ~~[illegible]~~ herm. form. Is there a kind of compatibility between these structures?

$\mathbb{C}^2$ equipped with complex volume $\Lambda^2\mathbb{C}^2 \rightsquigarrow \mathbb{C}$

$\mathbb{C}^2$ ———— pos herm. form ~~[illegible]~~

Consider again $V = \mathbb{C}^n$

$V^\wedge = \mathbb{C}^n$

~~[illegible]~~ Let $\mathbb{C}^{2n} = \begin{bmatrix} \mathbb{C}^n \\ \mathbb{C}^n \end{bmatrix}$ be endowed with $\begin{bmatrix} v_1 \\ \varphi_1 \end{bmatrix}^\dagger \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix}$ pos. herm. form

and with $\begin{bmatrix} v_1 \\ \varphi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix}$ $\mathbb{C}$-linear symp form.

(69) Consider a fd Hilbert space W equipped with a $\mathbb{C}$ linear map $W \xrightarrow{\sim} W^\wedge$ skew symmetric and nondegenerate. Choose a unit vector ε in W. Then you have $\varepsilon^\perp$ and ε^0 two hyperplanes, $\varepsilon \in \varepsilon^0$

$W = \mathbb{C}\varepsilon \oplus \varepsilon^\perp$ $\quad$ $\mathbb{C}\varepsilon \subset \varepsilon^0$. $\quad$ Point: $\mathbb{C}\varepsilon \subset \varepsilon^0 \subset W$

can be split using the ~~symplectic~~ scalar product

Say $\dim W = 2$. $\quad \|\varepsilon\| = 1$ $\qquad W = \underbrace{\mathbb{C}\varepsilon \oplus \varepsilon^\perp}_{\varepsilon^0}$

Start again: Choose a complex line $L \subset W$

Start again. W dim 2 Hilbert space equipped with a $\mathbb{C}$-linear symplectic form. Let L be a complex line in W. Then L and W/L are naturally dual, also can identify $L^\perp \xrightarrow{\sim} W/L$. So we have two $\perp$ lines $W = L \oplus L^\perp$, each with norms ~~...~~

Then $L \otimes L^\perp \xrightarrow{\sim} \Lambda^2 W = \mathbb{C}$, so you get a norm on $\mathbb{C}$ i.e. a pos. number.

---

good approach. Let W be a symplectic v.s. over $\mathbb{C}$ Recall standard canonical form is obtained by choosing $0 \neq e_1 \in W$, then choosing e_2 so that $\omega(e_1, e_2) = 1$, then splitting $W = (\mathbb{C}e_1 + \mathbb{C}e_2) \oplus (\mathbb{C}e_1 + \mathbb{C}e_2)^0$, and repeating the construction for the smaller symplectic space $(\mathbb{C}e_1 + \mathbb{C}e_2)^0$.

Next, suppose W equipped with pos herm form

What can you do? You have constructed a splitting of W into ~~orthogonal~~ perpendicular symplectic planes. So you

(70) List possible approaches. The problem concerns the compact form of the complex symplectic group $Sp(2n,\mathbb{C})$. $Sp(2n,\mathbb{C})$ = auto group of the Complex symplectic space $H(\mathbb{C}^n)$. General theory says maximal compact subgrps of $Sp(2n,\mathbb{C})$ $\exists$ and they are all conjugate. If you pick one K, then you can average to produce a pos herm form on $H(\mathbb{C}^n)$ fixed by K.

So in principle you should be able to find the desired compact form $Sp(2n) = K$ by producing a suitable pos herm. form on $H(\mathbb{C}^n)$.

Now you know that $H(\mathbb{C}^n) = H(\mathbb{C})^{\oplus n}$ so there should be an obvious candidate for an orthonormal basis in $H(\mathbb{C}^n)$.

Let's look at $H(\mathbb{C}) = \begin{bmatrix} \mathbb{C} \\ \mathbb{C} \end{bmatrix}$ IDEA instead of row + column vectors use upper + lower indices — tensor notation. Here's the program. You start with the $\mathbb{C}$-linear hyperbolic symplectic space $H(V)$. Better, you begin with symplectic space W over $\mathbb{C}$, and you construct a standard basis

---

W symplectic space over $\mathbb{C}$, choose $0 \neq \xi_1 \in W$, ~~let ξ_1 the~~ let $\xi_1^0 = \{w \mid \omega(w,\xi_1) = 0\}$, ω non deg $\Longrightarrow$ $\exists\, \xi_2 \in W$ s.t. $\omega(\xi_2,\xi_1) = 1$. Let $V = \mathbb{C}\xi_1 + \mathbb{C}\xi_2$

$$\begin{array}{ccc} V & \subset & W \\ \downarrow & & \downarrow S \\ V^* & \longleftarrow & W^* \end{array}$$

~~Other $\xi_1 \neq 0$, $\omega(\xi_1,\xi_2) = 1$~~

Choose $x_1 \neq 0$, choose ξ_1 : $\omega(x_1,\xi_1) = 1$

Choose $x_1 \neq 0$, verify $x_1^0/\mathbb{C}x_1$ symplectic quotient

(71) W, ω symp space fin. dim. ~~Choose~~ Let V be isot. subspace. There should be a direct way to see that V^0/V is symplectic

$$0 \subset V \subset V^0 \subset W$$

$$\begin{tikzcd} V \arrow[r, hook] & W \arrow[r, two heads] \arrow[d] & W/V \\ & W^* & \end{tikzcd}$$

$$\begin{tikzcd} V^0 \arrow[r, hook] & W^* \arrow[r, two heads] & V^* \\ V \arrow[u, hook] \arrow[r, hook] & W \arrow[u, "s"] \arrow[r, two heads] & W/V \arrow[u] \end{tikzcd}$$

$$\underbrace{0 \subset V}_{V} \subset \underbrace{V^0}_{V^0/V} \underbrace{\subseteq W}_{V^*}$$

inductive step. Choose $x_1 \neq 0$, $\exists\ \xi_1 \neq 0$, $(x_1, \xi_1) = 1$
two hyperplanes x_1^0, ξ_1^0

So given W complex symp v.s. you can split it into symplectic 2-planes, basis x_i, ξ_i. Take $n=1$, have $\mathbb{C}^2$ with $\begin{bmatrix} x_1 \\ \xi_1 \end{bmatrix}^\dagger \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ \xi_2 \end{bmatrix}$ pos. herm form and $\begin{bmatrix} x_1 \\ \xi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ \xi_2 \end{bmatrix}$ symp. form.

group of symmetries is $SU(2)$.

$$Sp(2n) = U(2n) \cap Sp(2n, \mathbb{C}) \subset GL(2n, \mathbb{C})$$

need to understand this a lot better!! Possible methods: $\mathbb{H}^n$? Lie alg.

$$X^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X = 0$$

$$\sim \begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} b & -a \\ d & -c \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$d = -a^t$, $c^t = c$, $b^t = b$. symmetric form

Also want X ~~skew-herm~~.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\dagger = \begin{bmatrix} a^\dagger & c^\dagger \\ b^\dagger & d^\dagger \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$

$a = -a^\dagger$ $\quad b^\dagger = -c$

$d = -d^\dagger$ $\quad c^\dagger = -b$

$b = b^t \Rightarrow b^\dagger = \bar{b}$

$\therefore \bar{b} = -c$

(72) Combine $a^\dagger = -a$, $d^\dagger = -d$ with $-d = a^t$

$d = -a^t = (a^\dagger)^t = \bar{a}$

So Lie alg ~~of~~ Sp(2n) seems to be $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

satisfying $d = \bar{a}$, $c = -\bar{b}$ i.e.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$ which ~~should~~ might yield the $\mathbb{H}$ connection

take $n=1$. $Sp(2) = SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \middle| \; |a|^2 + |b|^2 = 1 \right\}$.

$\therefore$ Lie $Sp(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \middle| \; a + \bar{a} = 0 \right\}$

Something's wrong: Repeat Lie$(Sp(2n, \mathbb{C})) = \left\{ X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \; X^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X = 0 \right\}$ inf form of $g^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

$\begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -b & a \\ -d & c \end{bmatrix} = \begin{bmatrix} -d & c \\ b & -a \end{bmatrix}$

$b^t = b$, $c^t = c$, $d = -a^t$ (i.e. the contragredient repn)

$\therefore n^2 + 2\frac{n(n+1)}{2} = 2n^2 + n = \frac{2n(2n+1)}{2}$

Next you want $X \in$ Lie$(U(2n))$: $X^\dagger + X = 0$.

$\begin{bmatrix} a^\dagger & c^\dagger \\ b^\dagger & d^\dagger \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$ $\quad a^\dagger = -a$, $c^\dagger = -b$, $b^\dagger = -c$ $\quad d^\dagger = -d$, $\overline{b^t = b}$, $\overline{c^t = c}$, $\overline{d = -a^t}$

$\bar{b} = -c$
$-b = \bar{c}$

$d = -a^t$
$d = -d^\dagger$ $\Rightarrow a^t = a^\dagger$

$b^t = b$, $c^t = c$, $c^\dagger = -b$, $b^\dagger = -c$ $\Rightarrow \bar{b} = -c$, $c = -\bar{b}$

$d = -a^t$, $a^\dagger = -a$ $\Rightarrow \bar{a} = d$

$\therefore X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$, $b = b^t$ $\quad n^2$

$b = b^t$ $\quad$ $b^\dagger = -c$ $\quad$ $b^\dagger = \bar b$ $\quad$ $-c = \bar b$

$c = c^t$ $\quad$ $c^\dagger = -b$ $\quad$ $c^\dagger = \bar c$ $\quad$ $-b = \bar c$

$b = b^t$ $\quad$ $c = -\bar b$ $\qquad X = \begin{bmatrix} a & b \\ -\bar b & -\bar a \end{bmatrix}$

$d = -a^t$ $\quad$ $a^\dagger = -a$ $\quad$ $-d = a^t = \bar a$

$a = -d^t$ $\quad$ $d^\dagger = -d$ $\quad$ $-a = d^t = \bar d$

$b^t = b$ $\qquad b^\dagger = -c \implies \bar b = -c$

$c^t = c$ $\qquad\qquad\qquad\qquad\qquad X = \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix}$

$d^t = -a$ $\qquad d^\dagger = -d \implies \bar a = d$

where $b = b^t$

$a^t = -a$

real dim of X is $n^2 + 2\frac{n(n+1)}{2} = n^2 + n^2 + n$

~~What about the~~ Conclusion is a nice ~~formula~~ picture for Lie $Sp(2n)$, namely $\left\{ \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix}, \ \underbrace{a^* = -a}_{a \in \text{Lie } U(n)}, \ b^t = b \right\}$

Contrag. rep. $X \mapsto -X^t$, $a \mapsto -a^t = -\overline{a^*} = -\overline{(-a)} = \bar a$

Q. Is $Sp(2n) \subset U(2n)$ the centralizer of something interesting? There's the question of the role of $\mathbb{H}$? Back to $n=1$. $\mathbb{H}(\mathbb{C})$ standard 2 dim'l symplectic space, action of $Sp(2) = SU(2)$.

Actually, now that you understand the Lie algebra of $Sp(2n)$ you should work out the Cartan subalgebra theory

$n = 1$. $\quad Sp(2) = SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix} \middle| \ |a|^2 + |b|^2 = 1 \right\}$.

Lie $Sp(2) = \left\{ \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix} \middle| \ a^* = -a \text{ i.e. } \bar a = -a \right\}$. Now

you expect $\mathbb{H} = \left\{ \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix} \middle| \ a, b \in \mathbb{C} \right\}$. $\quad \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix} \begin{bmatrix} c & d \\ -\bar d & \bar c \end{bmatrix}$

$$(a + bj)(c + dj) = ac + b\bar c j + adj + b\bar d(-1)$$

$$= (ac - b\bar d) + (ad + b\bar c)j$$

$$\begin{bmatrix} ac - b\bar d & ad + b\bar c \\ -\bar a \bar d - \bar b c & + \bar a \bar c - \bar b d \end{bmatrix}$$

(74) To learn about roots for $Sp(2n)$. You start with $H(\mathbb{C}^n) = H(\mathbb{C})^{\oplus n}$. You first have to understand roots for $n=1$, where $Sp(2) = SU(2)$ $= \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, |a|^2+|b|^2=1 \right\}$. Max torus T is $\left\{ \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \right\}$

Recall $\mathrm{Lie}\, Sp(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, a+\bar{a}=0 \right.$, Cartan subalg

$\mathbb{R}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, adjoint action of T: $\begin{bmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{bmatrix}\begin{bmatrix} i & \\ & -i \end{bmatrix}\begin{bmatrix} e^{-i\theta} & \\ & e^{i\theta} \end{bmatrix}$

$= \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\begin{bmatrix} ie^{-i\theta} & 0 \\ 0 & -ie^{i\theta} \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, Now you want the adjoint action of T on $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}$

$$\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\begin{bmatrix} 0 & \text{[illegible]} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\begin{bmatrix} 0 & e^{i\theta} \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}\begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} = \begin{bmatrix} 0 & e^{2i\theta}b \\ e^{-2i\theta}c & 0 \end{bmatrix}$$

So there are two roots for $Sp(2)$.

Next look at $Sp(2n)$ ~~and T = max torus is the~~ acting on $H(\mathbb{C}^n) = H(\mathbb{C})^{\oplus n}$. $T = \bigoplus$ of 2×2 blocks $\rho(\theta_1)\oplus \cdots \oplus \rho(\theta_n)$ where $\rho(\theta) = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$.

~~Recall Morse significance of roots. Suppose you have a 1-parameter subgp in the Cartan subalgebra, and a point p of this subgroup, then p is a nondegenerate critical point~~

Recall the Morse significant of roots. Take a vector in the Cartan subalgebra, and consider the corresponding geodesic segment starting at 0 and ending at p. This

(75) geodesic is not a critical ~~point~~ for the Morse theory of paths from O to p when ~~the~~ the line segment in the Cartan subalg does not cross ~~the~~ any root hyperplanes. In fact the index of this geodesic segment is the number of hyperplanes crossed.

Possibilities: You consider $H(\mathbb{C}^n) = H(\mathbb{C})^{\oplus n}$ the symplectic space acted on by $Sp(2n)$. For $n=1$ you have the 1-param subgp $\theta \mapsto \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} = \rho(\theta)$ going from 1 to -1. No hyperplanes should have been crossed since the centralizer of $\rho(\theta)$ doesn't jump up for $0<\theta<\pi$. Now ~~[illegible]~~ take $\bigoplus_1^n$ of this situation. Then you should get a nondegenerate critical submanifold of geodesics joining 1 to -1 in $Sp(2n)$.

How can we describe the space of these geodesics: Sphere centered at O in Lie $Sp(2n)$. NO you want to look at $X \in$ Lie $Sp(2n)$ such that $e^{\pi X} = -1$, the eigenvalues of X are $\pm i$. Can assume X is in the Cartan subalgebra. Now such an X in the Cartan subalg you've chosen is a direct sum of blocks of the form $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ or $\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$. But the Weyl group is $\Sigma_n \ltimes (\mathbb{Z}/2)^n$, which means you can arrange that $X = i\begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$.

$$\text{Lie } Sp(2n) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix},\ \begin{matrix} b = b^t \\ a^* = -a \end{matrix} \right\}$$

$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ seems the centralizer of X is $\begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -a^t \end{bmatrix}$

where

$$a^* = -a \implies a^t = -\bar{a}$$
$$\implies \bar{a} = -a^t$$

(76) $\left\{ \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} \;\middle|\; \begin{matrix} a^* = -a \\ a \text{ skew adj} \end{matrix} \right\}$

Let's review. $H(V) = \begin{bmatrix} V \\ V^\wedge \end{bmatrix}$, $\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = a_1^t b_2 + b_1^t a_2$

$O(2n, \mathbb{C})$ = autos of $H(V)$ respecting ~~symm~~ quadratic form

W $\mathbb{C}$-linear quadratic space.

Go back over the Sp theory. $H(V) = \begin{bmatrix} V \\ V^\wedge \end{bmatrix}$, $\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$, $a_1^t b_2 - b_1^t a_2$, $Sp(2n, \mathbb{C})$ = autos of $H(\mathbb{C}^n)$ respecting symp form.

pos. herm.

$U(2n)$ = autos respecting: $\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}^* \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = a_1^* a_2 + b_1^* b_2$

$\underline{Sp}(2n) = U(2n) \cap Sp(2n, \mathbb{C}) \subset GL(2n, \mathbb{C})$.

$Sp(2n, \mathbb{C}) = \left\{ g \in GL(2n, \mathbb{C}) \;\middle|\; g^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$.

$U(2n) = \left\{ g \in GL(2n, \mathbb{C}) \;\middle|\; g^* g = 1 \right\}$.

Lie algs easier to understand

$$X^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X = 0$$

$$\begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$$

$$\begin{bmatrix} -c^t & a^t \\ -d^t & b^t \end{bmatrix} + \begin{bmatrix} c & d \\ -a & -b \end{bmatrix} = 0 \iff \boxed{\begin{matrix} c = c^t & d = -a^t \\ b = b^t & \end{matrix}}$$

$\iff$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ -a & -b \end{bmatrix} \text{ is symm.}$$

$\mathrm{Lie}\, U(2n) = \left\{ g \in GL(2n, \mathbb{C}) \;\middle|\; X^* + X = 0 \right\}$

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = - \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}$$

(77) Calc. Lie $U(2n) \cap$ Lie $Sp(2n, \mathbb{C})$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} = \begin{bmatrix} b^* & d^* \\ -a^* & -c^* \end{bmatrix} \quad \text{symm ?}$$

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$$

$$\begin{bmatrix} b^* & d^* \\ -a^* & -c^* \end{bmatrix} + \underbrace{\begin{bmatrix} c & d \\ -a & -b \end{bmatrix}}_{\text{symm.}} = 0$$

$b = b^t$? ?

$c = c^t$

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{Lie } Sp(2n, \mathbb{C}) \iff \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ -a & -b \end{bmatrix} \text{ is symm.}$$

$$\begin{bmatrix} c & d \\ -a & -b \end{bmatrix} = \begin{bmatrix} c^t & -a^t \\ d^t & -b^t \end{bmatrix} \qquad \boxed{\begin{matrix} c = c^t & d = -a^t \\ b = b^t & \end{matrix}}$$

$X \in$ Lie $U(2n)$

$$\begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0 \qquad \begin{matrix} a = -a^* \\ b = -c^* \end{matrix}$$

$$\boxed{\begin{matrix} a^* + a = 0 \\ d^* + d = 0 \\ b^* + c = 0 \\ c^* + b = 0 \end{matrix}} \qquad \begin{matrix} a^* + a = 0. \\ \\ \bar{b} + c = 0 \\ \bar{c} + b = 0 \end{matrix} \qquad \begin{matrix} -d = a^t \\ -\bar{d} = a^* = -a \end{matrix}$$

$$\boxed{X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \qquad \begin{matrix} a^* = -a \\ b^t = b \end{matrix}}$$

$$n^2 + 2\frac{n(n+1)}{2} = 2n^2 + n$$

$$\cancel{X^* = \begin{bmatrix} a^* & -\bar{b}^* \\ b^* & \bar{a}^* \end{bmatrix} = \begin{bmatrix} -a & -b \\ \bar{b} & -\bar{a} \end{bmatrix}}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} = \begin{bmatrix} -\bar{b} & +\bar{a} \\ -a & -b \end{bmatrix}$$

78 Now need $\mathbb{H}$. Look at $n=1$. Then

$Sp(2,\mathbb{C}) = \left\{ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,\mathbb{C}) \;\middle|\; g^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$

$\Rightarrow \det(g)^2 = 1$ if $+1$, then all $g \in SL(2,\mathbb{C}) = Sp(2,\mathbb{C})$.

$Sp(2) = SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}$.

$\mathbb{H} = \mathbb{C} + \mathbb{C}j \qquad (a+bj)(c+dj) = ac + b\bar{c}j + adj + (-b\bar{d})$

$= (ac - b\bar{d}) + (b\bar{c} + ad)j$

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \begin{bmatrix} c & d \\ -\bar{d} & +\bar{c} \end{bmatrix} = \begin{bmatrix} ac - b\bar{d} & ad + b\bar{c} \\ -\bar{a}\bar{d} - \bar{b}c & \bar{a}\bar{c} - \bar{b}d \end{bmatrix}$$

$H(\mathbb{C})$
$\parallel$

So $\quad Sp(2) = SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, |a|^2 + |b|^2 = 1 \right\} \subset \mathbb{H}$

$H(\mathbb{C}^n) = H(\mathbb{C})^{\oplus n}$ $\quad$ n-dim'l v.s. over $\mathbb{H}$

You have $Sp(2n)$ acting on $H(\mathbb{C}^n)$ preserving pos herm form and the symp. form. Nice hom.

$Sp(2) \longrightarrow Sp(2n)$

<u>Question</u>. Use right mult of $\mathbb{H}$ on itself.

~~Does $\mathbb{H}$ act on $H(\mathbb{C}^n)$~~

$H(\mathbb{C})$ properties of. Something is fishy.

$H(\mathbb{C}) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, x, y \in \mathbb{C} \right\}$ fishy

You have $H(\mathbb{C}) = \begin{bmatrix} \mathbb{C} \\ \mathbb{C} \end{bmatrix}$

Lie $Sp(2)$

(19) There are puzzles about $H(\mathbb{C})$ and $\mathbb{H}$, which need clarification. Repeat. $H(\mathbb{C}) = \begin{bmatrix}\mathbb{C}\\ \mathbb{C}\end{bmatrix}$ with two structures

1) $\begin{bmatrix}a_1\\ b_1\end{bmatrix}^t \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}a_2\\ b_2\end{bmatrix} = a_1 b_2 - b_1 a_2$

2) $\begin{bmatrix}a_1\\ b_1\end{bmatrix}^{\dagger} \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}a_2\\ b_2\end{bmatrix} = \bar{a}_1 a_2 + \bar{b}_1 b_2$

You have subgroups of $GL(2,\mathbb{C})$ $\left|\begin{array}{l} Sp(2,\mathbb{C}) \text{ preserving the volume} \\ U(2) \text{ ——— pos. herm. form}\end{array}\right.$

and both structures are preserved by $SU(2) = Sp(2,\mathbb{C}) \cap U(2)$

$Sp(2,\mathbb{C}) = \left\{ g = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \in GL(2,\mathbb{C}) \;\middle|\; g^t \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} g = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\right\}$

$U(2) = \left\{ \text{———} \;\middle|\; g^* g = 1 \right\}.$

$$\cancel{\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}-b & -d\\ a & +c\end{bmatrix} = \begin{bmatrix}a & c\\ b & d\end{bmatrix}$$

$\begin{bmatrix}a & c\\ b & d\end{bmatrix}$ Let $g = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \in SU(2)$.

then $\begin{bmatrix}\bar{a} & \bar{c}\\ \bar{b} & \bar{d}\end{bmatrix} = g^* = g^{-1} = \begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$ $\quad d = \bar{a}$, $-b = \bar{c}$

so $g = \begin{bmatrix}a & b\\ -\bar{b} & \bar{a}\end{bmatrix}$ $\quad \det(g) = |a|^2 + |b|^2 = 1.$

Now maybe comes the point you missed, Sam ??

$\text{Lie } Sp(2n) = \left\{ X \in gl(2n,\mathbb{C}) \;\middle|\; X^t \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} + \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} X = 0 \right\}$

$X^* + X = 0.$

$$\begin{bmatrix}a^t & c^t\\ b^t & d^t\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} + \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix} = 0$$

$a^* + a = 0$
$b^* + c = 0$
$d^* + d = 0$

$$\begin{bmatrix}-c^t & a^t\\ -d^t & b^t\end{bmatrix} + \begin{bmatrix}c & d\\ -a & -b\end{bmatrix} = 0$$

$c = c^t$
$b = b^t$
$a^t + d = 0$

$d^* = a^t \quad \bar{d} = a$
$b + \bar{c} = 0 \quad d = \bar{a}$

(80) Again get Lie $Sp(2n) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \middle| \begin{array}{l} b = b^t \\ a^* + a = 0 \end{array} \right\}$

Look at what happens when you take two hyperbolic symplectic planes together.

Go back to $SU(2)$ acting on $H(\mathbb{C})$

Let $\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \in SU(2)$ let $\begin{bmatrix} x \\ y \end{bmatrix} \in H(\mathbb{C})$

$$Sp(2) = \left\{ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ g^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} -c & a \\ -d & b \end{bmatrix} \qquad \begin{bmatrix} -c & a \\ -d & +b \end{bmatrix}$$

Question. Does $\mathbb{H}$ act on $H(\mathbb{C})$?

$$(a + bj)(c + dj) = ac + b\bar{c}j + adj - b\bar{d}$$
$$= (ac - b\bar{d}) + (ad + b\bar{c})j$$

Recall $H(\mathbb{C}) = \begin{bmatrix} \mathbb{C} \\ \mathbb{C} \end{bmatrix}$ $\qquad \begin{bmatrix} x \\ \xi \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ \eta \end{bmatrix}$

compact Lie gp

$SU(2)$ acts on $H(\mathbb{C})$ respecting $\mathbb{C}$-linear structure, two forms. $\qquad \begin{bmatrix} x \\ \xi \end{bmatrix}^* \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ \eta \end{bmatrix}$

scalar mult by $\mathbb{C}$.

Aim: To see whether $H(V)$ V a $\mathbb{C}$ v.s. with pos. herm-form is naturally a vector space over $\mathbb{H}$ equipped with a positive (suitably hermitian) form.

Does $\mathbb{H}^n$ have such a form?

Look at $\mathbb{H}$ Let $q_1, q_2 \in \mathbb{H}$. Look at $q_1^* q_2$

where if $q_1 = a + bj$, $q_1^* = \bar{a} - j\bar{b} = \bar{a} - bj$

$a = t + xi$
$b = y + zi$

$$q = t + xi + yj + zk = a + bj$$
$$q^* = t - xi - yj - zk = \bar{a} - bj$$
$$q^*q = (\bar{a} - bj)(a + bj) = |a|^2 - bja + \bar{a}bj - bjbj$$
$$= |a|^2 - \cancel{b\bar{a}j} + \cancel{\bar{a}bj} - b\bar{b}(-1) = |a|^2 + |b|^2$$

<u>Question</u>: If V is a ~~Hilbert space~~ pos herm. v.s is there an $\mathbb{H}$ analog for $\mathbb{H} \otimes_{\mathbb{C}} V$?

$$\langle q_1 \otimes v_1 | q_2 \otimes v_2 \rangle = \langle v_1 | q_1^* q_2 \otimes v_2 \rangle = \underbrace{q_1^* q_2} \langle v_1 | v_2 \rangle \ ?$$

Maybe what you really want is just what you need to define a positive herm. form. Thus $q_1^* q_2$ needs to be a complex no.

Discussion. You have a construction $V \mapsto H(V)$ (in fact it's a functor for isos.) which assoc. a symplectic vector space/$\mathbb{C}$ to a $\mathbb{C}$-vector space V. In addition, if V is equipped with a pos herm. form, then there is an induced pos herm form on $H(V)$, and $V \mapsto H(V)$ is a functor for unitary isos. Compatible with $\oplus$ so that $\exists$ a $H(\mathbb{C}) \otimes_{\mathbb{C}} V \xrightarrow{\sim} H(V)$

Notice: There's a right action of $\mathbb{C}$ on $H(\mathbb{C})$. $H(\mathbb{C})$ should be an $\mathbb{H}, \mathbb{C}$ bimodule, ~~problem~~ probably the left $\mathbb{H}$ mult and some interesting homom. $\mathbb{C} \hookrightarrow \mathbb{H}$ followed by right mult. Recall that there are lots of subfields in $\mathbb{H}$ which are $\simeq \mathbb{C}$.

$$H(\mathbb{C}) = \begin{bmatrix} \mathbb{C} \\ \mathbb{C} \end{bmatrix} \qquad \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$ typical elt of $\mathbb{H}$ acting on $H(\mathbb{C})$

You want an operator on $H(\mathbb{C})$ probably not $\mathbb{C}$-linear

(82) which commutes with all $\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$

$$\mathbb{H} = \{x + yj \mid x, y \in \mathbb{C}\}.$$

$$(a+bj)(x+yj) = (ax - b\bar{y}) + (ay + b\bar{x})j$$

left mult by $\cancel{\begin{bmatrix} a & \\ & \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}$ got it wrong.

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let's get right + left straight. Point is that $H(\mathbb{C})$ is a $\mathbb{C}$-vector space and $SU(2)$ acts as $\mathbb{C}$-linear operators. So you want ~~quaternions~~ $\mathbb{H}$ to left act on $H(\mathbb{C})$ as a $\mathbb{C}$-vector space.

$\underset{?}{SU(2) \subset \mathbb{H}}$ $\qquad \mathbb{H} \ni x + jy$

$\begin{bmatrix} x \\ y \end{bmatrix} \in H(\mathbb{C}) \qquad \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cancel{ax} + by \\ -\bar{b}x + \bar{a}y \end{bmatrix}$

$a \quad b$

$$(a + jb)(x + jy) = (ax - \bar{b}y) + j(bx + \bar{a}y)$$

Somehow you have to organize this sensibly.
~~You feel that~~ $H(\mathbb{C})$ has a natural $SU(2)$ action
$H(\mathbb{C})$ is the basic rep of $SU(2)$. Can form
$H(\mathbb{C}^n) = H(\mathbb{C})^{\oplus n}$ rep of $SU(2)^{\times n}$

$$SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \ |a|^2 + |b|^2 = 1 \right\} \subset \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \in M_2\mathbb{C} \right\}.$$

(83) $SU(2)$ acts on $H(\mathbb{C})$ by $\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}\begin{bmatrix} x & y \\ -\bar{y} & \bar{x} \end{bmatrix} = \begin{bmatrix} ax - b\bar{y} & ay + b\bar{x} \\ -\bar{a}\bar{y} - \bar{b}x & \bar{a}\bar{x} - \bar{b}y \end{bmatrix}$$

$$(a + bj)(x + yj) = (ax - b\bar{y}) + (ay + b\bar{x})j$$

$$(a + jb)(x + jy) = (ax - \bar{b}y) + j(\bar{a}y + bx)$$

There is probably something simple happening.

Let's make a list of objects + properties, and try to put them in some order.

- $H(V)$ pos herm. form
$\mathbb{C}$-linear symplectic form
- $H(V)$ vector space over $\mathbb{H}$.

Somehow introducing $\mathbb{H}$ gets you into difficulty. The natural symmetry group is then $GL(n, \mathbb{H})$. For $n=1$ this $\mathbb{H}^\times$ which is 4 diml unlike $Sp(2) = SU(2)$ which is 3 diml. Also the Lie algebras are

$gl(n, \mathbb{H})$ $4n^2$ diml

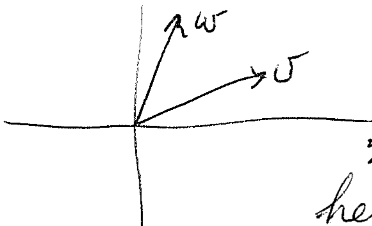
Lie $Sp(2n)$ $n(2n+1)$ diml ($\mathbb{R}$ sense)

Describe hope: $H(\mathbb{C}^n)$ is an n-diml vector space over $\mathbb{H}$ equipped with some sort of quaternionic inner product, whose auto gp is $Sp(2n)$.

You want some kind of $\mathbb{R}$-bilinear form

$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \longrightarrow L$ IDEA: You might use the quaternionic projective line $\mathbb{H}P^1$. This is a property of sfields. So review what happens for $\mathbb{R}, \mathbb{C}$. $\mathbb{R}$: Consider a real vector space V with Euclidean dot product, you have the projective space of lines. What is the goal? Distance between 2 points of the projective line?

$\mathbb{R}^2$ two $\neq 0$ vectors v, w then $\dfrac{|v \cdot w|}{\|v\| \cdot \|w\|} = \cos\theta$



$\mathbb{C}$ case $\mathbb{CP}_1$ = Riemann sphere S^2.

Again consider the positive hermitian product $v^t w$ which is ~~now~~ a complex numbers. You have 2 points on the Riemann sphere. The only geometric invariant around seems to be their angle.

You have $SU(2)$

Problem. Is there a notion of $\mathbb{H}$-module equipped with positive (suitably) hermitian form?

Restrict to dim 1. Look at field $\mathbb{R}$.

An $\mathbb{R}$ module of dim 1 is a line L and the appropriate bilinear form is a positive quadratic form; it leads to a unit sphere in L. A $\mathbb{C}$-module L of dim 1 has a unit sphere given by a positive ~~herm~~ form on L.

Next look at an $\mathbb{H}$-module of dim 1.

Proj. line over $\mathbb{H}$. , Proj space $\mathbb{HP}^n$

What might be nice about $\mathbb{HP}^n$ is its compactness.

Look at 1-diml subspaces of $\mathbb{H} \oplus \mathbb{H}$

Let's return to the problem of a good norm on ~~a~~ vector space over $\mathbb{H}$. This discussion of $\mathbb{HP}^1$ and $\mathbb{HP}^n$ may have clarified things.

(85) Aim to understand $Sp(2n) = U(2n) \cap Sp(2n,\mathbb{C})$.

Because you have $H(\mathbb{C}^n) = \begin{bmatrix}\mathbb{C}^n \\ \mathbb{C}^n\end{bmatrix}$ with

pos herm form $\begin{bmatrix}x_1 \\ y_1\end{bmatrix}^t \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}x_2 \\ y_2\end{bmatrix}$ symp. form $\begin{bmatrix}x_1 \\ y_1\end{bmatrix}^t \begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix}\begin{bmatrix}x_2 \\ y_2\end{bmatrix}$

$\underline{\mathcal{L}(U(2n))} \cap \mathcal{L}(Sp(2n,\mathbb{C}))$

$$X = \begin{bmatrix}a & b \\ c & d\end{bmatrix} \qquad X + X^* = 0 \qquad \begin{bmatrix}a & b \\ c & d\end{bmatrix} = -\begin{bmatrix}a^* & c^* \\ b^* & d^*\end{bmatrix}$$

$$X^t \begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix} + \begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix} X = 0$$

$$\begin{bmatrix}a^t & c^t \\ b^t & d^t\end{bmatrix}\begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix} + \begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix}\begin{bmatrix}a & b \\ c & d\end{bmatrix} = 0$$

$$\begin{bmatrix}-c^t & a^t \\ -d^t & b^t\end{bmatrix} + \begin{bmatrix}c & d \\ -a & -b\end{bmatrix} = 0$$

$c = c^t \qquad a^* + a = 0$

$b = b^t \qquad -b^* = c = -\bar{b}$

$-a^t = d$

$$\left\{ X = \begin{bmatrix}a & b \\ -\bar{b} & \bar{a}\end{bmatrix}, \quad a^* + a = 0, \; b^t = b \right\}$$

$\bar{d} = -\bar{a}^t = -a^* = a$

$d = \bar{a}$

$$n^2 + 2\frac{n(n+1)}{2} = 2n^2 + n.$$

$n=1 \quad \mathcal{L}(SL(2,\mathbb{C})) = \left\{\begin{bmatrix}a & b \\ c & -a\end{bmatrix}\right\}$

$n=1 \quad \mathcal{L}(SU(2)) = \left\{\begin{bmatrix}a & b \\ -\bar{b} & \bar{a}\end{bmatrix} \,\middle|\, \bar{a} + a = 0\right\}$

You now have $\mathcal{L}\,Sp(2n)$ under control, although the group itself is mysterious. The $\mathbb{H}$ link remains a puzzle, + probably is a phantom.

Why? You expected (case $n=1$) that $H(\mathbb{C})$ can be identified with $\mathbb{H}$, at least that $H(\mathbb{C})$ is an $\mathbb{H}$-module of rank 1.

You certainly have $SU(2)$ acting on $H(\mathbb{C})$, and $SU(2) =$ norm 1 subgroup of $\mathbb{H}^\times$??

86 $\quad \mathbb{H} = \{\underbrace{t + xi + yj + zk}_{q}\} \qquad q^* q = t^2 + x^2 + y^2 + z^2$

$a = t + xi$

$b = y + zi$

$$(a + bj)(c + dj) = (ac - b\bar{d}) + (ad + b\bar{c})j$$

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}\begin{bmatrix} c & d \\ -\bar{d} & \bar{c} \end{bmatrix} = \begin{bmatrix} ac - b\bar{d} & ad + b\bar{c} \\ -\bar{a}\bar{d} - \bar{b}c & \bar{a}\bar{c} - \bar{b}d \end{bmatrix}$$

Something NON UNITAL is occurring?

$H(\mathbb{C}) = \begin{bmatrix} \mathbb{C} \\ \mathbb{C} \end{bmatrix}$

Return to geodesics. $\quad \mathcal{L}Sp(2n) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \in \mathfrak{gl}(2n, \mathbb{C}) \;\; a^* = -a,\ b^t = b \right\}$

Cartan subalgebra where $b = 0$, a diagonal $\in i\mathbb{R}^n$. You know that every $X \in \mathcal{L}Sp(2n)$ is conjugate to ~~a diagonal~~ such a diagonal matrix. You want to pick something with large symmetry. $\begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix}$ or $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$? To decide look at $n = 1$. Max torus is $\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$, $0 \le \theta < \pi$

$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ should be conjugate in $\mathcal{L}SU(2)$

Let $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathcal{L}Sp(2n)$, ~~better let~~ $\cancel{X = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}}$

Find $\{X \in \mathcal{L}Sp(2n) \mid [X, J] = 0\}$

$X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad XJ = \begin{bmatrix} -b & a \\ -\bar{a} & -\bar{b} \end{bmatrix}$

$JX = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} = \begin{bmatrix} -\bar{b} & \bar{a} \\ -a & -b \end{bmatrix} \qquad X = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

b real symm. $\quad a$ real skewsymm.
$\therefore\ a + ib$ skew herm.

where $b = \bar{b} = b^t$
$\bar{a} = a = -a^* = -a^t$

87

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \begin{bmatrix} c & d \\ -\bar{d} & \bar{c} \end{bmatrix} = \begin{bmatrix} ac - b\bar{d} & ad - b\bar{c} \\ -\bar{a}\bar{d} - bc & \bar{a}\bar{c} - \bar{b}d \end{bmatrix}$$

$$\begin{bmatrix} c & d \\ -\bar{d} & \bar{c} \end{bmatrix} \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} = \begin{bmatrix} ca - d\bar{b} & d\bar{a} + cb \\ -\bar{d}a - \bar{c}\bar{b} & -\bar{d}b + \bar{c}\bar{a} \end{bmatrix}$$

Subtract: $\begin{bmatrix} [a,c] - (b\bar{d} - d\bar{b}) & \\ -\bar{a}\bar{d} + \bar{d}a - bc + \bar{c}b & \end{bmatrix}$

$b\bar{d} - d\bar{b} = bd^* - db^*$ because $b = b^t$, $d = d^t$

~~$(-\bar{a}\bar{d} + \bar{d}a)^t = -\bar{d}^* a^* + a^t \bar{d}^*$~~

$a^t = a^* = -\bar{a}$

$$(-\bar{a}\bar{d} + \bar{d}a)^t = a^t\bar{d} - \bar{d}\bar{a}^t$$

$$= -\bar{a}\bar{d} - \bar{d}(-a) = -\bar{a}\bar{d} + \bar{d}a$$

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \overset{?}{=} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \iff b = 0$$

So it looks like the stabilizer of $\begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix}$ is $\begin{matrix} \mathcal{L}U(n) & 0 \\ 0 & \end{matrix}$

$a \mapsto -a^t = (-a^*)^{\sim} = \bar{a}$

So it looks like the stabilizer of $J = \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix}$ is $U(n)$ embedded in $Sp(2n)$ via $a \mapsto \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}$

~~[illegible]~~ $\therefore$ getting $Sp(2n)/U(n) \xrightarrow{\sim} \Omega Sp(2n)$

(88) $SO(2n)$. Look at $H(\mathbb{C}) = \begin{bmatrix} \mathbb{C} \\ \mathbb{C} \end{bmatrix}$ $\begin{bmatrix} a \\ b \end{bmatrix}^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$

$O(2, \mathbb{C}) = \left\{ g \in GL(2, \mathbb{C}) \mid g^t g = 1 \right\}$ $\quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad g^tg=1 \Downarrow \det(g) = \pm 1$ $\quad$ If $+1$

$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $\quad$ $d = a$, $c = -b$, $b = -c$ $\quad$ $SO(2, \mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \ a^2 + b^2 = 1 \right\}$

If $\det = -1$

$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} -d & b \\ c & -a \end{bmatrix}$ $\quad g = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ $\quad \det = -(a^2+b^2) = -1$.

so $g^2 - 1 = 0$ $\quad \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \begin{bmatrix} a & b \\ b & -a \end{bmatrix} = \begin{bmatrix} a^2+b^2 & 0 \\ 0 & b^2+a^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

So much for the complex gp. $O(2, \mathbb{C})$. Now

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} = \begin{bmatrix} a+bi & a-bi \\ ia-b & -ai-b \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix}$$

So $SO(2, \mathbb{C}) \simeq \left\{ \begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix}, \ a^2 + b^2 = 1 \right\}$.

Consider a complex quadratic space W, choose a "standard" basis, so that

$$W = \begin{bmatrix} V \\ V^* \end{bmatrix} \quad \omega \quad \begin{bmatrix} v_1 \\ \varphi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix}$$

To study autos of W respecting the quad form

$$\left\{ g \in \underbrace{\mathrm{Aut}(W)}_{O(2n, \mathbb{C})} \ \middle| \ g^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

get $\det(g) = \pm 1$ + $n=1$ case

Easier is $\mathcal{L}\, O(2n, \mathbb{C}) = \left\{ X \in gl(W) \ \middle| \ X^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} X = 0 \right\}$

$\begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} + \begin{bmatrix} d & c \\ b & d \end{bmatrix} = 0 \quad X = \begin{bmatrix} a & b \\ & -a^t \end{bmatrix}$?

(89) $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad X^t + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} X \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0$

$$\begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} + \begin{bmatrix} d & c \\ b & a \end{bmatrix} = 0$$

$\therefore \quad c^t + c = 0 \qquad a^t + d = 0 \qquad X = \begin{bmatrix} a & b \\ c & -a^t \end{bmatrix}$

$\quad\;\; b^t + b = 0$

where $b^t + b = 0$

$c^t + c = 0$

Alt. $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad X^t + X = 0$

$$\begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$$

$\cancel{a^t = -a} \quad \cancel{c^t = b}$

$\cancel{b^t = c} \quad \cancel{d^t = -d}$

$a^t = -a \qquad d^t = -d$

$c = -b^t$

$X = \begin{bmatrix} a & b \\ -b^t & d \end{bmatrix}$ $\quad a^t = -a$

$d^t = -d$?

Aim: $O(2n, \mathbb{C})$. $W = \begin{bmatrix} V \\ V^* \end{bmatrix}$ $\quad \begin{bmatrix} v_1 \\ \varphi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix}$

You consider the hyperbolic quadratic space of dim $2n$.

$$\mathcal{L}O(2n, \mathbb{C}) = \left\{ X \in gl(W) \,\middle|\, X^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} X = 0 \right\}.$$

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\qquad \begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} + \begin{bmatrix} d & c \\ b & a \end{bmatrix} = 0$

So you find $\quad c^t + c = 0 \qquad a^t + d = 0$

$\qquad\qquad\quad b^t + b = 0 \qquad d^t + a = 0$. $\qquad X = \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}$

This means X consists of the $gl(V)$ action on V and V^* and two skew-symm. wings.

(90) So far you've looked at $O(2n,\mathbb{C})$, now you want the compact form. Proceed as in Sp case + look for pos. herm. form.

Take $n=1$. $H(\mathbb{C}) = \begin{bmatrix}\mathbb{C}\\ \mathbb{C}\end{bmatrix}$, $\begin{bmatrix}a_1\\ b_1\end{bmatrix}^t \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}a_2\\ b_2\end{bmatrix}$,

$$\mathcal{L}\,O(2,\mathbb{C}) = \left\{\begin{bmatrix}a & b\\ c & -a\end{bmatrix},\ b,c \text{ skew symm. } \therefore b=c=0\right\}.$$

$g \in O(2,\mathbb{C})$, $g = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$ s.t. $g^t\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}g = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$

$\Rightarrow \det(g)^2 = 1$. $g \in SO(2,\mathbb{C}) \Leftrightarrow \begin{bmatrix}a & c\\ b & d\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$

$\begin{bmatrix}c & a\\ d & b\end{bmatrix} = \begin{bmatrix}-c & a\\ d & -b\end{bmatrix} \Longleftrightarrow \begin{matrix}b=0\\ c=0\end{matrix}$ $\therefore g = \begin{bmatrix}a & 0\\ 0 & d\end{bmatrix}$ $ad=1$

Next introduce the pos. herm. form on $H(\mathbb{C}^n)$ $\begin{bmatrix}x_1\\ y_1\end{bmatrix}^*\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_2\\ y_2\end{bmatrix}$

Then $X^* + X = 0$ which means

for $X = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$ that $\begin{matrix}a^*+a=0 & c^*+b=0\\ b^*+c=0 & d^*+d=0\end{matrix}$ in addition to b,c skew symm, $d=-a^t$

$a = -a^* = \overline{-a^t} = \bar{d}$. So X respects both forms $\Leftrightarrow X = \begin{bmatrix}a & b\\ -\bar{b} & \bar{a}\end{bmatrix}$, $a^*+a=0$, $b^t+b=0$. Count

$\boxed{\text{Need } b=-b^t \Rightarrow -\bar{b} = b^* = c}$

real dims $n^2 + 2\frac{n(n-1)}{2} = 2n^2-n = \frac{2n(2n-1)}{2}$

So you now have some control over $\mathcal{L}\,O(2n)$.

But you should be able to reconcile the two answers:

$$\mathcal{L}\,O(2n) = \left\{X \in gl(2n,\mathbb{C}) = \begin{bmatrix}a & b\\ +\bar{b} & \bar{a}\end{bmatrix},\ \begin{matrix}a^*+a=0\\ b^t+b=0\end{matrix}\right\}$$

$$\mathcal{L}\,O(2n) = \left\{\underset{\begin{bmatrix}a & b\\ -b^t & d\end{bmatrix}}{X} \in gl(2n,\mathbb{R})\quad \begin{matrix}a^t+a=0\\ d^t+d=0\end{matrix}\right\}$$

$$2\frac{n(n-1)}{2} + n^2 = 2n^2-n$$

(9) $\mathcal{L}SO(2n) = \{X \in gl(2n,\mathbb{R}) \mid X^t + X = 0\}$

$X = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} = \begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix}$ $\quad a^t = -a \quad d^t = -d \quad b^t = -c$ $\quad \dim = 2n^2 - n$, $n^2 + 2\frac{n(n-1)}{2}$

$\mathcal{L}SO(2n) = \left\{ \begin{bmatrix} a & b \\ -b^t & d \end{bmatrix} \in gl(2n,\mathbb{R}),\ a^t + a = d^t + d = 0 \right\}$

~~Maximal~~ Cartan subalg.

$\begin{bmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & \lambda_2 \\ -\lambda_2 & 0 \end{bmatrix}$ $\qquad \begin{bmatrix} 0 & & & \lambda_1 \\ & & \lambda_2 & \\ & -\lambda_2 & & \\ -\lambda_1 & & & 0 \end{bmatrix}$

$\exp \theta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad 0 < \theta < \pi$

Centralizer of $\left[\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & -1 & & \\ -1 & & & \end{array}\right]$ Looks too hard.

Idea: a, d are independent ~~inf~~ orthogonal transf operating on a general $b : \mathbb{R}^n \to \mathbb{R}^n$, so there should be characteristic values the eigenvalues of $b^t b$. Canon form for b should be $\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ Then get $\left[\begin{array}{c|c} & \begin{matrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{matrix} \\ \hline \begin{matrix} -\lambda_1 & & \\ & \ddots & \\ & & -\lambda_n \end{matrix} & \end{array}\right]$

Why is $\begin{bmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & \lambda_2 \\ -\lambda_2 & 0 \end{bmatrix}$

conjugate to $\left[\begin{array}{cc|cc} & & \lambda_1 & \\ & & & \lambda_2 \\ \hline -\lambda_1 & & & \\ & -\lambda_2 & & \end{array}\right]$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \lambda_1 & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & -\lambda_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ -\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ -\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ -\lambda_1 & 0 & 0 & 0 \\ 0 & -\lambda_2 & 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \qquad e^{\theta X} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Centralizer of X in $SO(2n)$ should be $U(n)$.

Recap a bit. You study $\Omega SO(2n)$ via Morse theory on the space of paths from 1 to -1. You get nice nondeg critical submanifold $\simeq SO(2n)/U(n)$, which is the space of complex structures J on the Euclidean space $\mathbb{R}^{2n}$ such that J is orthogonal.

So now you look at Ω of the symmetric space $SO(2n)/U(n)$. You need the analog of the Lie alg in the compact group case.

(93) $Sp(2n)$ review. $V = \mathbb{C}^n$ $H(V) = \begin{bmatrix} V \\ V^* \end{bmatrix}$, $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$Sp(2n,\mathbb{C}) = \mathrm{Aut}\ H(V)$ preserving symp form.

$Sp(2n) = Sp(2n,\mathbb{C}) \cap U(2n) \subset GL(2n,\mathbb{C})$.

$\mathrm{Lie}\ Sp(2n,\mathbb{C}) = \left\{ X \in gl(2n,\mathbb{C}) \,\middle|\, X^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X = 0 \right\}$

$$\begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -b & a \\ -d & c \end{bmatrix} = \begin{bmatrix} -d & c \\ b & a \end{bmatrix}$$

$\therefore b = b^t,\ c = c^t,\ d = -a^t.$ $X^* + X = 0$ $\begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$

$a^* + a = 0$, $b = -c^*$
$c = -b^*$, $d = -d^*$ $\therefore a \in \mathcal{L}U(n)$: $a^* + a = 0$, $d^* + d = 0$ $d = -a^t$, $d^* = -\bar{a}$

$d = -a^t$, $\bar{d} = -a^* = a$

$$\boxed{\mathcal{L}Sp(2n) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* = -a \\ b = b^t \end{matrix} \right\}}$$

$c = -b^* = -\bar{b}^t = -\bar{b}$

$n^2 + 2\frac{n(n+1)}{2} = 2n^2 + n.$

Cartan subalg $\begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}$ a diagonal. $J = \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix}$

$e^{\theta J} = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$ $0 \le \theta \le \pi$. Lie centralizer of J

is $\left\{ \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}, a \in \mathcal{L}U(n) \right\}$

Conjugacy class of the Lie elt J

So next you want a symmetric space situation.

$BO \xmapsto{\Omega} O \supset SO \xmapsto{\Omega} O/U$

Let's try once more $H(V) = \begin{bmatrix} V \\ V^* \end{bmatrix}$, $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$.

$O(2n,V)$ = auto gp of $H(V)$ preserving symm-form

$\mathcal{L}O(2n,V) = \left\{ X \in Gl(2n,\mathbb{C}),\ X^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} X = 0 \right.$

$\begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} + \begin{bmatrix} d & c \\ b & a \end{bmatrix} = 0$ $b^t = -b,\ c^t = -c,\ -a^t = d$

(94) $\mathcal{L}O(2n,\mathbb{C}) \simeq \left\{ \begin{bmatrix} a & b \\ c & -a^t \end{bmatrix}, \; \begin{matrix} b^t = -b \\ c^t = -c \end{matrix} \right\}$ $\qquad n^2 + 2\frac{n(n-1)}{2}$

$$\begin{bmatrix} a^* & c^* \\ b^* & -\bar a \end{bmatrix} + \begin{bmatrix} a & b \\ c & -a^t \end{bmatrix} = 0 \qquad \begin{matrix} a^* + a = 0 \\ \bar a^t + \bar a = 0 \end{matrix} \qquad \begin{matrix} b = -c^* = +\bar c \\ b = -c^* = (-1)\bar c^t = (-1)(\overline{-c}) = \bar c \end{matrix}$$

$$\mathcal{L}O(2n) = \left\{ \begin{bmatrix} a & b \\ +\bar b & \bar a \end{bmatrix}, \; \begin{matrix} b^t = -b \\ a^* = -a \end{matrix} \right\} \qquad n^2 + n(n-1) = 2n^2 - n.$$

Again take $J = \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix}$, centralizer $\left\{ \begin{bmatrix} a & 0 \\ 0 & \bar a \end{bmatrix}, a^* = -a \right\}$

Conjugacy class of J is all complex structures.

Alternative real only version: $\Omega SO(2n)$. Have $\mathbb{R}^{2n}$ with $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1^t x_2 + y_1^t y_2$. $\quad X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0 \qquad \begin{matrix} a^t + a = 0 \\ c + b^t = 0 \\ d^t + d = 0. \end{matrix} \qquad \left\{ \begin{bmatrix} a & b \\ -b^t & d \end{bmatrix}, \; \begin{matrix} a^t + a = 0 \\ d^t + d = 0 \end{matrix} \right\}$$

$n(n-1) + n^2$ ✓ $\qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\qquad$ again get $O(2n)/U(n)$

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deal next with $O(2n)/U(n)$ which is a symmetric space. You want the loop space. But first you need analog of Lie G. Recall a compact Lie gp G is the symmetric space $G \times G/\Delta G$, where you have left + right mult symmetries identity of G is origin. Look for polar decomp.

What about

$$\mathcal{L}SO(2n) \ni X = \begin{bmatrix} a & b \\ -b^t & d \end{bmatrix} \quad \begin{matrix} a^t = -a \\ d^t = -d \end{matrix}$$

(95) Review symmetric spaces. Riemannian manifold (complete), at each point the reflection arising from $v \mapsto -v$ on the tangent space and the exponential map is an isometry. Ex G compact conn with left + right invariant metric, then $G \times G$, + flip should yield a symmetric space.

Ex. Grassmannian. $W \subset \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ $\quad F, \varepsilon$

$g = F\varepsilon$. A point W is the same as a $g \in U(V)$ such that $\varepsilon g \varepsilon = g^{-1}$. So you have the Grassmannian

You need more examples. e.g. $SO(2n)/U(n)$. should be analogous to $U(n) \times U(n) / \Delta U(n)$.

Consider then $SO(2n)/U(n)$

$$\mathcal{L}\, SO(2n) = \{ X \in gl(2n, \mathbb{R}) \mid X^t + X = 0 \}$$

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \qquad X = \begin{bmatrix} a & b \\ -b^t & d \end{bmatrix} \qquad \begin{array}{l} a^t + a = 0 \\ d^t + d = 0. \end{array}$$

$$JX - XJ = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ -b^t & d \end{bmatrix} - \begin{bmatrix} a & b \\ -b^t & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -b^t & +d \\ -a & -b \end{bmatrix} - \begin{bmatrix} -b & a \\ -d & -b^t \end{bmatrix} = 0 \Rightarrow \begin{array}{l} d = a \\ b = b^t \end{array}$$

$$X = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \qquad \begin{array}{l} a^t + a = 0 \\ b^t = b \end{array} \qquad (a+ib)^t = -a + ib$$

$$(a+ib)^* = a^* - ib^*$$
$$= a^t - ib^t$$
$$= -(a+ib)$$

(96) $SO(2n)/U(n)$. $\mathcal{L}SO(2n) = \{X \in gl(2n, \mathbb{R}) \mid X^t + X = 0\}$ i.e. $X = \begin{bmatrix} a & b \\ -b^t & d \end{bmatrix}$: $a^t + a = 0$, $d^t + d = 0$ $\quad$ $n \times n$ matrices

Let $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$JX = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ -a & -b \end{bmatrix}$$

$$XJ = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -b & a \\ -d & c \end{bmatrix}$$

So $[J, X] = 0$ iff $a = d$, $b = -c$

i.e. $X = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

$c = -b = -b^t \Rightarrow b$ symm.

Combining these two conditions ~~[illegible]~~ you find that the centralizer of J in $\mathcal{L}SO(2n)$ is $\left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : \begin{matrix} a^t + a = 0 \\ b^t = b \end{matrix} \right\}$.

Real dimension count $\frac{n(n-1)}{2} + \frac{n(n+1)}{2} = n^2$. This centralizer can be identified with $\mathcal{L}U(n)$, namely, a skew hermitian matrix has $\left| \begin{matrix} \text{skew symmetric real part} \\ \text{symmetric imag part} \end{matrix} \right.$ $\quad a + ib$

$$(a+ib)^* = a^* - ib^* = a^t - ib^t = -(a+ib).$$

Next you want a better model, one in which $U(n)$ is obvious. $H(V) = \begin{bmatrix} V \\ V^\wedge \end{bmatrix}$ $\quad \begin{bmatrix} v_1 \\ \varphi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix}$

X
$\parallel$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in so(2n, \mathbb{C})$ means $\begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$

$\begin{bmatrix} c^t & a^t \\ d^t & b^t \end{bmatrix} + \begin{bmatrix} c & d \\ a & b \end{bmatrix} = 0$ $\quad a^t + d = 0$, $c + c^t = 0$, $b + b^t = 0$

$X \in u(2n)$ means $0 = X^* + X = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \begin{matrix} a^* + a = 0 \\ c^* + b = 0 \\ b^* + c = 0 \\ d^* + d = 0 \end{matrix}$

$c = -b^* = +\bar{b}$ $\quad d = -a^t \Rightarrow d^* = -\bar{a}$, $d^* = -d$ $\Rightarrow d = \bar{a}$

$\therefore$ $X \in u(2n) \cap so(2n, \mathbb{C})$ means $X = \begin{bmatrix} a & b \\ +\bar{b} & \bar{a} \end{bmatrix}$ $\quad \begin{matrix} a^* + a = 0 \\ b^t + b = 0. \end{matrix}$

It seems too easy to make a mistake. Go over ~~it~~ again.

(97) $V=\mathbb{C}^n$ $\quad H(V)=\begin{bmatrix}V\\ V^\wedge\end{bmatrix}$ $\quad \begin{bmatrix}v_1\\ \varphi_1\end{bmatrix}^t\begin{bmatrix}0&1\\ 1&0\end{bmatrix}\begin{bmatrix}v_2\\ \varphi_2\end{bmatrix}$

$$so(2n,\mathbb{C})=\left\{X\in gl(2n,\mathbb{C}) ; X^t\begin{bmatrix}0&1\\ 1&0\end{bmatrix}+\begin{bmatrix}0&1\\ 1&0\end{bmatrix}X=0\right\}$$

$$\begin{bmatrix}a^t&c^t\\ b^t&d^t\end{bmatrix}\begin{bmatrix}0&1\\ 1&0\end{bmatrix}+\begin{bmatrix}0&1\\ 1&0\end{bmatrix}\begin{bmatrix}a&b\\ c&d\end{bmatrix}=\begin{bmatrix}c^t&a^t\\ d^t&b^t\end{bmatrix}+\begin{bmatrix}c&d\\ a&b\end{bmatrix}=0$$

$$so(2n,\mathbb{C})=\left\{\begin{bmatrix}a&b\\ c&-a^t\end{bmatrix} : \begin{matrix}d+a^t=0\\ b^t+b=0\\ c^t+c=0\end{matrix}\right\} \qquad (-a^t)^*=-\bar{a}$$

$$so(2n,\mathbb{C})\cap u(2n)=\left\{\begin{bmatrix}a&b\\ c&-a^t\end{bmatrix} : \begin{bmatrix}a&b\\ c&-a^t\end{bmatrix}+\begin{bmatrix}a^*&c^*\\ b^*&-\bar{a}\end{bmatrix}=0\right\}$$

$a+a^*=0$, $b+c^*=0$, $c=-b^*=+(b^t)^*=\bar{b}$,

$\Updownarrow$

$\bar{a}+a^t=0 \Rightarrow -a^t=\bar{a}$, $\therefore so(2n,\mathbb{C})\cap U(2n)=\left\{\begin{bmatrix}a&b\\ \bar{b}&\bar{a}\end{bmatrix} \middle| \begin{matrix}a^*+a=0\\ b^t+b=0\end{matrix}\right\}$

So $so(2n)=\left\{\begin{bmatrix}a&b\\ \bar{b}&\bar{a}\end{bmatrix}\in gl(2n,\mathbb{C}) \middle| \begin{matrix}a^*+a=0\\ b^t+b=0\end{matrix}\right\}$. You hope this model is better, because $U(n)$ is nicely embedded. $\mathbb{R}$-dim check: $n^2+2\frac{n(n-1)}{2}=2n^2-n$

Look at $O(2n)/U(n)$. A point of this space is an orthogonal transformation J on the Euclidean space $\mathbb{R}^{2n}$ satisfying $J^2=-1$, i.e. a complex structure on $\mathbb{R}^{2n}$ which is compatible with the Euclidean structure. Note that $J^2=-1$ is not far from $J^2=1$. The idea is to see whether you can extend your treatment of Grassmannians, where you have the basepoint ε and study another point F using the "midpoint" of the geodesic joining ε to F.

What can you do with 2 anti commuting complex structures? If you have $i^2=-1$, $j^2=-1$, and $ij=-ji$ then you are dealing with the quaternion gp of order 8. You really ought to understand first representations of i,j s.t. $i^2=j^2=-1$

(98) You should work in the appropriate orthogonal group. The group you start with is $SO(2n)$, which is the group of symmetries of $H(V)$ ~~and its~~ equipped with its two structures.

Consider the space of orth. transf J of square -1 on the Euclidean space $\mathbb{R}^{2n}$. $O(2n)$ should act transitively on this space and the isotropy group of the standard J namely $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ should be $U(n)$. You want to treat complex structures $J^2=-1$ in analogy with involutions $F^2 = 1$. (self adjoint: $F=F^*$)

Recall that for the Grassmannian situation you have ~~the basepoint~~

$$V = \begin{bmatrix} V_+ \\ V_- \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$ – this is the basepoint

and you consider another involution F. Set $g = F\varepsilon$, then $\varepsilon g \varepsilon = \varepsilon F \varepsilon \varepsilon = \varepsilon F = g^{-1}$. There's also the C.T.

$g = \frac{1+X}{1-X}$ defined when $g+1$ is invertible, and also

$g^{1/2} = \frac{1+X}{(1-X^2)^{1/2}}$. Our aim now is to see whether this generalizes, rather, that there is some analog for complex structures.

Let J, K be complex structures: $J^2 = K^2 = -1$

Put $g = JK$. Then $JgJ^{-1} =$
$J(JK)J^{-1} = (-1)K(-J) = KJ = JK$ since $KJJK = K(-1)K = (-1)K^2 = (-1)(-1) = 1$. Thus $JgJ^{-1} = g^{-1}$.
Similarly $K(JK)K^{-1} = KJ = (JK)^{-1}$. Thus conjugation by both J and K sends g to g^{-1}.

$g = JK$, $g^{-1} = KJ$. Maybe you should ask about central extensions of $\mathbb{Z}/2 \times \mathbb{Z}/2$. First look at the group generated by J, K with the relations

(99) Look at two complex structures J, K

J orthogonal: $J^t = J^{-1} \iff J^t J = 1$.
and $J^2 = -1$, also $J^t = -J$

Possibilities:

J orthogonal $\quad J^t = J^{-1}$
J skew-sym $\quad J^t = -J$
$\quad J^2 = -1$

~~Fix a base point i.e. your linear space is $\mathbb{C}^n$ with usual $\langle x, y\rangle = x^* y$ and J is mult by i~~

Idea last night: $(J+K)^2 = -1 + (JK + KJ) - 1$
$\qquad g + g^{-1}$

$J+K$ skew-symm $\Rightarrow (J+K)^2 \leq 0$. Also $g = JK$ is orthogonal. It seems clear that $\frac{g+g^{-1}}{2}$ will give the desired eigenvalues, just like $\frac{F\varepsilon + \varepsilon F}{2}$.

$J(JK+KJ) = -K + JKJ \qquad \frac{JK+KJ}{2} = \frac{g+g^{-1}}{2}$
$(JK+KJ)J = JKJ - K \qquad$ central, symm.

$n=1 \quad \mathbb{R}^2 \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Only Other poss is $-J$.

two cases $\quad K = J, -J$

$(J+K)^2 = -2 + g + g^{-1}$
$-(J+K)^2 = 2 - (g+g^{-1}) \geq 0$

$J^2 = K^2 = -1$
J, K skew symm.

$(J+K)^2 \leq 0 \qquad (J+K)^2 = -2 + g + g^{-1} \leq 0 \qquad \frac{g+g^{-1}}{2} \leq 1$

extreame cases are $\quad J = K \qquad g = -1$
$\qquad -J = K \qquad g = 1$

$(J-K)^2 = -2 - g - g^{-1} \leq 0 \qquad 2 + g + g^{-1} \geq 0 \qquad \frac{g+g^{-1}}{2} \geq -1$

(100) ~~[illegible]~~ V Euclidean space dim $2n$.

J, K orth complex structures on V

$$J^2 = -1 \qquad J^t = J^{-1} = -J$$

so for a complex structure J orth $\Longleftrightarrow$ J skew symm.

You want eigenvalue picture of K from the viewpoint of J. ~~[illegible]~~ ~~$g^{-1} = (-J)(-K) = JK$~~

$$g = JK, \qquad g^{-1} = (-K)(-J) = KJ$$

$$0 \leq -(J+K)^2 = 1 - JK - KJ + 1 = 2 - g - g^{-1}$$

$$JgJ^{-1} = J(JK)(-J) = KJ = g^{-1} \qquad g^t = g^{-1}$$

$$Kg\cancel{K}^{-1} = K(JK)K^{-1} = KJ = g^{-1}$$

so $\frac{g+g^{-1}}{2}$ is symmetric & commutes with J, K.

spectrum $\subset [-1, 1]$. Look at $g = +1$ eigenspace

i.e. $JK = 1$ $\quad J = K^{-1} = -K$.

Suppose that $\frac{g+g^{-1}}{2} = \lambda$, $-1 < \lambda < 1$. So what are you doing? You have V Euclidean space dim $2n$ with J, K skew-symm. $J^2 = K^2 = -1$.

$g = JK$, $g^{-1} = KJ$ orth

V Euclidean dim $2n$ $\qquad J, K$ skew-symmetric $\quad J^2 = K^2 = -1$

$J^t = -J = J^{-1}$ $\quad \therefore J$ orth

$g = JK$ $\qquad \frac{g+g^{-1}}{2}$ symmetric, $-1 \leq \lambda \leq 1$, central

What the minimum you get from an eigenvalue $\lambda \in [-1, 1]$

Assume $\frac{g+g^{-1}}{2} = \cos\theta$ $\qquad 0 < \theta < \pi$.

Suppose you complexify V and replace J, K by $-iJ, -iK$

s.a. involutions

(101) Look at $\mathbb{R}^4 = \mathbb{C}^2$, pick a standard matrix for J

$$\left[\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right]$$

Instead try

$$J = \left[\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline -1 & & & \\ & -1 & & \end{array}\right]$$

This suggests using the non real picture for $O(2n)$. IDEA: Cayley Transform for handling the group elts.

Nonreal model ~~$\mathfrak{L}O(2n) = \left\{ \begin{bmatrix} a & b \\ -b^t & -a \end{bmatrix} : c^t + a = 0 \right\}$~~

$$0 = \underbrace{\begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} + \begin{bmatrix} d & c \\ b & a \end{bmatrix} = 0$$

$$\mathfrak{L}O(2n,\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & -a^t \end{bmatrix} : \begin{array}{l} b^t + b = 0 \\ c^t + c = 0 \end{array} \right\} \qquad \begin{bmatrix} a^* & c^* \\ b^* & -a^{t*} \end{bmatrix} + \begin{bmatrix} a & b \\ c & -a^t \end{bmatrix} = 0$$

$$\mathfrak{L}O(2n,\mathbb{C}) \cap U(2n) = \left\{ \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} : \begin{array}{l} a^* + a = 0 \\ b^t + b = 0 \end{array} \right\}$$

$c^* = -b$

$\bar{c} = -b^t = b$

$d = -a^t$

$d^* = -d$

$\therefore d = -d^* = -(-a^t)^* = \bar{a}$

also $c^* = \bar{c}^t = -\bar{c}$

$-b \quad \therefore b = \bar{c}$

J is the basepoint complex structure having stabilizer $U(n)$; somehow you have to handle K.

(102) $J^t + J = 0$, $J^t J = 1$, $J^2 = -1$

Go back to Grass case. $F^2 = 1$, $\varepsilon^2 = 1$.

$$(F+\varepsilon)^2 = 1 + F\varepsilon + \varepsilon F + 1 = 2 + g + g^{-1}, \quad F = F^*, F = F^{-1}$$

Idea: Singularities of C.T. might be easy to find. $-1 + JK + KJ - 1 = (J+K)^2 \leq 0$

$$\frac{g+g^{-1}}{2} \text{ [redacted] } = 1 + (J+K)^2 \leq 1$$

$$-(J-K)^2 = +1 + JK + KJ + 1$$

$$0 \leq +2 + (g + g^{-1})$$

So you find $-1 \leq \frac{g+g^{-1}}{2} \leq 1$. You guess that the cases $\frac{g+g^{-1}}{2} = \pm 1$ are special

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~~Review: You have~~ $O(2,\mathbb{C}) \ni g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$g^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \det(g) = \pm 1$.

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$\det = 1$ $\quad g = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$

$\det = -1$ $\quad g = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$

$$\begin{bmatrix} c & a \\ d & b \end{bmatrix} = \begin{bmatrix} -c & a \\ d & -b \end{bmatrix} \qquad \begin{matrix} b = 0 \\ c = 0 \\ d = a^{-1} \end{matrix}$$

$\mathcal{L}O(2n,\mathbb{C})$: $\begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$

$$\begin{bmatrix} c^t & a^t \\ d^t & b^t \end{bmatrix} + \begin{bmatrix} c & d \\ a & b \end{bmatrix} = 0 \qquad \begin{matrix} c^t + c = 0 \\ b^t + b = 0 \\ d = -a^t \end{matrix}$$

$\begin{bmatrix} a & b \\ c & -a^t \end{bmatrix}$ $\qquad \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} + \begin{bmatrix} a & b \\ b & d \end{bmatrix} = 0 \qquad \begin{matrix} b^* + b = 0 \\ \shortparallel \\ -\bar{b} \end{matrix}$

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$\begin{bmatrix} a & b \\ \bar b & \bar a \end{bmatrix}$ $\qquad d = -a^t = -\overline{a^*} = \bar a$

So $\boxed{\mathcal{L}O(2n) = \left\{ \begin{bmatrix} a & b \\ \bar b & \bar a \end{bmatrix} : \begin{matrix} a^* + a = 0 \\ b^t + b = 0 \end{matrix} \right\}}$

Go back to $Sp(2n)$. $\quad \mathcal{L}Sp(2n,V) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} = X : X^tJ + JX = 0 \right\}$

$X^t = -JXJ^{-1} = JXJ$

$\begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$

$\begin{bmatrix} -c^t & a^t \\ -d^t & b^t \end{bmatrix} + \begin{bmatrix} c & d \\ -a & -b \end{bmatrix} = 0$

$d = -a^t$
$c = c^t$
$b = b^t$

$\begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$

$d^* = a^t \Rightarrow \bar d = a \Rightarrow d = \bar a$

$a^* + a = 0$
$b^* + c = 0$ $\quad \therefore \boxed{c = -\bar b}$
$\bar b$

[redacted]

$-d = +d^* = +a^t$

$\boxed{\mathcal{L}Sp(2n) = \left\{ \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix} : \begin{matrix} a^* + a = 0 \\ b^t = b \end{matrix} \right\}}$

You want the symmetric space which is the manifold of geodesics going from $+1$ to -1 in $Sp(2n)$. One idea is to treat $O(2n)$, $Sp(2n)$ similarly. ~~What's nice about the pictures is~~ Find Cartan subalg. For $n=1$ The max. torus is $\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$ for higher n use $\oplus$ to get $\begin{bmatrix} e^{i\theta} I_n & \\ & e^{-i\theta} I_n \end{bmatrix}$ $\qquad J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$X = \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix} \in \mathcal{L}Sp(2n)$ $\qquad \begin{matrix} a^* + a = 0 \\ b^t = b \end{matrix}$

$\frac{1}{i}(JX - XJ) = \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix} - \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\begin{bmatrix} a & b \\ \bar b & -\bar a \end{bmatrix} = \begin{bmatrix} a & -b \\ -\bar b & -\bar a \end{bmatrix}$ $\qquad b = 0$

(104) Repeat: You consider

$\mathcal{L}Sp(2n) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* + a = 0 \\ b^t = b \end{matrix} \right\}$. Calculate:

the centralize of $J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ is $\left\{ \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} : a^* + a = 0 \right\}$

It should be true that the centralizer of $J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ in $Sp(2n)$ is $\left\{ \begin{bmatrix} g & 0 \\ 0 & \bar{g} \end{bmatrix} : g \in U(n) \right\}$

$g^* = g^{-1}$

$\bar{g} = (g^{-1})^t$

contragredient

You have a homom. $U(n) \longrightarrow Sp(2n)$ which may be related to the functor

$V \longmapsto \mathbb{H} \otimes_{\mathbb{C}} V$. ??

Look at $n = 1$. $Sp(2) = SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \det = 1 \right\}$

Can you understand J's in $SU(2)$? There is the obvious one: $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ whose centralizer is $\begin{bmatrix} U(1) & 0 \\ 0 & U(1) \end{bmatrix}$.

It seems that you get a J for each line in $\mathbb{C}^2$. You need to understand this much better.

You want a much better description of elts of $Sp(2n)$. Hope: Set up something involving creation + annihilation operators. This should be related to b being symmetric. You have a vague recollection about isotropic subspaces $W < \mathbb{C}^{2n}$

IDEA Your $H(V)$ with symp and pos herm forms should be the space of BOSON creation + annihilation operators. Similarly for FERMION. The action of the Lie algebra symplectic resp orthogonal is given by the bracket with quadratic operators. This should yield a standard form for the orthogonal case.

105 Problem: Find a simple description of elements of $Sp(2n)$, such as unitary matrices of degree $2n$ with same property. Take $n=1$. $Sp(2n) = SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : |a|^2+|b|^2 = 1 \right\}$

Start with $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{U}(2n)$ $\qquad a + a^* = 0,\ d + d^* = 0,\ c = b^*$

$$\begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$X = JX^tJ \qquad X = -JX^tJ^{-1}$$

$$e^{hX} = e^{hJ(-X^t)J^{-1}} = J \underbrace{e^{h(-X^t)}}_{\left(e^{-hX}\right)^t} J^{-1}$$

So J intertwines g and $(g^t)^{-1}$

$\boxed{gJ = J(g^t)^{-1}}$ if you require $g^* = g^{-1}$, g unitary.

then you get $\boxed{gJ = J\bar{g}}$.

Look at $O(2n)$ case $\qquad X^t \overbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}^{F} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} X = 0$

$FXF = -X^t$, $FgF = (g^t)^{-1}$, and if g also required unitary then $FgF = \bar{g}$. What about $X^t\varepsilon + \varepsilon X = 0$

$\varepsilon X \varepsilon = -X^t$; $\varepsilon g \varepsilon = (g^t)^{-1}$ ~~also then~~ if g also unitary ~~then~~ $\boxed{\varepsilon g \varepsilon = \bar{g}}$ $\qquad (g^t)^{-1} = (g^{-1})^t = (g^*)^t = \bar{g}$

(106) $Sp(2) = SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \det = 1 \right\}$.

Pick J s.t. $J^* = -J = J^{-1}$ e.g. $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. Centralizer of $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ is the 1-parameter subgroup of $\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}$ s.t. $|a|^2 = 1$. conjugacy class of J can be identified with $SU(2)/\text{diagonal} \simeq$ Riemann sphere. Given $L \subset \mathbb{C}^2$ you define the corresp J to be i on L and $-i$ on $L^\perp$.

Note that you have now identified $Sp(2n)/U(n)$ when $n=1$. Next you want to understand the symmetric space structure. This should involve an involution (automorphism of order 2) on $Sp(2n)$ with fixed subgroup $U(n)$. This involution should be conjugation by J.

Let $\sigma : G \longrightarrow G$ be a group homom. such that $\sigma^2 = id$. Let G^σ be the subgroup of elements fixed by σ. What do you expect? $G^\sigma = \{ g \mid g^\sigma = g \}$ $= \{ g \mid g^\sigma g^{-1} = 1 \}$. What happens when G is abelian? Then normally you would expect an approximate splitting of G into $G^+ \times G^-$ where $G^+ = G^\sigma$ and $\sigma(g_-) = g_-^{-1}$. If $\sigma(g) = g^{-1}$, then $g = \sigma(\sigma g)$ $= \sigma(g^{-1}) = (\sigma g)^{-1}$. ?

Try $x(\sigma x)^{-1}$.

Example. $G = H \times H \quad \sigma(h_1,h_2) = (h_2,h_1)$

$$G^\sigma = \{(h_1,h_2) \mid (h_1,h_2) = (h_2,h_1)\} = \Delta H \subset G.$$

Look at (h_1,h_2) such that $(h_1,h_2)^{-1} = \sigma(h_1,h_2)$

i.e. $(h_1^{-1},h_2^{-1}) = (h_2,h_1) \implies h_1^{-1} = h_2 \Rightarrow h_2^{-1} = h_1.$

So you get a "transversal" set $T = \{(h,h^{-1})\} \subset G.$

What is the map $T \hookrightarrow G \to G/\Delta H$?

$$\begin{array}{ccccc} T & \hookrightarrow & G & \to & G/\Delta H \\ (h,h^{-1}) & & (h,h^{-1}) & & \shortparallel \\ & & & & H \end{array}$$

Seems to $h \mapsto h^2$.

Check: $\Delta H \to H\times H \longrightarrow (H\times H)/\Delta H$

$$(h_1,h_2) \longmapsto h_1h_2^{-1}$$

$$G^\sigma \hookrightarrow G \twoheadrightarrow G/G^\sigma \overset{?}{\hookrightarrow} G$$

$$g \longmapsto g(\sigma g)^{-1}$$

Let G act on itself by

$g * x = (\sigma g)x g^{-1}$ Then G^σ = stab. of 1

so you find $g \mapsto (\sigma g)g^{-1}$?

The interesting point here I think is that if you let G acts on itself via $g*x = gx(\sigma g)^{-1}$, then G^σ is the stabilizer of 1, and the map

$$gG^\sigma \longmapsto g(\sigma g)^{-1}$$

identifies G/G^σ with a subset of elements of G which are "inverted" by σ: $\sigma(g(\sigma g)^{-1}) = (\sigma g)g^{-1}.$

(108) So return to $SU(2)$ with σ given by conjugation with $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$:

$$\sigma \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \bar b & -\bar a \\ a & b \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} +\bar a & \bar b \\ -b & a \end{bmatrix}$$

fixed subgp is $\left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : \begin{array}{l} a, b \text{ real} \\ a^2 + b^2 = 1. \end{array} \right\}$

Better $J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ $\quad \sigma\begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix}\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} =$

$$= \begin{bmatrix} ia & ib \\ i\bar b & -i\bar a \end{bmatrix}\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} a & -b \\ \bar b & \bar a \end{bmatrix}$$ fixed subgp is $\begin{bmatrix} a & 0 \\ 0 & \bar a \end{bmatrix}$

What you ultimately want is the eigenvalue picture for the symmetric spaces. In the $SU(2)/U(1) = \mathbb{CP}^1 = S^2$ example $J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ corresponds to $0 \in S^2$ and there's given another complex structure: $K^* = -K = K^{-1}$. Once the basepoint J is fixed you can conjugate K by elts of the centralizer $U(1)$ (diag).

Need some formula describing possible K. Shift to hermitian involutions $-iJ = \varepsilon$, $-iK = F$. It seems you're back to the Grassmannian situation.

Recap. Your aim is to understand the compact symmetric space $Sp(2n)/U(n)$. Recall that $Sp(2n)$ $= \left\{ g \in U(2n) \;\middle|\; g^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$. $\mathcal{L}Sp(2n) =$

$$= \left\{ \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix} : \begin{array}{l} a^* + a = 0 \\ b^t = b \end{array} \right\}. \qquad n=1 \quad \left\{ \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix} ; \ a + \bar a = 0 \right\}$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \begin{bmatrix} -c & a \\ -d & b \end{bmatrix} = \begin{bmatrix} -c & a \\ -d & +b \end{bmatrix}$$

109 $\quad g^t \begin{bmatrix}0&1\\-1&0\end{bmatrix} g = \begin{bmatrix}0&1\\-1&0\end{bmatrix} \iff \det(g) = 1.$

$g^* g = \begin{bmatrix}1&0\\0&1\end{bmatrix} \qquad \begin{bmatrix}\bar a&\bar c\\ \bar b&\bar d\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$

$\begin{bmatrix}\bar a&\bar c\\ \bar b&\bar d\end{bmatrix} = \begin{bmatrix}d&-b\\-c&a\end{bmatrix} \qquad \begin{matrix} d = \bar a \\ c = -\bar b\end{matrix} \qquad g = \begin{bmatrix}a&b\\-\bar b&\bar a\end{bmatrix}$

You have $Sp(2n) = U(2n) \cap Sp(2n, \mathbb{C}) \subset GL(2n, \mathbb{C})$

$Sp(2n)/U(n) = \{ K^* = -K = K^{-1} \}$ in $H(\mathbb{C}^n)$

replace K by $-iK = F$.

Look closely at $n=1$. $\quad Sp(2) = \left\{ \begin{bmatrix}a&b\\-\bar b&\bar a\end{bmatrix}, \det = 1 \right\}$

$S^2 = Sp(2)/U(1)$ embedded $\begin{bmatrix}e^{i\theta}&0\\0&e^{-i\theta}\end{bmatrix}$ basepoint $\begin{bmatrix}i&0\\0&-i\end{bmatrix}$

Let's figure out what to do. You have $Sp(2n)/U(n)$ and you want its path space, an approximation that is, given by nice manifold of geodesics in the symm. space. You need an appropriate tangent vector to the Symspace, there should a conjugation moving this vector to the Cartan subspace which should the part of the maximal torus $Sp(2n)/U(n)$. What's the max torus of $Sp(2n)$? Should be contained in $\oplus\, SU(2)$?? Go back to $n=1$. $\quad SU(2)/U(1) = S^2$.

You need some understanding of symm. spaces