Remaining: To understand the Morita context describing your Morita equivalence.

New insight I think $\mathbb{C}\Gamma\otimes V$ being Γ-graded has a canonical partition of unity indexed by Γ
$\mathbb{C}\Gamma\otimes V = \bigoplus_s s\otimes V$, ~~as well as being~~ preserved by the Γ action. Then W being a summand (retract) of $\mathbb{C}\Gamma\otimes V$ inherits a partition:

$$\sum e_s = 1_{\mathbb{C}\Gamma\otimes V} \qquad e_s = s e_1 s^{-1} = s\varepsilon_1\eta_1 s^{-1}$$

yields
$$1_W = \tilde{\imath}\Big(\sum_s e_s\Big)\tilde{\jmath} = \sum_s \tilde{\imath}e_s\tilde{\jmath}$$
$$= \sum_s s\tilde{\imath}e_1\tilde{\jmath}s^{-1} = \sum_s s\tilde{\imath}\varepsilon_1\eta_1\tilde{\jmath}s^{-1} = \sum_s s i j s^{-1}.$$

The new insight seems to be to deal with $p(\mathbb{C}\Gamma\otimes V)$ as a retract (summand) of $\mathbb{C}\Gamma\otimes V$.
Does this clarify anything.

$$\sum e_s \qquad \begin{array}{c} \mathbb{C}\Gamma\otimes V \\ \tilde{\imath}\downarrow\uparrow\tilde{\jmath} \\ W \end{array} \qquad p = \tilde{\jmath}\tilde{\imath}$$

$$\mathbb{C}\Gamma\otimes V \xrightarrow{\tilde{\imath}} W \xrightarrow{\tilde{\jmath}} \mathbb{C}\Gamma\otimes V$$

So you will get $\qquad u = ts^{-1} \quad us = t \quad s = u^{-1}t$

$$t\otimes v \overset{\tilde{\imath}}{\longmapsto} \sum_s ts^{-1}\otimes p(s)v = \sum_t u\otimes p(u^{-1}t)v$$

So simply $\quad v \longmapsto \sum u\otimes p(u^{-1})v$

So what. Take $V = A$. You want to identify $p(\mathbb{C}\Gamma \otimes \tilde{A})$ with Bh.

Let's try to get a better picture of this ~~...~~ tautology. Suppose we start with a ~~...~~ retract W of the Γ-module $\mathbb{C}\Gamma \otimes V$. Up to isom such a retract is equivalent to an operator p on the Γ module $\mathbb{C}\Gamma \otimes V$ satisfying $p^2 = p$. You have

$$\mathrm{Hom}_\Gamma(\mathbb{C}\Gamma \otimes V, \mathbb{C}\Gamma \otimes V) = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}\Gamma \otimes V)$$

which naturally contains the operators of finite Γ support, that is $\mathbb{C}[\Gamma^{op}] \otimes \mathrm{End}(V)$. Note that $f : V \longrightarrow \mathbb{C}\Gamma \otimes V$ has the form

$$f(v) = \sum s \otimes f_s(v) \qquad \text{where } \{s \mid f_s(v) \neq 0\} \text{ is finite } \forall v$$

~~It's possible for~~ It's stronger to require that there is a finite set Φ containing the support of $f_s(v)$ for all v. V fin. dim'l $\Rightarrow$ finite Γ-support.

Using inverse convert right Γ mod to left one, get alg of ops. on the Γ-module $\mathbb{C}\Gamma \otimes V$ with fin Γ support is $\mathbb{C}[\Gamma] \otimes \mathrm{End}(V)$, with action

$$(s \otimes \theta)(t \otimes v) = ts^{-1} \otimes \theta v$$

$s^{-1} = t^{-1}u$
$u = ts^{-1}$
$us = t, \ s = u^{-1}t$

$$\sum_{s,t} (s \otimes \theta_{(s)})(t \otimes v_{(t)}) = \sum_{s,t} ts^{-1} \otimes \theta(s)\, v(t)$$

$$= \sum_{t,u} u \otimes \theta(u^{-1}t)\, v(t)$$

$$\boxed{(\theta v)(u) = \sum_t \theta(u^{-1}t)\, v(t).}$$

Now there appears to be ~~[illegible]~~ a subtle aspect to $\mathbb{C}\Gamma\otimes V$ that might deserve clarifying. $\mathbb{C}\Gamma\otimes V$ is Γ-graded, equivalently a $\hat{\Gamma}$-module. There are two structures: Γ and $\hat{\Gamma}$.

$$\mathrm{Hom}_{\Gamma}(\mathbb{C}\Gamma\otimes V, M) = \mathrm{Hom}_{\mathbb{C}}(V, M)$$

$$\mathrm{Hom}_{\hat{\Gamma}}\big(N, \mathbb{C}\Gamma\otimes V\big) = \mathrm{Hom}_{\mathbb{C}}(N, V)$$

You use the following. Given Γ-module W and a suitable linear map $W \xrightarrow{j} V$, then j coextends (lifts) uniquely to a Γ-module map $\tilde{j} : W \longrightarrow \mathbb{C}\Gamma\otimes V$ $\qquad \tilde{j}(w) = \sum s\otimes j s^{-1}w$

suitable means $\{s \mid j s^{-1}w \neq 0\}$ is finite $\forall w\in W$.

Note that on the B side you see Γ action on the A-side ~~[illegible]~~ a $\hat{\Gamma}$ action on A

$$A \longrightarrow \mathbb{C}\Gamma\otimes A$$
$$p_s \longmapsto s\otimes p_s$$

What does strike you as strange is

$$\mathrm{Hom}_{\hat{\Gamma}}(N, \mathbb{C}\Gamma\otimes V) = \prod_s \mathrm{Hom}(N_s, V)$$

$$= \mathrm{Hom}\Big(\bigoplus_s N_s, V\Big) = \mathrm{Hom}_{\mathbb{C}}(N, V)$$

Why? Look at $\quad \mathrm{Hom}\big(\mathbb{C}\Gamma\otimes V, \mathbb{C}\Gamma\otimes V\big)$

$$\mathrm{Hom}_{\Gamma}(\mathbb{C}\Gamma\otimes V, \mathbb{C}\Gamma\otimes V) = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}\Gamma\otimes V)$$

$$\mathrm{Hom}_{\hat{\Gamma}}(\mathbb{C}\Gamma\otimes V, \mathbb{C}\Gamma\otimes V) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\Gamma\otimes V, V)$$

~~[illegible]~~ There might be a way here to use topological vector space ideas. ? ?

Let W be a Γ-module retract of $\mathbb{C}\Gamma\otimes V$. This is equivalent to an **idempotent** p in

$$\mathrm{Hom}_{\Gamma}(\mathbb{C}\Gamma\otimes V, \mathbb{C}\Gamma\otimes V) = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}\Gamma\otimes V)$$

$$\mathbb{C}\Gamma\otimes \mathrm{Hom}_{\mathbb{C}}(V,V) \subset \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}\Gamma\otimes V)$$

remove ~~eliminate~~ via 5'.

So it seems that from the outset you want to insist that p ~~have finite~~ Γ support.

> There must be some [redacted] equivalence
>
> retract of $\mathbb{C}\Gamma\otimes V$ with support $\subset \Phi$ $\longleftrightarrow$ A-module structure on V

[redacted] Viewpoint now. [redacted] $\mathbb{C}\Gamma\otimes V$ Γ module

finite support

$\mathbb{C}\Gamma \otimes V$ Γ-module

$\mathbb{C}\Gamma \otimes \operatorname{End}(V)$ = ring of finite support operators on $\mathbb{C}\Gamma \otimes V$ commuting w Γ-action.

$(s \otimes \theta)(t \otimes v) = ts^{-1} \otimes \theta v$

A retract W of $\mathbb{C}\Gamma \otimes V$ is equivalent to a proj. p on $\mathbb{C}\Gamma \otimes V$. Assume support finite $\subset \Phi$ i.e.

$p = \sum s \otimes p(s) \in \mathbb{C}\Gamma \otimes \operatorname{End}(V)$, $p(s) = 0$ $s \notin \Phi$.

object: $\mathbb{C}\Gamma \otimes V$ the Γ-module gen. by the vector space V.

~~$\mathbb{C}\Gamma \otimes \operatorname{End}(V)$ t.p. alg operators~~

~~object: $s \otimes \theta \in \operatorname{End}(V)$ yield endo~~

~~$(s \otimes \theta)(t \otimes v)$ action of t.p. alg $\mathbb{C}\Gamma \otimes \operatorname{End}(V)$~~

~~a operator~~

an endo of this module is equivalent to a linear map $V \longrightarrow \mathbb{C}\Gamma \otimes V$.

11:00 $\mathbb{C}\Gamma \otimes V$ = the Γ-module gen. by the v.s. V

$$\sum_s s \otimes g(s) \in \mathbb{C}\Gamma \otimes \operatorname{End}(V)$$

Define action of $\mathbb{C}\Gamma \otimes \operatorname{End}(V)$ on $\mathbb{C}\Gamma \otimes V$ by

$u = ts^{-1}$
$s = u^{-1}t$

$$\left(\sum_s s \otimes g(s)\right) * \left(\sum_t t \otimes v(t)\right) = \sum_{s,t} ts^{-1} \otimes g(s)\, v(t)$$

$$= \sum_u u \otimes \sum_t g(u^{-1}t)\, v(t)$$

$$\boxed{(g * v)(s) = \sum_t g(s^{-1}t)\, v(t)}$$

Explain how the t.p. alg $\mathbb{C}\Gamma \otimes \operatorname{End}(V)$ is the alg of finite support endomorphisms of the Γ-module $\mathbb{C}\Gamma \otimes V$.

for yourself: $\mathrm{Hom}_\Gamma(\mathbb{C}\Gamma\otimes V, \mathbb{C}\Gamma\otimes V) = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}\Gamma\otimes V)$
element of latter is $T(v) = \sum_s s\otimes\theta_s(v)$, $\theta_s \in \mathcal{L}(V)$
$\{s \mid \theta_s \neq 0\}$ = Supp of T. The set of ~~[illegible]~~ endos with supp $\subset \Phi$ is an alg under comp.
It can be identified with the t.p. alg $\mathbb{C}\Gamma\otimes\mathcal{L}(V)$ action on $\mathbb{C}\Gamma\otimes V$ via

$$(s\otimes\theta)(t\otimes v) = ts^{-1}\otimes\theta v$$

At this point you ~~have~~ introduce ~~defined~~ the Γ module $\mathbb{C}\Gamma\otimes V$ and the algebra of endos with support in Φ finite

$$\mathcal{L}_\Phi(\mathbb{C}\Gamma\otimes V) = \bigoplus_{s\in\Phi}\mathbb{C}s\otimes\mathcal{L}(V)$$

bad

puzzle

$$\left(\sum_s s\otimes\theta(s)\right)\left(\sum_t t\otimes\rho(t)\right)$$
$$= \sum_u u\otimes\sum_{u=st}\theta(s)\rho(t)$$

Start again: An endo of the Γ-mod $\mathbb{C}\Gamma\otimes V$ is equiv. to a map $V \longrightarrow \bigoplus_{s\in\Gamma} s\otimes\theta_s(v)$. Look at fin. supp?

$s\mapsto s^{-1}t$, $ts^{-1}\mapsto ~~tt^{-1}~~s$

$$\left(\sum_s s\otimes f(s)\right) * \left(\sum_t t\otimes v(t)\right) = \sum_{s,t} ts^{-1}\otimes f(s)v(t)$$
$$= \sum_s s\otimes\sum_t f(s^{-1}t)\,v(t)$$

You are confused, Go back to

$$\begin{tikzcd} \mathrm{Hom}_\Gamma(\mathbb{C}\Gamma\otimes V, \mathbb{C}\Gamma\otimes V) \arrow[r, phantom, "="] & \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}\Gamma\otimes V) \\ \mathbb{C}\Gamma\otimes\mathcal{L}(V) \arrow[u] \arrow[ur, hook] & \end{tikzcd}$$

Let $F \in \lambda$, so that $F(t\otimes v) = tF(1\otimes v) =$
$= t\sum_s s\otimes f_s(v)$. Thus F equiv to $v \mapsto \sum_s s\otimes f_s(v)$
where $\forall v$ $\{s \mid f_s(v)\neq 0\}$ is finite. You are interested is finite support F, which means $\{s \mid f_s \neq 0\}$ is finite: Suppose then F, G have finite support

$$F(t\otimes v) = \sum ts'\otimes f_{s'}(v)$$
$$G(t\otimes v) = \sum ts\otimes g_s(v)$$
$$F(G(t\otimes v)) = F\Big(\sum_s ts\otimes g_s(v)\Big) = \sum_s F(ts\otimes g_s(v))$$
$$= \sum_{s,s'} tss'\otimes f_{s'}(g_s(v))$$

So if $F = \sum_{s'} s'\otimes f_{s'}$, $G = \sum_s s\otimes g_s$

then $FG = \sum ss'\otimes f_{s'}g_s$, so the alg of finite ~~rank ops~~ support endos is $\mathbb{C}\Gamma^{op}\otimes\mathcal{L}(V)$ which is isom. to $\mathbb{C}\Gamma\otimes\mathcal{L}(V)$, provided $(s\otimes f)*(t\otimes v) = ts^{-1}\otimes fv$

$$(f*v)(s) = \sum_t f(s^{-1}t)\,v(t)$$

So how do I clean up all this.

Regard $\mathbb{C}\Gamma \otimes V$ as a Γ-module via left mult. and as a module over the t.p. alg $\mathbb{C}\Gamma \otimes \mathcal{L}(V)$ via

$$(s\otimes f) * (t\otimes v) = ts^{-1} \otimes fv$$

or

$$\left(\sum_s s\otimes f(s)\right) * \left(\sum_t t\otimes v(t)\right) = \sum_s s\otimes\left(\sum_t f(s^{-1}t)\, v(t)\right)$$

$$\| $$

$$\sum_{u,t} tu^{-1} \otimes f(u)v(t)$$

$u = s^{-1}t$

$tu^{-1} = tt^{-1}s = s.$

The support of this operator: $(fv)(s) = \sum_t f(s^{-1}t)\, v(t)$ is $\Phi = \{ s \mid f(s) \neq 0 \}$.

Review the words. $\mathbb{C}\Gamma \otimes V$ the Γ-mod. gen. by V

Define $(s\otimes f) * (t \otimes v) = ts^{-1} \otimes fv$

~~Extends to~~ This extends to an action of the t.p alg $\mathbb{C}\Gamma \otimes \mathcal{L}(V)$ on $\mathbb{C}\Gamma\otimes V$

$$\mathbb{C}\Gamma \otimes V = \left\{ \sum_t t\otimes v(t) \;\middle|\; v : \Gamma \to V \text{ f.s.} \right\}.$$

$$s \sum_t t\otimes v(t) = \sum_t st \otimes v(t) = \sum_t t\otimes v(s^{-1}t)$$

$$\therefore\ (sv)(t) = v(s^{-1}t).$$

$$\mathbb{C}\Gamma\otimes\mathcal{L}(V) = \left\{ \sum_s s\otimes f(s) \;\middle|\; f : \Gamma \to \mathcal{L}(V) \text{ f.s.} \right\}.$$

$s \mapsto st^{-1}$

$$\Big(\sum_s s\otimes f(s)\Big)\Big(\sum_t t\otimes g(t)\Big) = \sum_{s,t} st \otimes f(s)g(t) = \sum_s s\otimes \sum_t f(st^{-1})g(t)$$

$$(fg)(s) = \sum_t f(st^{-1})g(t).$$ assoc.

$s \mapsto s^{-1}t \qquad ts^{-1} \mapsto t(s^{-1}t)^{-1} = s$

$$\Big(\sum_s s\otimes f(s)\Big) * \Big(\sum_t t\otimes v(t)\Big) = \sum_{s,t} ts^{-1}\otimes f(s)v(t) = \sum_s s\otimes \sum_t f(s^{-1}t)v(t)$$

Γ-module gen. by V: $\quad \mathbb{C}\Gamma\otimes V \qquad s(t\otimes v) = st\otimes v$

$\mathbb{C}\Gamma\otimes V$ is naturally a module over $\mathbb{C}\Gamma\otimes \mathcal{L}(V)$ by

$(s\otimes f)*(t\otimes v) = ts\otimes fv$

Γ-module gen by V : $\mathbb{C}\Gamma\otimes V$

$\mathbb{C}\Gamma^{op}\otimes \mathcal{L}(V)$ acts naturally as endos of this Γ module

$$\mathrm{Hom}_\Gamma(\mathbb{C}\Gamma\otimes V, \mathbb{C}\Gamma\otimes V) = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}\Gamma\otimes V)$$

Support of an Endo $f : \mathbb{C}\Gamma\otimes V \circlearrowleft$, $f(1\otimes x) = \sum s\otimes f_s(x)$

is $\{s \mid f_s \neq 0\}$. $\qquad f(t\otimes x) = \sum ts\otimes f(s)(x)$

Γ-module gen by V: $\quad \mathbb{C}\Gamma\otimes V \quad$ Γ operates by left mult. $\quad s(t\otimes v) = st\otimes v$. ~~One gets~~ Endos of this module arise from ~~right multiplication~~ ~~endos of~~ operators on V and ~~the~~ right multiplication in the group ring. which can be converted to a left action using inversion. These commute yielding a repn of the t.p. alg $\mathbb{C}\Gamma\otimes \mathrm{End}(V)$ on the Γ module $\mathbb{C}\Gamma\otimes V$ given by

$$(s\otimes f)*(t\otimes v) = ts^{-1}\otimes fv \qquad s,t \mapsto s^{-1}t, t$$

$$\Big(\sum_s s\otimes f(s)\Big)*\Big(\sum_t t\otimes v(t)\Big) = \sum_{s,t} ts^{-1}\otimes f(s)v(t)$$

$$= \sum_s s\otimes \sum_t f(s^{-1}t)v(t)$$

endo F of $\mathbb{C}\Gamma\otimes V$ $\quad F(t\otimes v) = t\sum_s s\otimes f(s)$

$$F(t\otimes v) = tF(1\otimes v) = t\sum_s s^{-1}\otimes f(s)v$$

~~Also~~ Define support of Γ = $\{s \mid f(s)\neq 0\}$.

Clear that ~~have~~ can identify $\mathbb{C}\Gamma\otimes \mathrm{End}(V)$ with the algebra of endos of $\mathbb{C}\Gamma\otimes V$ with finite support.

So now you want to regard the Γ-module $\mathbb{C}\Gamma\otimes V$ as equipped with $\mathbb{C}\Gamma\otimes\mathrm{End}(V)$ as operators. restrict attention to operators with finite support

~~Next define consider a retract of $\mathbb{C}\Gamma\otimes V$ a module to be~~ Concept of retract of an object X this is a triple (Y, i, j) where $Y \xrightarrow{i} X \xrightarrow{j} Y$ and $ji = 1_Y$. $\quad p = ij = p^2$ on X

A retract of X up to canon isom. is equivalent to $p: X\to X$ $\quad p^2=p$. Within modules where the image exists.

Point: Consider $p=p^2\in\mathbb{C}\Gamma\otimes\mathrm{End}(V)$

$$p = \sum s\otimes p(s) \qquad p(s)=0 \quad s\notin\Phi.$$

$$p^2=p \qquad p(s) = \sum_{s=tu} p(t)p(u)$$

~~You learn that~~

Equivalence between ~~...~~ A-module structure on a v.s. V and

$p\in\mathbb{C}\Gamma\otimes\mathrm{End}(V) \ni p^2=p \quad \mathrm{Supp}(p)\subset\Phi$

See if you can make things clear.

V v.s. $\quad \mathbb{C}\Gamma \otimes \mathrm{End}(V)$ = alg of endos of the Γ-module $\mathbb{C}\Gamma \otimes V$ with finite support.

An A-module st. on V is same as a $p \in \mathbb{C}\Gamma \otimes \mathrm{End}(V)$ sat. $\mathrm{Supp}(p) \subset \Phi$, $p^2 = p$.

Given an A-mod V, whence p on $\mathbb{C}\Gamma \otimes V$, let

$W = p(\mathbb{C}\Gamma \otimes V) \qquad W \xrightarrow{\alpha} \mathbb{C}\Gamma \otimes V \xrightarrow{\beta} W$

$\beta\alpha = 1$, $p = \alpha\beta$. $\quad \mathbb{C}\Gamma \otimes V$ is Γ-graded, $\therefore$ e_s proj onto $s \otimes V$, $\quad e_s = s e_1 s^{-1}$, $\quad \sum e_s = 1_{\mathbb{C}\Gamma \otimes V}$

Suppose given an A-mod V, i.e. $p(s) \in \mathcal{L}(V)$ $\forall s \in \Gamma$ sat $p(s) = 0$ for $s \notin \Phi$, $\quad p(s) = \sum_t p(st^{-1}) p(t)$.

Then you get an idemp op p on $\mathbb{C}\Gamma \otimes V$ w support Φ

$$p\Big(\sum_t t \otimes v(t)\Big) = \sum_{s,t} ts^{-1} \otimes p(s) v(t) = \sum_s s \otimes \sum_t p(s^{-1}t) v(t)$$

$s \longmapsto s^{-1}t$

$ts^{-1} \longmapsto tt^{-1}s$

support $\quad t \otimes v \longmapsto t \sum_s s \otimes f(s) v$

$$\mathrm{Hom}_\Gamma(\mathbb{C}\Gamma \otimes V, \mathbb{C}\Gamma \otimes V) = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}\Gamma \otimes V)$$

$$F \longmapsto \{v \longmapsto F(1 \otimes v)\}$$

$$F(1 \otimes v) = \sum_s s \otimes f(s) v$$

$$F(t \otimes v) = tF(1 \otimes v) = \sum_s$$

Again you find details you don't understand.

$\mathbb{C}\Gamma \otimes \mathcal{L}(V)$ acts on $\mathbb{C}\Gamma \otimes V$ via

$$(s \otimes f) * (t \otimes v) = ts^{-1} \otimes f v$$

Make $\mathbb{C}\Gamma\otimes\mathcal{L}(V)$ act on $\mathbb{C}\Gamma\otimes V$ by

$$\left(\sum_s s\otimes f(s)\right)*\left(\sum_t t\otimes v(t)\right) = \sum_{s,t} ts^{-1}\otimes f(s)v(t)$$

$s\mapsto s^{-1}t$

$ts^{-1}\mapsto tt^{-1}s = s$

$$= \sum_s s\otimes \sum_t f(s^{-1}t)\,v(t)$$

$$F \longmapsto \left\{v\mapsto F(1\otimes v) = \sum_s s\otimes F(s)v\right\}$$

$$\mathrm{End}_\Gamma(\mathbb{C}\Gamma\otimes V) = \mathrm{Hom}(V, \mathbb{C}\Gamma\otimes V)$$

$$\uparrow$$

$$\mathbb{C}\Gamma\otimes \mathrm{End}(V)$$

$$\ni$$

$\sum s\otimes f(s)$ becomes the operator

$$F(t\otimes v) = \sum_s ts^{-1}\otimes f(s)v$$

leading to the map $V\longrightarrow \mathbb{C}\Gamma\otimes V$ given by

$$v\mapsto F(1\otimes v) = \sum_s s^{-1}\otimes f(s)v$$

Do it this way. Define * action of $\mathbb{C}\Gamma\otimes \mathrm{End}(V)$ on the Γ-module $\mathbb{C}\Gamma\otimes V$. Formula

$$\left(\sum_s s\otimes f(s)\right)*\left(\sum_t t\otimes v(t)\right) = \sum_s s\otimes \sum_t f(s^{-1}t)\,v(t)$$

$$\boxed{(f*v)(s) = \sum_t f(s^{-1}t)\,v(t)}$$

~~Consider an orbit.~~ Let F be an Γ endo of $\mathbb{C}\Gamma\otimes V$

F is equiv. to the $\mathbb{C}$ linear map $v\mapsto F(1\otimes v)$ from V to $\mathbb{C}\Gamma\otimes V$. Write $F(1\otimes v) = \sum_s s\otimes f(s^{-1})v$

where the operators $f(s^{-1}) \in \mathrm{End}(V)$ sat $\forall v \in V$ $\{s \mid f(s^{-1})v \neq 0\}$ is finite

can say F describe by a family $f(s^{-1})$
you need to say what the support of f is

$$F(t\otimes v) = \sum_s ts^{-1} \otimes f(s)v$$

$$F(1\otimes v) = \sum_s s^{-1} \otimes f(s)v$$

Support $= \{s \mid f(s) \neq 0\}$. Then

Anyway it will be worth seeing if this pen performs

Define action of $\mathbb{C}\Gamma \otimes \mathrm{End}(V)$ on the Γ-module $\mathbb{C}\Gamma\otimes V$

$$(s\otimes f)*(t\otimes v) = ts^{-1}\otimes fv$$

$s \to s^{-1}t$
$ts^{-1} \to s$

$$\left(\sum_s s\otimes f(s)\right)*\left(\sum_t t\otimes v(t)\right) = \sum_{s,t} ts^{-1}\otimes f(s)\, v(t)$$

$$= \sum_s s \otimes \sum_t f(s^{-1}t)\, v(t)$$

$$\boxed{(f*v)(s) = \sum_t f(s^{-1}t)\, v(t)}$$

$$\mathrm{End}_\Gamma(\mathbb{C}\Gamma\otimes V) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}\Gamma\otimes V)$$

$$F \longmapsto F(1\otimes v) = \sum_s s^{-1}\otimes f(s)v$$

clarify. Define supp of F to be ~~$\{s \mid e_{s^{-1}} F \cdot v$~~
$\{s \mid f(s) \neq 0\}$.

$$f(s)v = e_{s^{-1}} F(1\otimes v)$$

Put into words. $\mathbb{C}\Gamma \otimes V$ = the Γ-module gen. by V. Endos of this module arise from right mult. in $\mathbb{C}\Gamma$ and $\mathrm{End}(V)$. Define action of t.p. alg $\mathbb{C}\Gamma \otimes \mathrm{End}(V)$ on $\mathbb{C}\Gamma \otimes V$ by

$$(s \otimes f) * (t \otimes v) = ts^{-1} \otimes fv$$

$$s \mapsto s^{-1}t \qquad ts^{-1} \mapsto tt^{-1}s = s$$

In general:

$$\left(\sum_s s \otimes f(s)\right) * \left(\sum_t t \otimes v(t)\right) = \sum_{t,s} ts^{-1} \otimes f(s)v(t)$$

$$= \sum_s s \otimes \sum_t f(s^{-1}t)\, v(t).$$

$$F \in \mathrm{End}_\Gamma(\mathbb{C}\Gamma \otimes V) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}\Gamma \otimes V)$$

$$F \longmapsto \{v \mapsto F(1 \otimes v)\} \qquad ?$$

$$\left(\sum_s s \otimes f(s)\right)* \longmapsto \left(v \mapsto \sum_s s^{-1} \otimes f(s)v\right)$$

$$\mathrm{Hom}_\Gamma(\mathbb{C}\Gamma \otimes V, \mathbb{C}\Gamma \otimes V) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}\Gamma \otimes V)$$

$$V \xrightarrow{f} \mathbb{C}\Gamma \otimes V$$

$$f(v) = \sum_s s^{-1} \otimes f(s)v \qquad \forall v \ \{s \mid f(s)v \neq 0 \text{ finite}\}$$

$$\mathbb{C}\Gamma \otimes \mathrm{Hom}(V,V) \longrightarrow \mathrm{Hom}_\Gamma(\mathbb{C}\Gamma \otimes V, \mathbb{C}\Gamma \otimes V) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}\Gamma \otimes V)$$

$$s \otimes \phi \longmapsto (s \otimes \phi) * (t \otimes v) = ts^{-1} \otimes \phi v \longmapsto (v \mapsto s^{-1} \otimes \phi v)$$

$$\sum s \otimes \phi(s) \longmapsto \left(v \mapsto \sum_s s^{-1} \otimes \phi(s) v\right)$$

$$\mathrm{Hom}_\Gamma(\mathbb{C}\Gamma\otimes V, \mathbb{C}\Gamma\otimes V) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}\Gamma\otimes V)$$

F $\qquad\qquad F(1\otimes v) = \sum_s s^{-1}\otimes f(s)v$

where $f:\Gamma\to \mathrm{End}(V)$

Define the support of F to be $\{s \mid f(s)\neq 0\}$.

Lecture. $\mathbb{C}\Gamma\otimes V$ the Γ-module gen. by V

endos of this modules arise from

- right mult on $\mathbb{C}\Gamma$
- operators on V

Action of the tensor prod alg $\mathbb{C}\Gamma\otimes \mathrm{End}(V)$ on $\mathbb{C}\Gamma\otimes V$ defined by

$$(s\otimes f)*(t\otimes v) = ts^{-1}\otimes fv$$

$$\left(\sum_s s\otimes f(s)\right)*\left(\sum_t t\otimes v(t)\right) = \sum_{s,t} ts^{-1}\otimes f(s)v(t)$$

~~substitute~~ $s\mapsto s^{-1}t,\ t\mapsto t$ $\qquad = \sum_s s\otimes \sum_t f(s^{-1}t)v(t)$

$ts^{-1}\to tt^{-1}s = s$

i.e. $$\boxed{(f*v)(s) = \sum_t f(s^{-1}t)\,v(t)}$$

Support $\quad F$ endo of $\mathbb{C}\Gamma\otimes V$ $\qquad v\longmapsto F(1\otimes v)$

$F(t\otimes v) = t\,F(1\otimes v)$ $\quad$ so F equiv to $V\to \mathbb{C}\Gamma\otimes V$

$v\longmapsto F(1\otimes v) = \sum_s s^{-1}\otimes f(s)v$ $\qquad f:\Gamma\to\mathrm{End}(V)$

Support of $F = \{s \mid f(s)\neq 0\}$.

So can identify $\mathbb{C}\Gamma\otimes \mathrm{End}(V)$ with the algebra of finite support endos on $\mathbb{C}\Gamma\otimes V$.

$A = \mathcal{P}_{\Gamma,\Phi}$ gen $\quad p(s),\ s \in \Gamma$

rel $\quad p(s) = 0,\quad s \notin \Phi$

$$p(s) = \sum_{s=tu} p(t)\,p(u) = \sum_u p(su^{-1})\,p(u)$$

An action of A on V is the same as an

$p \in \mathbb{C}\Gamma \otimes \mathrm{End}(V)$ $\qquad p = \sum_s s \otimes p(s)$ $\qquad \left|\ \begin{array}{l} \mathrm{Supp}\, p \subset \Phi \\ p = p^2 \end{array}\right.$

Prop: An A-module V is the same as a vector space equipped with an idempotent operator p on $\mathbb{C}\Gamma \otimes V$ with $\mathrm{Supp} \subset \Phi$.

$W = p(\mathbb{C}\Gamma \otimes V)$ $\qquad \beta\alpha = 1$

$$\begin{tikzcd} W \arrow[r, hook, "\alpha"] \arrow[rr, bend right, "1"'] & \mathbb{C}\Gamma \otimes V \arrow[r, two heads, "\beta"] & W \end{tikzcd}$$

(from A-mod V to p on $\mathbb{C}\Gamma \otimes V$ to retract W, α, β of $\mathbb{C}\Gamma \otimes V$)

Now to prove W is a firm B-module. You have $\beta\alpha = 1$

$\mathbb{C}\Gamma \otimes V$ has the partition of unity $\sum_s{}' e_s = 1$

$e_s = s e_1 s^{-1}$ = projection onto $s \otimes V$

also $\quad e_1 = i_1 j_1 : \mathbb{C}\Gamma \otimes V \twoheadrightarrow V \hookrightarrow \mathbb{C}\Gamma \otimes V$

$$\sum_s s \otimes v_s \longmapsto v_1, \quad v \longmapsto 1 \otimes v$$

So check ~~...~~ the conditions for being a firm B-module. $\quad h = \beta e_1 \alpha = (\beta i_1)(j_1 \alpha)$

$h_s = s h s^{-1} = \beta s e_1 s^{-1} \alpha = \beta e_s \alpha \qquad \sum h_s = \beta\alpha = 1$

$h s h = \beta e_1 \alpha s \beta e_1 \alpha = 0 \quad$ for $s \notin \Phi$.

$p(s)$ NO

V A-module, get p on $\mathbb{C}\Gamma \otimes V$

Let $W = p(\mathbb{C}\Gamma \otimes V)$ whence canon. Γ-module maps

$$\begin{tikzcd}
W \arrow[r, "\alpha"] \arrow[rr, bend left, "1_W"] \arrow[dr, "j"'] & \mathbb{C}\Gamma \otimes V \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] \arrow[rr, bend left, "p"] & W \arrow[r, "\alpha"] & \mathbb{C}\Gamma \otimes V \\
 & V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "\iota"'] & &
\end{tikzcd}$$

$\beta\alpha = 1_W$

$\alpha\beta = p$ on $\mathbb{C}\Gamma \otimes V$

$$\sum_s{}' s\,\varepsilon_1 \eta_1 s^{-1} = \sum_s e_s = \mathrm{id} \text{ on } \mathbb{C}\Gamma \otimes V$$

Let $h = \beta e_1 \alpha = \beta \varepsilon_1 \eta_1 \alpha$

$$\sum_s s h s^{-1} = \beta \sum_s (s\varepsilon_1\eta_1 s^{-1})\alpha = \beta\alpha = 1_W$$

$$hsh = \beta \overbrace{\varepsilon_1 \eta_1}^{e_1} \alpha s \beta \overbrace{\varepsilon_1 \eta_1}^{e_1} \alpha$$

$$\begin{aligned}
(\eta_1 s p \varepsilon_1)(v) &= \eta_1 s p(1 \otimes v) \\
&= \eta_1 s \sum_t t^{-1} \otimes p(t) v \\
&= \sum_t \eta_1 (s t^{-1} \otimes p(t) v) = p(s) v
\end{aligned}$$

Given a $^{\text{firm}}$ B-module W factor

$$h = \iota j : W \xrightarrow{j} V = hW \xrightarrow{\iota} W$$

$$\sum s h s^{-1} = \sum_s (s\iota)(j s^{-1}) \qquad W = \sum \textstyle{'} hW$$

$w =$

$j s^{-1} t$

Given W a finn B-module, factor
$$h = \iota j : W \xrightarrow{j} hW = V \xhookrightarrow{\iota} W$$

$$\sum_s shs^{-1}w = \sum_s (si)(js^{-1}w)$$ assumed to be a finite sum, si inj $\Rightarrow \{s \mid js^{-1}w \neq 0\}$ fin.

$$W \xrightarrow{\alpha} \mathbb{C}\Gamma \otimes V \xrightarrow{\beta} W \xrightarrow{\alpha} \mathbb{C}\Gamma \otimes V$$

$$\alpha(w) = \sum_s s \otimes js^{-1}w, \quad \beta\Big(\sum t \otimes v(t)\Big) = \sum_t t\iota \cancel{f}(t)$$

$$\beta\alpha \cancel{\alpha}(w) = \sum_S s\iota j s^{-1}w = \sum_s shs^{-1}w = w.$$

$$\alpha\beta\cancel{\alpha}\Big(\sum_t t \otimes v(t)\Big) = \cancel{\beta}\alpha\sum_t t\iota v(t)$$
$$= \sum_s s \otimes \sum_t \underbrace{js^{-1}t\iota}_{p(s^{-1}t)} v(t)$$

You are not doing this right. Start with W with ~~free~~ Γ, h acting
$$\begin{cases} hsh = 0 & s \notin \Phi \\ \sum shs^{-1}w = w & \forall w \in W \end{cases}$$
factor $h = \iota j$ with $V = hW$. Define $p(s) = js\iota$ on V
$$hsh = \underset{\text{inj}}{\iota}(js\iota)\underset{\text{surj}}{j} \Longrightarrow p(s) = 0 \quad s \notin \Phi.$$
$$\sum_t j(st^{-1})\iota j t \iota v = js\iota v$$

~~Is it simpler to say $V = hW$ $p(s)$~~

Given W ... $\sum_s shs^{-1}w = \sum_s (s\iota)(js^{-1}w)$ finite sum si inj

get $W \xrightarrow{\alpha} \mathbb{C}\Gamma \otimes V \xrightarrow{\beta} W$
$$\alpha(w) = \sum s \otimes js^{-1}w, \quad \beta\Big(\sum_s s \otimes v(s)\Big) = \sum s\iota v(s)$$
Then $\beta\alpha w = \sum s\iota j s^{-1}w = w$

Seems you can take $V = W$, $i = 1$, $j = h$.

$$W \xrightarrow{\alpha} \mathbb{C}\Gamma\otimes W \xrightarrow{\beta} W$$

$$w \longmapsto \sum s\otimes hs^{-1}w \longmapsto \sum shs^{-1}w$$

$$\sum_s s\otimes w(s) \longmapsto \sum_t t\,w(t) \longmapsto \sum_s s\otimes \sum_t hs^{-1}t\,w(t)$$

$$(pw)(s) = \sum_t \underbrace{p(s^{-1}t)}_{hs^{-1}t}\, w(t)$$

$$\sum_t \underset{p(t)}{ht}\ \underset{p(t^{-1}s)}{ht^{-1}s} = \underset{p(s)}{hs}.$$

Given W you factor $h = ij : W \overset{j}{\twoheadrightarrow} \underset{hW}{\underset{\parallel}{V}} \overset{i}{\hookrightarrow} W$

you get A acting on V via $p(s) = jsi$

$0 = hsh = \underset{inj}{i}\,\overbrace{jsi}^{p(s)}\,\underset{surj}{j} \Rightarrow p(s) = 0$ for $s \in \underline{\Phi}$.

Given V with A-action, form $\mathbb{C}\Gamma\otimes V$ Γ-mod

$p = \left(\sum s\otimes p_s\right)^*$ on $\mathbb{C}\Gamma\otimes V$. $W' = \mathrm{Im}(p)$.

$$\begin{tikzcd}
\tilde{W}' \arrow[r, "\alpha"] \arrow[dr, "j"'] & \mathbb{C}\Gamma\otimes V \arrow[r, two heads, "\beta"] \arrow[d, shift right, "\eta_1"'] & \tilde{W}' \\
& V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "i"'] &
\end{tikzcd}$$

$$h = ij = \beta\varepsilon_1\eta_1\alpha$$

$$hsh = \beta\varepsilon_1\underbrace{\boxed{\eta_1\alpha s\beta\varepsilon_1}}_{p(s)}\eta_1\alpha$$

You need an isom $W \longrightarrow \tilde{W}' = p(\mathbb{C}\Gamma\otimes hW)$

All you need to do is to construct the retraction

So define $W \xrightarrow{\alpha} \mathbb{C}\Gamma\otimes V \xrightarrow{\beta} W$

$$\alpha w = \sum_s s\otimes js^{-1}w$$

It gets clearer. Again.

Begin with W define $V = hW$, i, j, $p(s) = jsi$

gives functor $W \mapsto V$.

given V A-module define p on $\mathbb{C}\Gamma \otimes V$

$\tilde{W} = p(\mathbb{C}\Gamma \otimes V)$ with Γ action + $h = \beta\varepsilon_1\eta_1\alpha$

gives functor $V \mapsto \tilde{W}$.

need to check inverse to each other.

First construct isom. $W \xrightarrow{\sim} \tilde{W}$ by making W a retract of $\mathbb{C}\Gamma \otimes V$ and checking you have the right p.

$$W \xrightarrow{\alpha} \mathbb{C}\Gamma \otimes V \xrightarrow{\beta} W$$

$$\alpha w = \sum_s s \otimes j s^{-1} w, \quad \sum_t t \otimes v(t) \overset{\beta}{\longmapsto} \sum_t t i v(t)$$

$$\alpha\beta \sum_t t \otimes v(t) = \sum_s s \otimes \sum_t (j s^{-1} t i) v(t)$$

~~isom. the other way.~~

next Start with V reduced and construct isom $V \simeq h \underbrace{p(\mathbb{C}\Gamma \otimes V)}_{\tilde{W}}$

$$\begin{tikzcd} \tilde{W} \arrow[r, "\alpha"] \arrow[dr, "j"'] & \mathbb{C}\Gamma \otimes V \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & \tilde{W} \\ & V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "i"'] & \end{tikzcd}$$

$\mathrm{Ker}(i) = \bigcap \mathrm{Ker}\, p(s)$.

$$\mathrm{Ker}(i) = \{ v \mid p(1 \otimes v) = \sum_s s^{-1} \otimes p(s) v = 0 \}$$

$$j\tilde{W} = \eta_1 \, p(\mathbb{C}\Gamma \otimes V)$$
$$= \eta_1 \Big\{ \sum' s \otimes \sum_t p(s^{-1}t)\sigma(t) \;\Big|\; \sigma(t) \text{ arb.} \Big\}.$$
$$= \Big\{ \sum_t p(t)\sigma(t) \;\Big|\; \sigma(t) \text{ arb} \Big\} = \sum_{t\in\Gamma} p(t) V.$$

W firm B-module Γ action h $\left| \begin{array}{ll} hsh = 0 & s \in \Phi, \\ \sum shs^{-1} = 1 & \text{on } W. \end{array} \right.$

$h = ij : W \xrightarrow{j} hW \xrightarrow{i} W$

Put $p(s) = jsi$ on hW. Claim hW becomes an A-module $0 = hsh = ijsij = i\,p(s)\,j \Rightarrow p(s) = 0 \quad s \in \Phi,$

$$\sum_t p(st^{-1})p(t) = \sum_t j s \underbrace{t^{-1} i j t}\, i = jsi = p(s)$$

Thus get fun $W \longmapsto hW = V$

$AV = V$ $\quad w = \sum_s shs^{-1}w = \sum si\,js^{-1}w$

$W = \sum_s siV \Rightarrow V = jW = \sum_s jsiV = \sum_s p(s)V \Rightarrow V = AV$

${}_AV = 0.$ $\quad iv = \sum s\underbrace{ij s^{-1} i}_{p(s^{-1})} v$ $\quad p(s^{-1})v = 0 \quad \forall s \Rightarrow iv = 0 \Rightarrow v = 0$

V A-module $p = \sum_{s\in\Gamma} s \otimes p(s) \in \mathbb{C}\Gamma \otimes \operatorname{End}(V)$ acts on $\mathbb{C}\Gamma\otimes V$

Sat $p^2 = p$, $\{s \mid p(s) \neq 0\} \subset \Phi$.

$$p * \sum_t t \otimes \sigma(t) = \sum \cancel{ts}^{\,s} \otimes \sum_t p(s^{-1}t)\sigma(t)$$

~~Get functor $V \mapsto p(\mathbb{C}\Gamma\otimes V)$~~

Claim $p(\mathbb{C}\Gamma \otimes V)$ becomes a B-module. Γ action

Put $W = p(\mathbb{C}\Gamma \otimes V)$.

$$\begin{tikzcd}
W \arrow[r, hook, "\alpha"] \arrow[rr, bend left, "1_W"] \arrow[dr, "j"'] & \mathbb{C}\Gamma\otimes V \arrow[r, two heads, "\beta"] \arrow[d, shift right, "\eta_1"'] & W \\
& V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "\iota"'] &
\end{tikzcd}$$

$\beta\alpha = 1$
$\alpha\beta = (p\times)$.

$j = \eta_1\alpha$, $\iota = \beta\varepsilon_1$ $\qquad h = \iota j$

Leave out ι, j. $\qquad e_1 = \varepsilon_1\eta_1$ gives comp. of $1\otimes V$

$$\sum_s \beta s \varepsilon_1 \eta_1 s^{-1}\alpha = 1 \text{ on } \cancel{\mathbb{C}\Gamma\otimes V}\ W$$

$$\| $$

$$\sum s(\beta e_1 \alpha)s^{-1} \qquad h = \beta e_1 \alpha = \iota j$$

$$hsh = \iota(js\iota)j$$

$$js\iota v = \eta_1 \alpha s \beta \varepsilon_1 v = \eta_1 s p \overbrace{\varepsilon_1 v}^{1\otimes v} = p(s)v$$

$$p\varepsilon_1 v = \sum t^{-1}\otimes p(t)v$$

So W is a firm B-module. So you get functor

$V \longmapsto p(\mathbb{C}\Gamma\otimes V)$ $\qquad$ A-modules $\longrightarrow$ firm B-mods.

exact functor of V $\qquad$ kills $V \ni AV = 0$.

$$0 \to V' \to V \to V'' \to 0$$

$$0 \to p(\mathbb{C}\Gamma\otimes V') \to p(\mathbb{C}\Gamma\otimes V) \to p(\mathbb{C}\Gamma\otimes V'') \to 0$$

---

next point ~~suppose V reduced.~~

$$jW = \eta_1 p(\mathbb{C}\Gamma\otimes V) = \sum_t p(t)V$$

$$\eta_1\Big(\sum_s s\otimes \sum_t p(s^{-1}t)v(t)\Big) = \sum_t p(t)v(t)$$

$$i(v) = p(1\otimes v) = \sum_s s^{-1}\otimes p(s)v$$

$$\operatorname{Ker} i = \bigcap \operatorname{Ker} p(s).$$

~~$W \rightsquigarrow V = hW \rightsquigarrow p(\mathbb{C}\Gamma\otimes W)$~~

V red $\rightsquigarrow p(\mathbb{C}\Gamma\otimes V)$ on this the fact $h = ij \Rightarrow i$ inj j surj.

$$hp(\mathbb{C}\Gamma\otimes V) \simeq V.$$

Steps. Given W factor $h = ij$: $W \xrightarrow{j=h} hW \xhookrightarrow{i} W$

get reduced A-module hW $\quad p(s) = jsi$

Conversely given V get $p = \sum s\otimes p(s) \in \mathbb{C}\Gamma\otimes \operatorname{End}(V)$

get firm B-module $p(\mathbb{C}\Gamma\otimes V)$ with $h = ij$

Given W factor $h = ij$: $W \xrightarrow{j=h} hW \xhookrightarrow{i} W$.

$$w = \sum s i j s^{-1} w \Rightarrow \forall w \ \{s \mid js^{-1}w \neq 0\} \text{ finite}$$

$$W \xrightarrow{\alpha} \mathbb{C}\Gamma\otimes hW \xrightarrow{\beta} W$$

$$\alpha(w) = \sum_s s\otimes js^{-1}w, \quad \beta\Big(\sum_s s\otimes v(s)\Big) = \sum_s s i v(s)$$

α is the unique Γ-map $\ni \eta_1\alpha = j$

β ———————— $\ni \beta\varepsilon_1 = i$

$$p = \alpha\beta \qquad \alpha\beta\Big(\sum_t t\otimes v(t)\Big) = \sum_s s\otimes \sum_t \underbrace{js^{-1}t i}_{p(s^{-1}t)}\, v(t)$$

Work out the Morita context.

~~First step is to fo~~

$W \mapsto hW \qquad \{\text{firm } B \text{ mod}\} \longrightarrow \{\text{red } A \text{ mod}\}$

$V \mapsto p(\mathbb{C}\Gamma \otimes V) \qquad \{A \text{ mods}\} \longrightarrow \{\text{firm } B \text{ mods}\}$

~~There should be a firm~~ These functors should lead to firm bimodules. By exactness of $V \mapsto p(\mathbb{C}\Gamma \otimes V)$ you have $p(\mathbb{C}\Gamma \otimes \tilde{A})$ is A^{op} flat firm as well as B-firm. It should be true that

$$Bh = p(\mathbb{C}\Gamma \otimes \tilde{A}) \qquad hB = (\tilde{A} \otimes \mathbb{C}\Gamma)p$$

right A-flat firm

~~anyway~~

left A flat. firm.

$$V \mapsto p(\mathbb{C}\Gamma \otimes V) \simeq p(\mathbb{C}\Gamma \otimes \tilde{A}) \otimes_A V$$

$$\boxed{\begin{array}{l} j = \eta_1 \alpha \\ \hline i = \beta \varepsilon_1 \end{array}}$$

$$\begin{tikzcd}
 & \mathbb{C}\Gamma \otimes \tilde{A} \arrow[dr, "\beta"] \arrow[d, shift right, "\eta_1"'] & \\
p(\mathbb{C}\Gamma \otimes \tilde{A}) \arrow[ur, "\alpha"] \arrow[dr, "j"'] & \tilde{A} \arrow[u, shift right, "\varepsilon_1"'] \arrow[r, "i"'] & p(\mathbb{C}\Gamma \otimes \tilde{A}) \\
 & &
\end{tikzcd}$$

B as a left B-module is firm flat

$\Rightarrow$ hB is firm flat A-mod.

$$p(\mathbb{C}\Gamma \otimes V) = p(\mathbb{C}\Gamma \otimes \tilde{A}) \otimes_A V$$

$$\mathbb{C}\Gamma \overset{\rho}{\otimes} \boxed{\tilde{A} \otimes_A \tilde{A}} \overset{\rho}{\otimes} \mathbb{C}\Gamma$$

wait: a bimodule works 2 ways, one way for left and the other way for right.

How

Repeat: $A, B \qquad \Gamma, \Phi$

V A-module, let $E(V) = p(\mathbb{C}\Gamma \otimes V)$

$$\begin{tikzcd}
E(V) \arrow[r, "\alpha"] \arrow[dr, "j"'] & \mathbb{C}\Gamma \otimes V \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & E(V) \\
& V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "i"'] &
\end{tikzcd}
\qquad
\begin{array}{l} \beta\alpha = 1 \\ \alpha\beta = p \end{array}$$

$$\sum s\, \varepsilon_1 \eta_1 s^{-1} = 1 \quad \text{on } \mathbb{C}\Gamma \otimes V$$

$$\sum s \underbrace{(\beta \varepsilon_1 \eta_1 \alpha)}_{h} s^{-1} = 1 \quad \text{on } E(V)$$

$$hsh = \beta\varepsilon_1 (\eta_1 \alpha s \beta \varepsilon_1) \eta_1 \alpha$$

You think ε, η too much. So introduce

$$j = \eta_1 \alpha \qquad i = \beta \varepsilon_1$$

$$j s^{-1} = \eta_s \alpha \qquad s i = \beta \varepsilon_s \qquad \sum s i j s^{-1} = \sum \eta_s \varepsilon_s = 1$$

$$\eta_u p \varepsilon_t \sigma = \eta_u \sum_s t s^{-1} \otimes p(s) \sigma = p(u^{-1} t)$$

$u = t s^{-1} \quad u s = t \quad s = u^{-1} t$.

$$j s^{-1} t i = \eta_s \alpha \beta \varepsilon_t = \eta_s p \varepsilon_t = p(s^{-1} t)$$

Repeat. Given V put $E(V) = p(\mathbb{C}\Gamma \otimes V)$

$$\begin{tikzcd} E(V) \arrow[r, hook, "\alpha = \mathrm{in}"] \arrow[dr, "j = \eta_1 \alpha"'] & \mathbb{C}\Gamma \otimes V \arrow[r, two heads, "\beta = p"] \arrow[d, shift right, "\eta_1 s^{-1} = \eta_s"'] & E(V) \\ & V \arrow[u, shift right, "\varepsilon_t = t\varepsilon_1"'] \arrow[ur, "i = \beta\varepsilon_1"'] & \end{tikzcd}$$

$$\alpha = \sum_s \varepsilon_s \eta_s \alpha = \sum_s s \otimes j s^{-1}$$

$$\beta = \sum_t \beta \varepsilon_t \eta_t = \sum_t t i \eta_t$$

$$\beta\alpha = \sum_t t i j t^{-1} = 1$$

$$\alpha\beta = \sum_s \varepsilon_s j s^{-1} t i \eta_t = \sum_{s,t} \varepsilon_s p(s^{-1}t) \eta_t$$

So you learn that the clear way to proceed is to ~~study~~ treat $\mathbb{C}\Gamma \otimes V$ first, handle Γ-grading

$$\begin{tikzcd} \mathbb{C}\Gamma \otimes V \arrow[d, shift right, "\eta_s"'] \\ V \arrow[u, shift right, "\varepsilon_s"'] \end{tikzcd}$$

$$\varepsilon_s v = s \otimes v$$

$$\eta_s(t \otimes v) = \begin{cases} v & t = s \\ 0 & t \neq s \end{cases}$$

$e_s = \varepsilon_s \eta_s$ proj onto $s \otimes V$

$\sum \varepsilon_s \eta_s = 1$ on $\mathbb{C}\Gamma \otimes V$

Then Γ acts.

$$t\varepsilon_s = \varepsilon_{ts}$$

$$\eta_{ts} = \eta_s t^{-1}$$

$$t e_s t^{-1} = e_{ts}$$

$$\eta_t = \eta_1 t^{-1}$$

$$\varepsilon_t = t\varepsilon_1$$

Then define p on $\mathbb{C}\Gamma\otimes V$. Wait: you ~~[illegible]~~ now see how to handle ~~[illegible]~~ elements of $\mathbb{C}\Gamma\otimes V$ namely $\sum_s \varepsilon_s \eta_s \phi$
$= \sum_s s \otimes \phi(s)$.

$$E(V) \xrightarrow{\alpha = \text{in}} \overset{\phi}{\mathbb{C}\Gamma\otimes V} \xrightarrow{\beta = p} E(V)$$

$$\alpha\xi w = \sum_s \varepsilon_s \eta_s \,\alpha\xi w = \sum s \otimes \eta_1 s^{-1} \alpha\xi w$$

$$= \sum s \otimes j s^{-1} \xi w$$

$$\beta\phi = \beta \sum_s \varepsilon_s \eta_s \phi = \sum_s \beta s \varepsilon_1 \phi(s) = \sum_s s \iota \phi(s)$$

$$\beta\alpha\,[redacted]\,w = \sum_s s\, \overset{h}{i j}\, s^{-1}\,[redacted]\,w = [redacted]\,w$$

$$\alpha\beta\phi = \sum_s s \otimes \sum_t \underbrace{j s^{-1} t i}_{p(s^{-1}t)} \phi(t)$$

$$hsh = \iota p(s) j \qquad \therefore\ p(s) = 0 \implies hsh = 0$$

$\Longleftarrow$ true for ι inj, j surj.

Where to next?

~~[illegible] should [illegible]~~.

Given V you get retract W ~~[illegible]~~ of $\mathbb{C}\Gamma\otimes V$

Put into words. An A-module structure on V determines a retract W of the Γ-module $\mathbb{C}\Gamma\otimes V$ which is a firm B-module equipped with a fact $h = ij$. V red. $\Leftrightarrow$ i inj, j surj.

Conv. given a firm B-module W, you factor $h = ij$ i inj, j surj, get a red. A-module V, $p(s) = jsi$, together with an isom of W with $p(\mathbb{C}\Gamma\otimes V)$.

$$W \longmapsto hW \qquad \text{firm } B\text{-mods} \longrightarrow \text{red } A\text{mod}$$

$$V \longmapsto p(\mathbb{C}\Gamma\otimes V)$$

Can check $\underline{hp(\mathbb{C}\Gamma\otimes V)}$ canon. isom to V

Now how do you make progress on Mor cont. Discuss in general.

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} = \begin{pmatrix} A & \\ & \end{pmatrix}$$

$P = p(\mathbb{C}\Gamma\otimes\tilde{A})$ is a firm B-module which seems to have a canonical element $p(1\otimes 1)$

$= \sum_s s^{-1}\otimes p(s)$.

$F : Mod(R) \longrightarrow Mod(S)$ rt cont.

$F(R)$ $\qquad$ S, R bimod

$F(R)\otimes_R M \longrightarrow F(M)$ canon.

$\xi\otimes m \longmapsto F(r\mapsto rm)\xi$

~~86~~ The best idea seems to be to look at $\Phi = \{1\}$. $A = \mathbb{C}p_*$ $\boxed{p_*^2 = p_*}$

What is B? firm B-modules are Γ-mods W with h satisfying $hsh = 0$ $s \neq 1$. and $\sum shs^{-1} = 1$

~~factor $h = \eta \cdot W \to hW \to W$.~~

$$\sum \underbrace{hshs^{-1}}_{0 \text{ for } s\neq 1} = h \qquad h^2 = h.$$

So there is a Morita equivalence in this case ~~8~~ which is the standard one between vector spaces V and Γ modules with Γ-grading

$A = \mathbb{C}$ $\mathbb{C}\hat{\Gamma} = \bigoplus_{s\in\Gamma} \delta_s\mathbb{C}$ functions with fin. supp

$p = \mathbb{C}\Gamma$ $\Gamma \ltimes \hat{\Gamma}$

$V \longmapsto \mathbb{C}\Gamma \otimes V = W$ $\mathbb{C}\Gamma$

$V \longmapsto \mathbb{C}\Gamma \otimes V$ $B = \mathbb{C}\Gamma \otimes \mathbb{C}\hat{\Gamma}$

~~4~~ basis $s \otimes \delta_t$

$(s \otimes \delta_t)(u \otimes v)$?

$$\begin{array}{c} V \longmapsto \mathbb{C}\Gamma \otimes V \\ \eta_1 \downarrow \uparrow \varepsilon_1 \end{array}$$

$h = \varepsilon_1 \eta_1$

$shs^{-1} = \varepsilon_s \eta_s$ Looks good

How am I supposed to proceed. You need ~~$\Sigma\eta$~~,

$\mathbb{C}p_1$

firm B-mods = Γ-mods W with h

$hsh = 0 \quad s \neq 1$, $\sum_s shs^{-1} = 1_W$

functor $W \longmapsto hW$ ~~as vector space~~ as reduced $\mathbb{C}p_1$-module $p_1^2 = p_1$

other functor is $V \longmapsto \mathbb{C}\Gamma \otimes V$ obvious Γ action

and $h = \varepsilon_1 \eta_1$ $\quad shs^{-1} = (s\varepsilon_1)(\eta_1 s^{-1}) = \varepsilon_s \eta_s$ etc.

How do I ~~deter~~ find the M context?

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \qquad \begin{matrix} \cancel{W} \longmapsto P\otimes_A \cancel{W} \\ Q\otimes_B W \longleftarrow\!\shortmid\ \cancel{W} \end{matrix}$$

$P = \mathbb{C}\Gamma$ as $\mathbb{C}\Gamma, \mathbb{C}$ bimod

$Q = hB$

$$\mathcal{M}(A) \underset{Q\otimes_B -}{\overset{P\otimes_A -}{\rightleftarrows}} \mathcal{M}(B)$$

To get P you apply functor to $A = \mathbb{C}$ and you ~~find~~ $P = \mathbb{C}\Gamma$ $\quad \Gamma$ acts left mult.

To get Q you apply other functor to B so that $Q = hB$ as right B module. B is the crossproduct $\Gamma \ltimes C_{fin}(\Gamma)$. basis $s \otimes e_t = se_t$

$se_t = e_{st}s$

$se_{s^{-1}t} = e_t s$

$h = e_1$ $\quad$ ~~$e_1 se_t = e_1 e_{st} s$~~

~~$e_1 se_t = e_1 e_{s^{-1}t} s$~~

$$e_1 se_t = s e_{s^{-1}} e_t = \begin{cases} 0 & s^{-1} \neq t \\ se_{s^{-1}} & \text{if } s^{-1} = t. \end{cases} \qquad ?$$

Since Q is a right B-module use $C_{fin}(\Gamma) \rtimes \Gamma$

$hB = e_1 C_{fin}(\Gamma) \otimes \mathbb{C}\Gamma \simeq \mathbb{C}\Gamma$

$P = \mathbb{C}\Gamma$ as Γ-module mod with $h = e_1$

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to construct the Morita context corresp. to the Morita equiv. between $\mathbb{C}$ and finite Γ, $\hat{\Gamma}$ mods.

from B-module W has Γ-action and a Γ-grading $W = \bigoplus_t W_t$ related by $sW_t \subset W_{st}$. Then one has canon. isom $\mathbb{C}\Gamma \otimes W_1 \longrightarrow W$, which is the $\bigoplus_s$ of $s : W_1 \xrightarrow{\sim} W_s$.

The functors are $V \longmapsto \mathbb{C}\Gamma \otimes V$ where this is a B-module with the operator $s \in \Gamma$, e_t for $t \in \Gamma$.

$$B = \mathbb{C}\Gamma \otimes C_{fin}(\Gamma) = \bigoplus_{s,t} \mathbb{C}\, s e_t \qquad s e_t s^{-1} = e_{st}$$

Then B, A bimodule P is $\mathbb{C}\Gamma$ clearly.

The other functor $W \longmapsto e_1 W$ yields for $W = B$ as left B-module the A, B bimodule $e_1 B$

$$s e_t = e_{st} s \quad \text{is in } e_1 B \iff st = 1$$

so $$e_1 B = \bigoplus_t \mathbb{C}\, e_1 t$$

Composite functors. $V \longmapsto \mathbb{C}\Gamma \otimes V \longmapsto e_1 \mathbb{C}\Gamma \otimes V$

yields $$e_1 \mathbb{C}\Gamma \underset{\mathbb{C}\Gamma}{\otimes} \mathbb{C}\Gamma \longrightarrow e_1 \mathbb{C}\Gamma$$

$$e_1 t \otimes s \longmapsto e_1(ts)$$ mult in Γ followed by e_1

Now $W \longmapsto e_1 W \longmapsto \mathbb{C}\Gamma \otimes e_1 W \simeq W$

To understand better. Functors

$V \longmapsto \mathbb{C}\Gamma \otimes V$

$W \longmapsto e_1 W = W_1$

are inverse since $e_1 \mathbb{C}\Gamma = \mathbb{C}$

and $\mathbb{C}\Gamma \otimes W_1 \xrightarrow{\sim} W$

e_1 is the element of B which proj onto degree $= 1$ in the Γ grading

$\mathbb{C}\Gamma$ is a left B-module, right $\mathbb{C}$

$e_1 B$ the image of e_1 on B, that is, the projection on the degree 1 component. Use bases $s e_t$ for B.

$$B = \mathbb{C}\Gamma \otimes \bigoplus_t \mathbb{C} e_t$$

$$B_s = s \otimes \bigoplus_t \mathbb{C} e_t$$

$\therefore\ e_1 B =$

Two much confusion, $B = \mathbb{C}\Gamma \rtimes \mathbb{C}\hat{\Gamma}$

$\mathbb{C}\hat{\Gamma} = \bigoplus_{t \in \Gamma} \mathbb{C} e_t$ $\qquad e_t e_{t'} = \delta_{tt'} e_t$

$\mathbb{C}\Gamma = \bigoplus \mathbb{C} s$ $\qquad s e_t s^{-1} = e_{st}$

$\mathbb{C}\Gamma$ is an obvious firm B module

e_t is the projection on $\mathbb{C}t = (\mathbb{C}\Gamma)_t$

$\mathbb{C}\Gamma = B e_1$. $\mathbb{C}\hat{\Gamma}$ should be obviously a firm right B-module. Try $e_1 B = B_1 = 1 \otimes \mathbb{C}\hat{\Gamma}$.

yes. $(e_t) s = s e_{s^{-1}t}$

[redacted] Functors $V \longmapsto \mathbb{C}\Gamma \otimes V$ $\quad$ $\mathrm{Mod}(\mathbb{C}) \longrightarrow \mathrm{Form}(B)$

$W \longmapsto W_1 = h_1 W$

use h_s? No. $W_1 = e_1 W$

Obvious isoms. $\quad (\mathbb{C}\Gamma \otimes V)_1 = 1 \otimes V \simeq V$

$\mathbb{C}\Gamma \otimes W_1 \xrightarrow{\sim} W.$

You want the Morita context behind this.

$P = \mathbb{C}\Gamma$ $\quad$ left B-module structure + right $\mathbb{C}$.

$Q =$ [redacted] $B_1 = 1 \otimes \mathbb{C}\hat{\Gamma} = e_1 B$

which means it is obvious a right B-module

Now $Be_1 = \mathbb{C}\Gamma \otimes (\mathbb{C}\hat{\Gamma})e_1 \simeq \mathbb{C}\Gamma$. As left B-module B splits into $\mathbb{C}\Gamma \otimes e_t = Be_t$.

Now comes the crunch, namely $P \otimes_A Q \xrightarrow{\sim} B$

$Q \otimes_B P \xrightarrow{\sim} A$. Now the should follow from

$$V \longmapsto \mathbb{C}\Gamma \otimes V \longmapsto (\mathbb{C}\Gamma \otimes V)_1 \simeq V$$

$$V \longmapsto \mathbb{C}\Gamma \otimes_A V \longmapsto \mathbb{C}\hat{\Gamma} \otimes_B \mathbb{C}\Gamma \otimes V$$

The isom. $\mathbb{C}\hat{\Gamma} \otimes_B \mathbb{C}\Gamma \xrightarrow{\sim} \mathbb{C}$?

$\langle e_t, s \rangle = ?$

$e_1 B \otimes_B Be_1$

Repeat: Functors $V \longmapsto \mathbb{C}\Gamma \otimes V$.

$$W \longmapsto W_1 = e_1 W$$

Here $B = \mathbb{C}\Gamma \ltimes \mathbb{C}\hat{\Gamma}$, firm B-modules are Γ-modules W with h such that $shs^{-1} = e_s$, $s \in \Gamma$ are mutually annihilating projections with sum $= 1$. Γ action and Γ grading: $W = \oplus W_t$ such that $sW_t = W_{st}$.

So $P = \mathbb{C}\Gamma$ with obvious left B mult.

It should be clear that $Be_1 = \mathbb{C}\Gamma \otimes \boxed{(\mathbb{C}\hat{\Gamma})e_1} \simeq \mathbb{C}\Gamma$.

$$W_1 = Q \otimes_B W \qquad Q = e_1 B$$

Note that B is Γ graded, as it's a crossproduct.

$\Phi = \{1\}$. firm B-module is Γ-module W with an h $\ni$ $hsh = 0$ $s \neq 1$, $\sum shs^{-1} = 1$

$\Rightarrow$ $shs^{-1} = e_s$ annih projection. $W = \bigoplus_s \underbrace{(shW)}_{= W_s}$

Case $\Phi = 1$. $B = \Gamma \ltimes C_f(\Gamma) = \bigoplus_{s,t} sh_t$

A firm B mod is a Γ-mod W with h such that $hsh = 0$, $s \neq 1$ and $\sum shs^{-1} = 1$. Then $shs^{-1} = h_s$ are ann. idemp. Equiv.

$$V \longmapsto \mathbb{C}\Gamma \otimes V \qquad \text{obvi } \Gamma \text{ action, } h_t = \varepsilon_t^{d} \eta_t^{f}$$

$$W \longmapsto hW = W_1 \quad \text{for the } \Gamma \text{ grading}$$

Point: The partition $\sum h_s = 1$ is a Γ grading in the disjoint case.

~~Now take~~ functors $V \longmapsto \mathbb{C}\Gamma \otimes V$

$W \longmapsto hW$

canonical isos ~~[illegible]~~ $h(\mathbb{C}\Gamma \otimes V) \simeq V$.

$$\mathbb{C}\Gamma \otimes hW \xrightarrow{\sim} W$$

bimodule $P = \mathbb{C}\Gamma$ left B, right $A = \mathbb{C}$

$Q = hB$ left $\mathbb{C}$, right B

You ~~should~~ now have canon isos. $V \simeq Q \otimes_B P \otimes_A V$

$W \simeq P \otimes_A Q \otimes_B W$

$$Q \otimes_B P = hB \otimes_B \mathbb{C}\Gamma \xrightarrow{\sim} h\mathbb{C}\Gamma = \mathbb{C}$$

$hB = \cancel{\bigoplus_t \mathbb{C} h_t} = \bigoplus_t \mathbb{C}(1 \otimes h_t)$. You want to

consider $hB \times \mathbb{C}\Gamma \longrightarrow h\mathbb{C}\Gamma = \mathbb{C}1$ B-mult. on $\mathbb{C}\Gamma$ restricted to hB.

$$\langle \cancel{1\otimes} h_t, s \rangle = \begin{cases} 0 & t \neq s \\ 1 & t = s. \end{cases}$$

delet.

At the moment you have ~~$P = \mathbb{C}\Gamma$, $Q = \mathbb{C}$~~

$P = \mathbb{C}\Gamma = \bigoplus_s \mathbb{C}s$, $Q = \bigoplus_t h_t$ $\qquad h_t s = s h_{s^{-1}t}$?

left B-mod $\qquad$ right B-mod

$$Q = hB = h \bigoplus_{s,t} s\, h_t \mathbb{C} \simeq \bigoplus h_t \mathbb{C}$$

~~it seems that~~ What is right mult by

u on Q

B has basis sh_t and mult.

$$sh_t\, s'h_{t'} = ss'h_{s'^{-1}t}h_{t'} = 0 \quad \text{if } s'^{-1}t \neq t'$$

M has basis $h_t s$ with cross product

$$h_t s\, h_{t_1} s_1 = h_t h_{st_1} ss_1$$

~~[illegible]~~ $\checkmark$

$\Phi = 1$. $\quad B = \mathbb{C}\Gamma \otimes \mathbb{C}\hat{\Gamma} = \bigoplus_{s,t} \mathbb{C} s h_t$

firm B-module $= \Gamma$ module W with h sat.

$hsh = 0$, $s \neq 1$, $\sum shs^{-1} = 1$.

$$hB = \bigoplus_t \mathbb{C}ht = h \otimes \mathbb{C}[\Gamma] \quad ?$$

$$Bh = \bigoplus_s \mathbb{C}sh = \mathbb{C}[\Gamma] \otimes h$$

So what next? Functors

$$V \longmapsto \mathbb{C}\Gamma \otimes V \longmapsto h(\mathbb{C}\Gamma \otimes V) = 1 \otimes V$$

$$W \longmapsto hW \longmapsto \mathbb{C}\Gamma \otimes hW \xrightarrow{\sim} W$$

$hW \leftarrow hB \otimes_B W$ since $h^2 = h$.

$P = \mathbb{C}\Gamma$ left B, right $\mathbb{C}$ bimodule

$Q = hB = h \otimes \mathbb{C}\Gamma$ right B, left $\mathbb{C}$ bim-

$$W \mapsto hW = Q \otimes_B W \longmapsto \mathbb{C}\Gamma \otimes hW = P \otimes_A Q \otimes_B W$$

$V \longmapsto \mathbb{C}\Gamma \otimes V \longmapsto h(\mathbb{C}\Gamma \otimes V) = 1 \otimes V$

$W \longmapsto hW \longmapsto \mathbb{C}\Gamma \otimes hW \xrightarrow{\sim} W$

$P = \mathbb{C}\Gamma \quad , \quad Q = hB$

Check that $Bh \simeq \mathbb{C}\Gamma$ canonically.

$$\underset{\mathbb{C}\Gamma \otimes (\mathbb{C}\hat{\Gamma})h \simeq \mathbb{C}\Gamma \otimes h}{\overset{\|}{Bh}}$$

So using the basis $s h_t$ for B, you see that $Bh = \mathbb{C}\Gamma \otimes h$, ~~and~~ $Bh \simeq \bigoplus_s \mathbb{C}sh$

Using $h_{st}s$ for B, you see that

$hB = \bigoplus_s \mathbb{C}hs = h \otimes \mathbb{C}\Gamma$

$V \longmapsto \mathbb{C}\Gamma \otimes V$ $\qquad P = \mathbb{C}\Gamma = \bigoplus_s \mathbb{C}s$

$W \longmapsto hW \simeq hB \otimes_B W$ $\qquad Q = hB = \bigoplus_t \mathbb{C}ht = h \otimes \mathbb{C}\Gamma$

$P \otimes_A Q = \mathbb{C}\Gamma \otimes h \otimes \mathbb{C}\Gamma$

$= \bigoplus_{s,t} \mathbb{C} \cdot sht$

$W \longmapsto hW = hB \otimes_B W \longmapsto \mathbb{C}\Gamma \otimes hB \otimes_B W \xrightarrow{\sim} W$

You need to u

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functors $\quad V \mapsto \mathbb{C}\Gamma \otimes V, \quad W \mapsto hW = hB \otimes_B W$

bimodules $\quad P = \mathbb{C}\Gamma \qquad Q = hB = \bigoplus_t \mathbb{C}ht$

composite fun*s* $\quad Q \otimes_B P = hB \otimes_B \mathbb{C}\Gamma \xrightarrow{\sim} h(\mathbb{C}\Gamma) = \mathbb{C}$

$\qquad ht \otimes s \qquad ht \otimes s \longmapsto h(ts) = \begin{cases} 1 & \text{if } ts = 1 \\ 0 & \text{if } ts \neq 1 \end{cases}$

$P \otimes_A Q = \mathbb{C}\Gamma \otimes hB \xrightarrow{\sim} B$

$\qquad s \otimes ht \longmapsto sht = h_s(st)$ basis for $\mathbb{C}\hat{\Gamma} \rtimes \Gamma$.

so what to do next. Review case $\Phi = 1$.

$$V \longmapsto \mathbb{C}\Gamma \otimes V \longmapsto h(\mathbb{C}\Gamma \otimes V) = 1 \otimes V$$

$$W \longmapsto hW \longmapsto \mathbb{C}\Gamma \otimes hW \xrightarrow{\sim} W$$

$$P = \mathbb{C}\Gamma \qquad Q = hB = \bigoplus_t \mathbb{C}ht$$

Stop and look again at the formalism ~~trying~~ keeping left + right [redacted] evenly.

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \qquad \mathcal{M}(A) \underset{Q \otimes_B -}{\overset{P \otimes_A -}{\rightleftarrows}} \mathcal{M}(B)$$

$$\mathcal{M}(A^{op}) \underset{- \otimes_B P}{\overset{- \otimes_A Q}{\rightleftarrows}} \mathcal{M}(B^{op})$$

P is the B-module corresp to the A-module A

P —— A^{op}-module —— B^{op}-module B

Q —— A- —— B —— B

Q —— B^{op} —— A^{op} —— A

Thus you have two descriptions of each bimodule.

Begin with the functors $V \mapsto p(\mathbb{C}\Gamma \otimes V)$ and $W \longmapsto hW$. These yield the bimodules

$p(\mathbb{C}\Gamma \otimes \tilde{A}) = P$ and $hB = Q$. ~~These~~ Now you have canon. isos.

$$Q \otimes_B P \otimes V = hB \otimes_B p(\mathbb{C}\Gamma \otimes \tilde{A}) \otimes V \longrightarrow hp(\mathbb{C}\Gamma \otimes V) = V$$

Start $\Phi = 1$ case again

$$V \longmapsto \mathbb{C}\Gamma \otimes V \longmapsto h(\mathbb{C}\Gamma \otimes V) = 1 \otimes V$$

here h is

$V \longmapsto \mathbb{C}\Gamma \otimes V$

$W \longmapsto hW$

~~The~~ important is the fact that hW comes with i,j $\quad W \xrightarrow{j} hW \xrightarrow{i} W$

Important seems to be that hW comes with i,j and similarly $W = \mathbb{C}\Gamma \otimes W$ comes with maps

$W \xrightarrow{\alpha} \mathbb{C}\Gamma \otimes V \xrightarrow{\beta} W \qquad \beta\alpha = 1, \ \alpha\beta = p.$

Repeat the details of the M equiv.

Given a firm B-module W, that is, a Γ-module with $\mathbb{C}$-linear op h sat $hsh = 0$ $s \notin \Phi$, $\sum shs^{-1} = 1$ on W. Let $V = hW$, let $i: V \hookrightarrow W$ be the inclusion, $j: W \twoheadrightarrow V$ be the surj ind by h. Then V equipped with $p(s) = jsi$ is a reduced A-mod.

~~Given an A-mod V, let form the Γ-module ℂΓ⊗V gen. by V. Let p be the op. on ℂΓ⊗V~~

~~$p\left(\sum_t t \otimes v(t)\right) = \sum_s s \otimes \sum_t p(s^{-1}t)\, v(t)$. P comm. with Γ action~~

To understand properly $h: W \xrightarrow{j} V \hookrightarrow W$, exhibit

$$W \xrightarrow{\alpha} \mathbb{C}\Gamma \otimes V \xrightarrow{\beta} W$$

$$\alpha w = \sum_s s \otimes j s^{-1} w, \qquad \beta\left(\sum_t t \otimes v(t)\right) = \sum_t t\, i\, v(t)$$

$$\beta\alpha w = \sum s\, i\, j\, s^{-1} w = w$$

$$\alpha\beta(t \otimes v) = \alpha(t\, i\, v) = \sum_s s \otimes \underbrace{(j s^{-1} t\, i)}_{p(s^{-1}t)} v$$

How do you write this up? From the partition you ~~get~~ see W is a retract of the Γ-module gen. by V, for the projection

$$p(t\otimes v) = \sum_s s \otimes p(s^{-1}t)v$$

$$p(ut\otimes v) = \sum_s s\otimes p(s^{-1}ut)v = \sum_s us\otimes p(s^{-1}t)v = u\,p(t\otimes v)$$

$s \mapsto us$

$$\cancel{pp(t\otimes v) = \sum_s u\otimes p(u^{-1}t)}$$

$$p(t\otimes v) = \sum_s ts^{-1} \otimes p(s)v$$

$$pp(t\otimes v) = \sum_{s,u} ts^{-1}u^{-1} \otimes p(u)p(s)v \qquad s \to u^{-1}s$$

$$= \sum_{s} ts^{-1} \otimes \underbrace{\sum_u p(u)p(u^{-1}s)}_{p(s)}.$$

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flow. W choose $h = \iota j : W \twoheadrightarrow V \hookrightarrow W$

define $W \xrightarrow{\alpha} \mathbb{C}\Gamma\otimes V \xrightarrow{\beta} W$

show $\beta\alpha = 1_W$, $\alpha\beta = p$ on $\mathbb{C}\Gamma\otimes V$

~~It obvious helps to understand the flow and ... put the~~

It should help to put the flow into lemmas, assertions.

Given firm B-mo

Assertions finite B-module = Γ mod. W equipped with h sat. $hsh = 0 \quad s \notin \Phi$, $\sum shs^{-1} = 1_W$.

Given W, factor $h = ij : W \xrightarrow{j} V \xrightarrow{i} W$

j surj + i inj. Thus V = image of h, i is the inclusion map and $j = h$.

Ass. V, equipped with $p(s) = jsi$ for $s \in \Gamma$ is ~~an~~ reduced A-module.

~~Let $\mathbb{C}\Gamma \otimes V$ be the Γ-module gen. by V~~

Next embed Γ-mod W as a retract (or summand) of the free Γ-module $\mathbb{C}\Gamma \otimes V$ gen. by V.

$$W \xrightarrow{\alpha} \mathbb{C}\Gamma \xrightarrow{\beta} W$$

$$\alpha w = \sum_s s \otimes j s^{-1} w \quad , \quad \beta\Big(\sum_t t \otimes v(t)\Big) = \sum_t t\, i\, v(t)$$

$$\beta\alpha = 1_W, \quad \alpha\beta = p \qquad p\Big(\sum t \otimes v(t)\Big) =$$

Can you prove $Bh \simeq p(\mathbb{C}\Gamma \otimes A)$? Establish a canon isom. If so then you have $hBh = A/_A A$ and by symmetry you might have $hBh = A/A_A$ so that $_A A = A_A$.

So one thing to do is to define a ring structure on $hBh = hB \otimes_B Bh$. $\Gamma = \Phi$ fin case B is unital. ~~You have unital simplex~~ ;

$Bh \simeq p(\mathbb{C}\Gamma \otimes A)$ Why this should be true: $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ A considered as left A-module corresponds to the left B-module $P = p(\mathbb{C}\Gamma\otimes A)$. B considered as right B module corresponds to the right A-module

$$P = P \otimes_B B \quad \text{and} \quad Bh.$$

Say this better.

the A module $A \rightsquigarrow$ the B-module $p(\mathbb{C}\Gamma\otimes A)$ and P

the B^{op} —— $B \rightsquigarrow$ the A^{op}-module Bh and P

Set up a map, candidate for identifying Bh with a Γ-module retract of $\mathbb{C}\Gamma\otimes A$. You want to identify hBh + A. You should have

$$hB \otimes_B Bh \twoheadrightarrow hBh$$

$B = D \rtimes \Gamma$ $\qquad$ D has gen. h_s $s\in\Gamma$

$h_s h_t = 0 \quad s^{-1}t \notin \Phi$

$B = \bigoplus_s Ds$

$$h_s = \sum_t h_s h_t = \sum_t h_t h_s$$

$h = h_1$. Look at $Bh_1 = \mathbb{C}\Gamma \otimes Dh_1$.

$$Bh_1 = \bigoplus s \otimes Dh_1$$

$$t(s\otimes dh_1) = ts \otimes dh_1$$

$$h_1(s\otimes dh_1) = s\otimes h_{s^{-1}} dh_1$$

seems that $h_1 B h_1 = \bigoplus s^{-1} \otimes h_s D h_1$

$B = \mathbb{C}\Gamma \tilde{\otimes} D$ ~~$s d t d$ st~~

Do the other way $B = D \rtimes \Gamma$

$$h_1 \Big(\sum_s d_s s\Big) h_1 = \sum_s h_1 d_s h_s s$$

$$h_1 B h_1 = \bigoplus_s h_1 D s h_1 = \bigoplus_s (h_1 D h_s) s$$

$$B = D \rtimes \Gamma = \bigoplus_s D s \qquad d s\, d_1 t = d (s * d_1) st.$$

$$h_1 B h_1 = \bigoplus_s (h_1 D h_s) s$$

Now you need A to act on this both on the left and right. Should be clear

~~$p(s) h w = j s i h w = j s h w = h s h w$~~

~~$x h p(s) = x \, j i$~~

~~$h = j i : X \xrightarrow{j} X h \xrightarrow{i} X \qquad p(s) = i s j$~~

$$p(s) h w = j s i h w = j s h w = h s h w$$

$$x h p(s) = x h i s j = x h s h$$

Obviously bimodule map:

$$(p(s) h b h) p(t) = (s h b h) \cancel{t h = s h b h t h}$$

$$= (h s h b h) t h = h s h b h t h$$

$$B = D \rtimes \Gamma = \bigoplus_s Ds$$

$$h_1 B h_1 = \bigoplus_s h_1 D s h_1 = \bigoplus_s (h_1 D h_s) s$$

Now hBh is an A-bimodule with

$$p(s)\, hbh = \overset{s}{\underset{\text{inc.}}{i}}\, hbh = hs\,bh \qquad X \overset{j}{\twoheadrightarrow} Xh \overset{i}{\hookrightarrow} X$$

$$hbh\, p(s) = hbh\, i s j = hbhsh$$

so it is an obvious A-bimodule.

What are you doing really?

Can you get $\quad Bh \simeq p(\mathbb{C}\Gamma \otimes A)$ $\quad$ A^{op} module corresp to the B^{op} module p.

Both are firm B modules, so applying should give $\quad hBh = A \otimes_A A$ by symm. $= A/A_A$

What are you doing really?

$$p(s) = \sum_t p(t)\, p(t^{-1}s)$$

$$A \subset \sum p(t) A$$

You know Bh is B firm, so it should follow that Bh is A^{op} firm.

Try to start with dual pair over A namely $\quad p(\mathbb{C}\Gamma \otimes A)\,,\ (A \otimes \mathbb{C}\Gamma)p$

Review. Consider case $\Gamma = \Phi$ finite

$B = \mathcal{E} \rtimes \Gamma$ is unital

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

Then P is a fin. proj. A^{op}-mod

Q ———— A-mod

$\langle\ ,\ \rangle : Q \times P \longrightarrow A$ perfect duality

$$B \xrightarrow{\sim} \mathrm{Hom}_{A^{op}}(P,P), \quad B^{op} \xrightarrow{\sim} \mathrm{Hom}_A(Q,Q)$$

~~to what~~ Why. $1 = \sum p_s q_s$ ~~is nuclear~~

1_P is nuclear $\quad p = \cancel{p_s \langle q_s, p\rangle} \sum_s p_s (q_s p)$

$$P \otimes_A \mathrm{Hom}_{A^{op}}(P,A) \longrightarrow \mathrm{Hom}_{A^{op}}(P,P)$$

$$p_s \qquad (q_s \,\cdot\,)$$

In general for any A^{op} module N, have canon

$$N \otimes_A \mathrm{Hom}_{A^{op}}(P, \tilde{A}) \longrightarrow \mathrm{Hom}_{A^{op}}(P,N)$$

$$\sum \phi(p_s) q_s p = \phi(p)$$

So in this situation you want to clearly understand the role of p.

$$\begin{pmatrix} A & \cancel{Y} \\ \cancel{X} & B \end{pmatrix} \qquad \begin{pmatrix} \tilde{A} & A\Gamma \\ \Gamma A & \Gamma A \Gamma \end{pmatrix} \qquad \begin{matrix} 1\times 1 & 1\times n \\ n \times 1 & n\times n \end{matrix}$$

Notation is screwing up.

$$\begin{matrix} A & (A\Gamma)p \\ p(\Gamma A) & \end{matrix} \qquad \begin{matrix} A & A\Gamma \\ \Gamma A & \end{matrix}$$

$$\begin{pmatrix} A & A\Gamma \\ \Gamma^{-1}A & \Gamma^{-1}A\Gamma \end{pmatrix} \qquad \begin{matrix} A & A \iota s \\ j s^{-1} & 0 \end{matrix} \quad ?$$

$$\begin{pmatrix} A & (A\Gamma)p \\ p(\Gamma A) & p(\Gamma A\Gamma)p \end{pmatrix}$$

There are many questions. Let do $\Phi = 1$ again. $A = \mathbb{C}$

$B = \hat{\Gamma} \rtimes \Gamma = \mathbb{C}\hat{\Gamma} \otimes \mathbb{C}\Gamma$

$\mathbb{C}\hat{\Gamma} = \bigoplus_{t \in \Gamma} e_t \mathbb{C}$ min. idemp.

$s e_t s^{-1} = e_{st}$. In this case fin B-module = Γ-module W with Γ-grading $W = \bigoplus_t W_t$ $\quad sW_t \subset W_{st}$.

The functors giving the M eq are

$V \longmapsto \mathbb{C}\Gamma \otimes V$ with left Γ-mult, Γ-grading $(\mathbb{C}\Gamma \otimes V)_s = s \otimes V$.

$W \longmapsto W_1 = e_1 W$

composite function

$V \longmapsto \mathbb{C}\Gamma \otimes V \longmapsto (\mathbb{C}\Gamma \otimes V)_1 = 1 \otimes V \simeq V$

this bit here amounts to V being the retract of $\mathbb{C}\Gamma \otimes V$ corresp. to the projection e_1, i.e. the maps

$$\begin{matrix} \mathbb{C}\Gamma \otimes V \\ \eta_1 \downarrow \uparrow \varepsilon \\ V \end{matrix}$$

$W \longmapsto W_1 \longmapsto \mathbb{C}\Gamma \otimes W_1 \xrightarrow{\sim} W$

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adjoint funs. $V \longmapsto X \otimes_A V$ $\qquad Y \otimes_B W$

$\mathrm{Hom}_B(X \otimes_A V, W) \qquad \mathrm{Hom}_A(V, \mathrm{Hom}_B(X, W))$

First you must understand $\Phi = \{1\}$: $A = \mathbb{C} = B = \left(\bigoplus_t \mathbb{C}e_t\right) \otimes \mathbb{C}\Gamma$ $\quad e_t$ orthon idempotents, $se_ts^{-1} = e_{st}$

firm B-mod $=$ Γ-mod W with Γ-grad $W = \bigoplus_t W_t$ sat $sW_t \subset W_{st}$. Functors giving Meq

$$V \longmapsto \mathbb{C}\Gamma \otimes V, \ \underset{\text{mult}}{\Gamma\text{left}}, \ (\mathbb{C}\Gamma \otimes V)_t = t \otimes V$$

$$W \longmapsto W_1$$

composite fns. $\quad V \mapsto \mathbb{C}\Gamma \otimes V \longmapsto (\mathbb{C}\Gamma \otimes V)_1 = 1 \otimes V \simeq V$

iso $V \xrightarrow{\sim} (\mathbb{C}\Gamma \otimes V)_1$ given by $V \xrightarrow{\varepsilon_1} \mathbb{C}\Gamma \otimes V \xrightarrow{\eta_1} V$

$W \longmapsto W_1 \longmapsto \mathbb{C}\Gamma \otimes W_1 \overset{?}{\xrightarrow{\sim}} W$. Here you have

$e_1 = \iota j : W \xrightarrow{j} W_1 \xrightarrow{\iota} W$ yielding $W \xrightarrow{\alpha} \mathbb{C}\Gamma \otimes W_1 \xrightarrow{\beta} W$

where α, β are inverse isos.

To generalize.

firm B-module $=$ Γ-module W with operators $h_t, t \in \Gamma$

$sh_ts^{-1} = h_{st}$, $\sum_t h_t = 1$, $\quad h_th_s = 0 \quad t^{-1}s \notin \Phi$.

Functors giving M. eq.

$$p(\mathbb{C}\Gamma \otimes V) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathbb{C}\Gamma \otimes V$$

$V \mapsto X(V) =$ retract of the Γ-module $\mathbb{C}\Gamma \otimes V$ with $h = \iota j \quad i = \eta_1\alpha, \ j = \beta\varepsilon_1$

$W \mapsto Y(W) = hW$, with $p(s) = jsi$

where $h = \iota j : W \xrightarrow{j = h} hW \xrightarrow{\iota = \text{inc}} W$

$$\begin{tikzcd} X(V) \arrow[r, "\alpha"] \arrow[dr] & \mathbb{C}\Gamma \otimes V \arrow[r, "\beta"] \arrow[d] & X(V) \\ & V \arrow[ur, "i"] & \end{tikzcd}$$

equiv $\quad V \longmapsto X(V) \longrightarrow hX(V) = V$

V red.

canon. isom. from j and the fact that j is surj, i injective when V red.

$W \mapsto hW \longmapsto \underline{X(hW)}$

Repeat In the $\Phi = \{1\}$ case ~~[illegible]~~ $B = \mathbb{C}\hat{\Gamma} \rtimes \Gamma$

This has two Γ gradings at least. Recall B has the basis $e_t s$ and cross product $s e_t = e_{st} s$. You want to obtain B from a dual pair over $A = \mathbb{C}$. $P = \mathbb{C}\Gamma$. It

Your problem or mistake. You want ~~[illegible]~~ ~~to resemble~~ $B = \mathbb{C}\Gamma \rtimes \hat{\Gamma}$ to resemble $B = P \otimes Q$. But ~~[illegible]~~ in the crossproduct the relation is $e_t s \, e_{t_1} s_1 = e_t e_{st_1} s s_1$ whereas the mult in B is $pq \, p_1 q_1 = p \underbrace{\langle q, p_1 \rangle}_{\text{scalar}} q_1$

Go back to functors $V \longmapsto \mathbb{C}\Gamma \otimes V$

$\therefore$ $P = \mathbb{C}\Gamma$ with left B from the $^{\text{left}}\Gamma$ action and the Γ-grading. $W \longmapsto e_1 W = e_1 B \otimes_B W$

$\therefore$ ~~[illegible]~~ $Q = e_1 B = \bigoplus_t e_1 t \mathbb{C}$ with right B mult.

$$(e_1 t)(e_s u) = e_1 e_{ts} tu = \begin{cases} e_1 tu & \text{if } ts = 1 \\ 0 & \text{if not} \end{cases}$$

So you need $\langle q, p \rangle$

$$\langle e_1 t, s \rangle = ?$$

$$V \longmapsto \mathbb{C}\Gamma \otimes V \longmapsto \underbrace{(\mathbb{C}\Gamma \otimes V)_1}_{e_1 B \otimes_B \mathbb{C}\Gamma \otimes V} \underset{\varepsilon_1}{\overset{\eta_1}{\rightleftarrows}} V$$

So it's clear that $\langle e_1 t, s \rangle = \eta_1(e_1 ts)$

$e_1 t \otimes s \otimes v \mapsto \eta_1(e_1 ts \otimes v)$

Check that it leads to the correct mult on

$B = P \otimes Q = \underline{\mathbb{C}\Gamma \otimes \mathbb{C}e_1\Gamma}$ ~~How are you~~

$$W \longmapsto e_1W \longmapsto \mathbb{C}\Gamma \otimes e_1W \xrightarrow{\sim} W$$

$$\begin{array}{ccccccc} B & \longmapsto & e_1B & \longmapsto & \mathbb{C}\Gamma \otimes e_1 B & \xrightarrow{\sim} & B \\ & & \| & & s \otimes e_1 t & & se_1 t \\ & & \mathbb{C}e_1\Gamma & & & & \end{array}$$

$$(se_1t)(s'e_1t') = s\langle e_1t, s'\rangle e_1t'$$

$$V \longmapsto p(\mathbb{C}\Gamma \otimes V) \longmapsto hp(\mathbb{C}\Gamma \otimes V)$$ retract

$$W \longmapsto hW \longmapsto p(\mathbb{C}\Gamma \otimes hW)$$

~~Given A-mod V you get a function~~

Ultimately what's important is a triple $(W, V, ?)$ where ? is the data you need to construct an isom ~~$p(\mathbb{C}\Gamma\otimes V) \to W$~~ $hW \xrightarrow{\sim} V$ equivalently an isom $W \xrightarrow{\sim} p(\mathbb{C}\Gamma \otimes V)$. Maybe adjunction arrows?

So how does it go? The basic object should ~~be a~~ consist of a firm B-module W, a reduced A-module V, and a factorization of the op $h = ij : W \xrightarrow{j} V \xhookrightarrow{i} W$ where i inj, j surj. These form a cat in an obvious way, and there are forgetful function to firm B and red A modules which are equivalences

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "j"'] & \mathbb{C}\Gamma \otimes V \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & W \\ & V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "i"'] & \end{tikzcd}$$

Idea: The category of (V, W, i, j) should turn out to be reduced modules for the Morita context

$\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$ $\begin{pmatrix} V \\ W \end{pmatrix}$ $j\sigma^{-1} \in Y$, $t\iota \in X$

XY contains $th\sigma^{-1}$

YX " $j\sigma^{-1}t\iota = p(\sigma^{-1}t)$

Discuss the problem. You have a category of (V, W, ι, j) ~~W is Γ-module,~~

You have a category of "modules", you would like to find the ring behind these modules, this means endos of the forgetful functor

$(V, W, \iota, j) \longmapsto \begin{matrix} V \\ \oplus \\ W \end{matrix}$ Problem - how to get around absence of a 1.

$\mathrm{Mod}(R) \longrightarrow \mathcal{A}b$

$T : M \longrightarrow M$ endo of the forget R-mult.

look at $T : R \longrightarrow R$ commutes with right mult.

$T(r) = T(1r) = T(1)r$

For idempotent rings ~~you need~~ you get multipliers

$\mathcal{M}(A) \longrightarrow \mathrm{Mod}(\mathbb{Z})$

$\mathrm{Hom}_{A^{op}}(A, A)$ left mult.

$A \otimes_A M = M$

Question: $\Gamma = \Phi$ finite. $\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$

B is unital, but A is not. You would like to understand firm modules over C.

$\begin{pmatrix} ReR & Re \\ eR & eRe \end{pmatrix}$? $\begin{pmatrix} yx & y \\ x & xy \end{pmatrix}$

You want a ~~simple~~ ~~...~~ example of $\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$

generators x, y $\quad (xy)x = x$ $\quad$ Four elts
$y(xy) = y$

$\begin{pmatrix} yx & y \\ x & xy \end{pmatrix}$

Recall ~~...~~ that you wanted to understand the case ~~$\Phi = \{1\}$, where~~ where idempotent e is factored $e = xy$ ~~$\Phi = \{1\}$~~. To look at $W \longrightarrow V \longrightarrow W$

$\begin{pmatrix} A & Y \\ X & B \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix}$ $\quad W \underset{y}{\overset{x}{\rightleftarrows}} W$

assume that $yx = e$ is idempotent

Point: $A = \mathbb{C}$ but you take a V which is not reduced.

Take Γ arb. but $\Phi = \{1\}$. Then $A = \mathbb{C}e$
$e = p(1)$ and

I am confused. But take V to be a $\mathbb{C}e$ module, form $W = p(\mathbb{C}\Gamma \otimes V) = \mathbb{C}\Gamma \otimes eV$. Then you should have maps $W \xrightarrow{j} V \xrightarrow{i} W$

$$\begin{tikzcd} \mathbb{C}\Gamma \otimes eV \arrow[r, "\alpha"] \arrow[d] \arrow[dr, bend left, "j"] & \mathbb{C}\Gamma \otimes V \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & \mathbb{C}\Gamma \otimes eV \\ eV \arrow[r, hook] & V \arrow[r] \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, bend right, "i"] & eV \arrow[u, hook] \end{tikzcd}$$

review the problem. You construct the M. eq. using (V, W, i, j) where

$ij = h$
j surj
i inj

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "j"'] & \mathbb{C}\Gamma \otimes V \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & W \\ & V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, hook, "i"'] & \end{tikzcd}$$

$\beta\alpha = 1_W$
$\alpha\beta = p$

Show $\forall$ red A-mod V $\exists!$ form B W, i, j
Show $\forall$ form B-mod W $\exists!$ red V st fact.

So now you want to identify (V, W, i, j) with certain (reduced) modules for the M. context

$$\begin{pmatrix} A & Y \\ X & B \end{pmatrix} \cdot \begin{pmatrix} V \\ W \end{pmatrix}$$

X contains ti
Y contains js^{-1}

You want to construct the Morita context which should be generated in some sense by elements $ti : V \to W$ $t \in \Gamma$ and $js^{-1} : W \to V$. ~~In fact the generators seem to be simply~~. These are ~~the~~ generators. It seems that Γ sits in the multiplier ring.

It looks like you want to define the Morita context in the same fashion as for A, B separately, namely via generators and relations.

So you consider generators i_t, j_s

$$\begin{pmatrix} A & Y \\ X & B \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix}$$

relations $yy = 0$

$xx = 0$

$$x_t = \sum_s x_s y_s x_t$$

$$y_t = \sum_s y_t x_s y_s$$

Again
$$\begin{pmatrix} A & Y \\ X & B \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix} \qquad \begin{aligned} x_t &= ti \\ y_s &= js^{-1} \end{aligned}$$

relations: $y_s x_t = 0$ if $s^{-1}t \notin \Phi$

$$\sum_s x_s y_s x_t = x_t \qquad y_s y_t = 0$$

$$x_s x_t = 0$$

$$\sum_t y_s x_t y_t = y_s$$

The ring is idempotent ~~Obvious action of Γ~~

local left units on X

~~local right~~ $\sum_s x_s y_s$ is a local unit in B

and X is a firm left B-module

Y — right B-module.

The above B is clearly a quotient of $\Gamma \ltimes \mathcal{E}_{\Gamma,\Phi}$.

Construct the Morita context $\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$ $\begin{pmatrix} V \\ W \end{pmatrix}$

via generators and relations.

Generators x_t $(= ti : V \to W)$, y_s $(= js^{-1} : W \to V)$

for $t, s \in \Gamma$, Relations. $y_s x_t = 0$ for $s^{-1}t \notin \Phi$

You forgot $x_t x_{t'} = 0$, $y_s y_{s'} = 0$.

"Completeness" relation: $\sum_s x_s y_s x_t = x_t$

$$\sum_t y_s x_t y_t = y_s$$

~~It should be clear that~~

~~B = ... $x_t y$ = $\sum$~~

You've defined a ring C. What about its modules?

So C idempotent, better question is whether C strictly idemp.

$A^2 = A = YX$ $AY = YB = Y$? $Y = BY = YXY$

$XA = BX = X$ $XY = B^2 = B$

$x_t = ti : V \to W$ $y_s x_t = js^{-1}ti = p(s^{-1}t)$

$y_s = js^{-1} : W \to V$ $\sum_s x_s y_s = \sum_s s i j s^{-1}$

Consider a C module M. ~~Look at~~ ...

Embed C into $\begin{pmatrix} \tilde{A} & Y \\ X & \tilde{B} \end{pmatrix}$. ~~So you have a~~

~~reduced~~ Say M is a reduced C-module. This should be independent of the embedding. So you should get splitting $M = \begin{pmatrix} V \\ W \end{pmatrix}$

OKAY $\begin{pmatrix} A & Y \\ X & B \end{pmatrix} \hookrightarrow \begin{pmatrix} \vec{A} & Y \\ X & \vec{B} \end{pmatrix} \twoheadrightarrow \begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix}$

So a firm or reduced C-module splits into $\begin{pmatrix} V \\ W \end{pmatrix}$ consistent with matrix mult.

~~Hmm~~, There ~~is~~ should be a way to understand this in terms of a grading. When you write down generators and relations there should be something like what you did for $A = \mathcal{P}_{\Gamma, \Phi}$.

It looks as if there's a quiver involved. There are four vertices, two incoming two outgoing

$$\begin{tikzcd} V & V \arrow[l, "A"'] \arrow[dl, "X"] \\ W & W \arrow[l, "B"] \arrow[ul, "Y"'] \end{tikzcd} \qquad \begin{pmatrix} A & Y \\ X & B \end{pmatrix}\begin{pmatrix} V \\ W \end{pmatrix}$$

So what are you doing? You write down generators and relations.

$$x_t (= ti) \qquad y_s (= js^{-1})$$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$A_{11}A_{11}$, $A_{12}A_{21}$

$A_{11}A_{12}$, $A_{12}A_{22}$

$A_{21}A_{12}$, $A_{22}A_{21}$

$A_{22}A_{22}$, $A_{21}A_{12}$

somehow like a group.

Too many A's.

So you want to make a map

$$A \longrightarrow M_2(\mathbb{C}) \otimes A$$

$$\begin{pmatrix} A & Y \\ X & B \end{pmatrix} \longmapsto \left(\begin{array}{c|c} \begin{matrix} A & 0 \\ 0 & 0 \end{matrix} & \\ \hline & \end{array}\right)$$

$$\begin{tikzcd} A \arrow[r] \arrow[d, equal] & \mathbb{C}\Gamma \otimes A \arrow[d, equal] \\ \bigoplus A_s \arrow[r, "\varepsilon_s"] & \bigoplus s \otimes A \end{tikzcd}$$

You seem to be dealing with a coalgebra?

$M_2(\mathbb{C})$ has the basis e_{ij} $\quad i,j = 1,2$

$$\begin{pmatrix} a & y \\ x & b \end{pmatrix} \longmapsto \begin{pmatrix} e_{11}\otimes a & e_{12}\otimes y \\ e_{21}\otimes x & e_{22}\otimes b \end{pmatrix} \qquad 0$$

$$\begin{pmatrix} a & y \\ x & b \end{pmatrix}\begin{pmatrix} a' & y' \\ x' & b' \end{pmatrix} = \begin{pmatrix} aa' + yx' & ay' + yb' \\ xa' + bx' & xy' + bb' \end{pmatrix}$$

~~[illegible]~~ partial monoid, path alg. ?

~~[illegible]~~ Functor $A \longmapsto M_2 \otimes A$, can apply it twice, $M_2 \otimes M_2 \otimes A$ isom to $M_4 \otimes A$.

$$A \longmapsto \cancel{}\; V \otimes A \otimes V^*$$
$$\longmapsto V \otimes V \otimes A \otimes V^* \otimes V^*$$

e_{ij} $\quad$ ~~[illegible]~~ $\quad i,j = 1,2$

---

unital ring R + idempotent e yields ~~[illegible]~~ M context.

$$\begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix} = \begin{pmatrix} eRe & eRe^{\perp} \\ e^{\perp}Re & e^{\perp}Re^{\perp} \end{pmatrix}$$

Question. Is it possible to interpret a Morita context as a kind of graded ring, where $\mathbb{C}\Gamma$ is replaced by $M_2\mathbb{C}$? No you want $\mathbb{C}e$.

~~Notation Let~~ Consider an extension of rings

$$0 \longrightarrow C \longrightarrow B \longrightarrow \mathbb{C}e \longrightarrow 0$$

Adjoing an identity

$$0 \longrightarrow C \longrightarrow \tilde{B} \longrightarrow \widetilde{\mathbb{C}e} \longrightarrow 0$$

Assume given a lifting of e ~~into~~ to an idempotent in B. Then you are in the situation of a unital ring R with idemp. e, whence you have the M. cont.

$$\begin{pmatrix} eRe & eRe^{\perp} \\ e^{\perp}Re & e^{\perp}Re^{\perp} \end{pmatrix}$$

~~Conversely Suppose~~ You want to specify a cat of M. cont.

$\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$ $\begin{pmatrix} V \\ W \end{pmatrix}$ gen. $x_t\ (= ti)$, $y_s\ (= js^{-1})$

$$y_s x_t = 0 \quad \text{for} \quad s^{-1}t \notin \Sigma$$

$x_t x_s = 0$

$y_t y_s = 0$

$$\left.\begin{aligned} \sum_s x_s y_s x_t &= x_t \\ \sum_t y_s x_t y_t &= y_s \end{aligned}\right\} \quad \sum x_s y_s = 1$$

Γ acts on C.

$$u * x_t = x_{ut}$$
$$u * y_s = y_{us} \quad (=$$

$\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$ $\begin{pmatrix} V \\ W \end{pmatrix}$

$$\begin{array}{ccc} \mathrm{Mod}(A) & & \mathrm{Firm}(B) \\ V & \longmapsto & X \otimes_A V \\ Y \otimes_B W & \longleftarrow\!\!\!\shortmid & W \end{array}$$

$$V \longmapsto X \otimes_A V \longmapsto Y \otimes_B X \otimes_A V \xrightarrow{\sim} V$$

$$y \otimes x \otimes v \longmapsto (yx)v$$

$\beta\alpha = 1_W, \; \alpha\beta = p$

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "j"'] & C\Gamma \otimes V \arrow[r, "f"] \arrow[d, shift right, "\eta_1"'] & W \\ & V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "i"'] & \end{tikzcd}$$

$ij = h$

$\alpha w = \sum_s s \otimes js^{-1}w$

$t\iota$ $\quad js^{-1}$ $\quad \boxed{\text{~~to what~~}}$

Define C to gen. $x_t \; (= ti)$, $y_s (= js^{-1})$

relations. $y_s x_t = 0$ for $s^{-1}t \notin \Phi$.

$x_t x_{t'} = 0$

$y_s y_{s'} = 0$

$$\boxed{\sum_s x_s y_s x_t = x_t, \quad \sum_t y_s x_t y_t = y_s}$$

~~A has elts~~ $\cancel{y_s x_t}$

~~group Γ to be~~

Γ act $\quad u*(x_t) = x_{ut} \quad u*(y_s) = y_{us}$

$y_{us} x_{ut} = 0 \quad (us)^{-1}(ut) = s^{-1}t \notin \Phi$

$\cancel{y_{us} x_{ut}}$

$y_s x_t = js^{-1}ti$

$\begin{pmatrix} & j\Gamma \\ \Gamma i & \Gamma \end{pmatrix}$ .. So try to construct the Mor context via gens + relations $x_t = ti$, $y_s = js^{-1}$

$js^{-1}ti = p(s^{-1}t)$. So impose the condition that $y_s x_t$ ~~is independ~~ depends only on $s^{-1}t$

Then your Morita context should admit a Γ-action

generators x_t, y_s $\quad x_t x_{t'} = 0, \; y_s y_{s'} = 0$

$y_s x_t$ depends only on $s^{-1}t$ and $= 0$ for $s^{-1}t \notin \Phi$.

$$\sum_s x_s y_s x_t = x_t \qquad \sum_t y_s x_t y_t = y_s$$

$$\begin{pmatrix} A & \overset{jB}{\shortparallel} Y \\ \underset{\underset{Bi}{\shortparallel}}{X} & \cancel{B} \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix}$$

. Probably the Γ action allows cross product ~~with~~ ?

Let's begin with the nice modules V, W Γ-module, ~~$W \xrightarrow{} W \xrightarrow{} W$~~

You are in a GNS framework of some sort. What seems to be important is the special case of GNS involving projections

Go back over GNS ideas. Old notation $\rho : A \longrightarrow B$ $\quad \rho(1) = 1$. Consider M, N, i, j where $M \in Mod(A)$, $N \in Mod(B)$, $\quad i : N \longrightarrow M$, $j : M \longrightarrow N$ such that $j(a\, i(n)) = \rho(a) n$.

Better maybe to begin with Hilb. space picture Γ act on H, subspace $V \underset{i}{\subset} H$ then you get $\rho(s) = i^* s i$, $\quad \rho : \Gamma \longrightarrow \mathcal{L}(V)$ completely positive map

Look at examples!! $\mathbb{Z}$ a positive def. function $\rho : \mathbb{Z} \to \mathcal{L}(H)$, $\mathbb{Z}$ can be any abelian group, should be equivalent to a positive hermitian operator function on $\hat{\mathbb{Z}}$

If Γ abelian, then $\rho : \Gamma \to \mathcal{L}(H)$ is completely positive when $\hat{\rho}$ is a function on $\hat{\Gamma}$ with ~~val~~ pos. herm. values. — this includes measure on $\hat{\Gamma}$ with pos. herm. values. Stick to finite support where $\hat{\rho}$ is a fn. ~~[illegible]~~ Easiest way to arrange this is to have ~~proj~~ its values in projections.

Let W be a ~~[illegible]~~ unitary representation of the group Γ, $i : V \hookrightarrow W$, then $\rho(s) = i^* s i$, $\rho : \Gamma \to \mathcal{L}(V)$ is completely positive, which means that you can use ~~[illegible]~~ ρ to construct a pos. inner product on $\mathbb{C}\Gamma \otimes V$

$$\Big\| \sum_s s \otimes v(s) \Big\|^2 = \Big\| \sum_s s\, i\, v(s) \Big\|^2$$

$$= \Big\langle \sum_s s\, i\, v(s), \sum_t t\, i\, v(t) \Big\rangle = \sum_{s,t} \langle v(s), \underbrace{i^* s^{-1} t\, i}_{\rho(s^{-1}t)}\, v(t) \rangle$$

The general kernel $\rho(s^{-1}t)$ should be ≥ 0

You have an ~~example~~ special case now where the kernel is idempotent. Meaning? ~~[illegible]~~ Perhaps you want

$$W \xrightarrow{\alpha = \beta^*} \mathbb{C}\Gamma \otimes V \xrightarrow{\beta} W$$

When is α an isometry

$$\| \alpha w \|^2 = \Big\| \sum_s s \otimes i^* s^{-1} w \Big\|^2 = \sum_s \langle i^* s^{-1} w, i^* s^{-1} w \rangle$$

$$= \sum_s \langle w, (s\, i\, i^* s^{-1})\, w \rangle$$

~~The most~~ Go back to your Morita context.

$x_t = ti : V \longrightarrow W \qquad y_s = js^{-1} : W \longrightarrow V$

these are the generators.

First aim. To find what should be the good C-modules. Such a module should consist of a Γ-module W, a C-module V, and linear maps $j : W \longrightarrow V$, $i : V \longrightarrow W$ satisfying (j surj, i inj)

$$\sum_s s i j s^{-1} = 1_W$$

$$j s^{-1} t i = 0 \quad \text{for} \quad s^{-1}t \notin \Phi .$$

~~Here is a candidate for $\mathcal{C}$, namely generators x_t, y_s~~

Repeat: "good" C-module $\left(V, W, \begin{matrix} \iota : V \to W \\ j : W \to V \end{matrix}\right)$

V vector space, W is a Γ-module, ι inj, j surj

$j s^{-1} t \iota = 0 \quad$ for $\quad s^{-1}t \notin \Phi$

$$\sum_s s \iota j s^{-1} = 1_W$$

Now you define C; generators x_t, y_s $\qquad x_t = t\iota$, $y_s = js^{-1}$ on (V, W, ι, j)

relations $\qquad x_t x_{t'} = 0$, $y_s y_{s'} = 0$

$p(s^{-1}t) = y_s x_t$ depends on $s^{-1}t$ and is zero when $s^{-1}t \notin \Phi$

$$\sum_s x_s y_s x_t = x_t , \quad \sum_t y_s x_t y_t = y_s$$

~~What I need~~. $C = \begin{pmatrix} A' & Y \\ X & B' \end{pmatrix}$

A' generated by $p(s^{-1}t) = y_s x_t$. satisfying

$$\sum_t p(s^{-1}t)\,p(t^{-1}u) = \sum_t y_s x_t y_t x_u = y_s x_u = p(s^{-1}u)$$

$\therefore$ You have a ~~hom~~ surjective homom. $A \twoheadrightarrow A'$ sending $y_s x_t$ to $p(s^{-1}t)$.

B' generated by $x_t y_s \approx t i j s^{-1}$

$$x_t y_s = h_t t s^{-1}$$

Θ

first you find $\mathcal{E}_{\Gamma,\Phi} \longrightarrow B'$.

$$h_s \longmapsto x_s y_s$$

$\underbrace{h_s h_t}_{} = 0$ for $s^{-1}t \notin \Phi$

$x_s y_s x_t y_t$ $\qquad \sum_t h_s h_t = h_s$, $\sum_s h_s h_t = h_t$

$C = \begin{pmatrix} A' & Y \\ X & B' \end{pmatrix}$ gen x_t, y_s $\quad t, s \in \Gamma$

rel $x_t x_{t'} = 0 = y_s y_{s'}$

$y_s x_t$ depends only on $s^{-1}t$, $\quad y_s x_t = 0$ for $s^{-1}t \in \Phi$

$$\sum_s x_s y_s x_t = x_t, \qquad \sum_t y_s x_t y_t = y_s$$

Suppose given (V, W, i, j)

$j s^{-1} t i = 0$ for $s^{-1}t \in \Phi$

$\sum s i j s^{-1} = 1$ on W

i inj, j surj

Claim

W is a Γ-mod
V is a v.s.
$i: V \rightarrowtail W$, $j: W \twoheadrightarrow V$ are Clenian maps.
$\binom{V}{W}$ naturally a C-mod.
$x_t = ti: V \longrightarrow W$
$y_s = js^{-1}: W \longrightarrow V$

B has $\overset{\text{left}}{\text{local}}$ units means:

$\forall\, b_1, \dots, b_n \ \exists\, b' \ni \quad b' b_i = b_i$ (enough to check for $n = 1$.)

$\Longrightarrow \quad B \otimes_B M \longrightarrow M$ is injective $\forall$ B-module M.

$$\begin{array}{ccc} \sum b_i \otimes m_i & \longmapsto & \sum b_i m_i = 0 \\ \downarrow{\scriptstyle b'} & & \\ \sum b_i \otimes m_i = \sum_i b' b_i \otimes m_i & = & b' \otimes \overset{0}{\sum_i b_i m_i} \end{array}$$

(M firm $\Leftrightarrow$ $BM = M$)

$\Downarrow$

M red.

$m \in M = BM$, $Bm = 0$

$m = \sum b_i m_i$

$b' m = \sum b' b_i m_i = m$

Given V red A-module, $\quad W = p(\mathbb{C}\Gamma \otimes V)$

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, bend right, "j"'] & \mathbb{C}\Gamma \otimes V \arrow[r, "\beta"] \arrow[d, shift right, "\eta_1"'] & W \\ & V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "i"'] & \end{tikzcd}$$

$\beta\alpha = 1_W$

$\alpha\beta = p$

$x_t = t i \quad , \quad y_s = j s^{-1}$ defines gens on $\begin{pmatrix} V \\ W \end{pmatrix}$

$$\begin{pmatrix} p(s^{-1}t) = j s^{-1} t i & y_s = j s^{-1} \\ x_t = t i & x_t y_s = t i s^{-1} \end{pmatrix} \text{ on } \begin{pmatrix} V \\ W \end{pmatrix}$$

$$A \longrightarrow \begin{pmatrix} A' & Y \\ X & B' \end{pmatrix} \qquad p(s^{-1}t) \longmapsto \overset{j s^{-1} t i}{y_s x_t}$$

Problem: A Morita $\overset{\text{context}}{\cancel{\text{is}}}$ a kind of graded ring. ~~This question~~ Is there a tensor category behind this.

Problem: a Morita context is a kind of graded ring. Is there a Hopf algebra picture. Look at graded vector spaces ~~with~~ indexed by the set with 4 elts. $\left\{ \begin{smallmatrix} a & y \\ x & b \end{smallmatrix} \right\}$; call this $\square$

A graded vector space with respect to $\square$ is a counital comodule for,

$$\begin{pmatrix} A & Y \\ X & B \end{pmatrix}$$

Set this up more generally. Set Γ and Γ-grading $V = \bigoplus_{s \in \Gamma} V_s$. Partially def'd mult on Γ, namely a

$$\begin{array}{c} \Delta \subset \Gamma \times \Gamma \\ \downarrow \mu \\ \Gamma \end{array}$$

$\Gamma = \{ e_{ij} \mid i,j = 1,2 \}$

$$\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \otimes \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$$

$$= \begin{pmatrix} (V_{11} \otimes W_{11}) \oplus (V_{12} \otimes V_{21}) & (V_{11} \otimes W_{12}) \oplus (V_{12} \otimes W_{22}) \\ (V_{21} \otimes W_{11}) \oplus (V_{22} \otimes W_{21}) & (V_{21} \otimes W_{12}) \oplus (V_{22} \otimes W_{22}) \end{pmatrix}$$

$$(V \otimes W)_{ik} = \bigoplus_{j} V_{ij} \otimes V_{jk}$$

You seem to be making some progress. ~~Start~~ again. You want to treat a Morita context as a type of graded ring. A M. context is a ring A which has a grading indexed by the set of pairs (i,j), $1 \le i,j \le 2$. So

$$A = \bigoplus_{1 < i,j \le 1} A_{ij}$$

and the product $A_{ij} A_{i'j'}$ is 0 unless $j = i'$. The non trivial products are. $A_{ij} \cdot A_{jk} \subset A_{ik}$

Let $\Gamma = \{(i,j) \mid 1 \le i,j \le 2\}$. Then A as abelian group is graded wrt Γ

$$A = \bigoplus_{s \in \Gamma} A_s$$

not all product occur. Theres a subset $\Gamma \subset \Gamma \times \Gamma$ consisting of pairs $((i,j),(j,k))$ which can be composed.

You have the set Γ of ordered pairs (i,j) which gives the category of $\hat{\Gamma}$-modules, i.e. vector spaces graded wrt Γ $\quad V = \bigoplus_{(i,j) \in \Gamma} V_{ij}$

You also have a tensor product operation on such $\hat{\Gamma}$-modules

$$(V \otimes W)_{(i,k)} = \bigoplus_j V_{(i,j)} \otimes W_{(j,k)}$$

The Γ grading on V is equivalent to a counital comodule structure

$$V \xrightarrow{\Delta} \mathbb{C}\Gamma \otimes V$$
$$V_{(i,j)} \longmapsto (i,j) \otimes V$$

$$\Gamma = \{(i,j) \mid 1 \le i,j \le 2\}.$$

$$V = \bigoplus_{(i,j)} V_{ij}$$

$$W = \bigoplus_{(i,j)} W_{ij}$$

$$V \otimes W = \bigoplus_{(i,j)} \bigoplus_{(k,l)} V_{ij} \otimes W_{kl}$$

You want a situation where Γ-graded makes sense. This means

$$A = \bigoplus_{(i,j)} A_{(i,j)}$$

and
$$A_{(i,j)} \cdot A_{(k,l)} \begin{cases} = 0 & \text{if } j \neq k \\ \subset A_{(i,l)} & \text{if } j = k \end{cases}$$

Check: A Morita context is an alg A split into 4 subspaces $A = \bigoplus_{(i,j)} A_{(i,j)}$ such that if an element $a = \bigoplus_{(i,j)} a_{(i,j)}$ $\quad a_{(i,j)} \in A_{(i,j)}$ is written as a 2×2 matrix $\begin{pmatrix} a_{(1,1)} & a_{(1,2)} \\ a_{(2,1)} & a_{(2,2)} \end{pmatrix}$

then the mult in A is consistent with matrix mult. i.e. $(aa')_{(i,k)} = \sum_j a_{(i,j)} a'_{(j,k)}$

Amounts to $\quad A_{(i,j)} A_{(k,l)} \subset \begin{cases} 0 & j \neq k \\ A_{(i,l)} & j = k. \end{cases}$

Let's study the tensor product operations. $(V\otimes W)_{ik} = \bigoplus_j V_{ij} \otimes W_{jk}$

on vector spaces graded with respect $\{(i,j)\}$

You believe that a Morita context is algebra in this tensor category.

There is a resemblance with path algebra of a category. Generalization should be?

Treat $\Gamma = \{(i,j) \mid 1 \le i,j \le 2\}$ as a category? Unique arrow $j \to i$ for each (i,j)

You want an alg + coalg treatment

$$V \longrightarrow \mathbb{C}\Gamma \otimes V, \quad W \longrightarrow \mathbb{C}\Gamma \otimes W$$

$$V\otimes W \longrightarrow \underbrace{\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma}_{\mathbb{C}(\Gamma\times\Gamma)} \otimes V \otimes W$$

$$v_s \otimes w_t \longmapsto \bigoplus_{s,t} (s\otimes t \otimes v_s \otimes w_t)$$

kill parts not composable

$$\bigoplus_u u \otimes \sum_{u=st} v_s \otimes w_t$$

So it seems that you have a map

$$\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma$$

$$s \otimes t \longmapsto \begin{cases} 0 & \text{if } st \text{ not defined} \\ st & \text{is } \underline{\qquad} \end{cases}$$

In the Morita context case you have the coalgebra $\mathbb{C}[\Gamma]$ corresponding to Γ-grading

$$V \longrightarrow \mathbb{C}\Gamma \otimes V$$
$$v_s \longmapsto s \otimes v_s$$

Then $V \otimes W$ is naturally graded wrt $\Gamma \times \Gamma$:

$$V \otimes W = \bigoplus_{s,t} V_s \otimes W_t$$

$$V \otimes W \longrightarrow \mathbb{C}[\Gamma \times \Gamma] \otimes V \otimes W$$
$$v_s \otimes w_t \longmapsto s \otimes t \otimes v_s \otimes w_t$$

~~and you use~~ You now use a suitable map $\mathbb{C}[\Gamma \times \Gamma] \longrightarrow \mathbb{C}\Gamma$.

Point: You should be able to recover the set Γ from the coalgebra $\mathbb{C}\Gamma$. You guess that the counit plays a role.

Take $\Delta : \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma$
$$s \longmapsto s \otimes s$$

Idea is to look at the comm alg. $\mathbb{C}\Gamma^*$

$$(\mathbb{C}\Gamma)^* = \{f : \Gamma \to \mathbb{C} \text{ under mult.}\} = \bigoplus_{s \in \Gamma} \mathbb{C}e_s$$

Stick to finite sets Γ. There are nonunital alg homos.

$$\bigoplus_{s \in \Gamma} \mathbb{C}e_s \longrightarrow \bigoplus_{t \in \Gamma'} \mathbb{C}e_t$$

corresp to

$$\begin{array}{c} S \subset \Gamma' \\ \downarrow \\ \Gamma \end{array}$$

Yesterday made progress on ~~[illegible]~~ extending Γ-graded algebras to Morita contexts. Maybe this ~~[illegible]~~ is related to generalizing BC conjecture from groups to groupoids, then to coarse geometric objects.

Study example ~~for~~ of Mor. contexts. A Mor. cont. is a ring with a particular kind of grading

$A = \bigoplus_{s \in \Gamma} A_s$ where $\Gamma = \{(i,j) \mid i, j \in \{1,2\}\}$.

$A = \bigoplus_{(i,j)} A_{ij}$. The product and the grading are related by $A_{ij} A_{kl} \subseteq A_{il}$ if $j = k$ and $= 0$ otherwise. Your Γ is the set of arrows in ~~the~~ the groupoid with the objects 1, 2 and a unique map between any two objects. It looks like you can define a Γ graded algebra for any groupoid in the above way, namely, ~~when the~~ A has a grading $A = \bigoplus_{s \in \Gamma} A_s$ indexed by the arrows ~~satisfying~~ $A_s A_t \subset A_{st}$ where st is defined and $A_s A_t = 0$ otherwise.

~~Next you want associate to~~

If Γ is a groupoid, ~~[illegible]~~ let $\mathbb{C}\Gamma$ ~~[illegible]~~ be its path algebra. If A is Γ-graded alg, then there should be a homom.

$$A \longrightarrow \mathbb{C}\Gamma \otimes A$$

$$A_s \ni a_s \longmapsto s \otimes a_s$$

Is there an alg map $\mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma$?

Recall that a comodule ~~structure~~ structure on a vector space V for the coalgebra $\mathbb{C}\Gamma$, Γ a set, is NO equivalent to a family e_s, $s \in \Gamma$ of annihilating idempotent operators on V.

$$V \xrightarrow{\Delta} \mathbb{C}\Gamma \otimes V$$

need $\forall v$ $s \mapsto e_s v$ has fin. support.

$$\Delta v = \sum_s s \otimes e_s v$$

$$(\Delta \otimes 1)\Delta v = \sum_s s \otimes s \otimes e_s v \qquad e_t e_s = e_s \text{ if } t=s$$
$$= 0 \quad t \neq s.$$

$$(1 \otimes \Delta)\Delta v = \sum_{s,t} s \otimes t \otimes e_t e_s v$$

Say it correctly. A comodule structure on V for $\mathbb{C}\Gamma$ is equivalent to a family e_s, $s \in \Gamma$ of ann. projections on V such that $\forall v$ $\{s \in \Gamma \mid e_s v \neq 0\}$ is finite. This finiteness condition allows to define $e = \sum_{s \in \Gamma} e_s$ on V which is just $(\eta \otimes 1)\Delta : V \to V$. e is an idempotent whose image is $\sum_s e_s V$

Be precise about this.

Conclusion: A $\mathbb{C}\Gamma$-comodule V is the same as a graded vector space $V = \bigoplus_s V_s \oplus V_\infty$, the grading indexed by $\Gamma \cup \{\infty\}$. V is counital $\Leftrightarrow V_\infty = 0$.

You want to figure out what you need to define the algebra structure on $\mathbb{C}\Gamma$. Something like an assoc. partially defined product

First you define Γ graded algebra. $A = \bigoplus_{s \in \Gamma} A_s \oplus A_\infty$. Maybe you want to begin with

$$A \longrightarrow \mathbb{C}\Gamma \otimes A$$
$$a_s \longmapsto s \otimes a_s$$

A Γ-graded algebra is a $\mathbb{C}\Gamma$-comodule such that the coproduct

$$A \longrightarrow \mathbb{C}\Gamma \otimes A$$

respects the algebra structure

The good theory should proceed as follows. Given Γ some sort of partial monoid giving ~~rise~~ to a Hopf algebra $\mathbb{C}\Gamma$. ~~What will happen~~ A Γ-graded algebra is an alg A equipped with a Γ (maybe Γ_+) grading, equiv a coproduct

$$\begin{array}{ccc} A & \longrightarrow & \mathbb{C}\Gamma \otimes A \\ a_s & & s \otimes a_s \end{array}$$

which is also an alg map.

tensor product on Γ_+ graded modules

$$V \longrightarrow \mathbb{C}\Gamma \otimes V, \qquad W \longrightarrow \mathbb{C}\Gamma \otimes W$$

$$\begin{tikzcd} V \otimes W \arrow[r] & \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \otimes V \otimes W \arrow[d] \\ & \mathbb{C}\Gamma \otimes V \otimes W \end{tikzcd}$$

Let Γ be a group, let V, W be weakly Γ graded vector spaces.

$$V = \bigoplus_s V_s \oplus V_\infty$$

$$W = \bigoplus_t W_t \oplus W_\infty$$

$$V \otimes W = \bigoplus_{s,t} V_s \otimes W_t \quad \oplus \; V_\infty \otimes \bigoplus_t W$$

$$\bigoplus_s V_s \otimes W_\infty \quad \oplus \quad V_\infty \otimes W_\infty$$

so you have something graded wrt $\Gamma_+ \times \Gamma_+$. Push forward using the product st in Γ and sending the rest $(\infty \times \Gamma) \cup (\Gamma \times \infty)$ to ∞. In other words you extend the product $\Gamma \times \Gamma \longrightarrow \Gamma$ in the group Γ to $\Gamma_+ \times \Gamma_+ \longrightarrow \Gamma_+$

Let V, W be $\mathbb{C}\Gamma$ - comodules (Γ_+-graded)

$$V \longrightarrow \mathbb{C}\Gamma \otimes V \qquad V = \bigoplus_{s \in \Gamma_+} V_s$$

$$v_s \longmapsto s \otimes v_s \qquad \Gamma_+ = \Gamma \sqcup \{0\}$$

$$V \otimes W \longrightarrow \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \otimes V \otimes W \longrightarrow \mathbb{C}\Gamma \otimes V \otimes W$$

$$v_s \otimes w_t \longmapsto s \otimes t \otimes v_s \otimes w_t \longmapsto st \otimes v_s \otimes w_t$$

$$A = \bigoplus_{\Gamma_+} A_s$$

Go over the proof that $A = \mathcal{P}_{\Gamma,\Phi}$ is a Γ-graded algebra ~~nothing~~

$$A \xrightarrow{\Delta} \mathbb{C}\Gamma \otimes A \overset{\Delta\otimes 1}{\underset{1\otimes\Delta}{\rightrightarrows}} \mathbb{C}\Gamma\otimes\mathbb{C}\Gamma\otimes A$$

$$p(s) \qquad s \otimes p(s)$$

~~$\mathbb{C}$ is a Morita~~

Morita context = algebra graded wrt full groupoid having objects $\{1,2\}$.

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad ij$$

$$\mathbb{C}\Gamma = \begin{pmatrix} \mathbb{C}e_{11} & \mathbb{C}e_{12} \\ \mathbb{C}e_{21} & \mathbb{C}e_{22} \end{pmatrix} = M_2\mathbb{C}$$

Back to your Morita context.

$$\begin{pmatrix} A & Y \\ X & B \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \qquad \mathcal{J}\Gamma$$

Question: Look at $\Gamma i \quad \Gamma h \Gamma$

Idea

You want to get some feeling for the possible Γ yielding a "groupoid algebra" $\mathbb{C}\Gamma$. What data? Set Γ with correspondence

$T \subset \Gamma\times\Gamma$ yielding an associative product on $\mathbb{C}\Gamma$ $\qquad \Gamma_+ \wedge \Gamma_+ \longrightarrow \Gamma_+$

$$\begin{array}{c} T \\ \downarrow \\ \Gamma \end{array}$$

Go over things. Given $V \longrightarrow \mathbb{C}\Gamma \otimes V$, $W \longrightarrow \mathbb{C}\Gamma \otimes W$ comodule structures, get $V \otimes W \longrightarrow \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \otimes V \otimes W$

If V is graded wrt Γ_+, W is graded wrt Γ_+ then $V \otimes W$ is graded wrt $(\Gamma \times \Gamma)_+ = \Gamma_+ \wedge \Gamma_+$

Let $\Gamma = \{g\}$ $\qquad \Delta g = g \otimes g$

$$V \longrightarrow \mathbb{C}\Gamma \otimes V$$

$$\Delta v = g \otimes ev \underset{1 \otimes \Delta}{\overset{\Delta \otimes 1}{\rightrightarrows}} \begin{matrix} g \otimes g \otimes ev \\ g \otimes g \otimes eev \end{matrix} \qquad e^2 = e$$

$$V = \begin{matrix} eV \\ \oplus \\ e^\perp V \end{matrix} \begin{matrix} \xrightarrow{\sim} \\ \\ \xrightarrow[g \otimes e]{} \end{matrix} \begin{matrix} g \otimes eV \\ \oplus \\ g \otimes e^\perp V \end{matrix}$$

0

So what are the possible alg structures on $\mathbb{C}\Gamma$ $\qquad \Gamma = \{g\}$.

criteria. $\quad$ Because $\quad \Delta s = s \otimes s$

you want

Given $\quad \Delta : \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \qquad \Delta s = s \otimes s$

Let $\xi \in \sum_s c_s s \in \mathbb{C}\Gamma$ satisfy $\Delta \xi = \xi \otimes \xi$

$$\sum_s c_s (s \otimes s) = \sum_{s,t} c_s c_t (s \otimes t) \implies \begin{matrix} c_s c_t = 0 \\ \text{for } s \neq t \\ c_s^2 = c_s \end{matrix}$$

Thus ~~to regain~~ a coalgebra map $\theta : \mathbb{C}S \longrightarrow \mathbb{C}T$ is the same as a map $S_+ \longrightarrow T_+$. Thus an algebra structure on the coalg $\mathbb{C}\Gamma$, i.e. a product $\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma$ which is a coalg map should be the same as a map $(\Gamma \times \Gamma)_+ \longrightarrow \Gamma_+$ preserving 0

Start again: S set $\mathbb{C}S$ coalg $\Delta s = s \otimes s$. coass, cocomm, counital, V comodule for $\mathbb{C}S$, equivalent to a grading of V indexed by $S_+ = S \cup \{pt\}$.

Ex. $\mathbb{C}[pt]$ $\Delta(pt) = pt \otimes pt$

coalg morphisms $\mathbb{C}[pt] \longrightarrow \mathbb{C}S$ correspond to $S \cup \{0\}$. Seems that cat of coalgs $\mathbb{C}S$ is equivalent to cat of ptd sets.

$$\{\text{Coalgs}\}(\mathbb{C}S, \mathbb{C}T) = (\text{ptd sets})(S_+, T_+)$$

$$\text{Coalgs}(\mathbb{C}S \otimes \mathbb{C}S, \mathbb{C}S) = (\text{ptd sets})(\underbrace{(S \times S)_+}_{S_+ \wedge S_+}, S_+)$$

For a product $\mathbb{C}S \otimes \mathbb{C}S \longrightarrow \mathbb{C}S$ preserving the coalg structures you need an assoc. operation $S_+ \wedge S_+ \longrightarrow S_+$