23, Review: critical point of $\frac{1}{2}x^tAx$ subject to a linear constraint $y^tx = c$. Let

$F = \frac{1}{2}x^tAx + \lambda(c - y^tx)$ $dF = x^tA\,dx - \lambda y^t dx + d\lambda(c - y^tx) = 0$

i.e. $Ax = \lambda y$ and $c = y^tx$.

$\Longrightarrow$ $x = \lambda A^{-1}y$, $c = \lambda y^tA^{-1}y$, $\lambda = \dfrac{c}{y^tA^{-1}y}$, $x = \dfrac{c}{y^tA^{-1}y}A^{-1}y$

critical value $\dfrac{1}{2}\dfrac{c}{y^tA^{-1}y}\, y^t\cancel{A^{-1}}\cancel{A}\,\dfrac{c}{y^tA^{-1}y}A^{-1}y = \dfrac{1}{2}\dfrac{c^2}{y^tA^{-1}y}$ ← critical point

---

Review response to an external emf attached from a node α to the ground O. This situation can be handled entirely by means of $\bar{C}^0$, the node voltage space, equipped with the power form. In the notation above $x \in \bar{C}^0$, $\frac{1}{2}x^tAx$ = power form, $x \mapsto y^tx$ is $x = x(\alpha)$. | What to say next? Full phase space picture

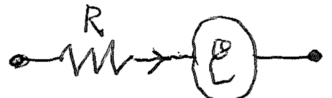
$$\begin{tikzcd}
\mathbb{R} & \bar{C}^0 \arrow[l, Leftarrow] \arrow[r, leftrightarrow, "\delta"] \arrow[d, "\delta^t R^{-1}\delta"'] & C^1 \arrow[r] \arrow[d, "R^{-1}"] & H^1 \\
\mathbb{R} \arrow[r, hook] & \bar{C}_0 & C_1 \arrow[l, Leftarrow, "\delta^t"'] & H_1 \arrow[l, leftrightarrow]
\end{tikzcd}$$

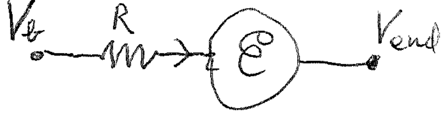
describe what's happening. You have the network X with **external emf** applied between node α and node O. You are confident that the response to this external emf is **a node potential** φ which satisfies a Poisson's equation, more precisely φ is harmonic away from the nodes α, O. | Review the problem. You have a conn **R-network** X with two $\neq$ nodes A, O specified. You want to fix V_A, $V_0 = 0$ but allow a node current going in at A coming ~~out~~ out at B. Apparently this works, and you can do a few examples. You even have some ideas of how the node current arises via Lagrange multipliers.

Idea: Go back to the problem of finding the stationary value of $\frac{1}{2}x^tAx$ subject to the condition $c = y^tx$. This formulation uses only the "voltage" picture. Suppose you use the Lagrange multiplier method:

p3

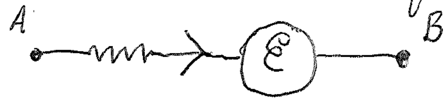
$F = \frac{1}{2}x^tAx + \lambda(c - y^tx)$, $\partial_x F = Ax - \lambda y = 0$
and $\partial_\lambda F = c - y^tx = 0$. So $x = \lambda A^{-1}y$, and
$c = y^tx = \lambda y^tA^{-1}y$, $\boxed{\lambda = \frac{c}{y^tA^{-1}y}, \quad x = \frac{c}{y^tA^{-1}y}A^{-1}y}$

Probably what you want is a symplectic interpretation of what's happening. Clearly λy is a dual variable to x. Maybe c and λ^{-1} are dual, or maybe there's a better interpretation using "affine" ideas. What you should be able to do is to find a symplectic ~~double~~ of x^tAx, $c - y^tx$ involving "voltage", "current" variables. Let's review the picture of a connected R-network equipped with a pair of nodes $A \neq O$ = the ground. Then you have voltage space $\bar{C}^0$ equipped with a pos. def. quadratic form $\delta^t R^{-1} \delta$ which is the restriction of the power form on C^1. You also have a surjection $\gamma : \bar{C}^0 \longrightarrow \mathbb{R}$, $\varphi \mapsto \varphi(A) - \varphi(0) = \varphi(A)$. For each $c \in \mathbb{R}$, $\gamma^{-1}\{c\}$ is the set of node potentials $\varphi \in \bar{C}^0$ such that $\varphi(A) = c$, i.e. such that c is the voltage drop from A to O. This is an inhomogeneous condition. Compare it to a Thevenin condition in which Ohm's Law for an R-edge: $V = RI$, is allowed to become inhomogeneous: $V = +RI - \mathcal{E}$?  ?
(Back to sign problems). Think of the edge as a real battery, $\mathcal{E}$ pure emf, R internal resistance. If you think of the edge as a pure emf in series with an internal resistance, i.e. a real battery, then you expect the current to flow toward the positive ~~direction of the~~ terminal. $V_{\text{beginning}} - RI + \mathcal{E} = V_{\text{end}}$

γ3  $V_b - RI + \mathcal{E} = V_{end}$

$\therefore$ $V_{edge} = -RI + \mathcal{E}$, which has not the sign you expect, want.

Idea: What is the power in the non homogeneous situation? You are reminded of the momentum of a charged particle in an EM field, something like $p - eA$.

Look at  $\varphi(A) - RI + \mathcal{E} = \varphi(B)$

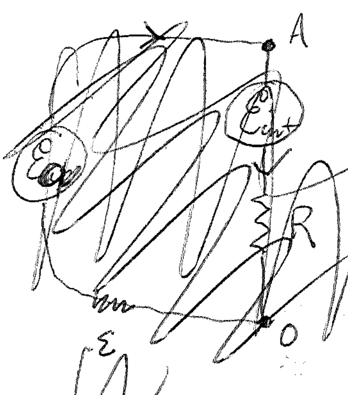
The voltage drop $V = \varphi(A) - \varphi(B)$ ~~is~~ $-RI + \mathcal{E}$. NO

Suppose $\mathcal{E} = 0$. Then the voltage drop is $V = \varphi(A) - \varphi(B) = -RI$ whence $\varphi(B) > \varphi(A)$. So this must be the mistake you have been making. How to clarify?

$$V_A - V_C = RI \qquad V_C - V_B = -\mathcal{E}$$

When you add you get $V_A - V_B = RI - \mathcal{E}$

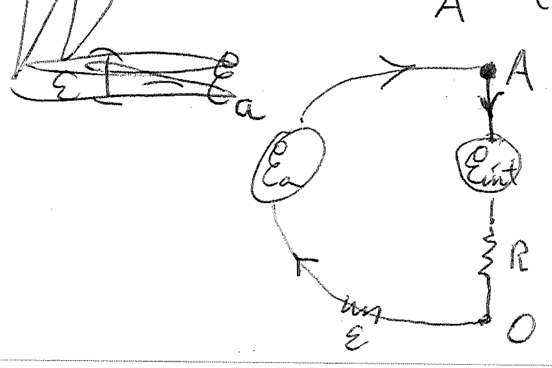
$\therefore$ It looks like you should introduce a new sign convention for the edge emf: the voltage drop for an emf $= \mathcal{E}$ is $-\mathcal{E}$



$\cancel{V_A - V_0 = -\mathcal{E}_{int} + RI}$

$\cancel{V_0 - V_A = -\mathcal{E}_a + \varepsilon I}$

$\cancel{V_A - (V_A + \mathcal{E}_{int}) + RI = V_0}$



$-\mathcal{E}_{int} + RI + \varepsilon I -$

83 Start again, but avoid the sign difficulties by setting up the linear algebra together with quadratic form. [Begin with a connected R-network, linear algebra, better linearization:

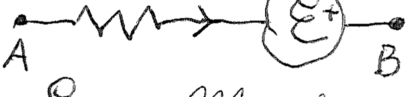
$$\begin{tikzcd}
\bar{C}^0 \arrow[r, hook] & C^1 \arrow[r, two heads] \arrow[d, "R^{-1}"] & H^1 \\
\bar{C}_0 & C_1 \arrow[l, two heads] & H_1 \arrow[l, hook']
\end{tikzcd}$$

The cochain s.e.s. and the chain s.e.s. are naturally dual, R^{-1} is a pos. def quadratic form on C^1, it gives the power of any edge voltage configuration. This quadratic form induces an orthogonal splitting of the cochain s.e.s., and also of the chain s.e.s., these splittings are compatible with the duality.

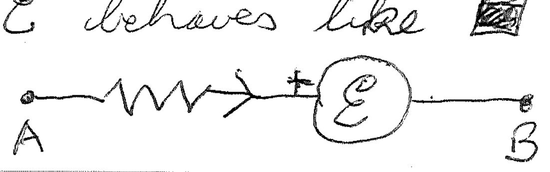
Next discuss circuit equations. A state of the network consists of a $V \in \bar{C}^0$ and an $I \in H_1$, i.e., edge voltages which come from a node potential and loop currents (= 1-cycles). You can replace I by RI which $\in (\bar{C}^0)^\perp$. So the splitting $C^1 = \bar{C}^0 \oplus (\bar{C}^0)^\perp$ sets up a 1-1 correspondence between C^1 and states of the network:

$$\begin{pmatrix} \bar{C}^0 \\ H_1 \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}} \begin{pmatrix} \bar{C}^0 \\ (\bar{C}^0)^\perp \end{pmatrix} = C^1$$

So $\forall\, \mathcal{E} \in C^1$ you get $\mathcal{E} = V + RI$ NO for unique $V \in \bar{C}^0$, $I \in H_1$. (Signs?



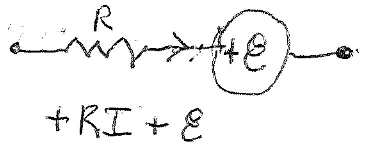
, V is the voltage drop $V_A - V_B = RI - \mathcal{E}$. Maybe it's simpler to put the emf in the opposite direction, so $\mathcal{E}$ behaves like RI. Try this: $V_A - V_B = RI + \mathcal{E}$



)

Back to starting point: connected R-network

$$\begin{tikzcd} V \in \bar{C}^0 \arrow[r, hook] & C^1 \arrow[r, two heads] & H^1 \\ \bar{C}_0 & C_1 \arrow[l, two heads] \arrow[u, "R"] & H_1 \ni I \arrow[l, hook] \end{tikzcd}$$



Inhomogeneous eqn $V + RI = \mathcal{E}$.

Next you would like to discuss an external emf. You have $\gamma : \bar{C}^0 \longrightarrow \mathbb{R}$ given, you want to restrict to $V \in \bar{C}^0$ s.t. $\gamma(V) = c$. So if $\gamma(V) = V_A - V_B$ you are fixing the voltage drop from A to B. This condition is inhomogeneous.

How to study this? You have $\bar{C}^0$ equipped with pos. quad form $\delta^t R^{-1} \delta : \bar{C}^0 \longrightarrow \bar{C}_0$. Write x for an element of $\bar{C}^0$, $A = \delta^t R^{-1} \delta$, and y^t for $\gamma : \bar{C}^0 \to \mathbb{R}$. Want stationary value of $x^t A x$ subject to $y^t x = c$. So far no real use of C^1.

You want to use the augmented network, which should amount to combining $\delta : \bar{C}^0 \longrightarrow C^1$ with $\gamma : \bar{C}^0 \longrightarrow \mathbb{R}$ to get

$$\bar{C}^0 \xrightarrow{\begin{pmatrix}\delta \\ \gamma\end{pmatrix}} \begin{pmatrix} C^1 \\ \mathbb{R} \end{pmatrix}$$ which is clearly 1-1 as δ is.

Next you need a quadratic form on $\begin{pmatrix} C^1 \\ \mathbb{R}\end{pmatrix}$ ~~extending~~ which pulls back via $\begin{pmatrix}\delta \\ \gamma\end{pmatrix}$ to the given form on $\bar{C}^0$, equivalently, you want a quad form Q on $\begin{pmatrix} C^1 \\ \mathbb{R}\end{pmatrix}$ such that $\begin{pmatrix}\delta \\ \gamma\end{pmatrix}^t Q \begin{pmatrix}\delta \\ \gamma\end{pmatrix} = \delta^t R^{-1} \delta$

~~[illegible]~~ Repeat. You begin with:

$$\begin{array}{ccc} \bar{C}^0 & \overset{\delta}{\hookrightarrow} & C^1 \\ \gamma\downarrow & & \\ \mathbb{R} & & \end{array}$$

together with the quadratic form R^{-1} on C^1. You want to obtain a map $\bar{C}^0 \xrightarrow{\binom{\delta}{\gamma}} \begin{pmatrix} C^1 \\ \mathbb{R} \end{pmatrix}$ together with a positive quadratic form on ↗ with appropriate properties. Everything ultimately should result from the quad form on C^1.

Focus upon the data, namely, the filtration

$$0 \subset \operatorname{Ker}\gamma \underset{\text{codim } 1}{<} \bar{C}^0 \overset{\delta}{\hookrightarrow} C^1$$

It seems that $(\operatorname{Ker}\gamma)^\perp$ is your space $\begin{pmatrix} C^1 \\ \mathbb{R} \end{pmatrix}$. It ? contains the line $\bar{C}^0 \cap (\operatorname{Ker}\gamma)^\perp$.

You should begin with splitting this filtration

$$\operatorname{Ker}\gamma \oplus \mathbb{R} \oplus \underbrace{(\delta\bar{C}^0)^\perp}_{\operatorname{Ker}\delta^*}$$

---

begin again

$$\begin{array}{ccccccc} K & \to & \bar{C}^0 & \overset{\delta}{\hookrightarrow} & C^1 & \to & H^1 \\ & & \downarrow\gamma & & & & \\ & & \mathbb{R} & & C^1/K & & \end{array}$$

What ~~is~~ you aim? You have

$$\begin{array}{ccc} \bar{C}^0 & \overset{\delta}{\hookrightarrow} & C^1 \\ \downarrow\gamma & & \\ \mathbb{R} & & \end{array} \qquad \delta^*\delta = 1 \qquad (\delta\bar{C}^0)^\perp \simeq H^1$$

OKAY it seems that instead of a 1-step filtration $\bar{C}^0 \subset C^1$, you have a 2-step filtration $0 \subset \operatorname{Ker}\gamma \subset \bar{C}^0 \subset C^1$ and a corresponding chain of ~~quotient~~ spaces of C^1.

See if you have the answer, namely you do not expect $\bar{C}^0$ to change ?

η3

Begin again
$$\begin{tikzcd} \bar{C}^0 \arrow[r, hook, "\delta"] \arrow[d, "\gamma"'] & C^1 \\ \mathbb{R} & \end{tikzcd}$$

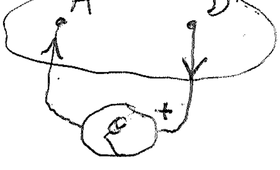
You have 2 step filtration of C^1: $0 \subset \operatorname{Ker}\gamma \subset \bar{C}^0 \subset C^1$ which you split into orthogonal layers. Use the fact that δ is isometric: $\delta^*\delta = 1_{\bar{C}^0}$. It should be true that $\operatorname{Ker}\delta^* = (\bar{C}^0)^\perp$. Also if the quadratic form on $\mathbb{R}$ is the pushforward via γ of the **quad.** form on $\bar{C}^0$ it should be true that $\gamma\gamma^* = 1$. You want to compare the above diagram with $\bar{C}^0 \xrightarrow{\binom{\delta}{\gamma}} \binom{C^1}{\mathbb{R}}$. Ideas: You **don't** change the node potential space.

Given
$$\begin{tikzcd} \bar{C}^0 \arrow[r, hook, "\delta"] \arrow[d, "\gamma"'] & C^1 \\ \mathbb{R} & \end{tikzcd}$$
together with a pos. def form on C^1. You want the linear analog of attaching an edge to the network. This is clearly like having $\bar{C}^0 \xrightarrow{\binom{\delta}{\gamma}} \binom{C^1}{\mathbb{R}}$. So there's an obvious equivalence between the two diagrams. Note that $\gamma: \bar{C}^0 \to \mathbb{R}$ can replaced by any 1-diml quotient space $\gamma: \bar{C}^0 \twoheadrightarrow L$ of $\bar{C}^0$. At this stage there is no significance to $1 \in \mathbb{R}$.

Next consider the pos. quad form $R^{-1}: C^1 \to C_1$ on C^1. Using this form ~~one~~ gets an orthogonal ~~splitting~~ of the filtration $0 \subset \operatorname{Ker}\gamma \subset \bar{C}^0 \xhookrightarrow{\delta} C^1$. There are three layers $\operatorname{Ker}\gamma$, $\bar{C}^0/\operatorname{Ker}\gamma \xrightarrow{\sim} L$, $(\delta\bar{C}^0)^\perp = \operatorname{Ker}\{\delta^*: C^1 \to \bar{C}^0\}$

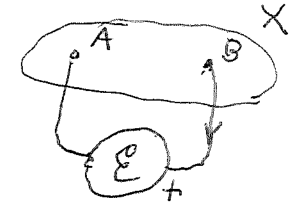
Look at L

Go back to the inhomogeneous problem in which an emf is connected between two nodes of a connected R-network.



$$V_A - V_B = R_{int} I$$
$$V_B - V_A = \mathcal{E}$$
$$V - RI = \mathcal{E}$$

θ3 Go back to a connected R-network X with an emf attached between 2 nodes.



[redacted] Find the equations ~~conmolven~~ [redacted] determining the state of the network, i.e. the states V_σ, I_σ for each edge in [redacted] X, [redacted] i.e. an elt $\begin{pmatrix} V \\ I \end{pmatrix}$ of $\begin{pmatrix} C^1X \\ C_1X \end{pmatrix}$.

You have $2e$ variables. The Kirchhoff ~~constraints~~ current constraint is weakened to allow a node current in at A and out at B. You also have the condition $V_B - V_A = \mathcal{E}$ [redacted] with $\mathcal{E}$ fixed. So the number of equations = the number of unknowns.

Next you should check in detail.

$$\begin{tikzcd}
\mathcal{E} \in \mathbb{R} & \bar{C}^0X \arrow[l, two heads, "\gamma"'] \arrow[r, hook] & C^1X \arrow[r, two heads] \arrow[d, phantom, "\wr"] & H^1X \\
\mathbb{R} \arrow[r, hook, "\gamma^t"] & \bar{C}_0X & C_1X \arrow[l, two heads] & H_1X \arrow[l, hook']
\end{tikzcd}$$

Here γ sends $\varphi \in \bar{C}^0X$ to $\varphi(B) - \varphi(A)$, and γ^t is the 0-current $[B] - [A]$. There's nothing new here it seems.

But let's be careful. You've made a **diagram** which is the reduced cochain complex of X, together with the external node information added. [redacted] What are the appropriate linear (inhomogeneous) equations?

Without the external node info you have the homogeneous linear eqns. $V \in \bar{C}^0X$, $I \in$ [redacted] H_1X, $V = RI$ which has only the soln. $V = I = 0$. There is an inhom. version $V \in \bar{C}^0(X)$, $I \in$ [redacted] H_1X, $V - RI = \mathcal{E}_{int}$ for any $\mathcal{E}_{int} \in C^1X$.

i3 But this "internal" system of eqns using edge emf's has $2e$ equations and $2e$ unknowns.

Review the situation. X conn R-network, A B two nodes of X, $\gamma : \bar{C}^0 X \longrightarrow \mathbb{R}$ $\quad \gamma : \varphi \longmapsto \varphi(B) - \varphi(A)$

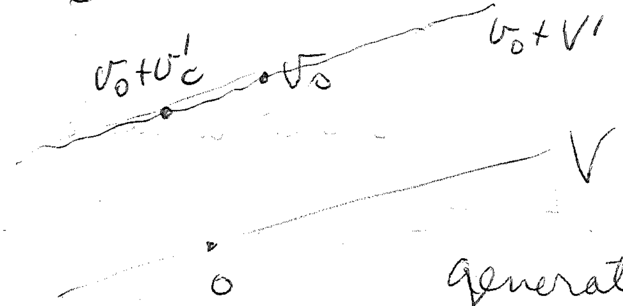
$$\begin{array}{ccccc} \gamma^{-1}\{\mathcal{E}_a\} & \subset & \bar{C}^0 X & \longrightarrow & C^1 X \\ & & \downarrow \gamma & & \\ \mathcal{E}_a & \in & \mathbb{R} & & \end{array}$$

$\gamma^{-1}\{\mathcal{E}_a\}$ is a torsor / coset for the vector space $\operatorname{Ker}\gamma$. You want a stationary point for the power on this coset, this means the variation in directions from $\operatorname{Ker}\gamma$ is zero. What does this means?

Change notation. $V' \hookrightarrow V \twoheadrightarrow V''$

$$\begin{array}{c} V \\ \downarrow A \\ V^* \end{array} \qquad A \text{ pos def}$$

Let $v_0 \in V$; to minimize $(v_0 + v')^t A (v_0 + v')$ as v' ranges over V'. Variation 1st order is $(\delta v')^t A (v_0 + v'_c) = 0$ i.e. $v_0 + v'_c \perp V'$ wrt A.



The confusion here may be due to the fact that the coset generates a vector space of dimension equal to 1 higher. So when you look for $V' \hookrightarrow V \twoheadrightarrow V''$ the V' is $\operatorname{Ker}\gamma$

---

minimize $\frac{1}{2} x^t A x$ subject to $c = y^t x$

$F = \frac{1}{2} x^t A x + \lambda (c - y^t x)$ $\qquad \nabla_x F = Ax - \lambda y = 0$

symplectic viewpoint $\qquad \partial_\lambda F = c - y^t x = 0$

K3 ~~Review the Legendre Transform. Example~~
~~$F(x) = \frac{1}{2}x^2$. Introduce a dual variable ξ to x,~~
~~[illegible]~~
~~[illegible] For each value~~
~~of ξ the critical points of~~ [redacted] ~~$\xi x - F$ wrt~~
~~the variable x are given by $\xi = F'(x)$. Assume~~
~~$F'' \neq 0$ invoke IFT to view x as a function~~
~~of ξ. let $\hat{F}(\xi)$ be the critical value~~

Review the Legendre Transform. Consider phase space for the real line with dual coordinates x, ξ. Let F be a (suitable) fn of x, e.g. $F = \frac{1}{2a}x^2$. Form the function $\xi x - F$ [redacted] on phase space and perform the following push-forward to the ξ line. Let $\hat{F}$ be the function of [redacted] ξ which gives the critical value of $\xi x - F$ as a function of x. When $F = \frac{1}{2a}x^2$ the critical ~~[redacted]~~ point of $\xi x - \frac{1}{2a}x^2$ is $\xi = \frac{x}{a}$, [redacted] or $x = a\xi$ and the critical value is $\xi a \xi - \frac{1}{2a}a^2\xi^2 = \frac{1}{2}a\xi^2$.

General F case. $\Phi(x,\xi) = x\xi - F(x)$, the critical point wrt x is $\xi = \frac{dF}{dx}$, one assumes $\frac{d^2F}{dx^2} \neq 0$ so that x becomes a fn of ξ via the IFT. Then $\hat{F} = x\xi - F$ is a function of ξ, well-defined at least locally. Then

$$\frac{d\hat{F}}{d\xi} = x + \xi\frac{dx}{d\xi} - \frac{dF}{dx}\frac{dx}{d\xi} = x$$

13 Go back to conn R-network X with external nodes pair A, B. You want the response to an emf $\mathcal{E}_a$ attached to these nodes. This is an inhomogeneous linear system of equations.

~~X has e edges, a state space of dim 2e,~~

Where to start? With the linearization of X:

$$\bar{C}^0 X \qquad C' X$$

Question: Is there a link between Lagrange multipliers and Legendre transform?

Look at the situation you have, namely a connected R-network with an inhomogeneous linear condition $V_B - V_A = \mathcal{E}_a$ on the space $\bar{C}^0 X$ of node potentials.

Lagrange multiplier method enables you to handle the constraint with ?

Repeat. Constraint $V_B - V_A = \mathcal{E}_a$ means

Set up intelligently. Basic object is a v.s. with pos def quadratic form. Operations of restriction to a subspace, push forward to a quotient spaces. At some stages you might have dilation, e.g. in the CL case.

But now you want inhomogeneous constraints, i.e. fixing value(s) of voltage variable(s).

$\mu 3$

Let's consider a simple situation.

X space with pos. quad form.

$\gamma : X \twoheadrightarrow \mathbb{R}$ linear fnl $\neq 0$, $K = \text{Kernel } \gamma$.

Aim to link Lagrange mult. & Legendre Transform

You start with X equipped with

$\frac{1}{2}x^tAx$ ~~constraint~~ Do L.T. $\xi^t x - \frac{1}{2}x^tAx$

critical point x corresp. to ξ is $\xi^t = x^tA$, $\xi = Ax$

$x = A^{-1}\xi$ $\quad \xi^tA^{-1}\xi - \frac{1}{2}\underbrace{(A^{-1}\xi)^tA(A^{-1}\xi)}_{\xi^tA^{-1}\xi} = \frac{1}{2}\xi^tA^{-1}\xi.$

But what about Lagrange multipliers?

$$F = \frac{1}{2}x^tAx + \lambda(c - y^tx) \qquad c, y \text{ fixed}$$

$$\nabla_x F = Ax - \lambda y \doteq 0 \qquad x = \lambda A^{-1}y$$

$$\partial_\lambda F = c - y^tx = 0 \qquad c = \lambda y^tA^{-1}y$$

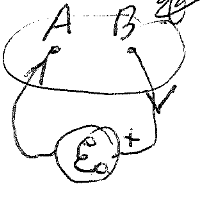
$$x^tAx = \lambda c \qquad \lambda = \frac{c}{y^tA^{-1}y}$$

$$= \frac{c^2}{y^tA^{-1}y}$$

~~X connected R-network, $\bar{C}^0 \xrightarrow{\delta} C^1$ Take a quotient space $\bar{C}^0 \to L$~~

X conn. R-network: $\bar{C}^0 \xhookrightarrow{\delta} C^1$ with pos. quad form R^{-1}

Consider an inhomogeneous linear problem, where you constrain $\varphi \in \bar{C}^0$ to satisfy $\varphi(B) - \varphi(A) = \mathcal{E}_a$



$$\varphi(A) - \varphi(B) = R_{int} I$$

$$\varphi(B) - \varphi(A) = \mathcal{E}_a$$

v3 More generally you can consider a quotient space $\gamma : \bar{C}^0 \twoheadrightarrow L$ together with $\mathcal{E}_a \in L$. Thus the possible $\varphi \in \bar{C}^0$ you allow are $\varphi \in \gamma^{-1}\{\mathcal{E}_a\}$, which is a coset for $\mathrm{Ker}\,\gamma$, call this K.

Consider a quadratic space $W \xrightarrow{q} W^*$
a linear functional $f : W \longrightarrow \mathbb{R}$

$\mathrm{Ker}(f) \longrightarrow W \xrightarrow{f} \mathbb{R}$. Interesting is the critical value of q on the hyperplane $f^{-1}(x)$ $x \in \mathbb{R}$. which should be a quadratic function of x.

Recall: $X \hookrightarrow W \twoheadrightarrow Y$ with $q>0$ on W.

split sequence get q given by

$$\begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$X^\perp = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, \begin{pmatrix} \delta x \\ 0 \end{pmatrix}^t \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0, \ \forall \delta x \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, ax + by = 0 \right\} = \begin{pmatrix} -a^{-1}b \\ 1 \end{pmatrix} y$$

$$\begin{pmatrix} -b^* a^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -a^{-1}b \\ 1 \end{pmatrix} = \begin{pmatrix} -b^* a^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -ca^{-1}b + d \end{pmatrix}$$

$$x^t a x + y^t (d - ca^{-1}b) y$$

§3

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$+\begin{pmatrix} -ca^{-1} & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -a^{-1}b \\ 1 \end{pmatrix} \qquad ?$$

Better

$$\begin{pmatrix} 1 & 0 \\ -ca^{-1} & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -a^{-1}b \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -ca^{-1} & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ c & -ca^{-1}b+d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -ca^{-1}b+d \end{pmatrix}$$

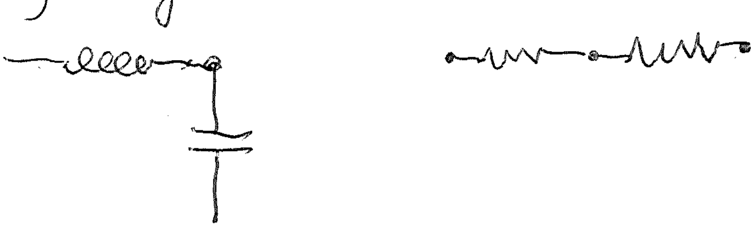
Let's see if we can apply this. Go back to

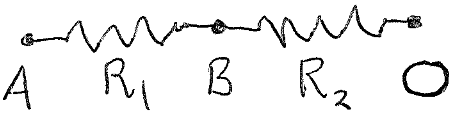
$$\begin{array}{ccccc} K = \operatorname{Ker}\gamma & \hookrightarrow & \bar{C}^0 & \longrightarrow & C^1 \\ & & \big\downarrow{\scriptstyle \gamma} & & \\ & & R & & \end{array}$$

You want to split $\bar{C}^0$ into $\begin{array}{ccc} K & & R \\ \| & & \| \\ X & \oplus & Y \end{array}$

You are probably being stupid because the norm on these spaces $K, \bar{C}^0, \gamma\bar{C}^0$ come from the scalar product on C^1. Think Euclidean!

Let's review problems & ideas. Maybe another example, say a tree e.g.



03 Consider 

$$\begin{array}{ccc} \bar{C}^0 X & \xrightarrow{\delta} & C^1 X \\ \| & & \| \\ \left\{ \begin{pmatrix} \varphi_A \\ \varphi_B \end{pmatrix} \right\} & & \left\{ \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \right\} \end{array}$$

$$\delta\varphi = \begin{pmatrix} \varphi(A) - \varphi(B) \\ \varphi(B) \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

$$\frac{1}{2R_1} V_1^2 + \frac{1}{2R_2} V_2^2$$

$$\frac{1}{2R_1}(\varphi_A - \varphi_B)^2 + \frac{1}{2R_2}\varphi_B^2 \quad \text{power form}$$

on $\bar{C}^0 X$. Next you want $\gamma : \bar{C}^0 X \to \mathbb{R}$

$\gamma : \varphi \longmapsto \varphi_A$. $\qquad K = \operatorname{Ker} \gamma = \{\varphi \mid \varphi_A = 0\}$.

$$\begin{array}{ccccc} K & \hookrightarrow & \bar{C}^0(X) & \xrightarrow{\gamma} & \mathbb{R} \\ & & \Psi & & \\ \begin{pmatrix} 0 \\ \varphi_B \end{pmatrix} & & \begin{pmatrix} \varphi_A \\ \varphi_B \end{pmatrix} & \longmapsto & \varphi_A \end{array}$$

$$\begin{bmatrix} \varphi_A \\ \varphi_B \end{bmatrix}^t \begin{bmatrix} \frac{1}{R_1} & -\frac{1}{R_1} \\ -\frac{1}{R_1} & \frac{1}{R_1} + \frac{1}{R_2} \end{bmatrix} \begin{bmatrix} \varphi_A \\ \varphi_B \end{bmatrix}$$

$$\text{Power}(\varphi) = \frac{1}{2R_1}(\varphi_A - \varphi_B)^2 + \frac{1}{2R_2}\varphi_B^2$$

Power (φ) restricted to Ker γ
$$= \frac{1}{2R_1}\varphi_B^2 + \frac{1}{2R_2}\varphi_B^2 = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\varphi_B^2$$

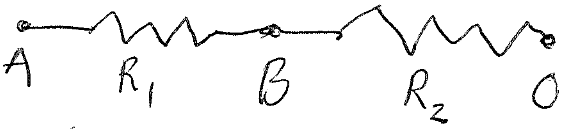
orth comp of $\left\{ \begin{bmatrix} 0 \\ \varphi_B \end{bmatrix} \right\}$ is: $\quad -\frac{1}{R_1}\varphi_A + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\varphi_B = 0$

$$\frac{\varphi_B - \varphi_A}{R_1} + \frac{\varphi_B}{R_2} = 0 \qquad\qquad \frac{\varphi_A - \varphi_B}{R_1} = \frac{\varphi_B}{R_2}$$

$\pi 3$

$$\begin{bmatrix} \varphi_A & \varphi_B \end{bmatrix} \begin{bmatrix} \frac{\varphi_A - \varphi_B}{R_1} \\ 0 \end{bmatrix} = \frac{\varphi_A(\varphi_A - \varphi_B)}{R_1} = \frac{\varphi_A \varphi_B}{R_2}$$

Repeat the calculation



$$\bar{C}^0 \ni \begin{pmatrix} \varphi_A \\ \varphi_B \end{pmatrix} \xmapsto{\delta} \begin{pmatrix} \varphi_A - \varphi_B \\ \varphi_B \end{pmatrix} \in C^1$$

$$\left\| \begin{pmatrix} \varphi_A - \varphi_B \\ \varphi_B \end{pmatrix} \right\|^2 = \frac{1}{R_1}(\varphi_A - \varphi_B)^2 + \frac{1}{R_2}\varphi_B^2$$

$$= \begin{bmatrix} \varphi_A \\ \varphi_B \end{bmatrix}^t \begin{bmatrix} \frac{1}{R_1} & -\frac{1}{R_1} \\ -\frac{1}{R_1} & \frac{1}{R_1} + \frac{1}{R_2} \end{bmatrix} \begin{bmatrix} \varphi_A \\ \varphi_B \end{bmatrix}$$

norm squared on $\bar{C}^0$.

The next point is the map $\gamma : \bar{C}^0 \longrightarrow \mathbb{R}$
$\varphi \longmapsto \varphi_A$

$$\operatorname{Ker}\gamma = \{\varphi \in \bar{C}^0 \mid \varphi_A = 0\} = \left\{ \begin{pmatrix} 0 \\ \varphi_B \end{pmatrix} \in \bar{C}^0 \right\}$$

$$(\operatorname{Ker}\gamma)^\perp = \left\{ \begin{bmatrix} \varphi_A \\ \varphi_B \end{bmatrix} \,\middle|\, -\frac{\varphi_A}{R_1} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\varphi_B = 0 \right\}$$

$$\frac{\varphi_A - \varphi_B}{R_1} = \frac{\varphi_B}{R_2} \qquad \frac{\varphi_A}{R_1} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\varphi_B$$

norm sqrd on $(\operatorname{Ker}\gamma)^\perp$

$$\varphi_A = R_1\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\varphi_B$$

$$\varphi_A \frac{(\varphi_A - \varphi_B)}{R_1} = \frac{\varphi_A \varphi_B}{R_2}$$

Positive?

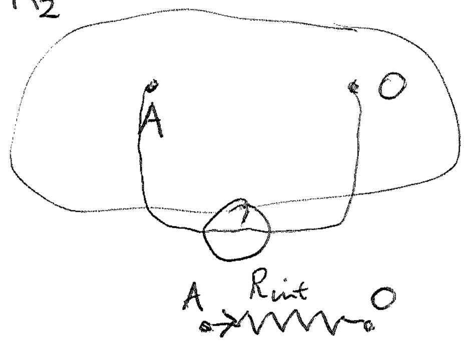
$$\frac{\varphi_A \varphi_B}{R_2} = \frac{\varphi_B}{R_2} R_1 \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\varphi_B > 0$$

p 3

$$\frac{\varphi_A - \varphi_B}{R_1} = \frac{\varphi_B}{R_2}$$

$$\frac{\varphi_A}{R_1} = \frac{\varphi_B}{R_1} + \frac{\varphi_B}{R_2} = \frac{R_2+R_1}{R_1 R_2}\varphi_B$$

$$\varphi_A = \frac{R_1+R_2}{R_2}\varphi_B$$



$$\frac{\varphi_A \varphi_B}{R_2} = \frac{R_1+R_2}{R_2^2}\varphi_B^2 > 0$$

OK this seems to work, but the situation is still opaque. What's a way to increase understanding? Idea: Finding the critical point, i.e. the orthogonal complement to $\operatorname{Ker}\gamma$ in $\bar{C}^0$ somehow introduces the current condition $\frac{\varphi_A - \varphi_B}{R_1} = \frac{\varphi_B}{R_2}$ i.e. $I_1 = I_2$ this is the Kirchhoff current condition at the node B.

[redacted]

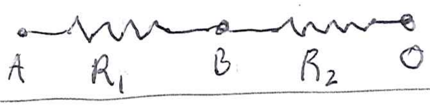
Assume you have a conn. R-network equipped with a ground O and a node $A \neq O$. You attach an emf E_a from O to A. Your problem is to ~~understand the circuit equations and understand~~ calculate the state of the network with the attached emf. Difficulties:

You feel that it should be ~~enough~~ to work with node voltages i.e. $\varphi \in \bar{C}^0$ and the positive definite forms induced via $\delta : \bar{C}^0 \hookrightarrow C^1$. $\delta^t R^{-1} \delta$

But currents popup naturally.

σ3 How should you handle the inhomogeneous condition $\varphi_A = \varphi_0^{\prime\prime 0} + \mathcal{E}_a$

Start again: conn. R-network w 2 nodes A, O with attached emf $\mathcal{E}_a$. To calculate the state of the network. Inhomogeneous condition $\varphi_A - \varphi_0 = \mathcal{E}_a$.
You have $V-1$ voltage variables φ_N, N node $\neq 0$.
So you have a positive symm. form on $\tilde{C}^0$ which has dim $V-1$.

Old problem - augmented graph
Yesterday's example 

Tree

Program: Category of quadratic spaces. Objects are vector spaces over $\mathbb{R}$ equipped with positive quad form. Q-category arising from induced quad form on subquotients.

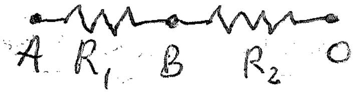
IDEA: $\exists$ L-version involving complexes, which perhaps generalizes what you are doing with cochains on a graph

Your program should be to understand why the dual framework of chains on the graph arises naturally in the calculations. This is physics philosophy maybe: introducing phase space and the Hamiltonian picture.

For R networks there is only statics and no dynamics, but CL networks have dynamics via Cayley Transform!

ī3 Start with a quadratic space and review pushing the quadratic form to a quotient space. Take yesterday's example

$$\bar{C}^0 \ni \begin{pmatrix}\varphi_A\\ \varphi_B\end{pmatrix} \xmapsto{\delta} \begin{pmatrix}\varphi_A-\varphi_B\\ \varphi_B\end{pmatrix} \in C^1$$



The power form on C^1 is $\begin{pmatrix}V_1\\ V_2\end{pmatrix} \longmapsto \frac{V_1^2}{R_1} + \frac{V_2^2}{R_2}$ better

$$\begin{pmatrix}V_1\\ V_2\end{pmatrix}^t \begin{pmatrix}\frac{1}{R_1} & 0\\ 0 & \frac{1}{R_2}\end{pmatrix}\begin{pmatrix}V_1\\ V_2\end{pmatrix}, \text{ restrict to } \underbrace{\begin{pmatrix}\varphi_A\\ \varphi_B\end{pmatrix}^t \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{R_1} & \\ & \frac{1}{R_2}\end{pmatrix}\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}\varphi_A\\ \varphi_B\end{pmatrix}}$$

$$\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{R_1} & -\frac{1}{R_1}\\ 0 & \frac{1}{R_2}\end{pmatrix} = \begin{pmatrix}\frac{1}{R_1} & -\frac{1}{R_1}\\ -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_2}\end{pmatrix}$$

$$\begin{pmatrix}\varphi_A\\ \varphi_B\end{pmatrix}^t \begin{pmatrix}R_1^{-1} & -R_1^{-1}\\ -R_1^{-1} & R_1^{-1}+R_2^{-1}\end{pmatrix}\begin{pmatrix}\varphi_A\\ \varphi_B\end{pmatrix}$$

power form on $\bar{C}^0$

Review the situation: $\bar{C}^0$ w Power form and $\bar{C}^0 \xrightarrow[\gamma]{} \mathbb{R}$

$\gamma : \varphi \longmapsto \varphi_A$

$$K \ni \begin{pmatrix}0\\ \varphi_B\end{pmatrix} \xmapsto{P} \left(\frac{1}{R_1}+\frac{1}{R_2}\right)\varphi_B^2$$

$K = \operatorname{Ker}\gamma$ restriction of P to $K \subset \bar{C}^0$

$$K^\perp = \left\{ \begin{pmatrix}\varphi_A\\ \varphi_B\end{pmatrix} \,\middle|\, \frac{\varphi_A}{R_1} = \left(\frac{1}{R_1}+\frac{1}{R_2}\right)\varphi_B \right\}$$

$$\varphi_A = R_1\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\varphi_B$$

$$\varphi_A^t\left[R_1^{-1}\varphi_A - R_1^{-1}\varphi_B\right] = \frac{\varphi_A}{R_1}\left[\varphi_A - \varphi_A\frac{1}{R_1\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\right]$$

$$\frac{\varphi_A^2}{R_1^2}\left(R_1 - \frac{1}{\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\right) = \frac{\varphi_A^2}{R_1}\left(\frac{[illegible]}{\frac{R_2}{R_1}+1}\right) = \frac{\varphi_A^2}{R_1+R_2}$$

43

This calculation is awkward, and it might tell you how to do things simpler by passing to phase space.

So begin with

$$V' \rightarrowtail V \twoheadrightarrow V''$$

Let's begin with a Lagrange multiplier example. X vector space with quad form $\frac{1}{2} x^t A x$ $\quad A : X \to X^*$ and a nonzero linear functional $y^t x$ $\quad y \in X^*$.

You want the push forward quadratic form for A under the map $X \xrightarrow{y^t} \mathbb{R}$. This is a simple Lagrange transform case, namely let $c \in \mathbb{R}$

$$F = \frac{1}{2} x^t A x + \lambda (c - y^t x)$$

$$\nabla_x F = Ax - \lambda y = 0 \qquad \frac{\partial F}{\partial \lambda} = c - y^t x = 0$$

(Notice that one has two new variables λ, c here which is a puzzle.)

Continue with Lagrange method, which should mean to eliminate the variables $\underline{x}, \lambda$. (Note: F has variables $\underline{x}, \lambda$ hence $n+1$ real variables $n = \dim X$. c, y are constants. $\nabla_x F = 0$, $\partial_\lambda F = 0$ are $n+1$ eqns.

Use $Ax = \lambda y$ to get $x = \lambda A^{-1} y$ and $c = y^t(\lambda A^{-1} y) = \lambda\, y^t A^{-1} y$. Thus x has been eliminated, and also λ; $\lambda = \dfrac{c}{y^t A^{-1} y}$ so we get the critical point $x = \dfrac{c}{y^t A^{-1} y} A^{-1} y$ and critical value

$$\frac{1}{2} x^t A x = \frac{1}{2} x^t \frac{c}{y^t A^{-1} y}\, y = \frac{1}{2} \frac{c^2}{y^t A^{-1} y}$$

$\varphi 3$ So you've just done Lagrange multiplier method, but not Legendre transform, which should proceed as follows: Consider for each $y \in X^*$ the fn

$$y^t x - \frac{1}{2} x^t A x$$

and find its critical point and critical value

crit pt. $\quad y^t - x^t A = 0 \quad$ or $\quad Ax = y, \quad x = A^{-1}y$

crit value $\quad y^t A^{-1} y - \frac{1}{2}(A^{-1}y)^t A (A^{-1}y) = \frac{1}{2} y^t A^{-1} y$

So the Legendre T of $\frac{1}{2}x^t A x$ is $\frac{1}{2} y^t A^{-1} y$.

Is there any relation to the push forward of A via y?

Repeat. X v.s. equipped with pos def $x^t A x$, let $y \in X^*$ $y \neq 0$, so that $\gamma : x \mapsto y^t x$, $X \twoheadrightarrow \mathbb{R}$ is onto, One has push forward $\gamma_*(A)$ defined by restricting A to $(\mathrm{Ker}\,\gamma)^\perp$, and then using $(\mathrm{Ker}\,\gamma)^\perp \xrightarrow{\sim} \mathbb{R}$.

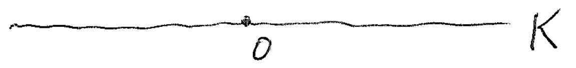
X becomes Euclidean space, $\exists!\ x_0$ such that $x_0^t A x = y^t x \quad \forall x$

Maybe $X = \mathbb{R}^n$ column vectors, then scalar product is $\langle y, x \rangle = y^t x$. Given $A = A^t > 0$ and $y \in X$, $y \neq 0$, get $X \xrightarrow{y^t} \mathbb{R}$ onto.

$K = \{ x \in X \mid y^t x = 0 \}$. Want to minimize $\frac{1}{2} x^t A x$ on $\{ x \mid c = y^t x \}$

$$\begin{array}{c} y^t x \\ \parallel \\ y^t A^{-1} A x \end{array}$$

X3 $\quad X = \mathbb{R}^n$ column vectors equipped with usual scalar product $(x, y) = x^t y = \sum_i x_i y_i$ and norm $\|x\| = (x,x)^{1/2}$. Consider a nonzero linear functional ξ on X, i.e. $\xi : X \longrightarrow \mathbb{R}$ is linear and onto. Get hyperplane $K = \{x \mid \xi(x) = 0\} = \xi^{\perp}$ when you identify $\xi \in X^*$ with the vector $\xi \in X$ such that $\xi(x) = (\xi, x) = \xi^t x$



You have an orthogonal splitting

$$X = K \oplus \mathbb{R}\xi$$

What is the scalar product on $\mathbb{R}\xi$? Ans.

$(c\xi, c'\xi) = cc' \|\xi\|^2$.

Next let's do the same calculation with the scalar product $(x, y)_A = x^t A y$, where $A = A^t > 0$. Let $\xi : X \longrightarrow \mathbb{R}$, $\xi(x) = \xi^t x$ be a nonzero linear functional on X. Write $\xi^t x = \xi^t A^{-1} A x = (A^{-1}\xi, x)_A$ i.e. you represent the linear fnl ξ by the A-scalar product with $A^{-1}\xi$. One has an A-orthogonal splitting

$$X = \underset{\substack{\| \\ \operatorname{Ker} \xi}}{K} \oplus \mathbb{R}(A^{-1}\xi) \qquad K = \{x \mid x^t A A^{-1} \xi = 0\}$$

Now restrict $(\ ,\ )_A$ to $\mathbb{R}(A^{-1}\xi)$.

$$(cA^{-1}\xi, c'A^{-1}\xi)_A = (cA^{-1}\xi, A c' A^{-1}\xi) = cc'(\xi, A^{-1}\xi)$$

p3 Repeat what you did. First take the setting $X = \mathbb{R}^n$ with scalar product $(x, y) = x^t y$. Identify a linear fnl $\xi : X \to \mathbb{R}$ with the vector $\tilde{\xi}$ such that $\xi(x) = (\tilde{\xi}, x) = \tilde{\xi}^t x$, & drop the ~. Thus you have $X \xrightarrow{\sim} X^*$ sending y to $y^t = (x \mapsto y^t x)$.

Better: If $y \in X$, then $x \mapsto y^t x$ is a lin fnl on X, and one gets isom $X \to X^*$, $y \mapsto y^t$.

Now let $X \to \mathbb{R}$, $x \mapsto y^t x$ be a nonzero linear fnl. One has orthog splitting

$$X = K \oplus \mathbb{R}y \qquad \text{where } K = \operatorname{Ker} y^t = y^\perp$$

The push forward scalar product on $\mathbb{R}y$ is the restriction of (x, x') to the orth comp of K, i.e. $\mathbb{R}y$. $\therefore$

$$(cy, c'y) = cc' \|y\|^2$$

Now consider $X = \mathbb{R}^n$ with scalar product $(x, x')_A = (x, Ax')$ where A pos. def. Let y^t be a lin fnl. $y^t x = (y, x) = (y, A^{-1}Ax) = (A^{-1}y, Ax) = (A^{-1}y, x)_A$ So y^t is represented for the A-scalar prod by $A^{-1}y$. Next get A-orth splitting

$$X = K \oplus \mathbb{R}A^{-1}y \qquad K = \operatorname{Ker} y^t$$

The push forward scalar product on $\mathbb{R}A^{-1}y$ is

$$(cA^{-1}y, c'A^{-1}y)_A = cc'(A^{-1}y, AA^{-1}y) = cc'(y, A^{-1}y)$$

ω3

Now look at Legendre T.

$$\mathcal{L} = y^t x - \frac{1}{2} x^t A x$$

Let y be fixed. Then $\mathcal{L}$ has ~~the~~ a unique critical point when

$y^t - x^t A = 0$ i.e. $Ax = y \iff x = A^{-1} y$

and the critical value is

$$\mathcal{L} = y^t A^{-1} y - \frac{1}{2} (A^{-1} y)^t \cancel{A A^{-1}} y = \frac{1}{2} y^t A^{-1} y$$

Let's now understand why $F = \frac{1}{2} x^t A x + \lambda (c - y^t x)$ yields something different. What you should have done earlier is to restrict $\frac{1}{2} x^t A x$ to the hyperplane $c = y^t x$, then found the critical point.

$n = 1$. $c = yx$, $x = \frac{c}{y}$, $F = \frac{1}{2} \frac{c^2 A}{y^2} = \frac{1}{2} \frac{c^2}{y^t A^{-1} y}$

$\partial_x F = Ax - \lambda y = 0$, $\partial_\lambda F = c - yx = 0$

$x = \frac{\lambda y}{A} = \frac{c}{y}$ $\qquad F = \frac{1}{2} \left(\frac{c}{y}\right)^2 A$

Review the calculation

$\nabla_x F = Ax - \lambda y = 0 \qquad \partial_\lambda F = c - y^t x = 0$

$x = \lambda A^{-1} y \qquad y^t x = \lambda y^t A^{-1} y = c \qquad \lambda = \frac{c}{y^t A^{-1} y}$

$x = \frac{c A^{-1} y}{y^t A^{-1} y}$, $F = \frac{1}{2} \frac{(y^t A^{-1} c) A (c A^{-1} y)}{(y^t A^{-1} y)^2} = \frac{1}{2} \frac{c^2}{y^t A^{-1} y}$