So tomorrow lecture? To explain the Morita equivalence. Here's what you can do?

Γ, Φ

$\mathcal{E}_{\Gamma,\Phi}$: alg gen by h_s, $s \in \Gamma$, rels $h_s h_t = 0$ $\quad s^{-1}t \notin \Phi$

$$\underbrace{\text{"}\sum_t h_t = 1\text{"}} \qquad \sum_t h_t h_s = h_s = \sum_t h_s h_t$$

$\mathcal{P}_{\Gamma,\Phi}$: alg gen by p_s, $s \in \Gamma$, rels $p_s = 0$ $\quad s \notin \Phi$

$$p_s = \sum_{s = tu} p_t p_u = \sum_{t \in \Gamma} p_t p_{t^{-1}s}$$

$A = A^2$ $\left(= \{ \sum_i a_i a_i' \in A \} \right)$

cat of A-modules $=$ cat of unitary ($1m = m$) $\tilde{A} = \mathbb{Z}1 \oplus A$ mod.

$\mathrm{Mod}(\tilde{A}) \,/\, \mathrm{Nil}(\tilde{A}, A)$

Pedersen–Weibel paper, idea is to use $\mathbb{Z}^n$, also filtrations to do lower K-theory. This seems to be a good way to learn some new stuff. List ideas.

Trees + Waldhausen, free product

Significance of control – why should ε matter, be useful.

Link to C^* theory?

Pedersen - Weibel version of negative K-theory.

It seems that if you ~~start with introducing~~ ~~...~~ introduce $\mathbb{Z}$ graded versions you get filtrations.

Where to start.

a filtered add. cat.

C_i $\mathbb{Z}^i$ graded objects + ~~...~~ bounded maps.

$$A = \bigoplus_{J \in \mathbb{Z}^i} A(J)$$

~~not direct with~~

$i=1.$ $A = \bigoplus_{n \in \mathbb{Z}} A_n$ $\mathbb{Z}$ graded module ~~maps~~

then bounded maps.

So the first thing that seems important is ~~that~~ to use ~~general the direct sum~~ infinite sequences of vector spaces V_n and bounded maps.

Consider $\mathbb{Z}$-graded fin. diml. vector spaces $(V_n)_{n \in \mathbb{Z}}$, bounded maps.

V metric space $C_V(a)$ $\bigoplus_{v \in V} A(v)$

Is there a refinement of Morita equivalence.

Think of a ~~vector space ...~~ graded vector space where the index set is a space X. So you have V_x $\forall x \in X$?

Pedersen (+ Weibel) have this interesting module situation. $\mathbb{Z}$-graded vector spaces $\bigoplus_{n\in\mathbb{Z}} V_n$ and maps of bounded degree.

Review. Pedersen + Weibel's $\mathcal{C}_1(\mathcal{A})$ where $\mathcal{A}$ is an additive category. The objects are sequences $(A_n)_{n\in\mathbb{Z}}$.

What is a morphism $(A_n) \longrightarrow (B_n)$? A morphism of degree 0 is a sequence of maps $u_n : A_n \longrightarrow B_n$ in A. A morphism of degree d is a sequence of maps $u_n : A_n \longrightarrow B_{n+d}$. Equivalently if you define translation $(TA)_n = A_{n-1}$ or $(A[1])_n = A_{n-1}$ then a degree d map from $A = (A_n)$ to $B = (B_n)$ is a map $T^dA = A[d] \longrightarrow B$. A bounded map from A to B should ~~be~~ split into a finite sum of homogeneous maps.

~~What~~ It seems ~~to happen is that~~ you have something like distributions + Schwartz kernel thm. Given $A = (A_n)$, $B = (B_n)$ you can form

$$\prod_{m,n\in\mathbb{Z}} \mathrm{Hom}(A_m, B_n) = \prod_m \mathrm{Hom}\Big(A_m, \prod_{\mathbb{Z}} B_n\Big)$$

$$= \mathrm{Hom}\Big(\bigoplus_{\mathbb{Z}} A_m, \prod_{\mathbb{Z}} B_n\Big)$$

Suppose given two objects (A_m), (B_n). You have the space of kernels

$$\prod_{m,n} \mathrm{Hom}(A_m, B_n) = \mathrm{Hom}\Big(\bigoplus_{\mathbb{Z}} A_m, \prod_{\mathbb{Z}} B_n\Big)$$

You want to describe kernels corresponding to "bounded" operators.

$$k = \big(k_{mn} \in \mathrm{Hom}(A_m, B_n)\big)$$

to be defined

Your problem is to define bounded operator. something like $\phi_{mn} \neq 0 \implies |m-n| \leq C$

$$(kf)(x) = \int k(x,y) f(y)$$

One requirement might by that k should determin maps $\bigoplus A_m \longrightarrow \bigoplus B_n$ and also $\prod A_m \longrightarrow \prod B_n$

$$\begin{tikzcd} & \mathrm{Hom}\big(\bigoplus A_m, \prod B_n\big) & \\ \mathrm{Hom}\big(\bigoplus A_m, \bigoplus B_n\big) \arrow[ur] & & \mathrm{Hom}\big(\prod A_m, \prod B_n\big) \arrow[ul] \end{tikzcd}$$

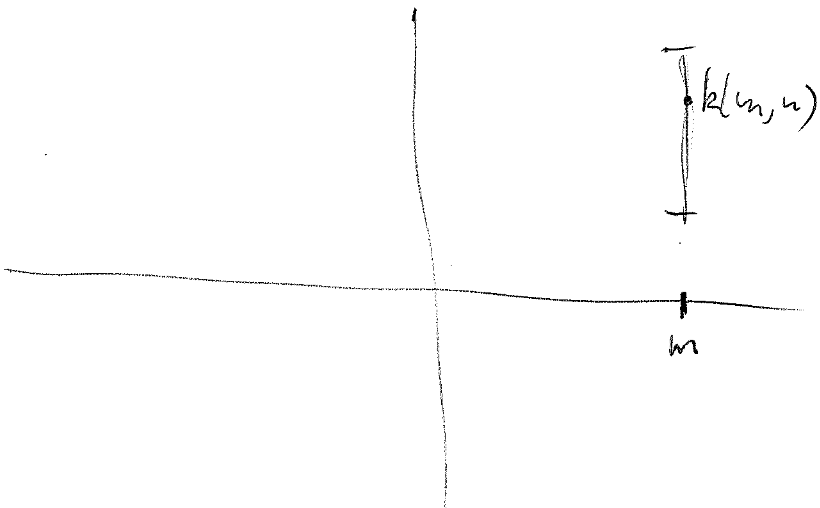
How to understand?

Given $(A_m)_{m\in\mathbb{Z}}$, $(B_n)_{n\in\mathbb{Z}}$ you have kernels

$$k = k_{(m,n)} \in \prod_{m,n} \mathrm{Hom}(A_m, B_n)$$
$$\| $$
$$\mathrm{Hom}\Big(\bigoplus_{\mathbb{Z}} A_m, \prod_{\mathbb{Z}} B_n\Big)$$

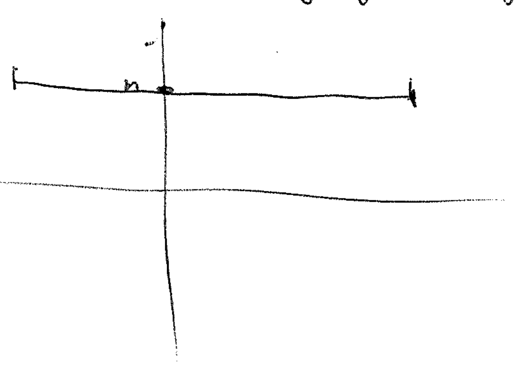
when does k map $\bigoplus A_m$ into $\bigoplus B_n$?

obvious sufficient condition is that $\forall m$ only finitely many $k_{(m,n)} : A_m \longrightarrow B_n$ are $\neq 0$.



when does k map $\prod A_m$ into $\prod B_n$

suff condition $\forall n$ only finitely many $k_{mn} : A_m \to B_n$ are $\neq 0$.



OKAY, you still don't have a good picture. PW introduce $\mathcal{C}_1$ the additive category of sequences $(V_n)_{n\in\mathbb{Z}}$ of f.d.v.s. and bounded maps for morphisms. What can you do with this?

You need examples. These should arise from the group $\mathbb{Z}$. ~~$\mathcal{C}_1$ is a noncommutative~~

There's this background, axiomatized by Karoubi, suspension of a ring, also the cone.

---

cone and suspension for a ring.

What is the point? Using periodicity to extend K_1, K_0 to negative degrees. Can define $K^{n}(X) = K^0(\Sigma^n X)$.

The other approach extends K_0 to lower K-theory, approach based on operators.

All of this is old.

ring A, cone ring $C(A)$

$$0 \longrightarrow A\otimes\mathcal{K} \longrightarrow \underbrace{C(A)}_{} \longrightarrow\!\!\!\!\rightarrow S(A) \longrightarrow 0$$

Morita equiv. $\wr$ A

has trivial K-theory by Eilenberg swindle.

get $K_0(S(A)) \xrightarrow{\sim} K_{-1}(A)$

$K_1(S(A)) \xrightarrow{\sim} K_0(A)$

Bass's version ~~[illegible]~~ involves

Oll

$$\begin{tikzcd} & A[z] \arrow[r, hook] & A[z,z^{-1}] \\ A \arrow[ur, hook] \arrow[dr, hook] & & \\ & A[z^{-1}] \arrow[uur, hook] & \end{tikzcd}$$

Today you will find an entrance to lower K-theory. You can try to list examples, to organize some of the ideas. But - where to begin?

Put an

You want to do the following, to somehow understand the finiteness obstruction. Start with perfect complex - the identity map is nuclear.

Where to start? Is there a geometric picture? Finiteness obstruction of Wall. ANR's. Verdier's article.

$$W \xrightarrow{\tilde{j}} \mathbb{C}[\Gamma] \otimes V$$

$$w \longmapsto \tilde{j}(w) = \sum_s s \otimes j s^{-1} w$$

to see that $s \mapsto j s^{-1} w$ has finite support.

because $\tilde{i}\tilde{j} : W \longrightarrow \mathbb{C}[\Gamma] \otimes V \longrightarrow W$ is the identity

W spanned by $t i v \quad v \in V$, ~~and~~ suffices to show

$\cancel{\mathrm{supp}(\tilde{j}(tiv))} = \{ s \mid j s^{-1} t i v \neq 0 \}$ finite

$h s^{-1} t h$

C_1 sequences (A_n) of f.d. vector spaces
morphisms are bounded maps.

Idempotent p_- on $C_1(A)$. An object is a graded f.d. v.s. $\bigoplus_{n\in\mathbb{Z}} A_n$, $p_- : \oplus A_n \longrightarrow \oplus A_n$ is 1 for $n \leq 0$.

What are important ideas? If V is a module then $\mathbb{C}[N] \otimes V = \mathbb{C}[T] \otimes V$ fits into an exact sequence

$$0 \longrightarrow \mathbb{C}[T] \otimes V \xrightarrow{T} \mathbb{C}[T] \otimes V \longrightarrow V \longrightarrow 0$$

So V is trivial in K-theory.

Cone and suspension. Mimic the extension

$$0 \longrightarrow \underset{\text{compact ops}}{\mathcal{K}} \hookrightarrow \underset{\text{bdd ops}}{\mathcal{L}} \longrightarrow \underset{\text{Calkin alg.}}{\mathcal{Q}} \longrightarrow 0$$

extension of C^* algs that links finite rank projections to invertible mod compacts (i.e. Fredholm operators).

Concrete version using Toeplitz algebra. Take the Hilbert space to be $L^2(S^1)$

Bass set up the whole business using poly ring.

Sticking point, namely, the canonical embedding $\mathbb{C}[\Gamma] \otimes V \longrightarrow C(\Gamma, V) = \mathrm{Map}(\Gamma, V)$.

First point: The are two Γ actions on $\mathrm{Map}(\Gamma, V)$ namely left regular repn $(L_t f)(s) = f(t^{-1}s)$
right ———— $(R_t f)(s) = f(st)$

Second point: You should use adjoint functors: If $j: H \hookrightarrow G$, then

$$\begin{tikzcd} \mathrm{Mod}_H \arrow[r, shift left=2, "j_!"] \arrow[r, shift right=2, "j_*"'] & \mathrm{Mod}_G \arrow[l, "j^*"'] \end{tikzcd}$$

$$j_! M = \mathbb{Z}[G] \otimes_H M$$
$$j_* M = \mathrm{Hom}_H(\mathbb{Z}[G], M)$$

You have $\mathrm{Hom}_G(j_! M, j_* M) = \mathrm{Hom}_H(j^* j_! M, M)$

where $j^* j_! M = \mathbb{Z}[G] \otimes_H M$ restricted to H.

~~there is a set of~~ This splits according to double cosets $H\backslash G/H$. There's an obvious H-bimodule map $\mathbb{Z}[G] \longrightarrow \mathbb{Z}[H]$ which is the identity on $H1H = H$ and zero on HgH for $g \notin H$.

So it seems you want to use the right regular rep of Γ on $C(\Gamma, V)$ along with the obvious left Γ action on $\mathbb{C}[\Gamma] \otimes V$

Relation to GNS. $\mathbb{Z}[G] \xrightarrow{\ \rho\ } \mathbb{Z}[H]$

$$\rho(g) = \begin{cases} 1 & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}$$

You look at $M \xrightarrow{j} N \xrightarrow{i} M$

where M is a G module, N an H-module

and $j a i n = \rho(a) n$ in particular $ji = 1_N$

You might look for a good M, given N.

$$A \otimes N \longrightarrow M \longrightarrow \mathrm{Hom}(A, N)$$

$$a \otimes n \longmapsto a i n \longmapsto (a' \mapsto j a' a i n)$$

Thus given an H-module N ?

Anyway: $\mathbb{C}[\Gamma] \otimes V \longrightarrow \mathbb{C}(\Gamma, V)$

$$\mathbb{C}(\Gamma, V) = j_*(V) = \mathrm{Hom}(\mathbb{C}[\Gamma], V)$$

So $V \hookrightarrow C(\Gamma, V)$ as functions with $\mathrm{Supp} \subset \{1\}$.

$$v \longmapsto \delta_1(s) v$$

$$t \otimes v \longmapsto \delta_1(st) v = \delta_{t^{-1}}(s) v$$

$$\sum_{t \in \Gamma} t \otimes f(t) \longmapsto \underbrace{\sum_{t \in \Gamma} \delta_{t^{-1}}(s) f(t)}_{s \longmapsto f(s^{-1})}$$

In other words

$$\mathbb{C}[\Gamma] \otimes V \longrightarrow C(\Gamma, V)$$

identifies $f(t)$ (of finite supp) with $f(s^{-1})$

$$j_! M = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M$$

$$j_* M = \mathrm{Hom}_H(\mathbb{Z}[G], M)$$

$$\mathrm{Hom}_G(j_! M, j_* M) = \mathrm{Hom}_H(j^* j_! M, M)$$

$$= \mathrm{Hom}_H(\underline{\mathbb{Z}[G]} \otimes_{\mathbb{Z}[H]} M, M)$$

$$\mathbb{Z}[G] = \mathbb{Z}[H] \oplus \mathbb{Z}[G-H]$$

$$\mathrm{Hom}_\Gamma(\mathbb{C}[\Gamma] \otimes V, \mathrm{Hom}_\mathbb{C}(\mathbb{C}[\Gamma], V))$$

$$\|$$

$$\mathrm{Hom}_\mathbb{C}(\mathbb{C}[\Gamma] \otimes V, V) \ni \delta_1(s) \cdot 1_V$$

$$\mathrm{Hom}_\mathbb{C}(\mathbb{C}[\Gamma], V) = \mathrm{Map}(\Gamma, V) \qquad (tf)(s) = f(st)$$

$$V \xrightarrow{\delta_1(s) 1_V} \mathrm{Hom}_\mathbb{C}(\mathbb{C}[\Gamma], V)$$

$$V \longrightarrow \delta_1(s) v$$

$$v \longmapsto \delta_1(s) v$$

$$t \otimes v \longmapsto \delta_1(st) v \qquad s \longmapsto f(s^{-1})$$

$$\|$$

$$\sum_t t \otimes f(t) \longmapsto \left(s \mapsto \sum_t \delta_1(st) \cancel{f}(t) \right)$$

$$\mathbb{C}[\Gamma] \otimes V \longrightarrow \mathrm{Map}(\Gamma, V)$$

$$s \otimes f(s) \longmapsto (s \longmapsto f(s^{-1}))$$

$$s \otimes f(t^{-1}s) \qquad s \longmapsto f((st)^{-1}) = f(t^{-1}s^{-1})$$

How to proceed?

$$v \longmapsto (s \mapsto \delta_1(s)v)$$

$$t\otimes v \longmapsto (s \mapsto \delta_1(st)\,v)$$

$$\sum_s s\otimes f(s) = \sum_t t\otimes f(t) \mapsto \Big(s \longmapsto \underbrace{\sum_t \delta_1(st) f(t)}_{f(s^{-1})}\Big)$$

$$\begin{tikzcd} W \arrow[r, "\tilde{j}"] \arrow[dr, "j"'] & \mathbb{C}[\Gamma]\otimes V \arrow[d, "\eta_1"] \\ & V \end{tikzcd}$$

$$\boxed{\tilde{j}w = \sum_s s\otimes f(s)}$$

$$jw = f(1)$$

$$\tilde{j}(tw) = t\tilde{j}(w) = \sum_s ts\otimes f(s)$$

$$j(tw) = \eta_1\tilde{j}(tw) = f(t^{-1}) \qquad \therefore f(s) = j(s^{-1}w)$$

$$\tilde{j}(w) = \sum_{s\in\Gamma} s\otimes j s^{-1}w$$

$$\begin{tikzcd} W \arrow[r, "\tilde{j}"] \arrow[dr, "j"'] & \mathrm{Map}(\Gamma, V) \arrow[d, "ev_1"] \\ & V \end{tikzcd}$$

$$\cancel{ev_1(\tilde{j}w) = jw}$$

$$\cancel{ev_1(\tilde{j}tw) = ev_1(t\tilde{j}w)}$$

$$(\tilde{j}w)(1) = jw$$

$$jtw = (\tilde{j}tw)(1) = (t(\tilde{j}w))(1) = \tilde{j}w(t)$$

$$\therefore \boxed{(\tilde{j}w)(s) = jsw} \qquad (\tilde{j}tw)(s) = jstw$$

$$jstw = (t\tilde{j}w)(s) = \tilde{j}w(st)$$

$$\begin{tikzcd} W \arrow[r, "\tilde{j}"] \arrow[dr, "j"'] & \mathrm{Map}(\Gamma, V) \arrow[d, "ev_1"] \\ & V \end{tikzcd}$$

$\tilde{j}$ is Γ-map and $ev_1 \tilde{j} = j$. $\boxed{jsw = (\tilde{j}w)(s)}$

$\tilde{j}sw = s\tilde{j}w \implies (\tilde{j}sw)(1) = (s\tilde{j}w)(1)$

Check. $\tilde{j}tw \overset{?}{=} t\tilde{j}w$

$$(\tilde{j}tw)(s) \overset{?}{=} (t\tilde{j}w)(s)$$

$$\begin{array}{ccc} (\tilde{j}tw)(s) & \overset{?}{=} & (t\tilde{j}w)(s) \\ \| & & \| \\ jstw & & (\tilde{j}w)(st) \\ & & \| \\ & & jstw \end{array}$$

Check.

$$\begin{tikzcd} & C(\Gamma, V) \arrow[d, "ev_1"] \\ W \arrow[ur, "\phi^\#"] \arrow[r, "\phi"] & V \end{tikzcd}$$

$\phi^\#(tw) = t\phi^\#(w)$ i.e.

$$(\phi^\#(tw))(s) = (t\cdot\phi^\#(w))(s)$$
$$= \phi^\#(w)(t^{-1}s)$$

Also $(\phi^\#(w))(1) = \phi(w)$

Put $s=1$

$\phi(tw) = \phi^\#(w)(t^{-1})$ Put $t^{-1}=s$

$\therefore$ $\phi^\#(w)(s) = \phi(s^{-1}w)$

~~[illegible]~~ What you need now is to review all the steps, and clean things up to the point where the arguments are straightforward.

fin B-module $W = \Gamma\text{-mod} + h \ni hsh = 0 \quad s \notin \Phi$
and $\sum shs^{-1} = 1_W$. $h = ij : W \xrightarrow{j} V \xhookrightarrow{i} W$

$p_s = jsi \qquad \sum p_t p_{t^{-1}s} v = \sum j t i j t^{-1} s i v = j \sum_t t h t^{-1} \underbrace{s i v}_{w}$

$i p_s j = hsh = 0, \; s \notin \Phi \qquad = j s i v = p_s v$

$\therefore p_s = 0.$

Converse direction. [redacted] Given p_s on V, define p on $\mathbb{C}[\Gamma] \otimes V$ by $p(\sum s \otimes f(s)) = \sum_s s \otimes \sum_t p_{s^{-1}t} f(t)$

$$(pf)(s) = \sum_t p_{s^{-1}t} f(t) = \sum_{tu^{-1}=s} p_u f(t)$$

Define p on $\mathbb{C}[\Gamma] \otimes V \simeq C_{fin}(\Gamma, V)$

by $(pf)(s) = \sum_{t \in \Gamma} p(s^{-1}t) f(t)$. p commutes with Γ action

$$(p(u \cdot f))(s) = \sum_t p(s^{-1}t) f(u^{-1}t) = \sum_t p(s^{-1}ut) f(t)$$

$$(u \cdot (pf))(s) = (pf)(u^{-1}s) = \sum_t p_{s^{-1}ut} f(t)$$

$$(pf)(u^{-1}s) = \sum_{t \in \Gamma} p(s^{-1}u\, u^{-1}t) f(u^{-1}t)$$

$$u(pf) = p(uf)$$

So how to organize this? Begin with $p(s) \in \mathcal{L}(V)$.

Something funny. You would like to start [redacted] with [redacted] the families $(si)_{s \in \Gamma}$ and $(js^{-1})_{s \in \Gamma}$, Other ideas $\mathbb{C}[\Gamma]\otimes A$ operates on $\mathbb{C}[\Gamma]\otimes V$

~~Review old ideas~~ $p = \sum_{s\in\Gamma} s\otimes p_s$ is a projection, idempotent elt of $\mathbb{C}[\Gamma]\otimes A$. Now the ring $\mathbb{C}[\Gamma]\otimes \mathrm{End}(V)$ should operate on $\mathbb{C}[\Gamma]\otimes V$. So what are you trying to find?

Review Γ grading, if Γ a set you have notion of Γ-graded vector space: $(V_s)_{s\in\Gamma}$, like a sheaf on Γ viewed as a space, you have $f_!, f^*, f_*$. $\otimes$ Category when Γ a group $\bigoplus_{s=tu} V_t\otimes W_u$ Γ-graded alg.

Given an A-module V, i.e. operator p_s, $s\in\Gamma$ sat. rels. you seem to get 2 things.

Γ-module projection on $\mathbb{C}[\Gamma]\otimes V$

Γ-graded projection (meaning?)

$\mathbb{C}[\Gamma]\otimes A$ operates on $\mathbb{C}[\Gamma]\otimes V$

you have $p = \sum s\otimes p_s$ idempotent in $\mathbb{C}[\Gamma]\otimes A$ hence a projection p on $\mathbb{C}[\Gamma]\otimes V$

Review the stages: Begin with fin B-mod W factor $h = ij : W \twoheadrightarrow V \hookrightarrow W$ get Γ-maps.

$W \xrightarrow{\tilde{j}} \mathbb{C}[\Gamma]\otimes V \xrightarrow{\tilde{i}} W$, $\tilde{i}\tilde{j} = 1_W$ $\therefore$

$$\begin{tikzcd}
\mathbb{C}[\Gamma]\otimes V \arrow[r, "\rho"] & \mathbb{C}[\Gamma]\otimes V \\
\sum_t t^{-1}\otimes f(t) \arrow[r, mapsto] & \sum_{s,t} st^{-1}\otimes \rho(s)f(t) \arrow[d, mapsto] \\
\sum_t t\otimes f(t) \arrow[u, mapsto] \arrow[r, squiggly] & \sum_{s,t} ts^{-1}\otimes \rho(s)f(t)
\end{tikzcd}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad \|$$

$ts^{-1} = u$

$$\cancel{\sum_{s,t} t^{-1}s^{-1}\otimes \rho(s)f(t)} \quad \text{~~replace s by s^{-1}~~}$$

$u = ts^{-1}$
$t^{-1}u = s^{-1}$
$u^{-1}t = s$

$$\sum_u{}' \, u \otimes \sum_{u=ts^{-1}}{}' \rho(s)f(t)$$

$$= \sum_u u \otimes \sum_t \rho(u^{-1}t)f(t)$$

$$= \sum_s s \otimes \sum_t \rho(s^{-1}t)f(t)$$

Right module case: a firm B^{op}-module should be a vector space W tog. w. operators $\cdot s: w \mapsto ws$ giving a right Γ-action and an op. $\cdot h: W \mapsto wh$ $\ni$ $whsh = 0$ $\forall w$, $s \notin \Phi$

$$\sum_s wshs^{-1} = w \qquad \forall w \qquad (\text{means } \Sigma \text{ is finite})$$

factor $\cdot h = \cdot ji : W \overset{j}{\twoheadrightarrow} V \overset{i}{\hookrightarrow} W$

next have right Γ-mod maps $\tilde{j} \to V\otimes\mathbb{C}[\Gamma] \nearrow \tilde{i}$

$(v\otimes s)\tilde{i} \longmapsto vis$ i.e. $\sum_s f(s)\otimes s \mapsto \sum_s f(s)is$

If you use $V \otimes \mathbb{C}[\Gamma] = \operatorname{Map}(\Gamma, V)$ fin supp.

$$\sum f(s) \otimes s \longleftrightarrow f$$

$$\cdot t \downarrow$$

then

$$\sum_s f(st^{-1}) \otimes s = \sum_s f(s) \otimes st \longleftrightarrow \boxed{(f \cdot t)(s) = f(st^{-1})}$$

What about $\tilde{j}$?

$w\tilde{j} \in \operatorname{Map}_{\text{fin}}(\Gamma, V)$

$(wt)\tilde{j} = (w\tilde{j})t$, $\quad w\tilde{j} \overset{ev_1}{\longmapsto} wj$

Suppose $w\tilde{j} = \sum_s f(s) \otimes s$

then $(wt)\tilde{j} = \sum_s f(s) \otimes st = \sum_s f(st^{-1}) \otimes s$

$$(wt)j = (wt)\tilde{j}\, ev_1 = f(t^{-1}) \qquad f(s) = ws^{-1}j$$

$$\boxed{w\tilde{j} = \sum_s ws^{-1}j \otimes s} \qquad (w\tilde{j})ev_s = ws^{-1}j$$

$$w\tilde{j}\tilde{\imath} = \sum_s ws^{-1}j\iota s = 1.$$

$$\left(\sum_t f(t) \otimes t\right)\tilde{\imath}\tilde{j} = \left(\sum_t f(t)\iota t\right)\tilde{j}$$

$$= \sum_t \sum_s f(t)\,\iota t s^{-1} j \otimes s$$

$$(f(t))_{t \in \Gamma} \longmapsto \left(\sum_t f(t)\, p(ts^{-1})\right)_{s \in \Gamma}$$

I feel that you are missing something, that you need some way of using $\sum_{s \in \Gamma} s \otimes s^{-1}$ in a systematic way

Go back to constructing the Morita context, the case Γ finite, etc., also graded vector spaces.

Look first at $\begin{pmatrix} B & ? \\ & A \end{pmatrix}$

Look at case of a firm dual pair $A \quad P \quad Q$ with A unital and

---

Look at firm $\begin{pmatrix} B & E \\ F & A \end{pmatrix}$ with B unital

B unital $\Longleftrightarrow$ $E \in \mathcal{P}(A^{op})$, $F \in \mathcal{P}(A)$ and $F \times E \longrightarrow A$ is a perfect duality.

there's a nuclearity condition!!

What you want to do? ~~is to~~

~~the idea investigate the~~

You want to connect free modules with basis indexed by a metric spaces, and the control theory of Quinn to ~~...~~

Your ultimate goal is to ~~understand~~ decipher the stuff Andrew knows. You want to understand why fg free modules with basis indexed by a metric space

---

Goal is to understand Andrew's lower K-theory, really the significance of free modules with basis indexed by a metric space. Your idea is to use nuclear maps, really to factorize nuclear maps through a finitely generated free modules

---

Back to your Morita equiv between simplest notation

$B = \mathcal{E}_{\Gamma,\Phi}$ and $A = \mathcal{P}_{\Gamma,\Phi}$.

Given firm B-mod W, i.e. with operators $s \in \Gamma$, h

you factor $h = ij : W \overset{j}{\twoheadrightarrow} V \overset{i}{\hookrightarrow} W$ $\qquad V = hW$

you construct Γ-maps

$$\begin{tikzcd}
W \arrow[r, "\tilde{j}"] \arrow[dr, bend right, "j"'] & \mathbb{C}[\Gamma]\otimes V \arrow[r, "\tilde{\imath}"] \arrow[d, shift right, "\eta_1"'] & W \\
 & V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, bend right, "i"'] &
\end{tikzcd}$$

$$\tilde{j}w = \sum_{s\in\Gamma} s \otimes j s^{-1} w \qquad \tilde{\imath} \sum_{s\in\Gamma} s \otimes f(s) = \sum_{s\in\Gamma} s\, i f(s)$$

~~If you understand the beginning~~

Review right module picture – same with the composition to the right: W has operators $\cdot s$, $\cdot h$ relations $whsh = 0$, $\sum_s w s h s^{-1} = w$.

factor $h = \cdot j i : W \xrightarrow{j} V \xrightarrow{i} W$

$$W \xrightarrow{\tilde{j}} V \otimes \mathbb{C}[\Gamma] \xrightarrow{\tilde{i}} W$$

$$w\tilde{j} = \sum_s w s^{-1} j \otimes s \qquad \left(\sum_t f(t) \otimes t\right)\tilde{i} = \sum_t f(t) i t$$

$$w\tilde{j}\tilde{i} = \sum_s w s^{-1} j i s = w$$

$$v\,p(s) = v i s j$$

what is $p = \tilde{i}\tilde{j} : V \otimes \mathbb{C}[\Gamma] \longrightarrow V \otimes \mathbb{C}[\Gamma]$

$$\sum_s f(s) \otimes s \xmapsto{\tilde{i}} \sum_s f(s) i s \mapsto \sum_s f(s) i s \sum_t t^{-1} j \otimes t$$

better

$$\sum_{s \in \Gamma} f(s) \otimes s \xmapsto{\tilde{i}} \sum_{s\in\Gamma} f(s) i s \longmapsto \sum_{t,s} f(s) \overbrace{i s t^{-1} j}^{p(st^{-1})} \otimes t$$

$$f \longmapsto (fp)(t) = \sum_s f(s)\, p(st^{-1})$$

and if you use left modules

$$\boxed{f \longmapsto (pf)(s) = \sum_t p(s^{-1}t) f(t)}$$

this is strange but if you try to change the defn of p ~~bad~~

to $p(t^{-1}s)$ (thereby keeping left invariance of this kernel) then you have $\sum p(t^{-1}s)f(t)$, not the usual convolution.

More details on the grading. ~~This time you need yield~~ You want the bimodules.

The structure should be very simple. What do you ~~need ... with~~ want?

Try to describe the puzzle; take Γ finite, $\Phi = \Gamma$.

Try to determine the firm bimodules behind the Morita equivalence

$$\begin{pmatrix} B & E \\ F & A \end{pmatrix} \qquad V \longmapsto E \otimes_A V = p(\mathbb{C}[\Gamma] \otimes V)$$

but the action of p is funny — it works internally

$$E = p(\mathbb{C}[\Gamma] \otimes A)$$ should be true

A or $\tilde{A}$ give same

Yes you want Γ to act by left mult on $\mathbb{C}[\Gamma]$ which means you want to involve right mult by s^{-1} on $\mathbb{C}[\Gamma]$. So p is maybe $\sum (\cdot s^{-1}) \otimes (p_s \cdot)$

$\mathbb{C}[\Gamma] \overset{P}{\otimes} A$? Let's see if you get a Γ-grading on E. First check this p works.

$$P: \mathbb{C}[\Gamma] \otimes A \longrightarrow \mathbb{C}[\Gamma] \otimes A$$

$$p\left(\sum_t t \otimes a(t)\right) = \sum_s \sum_t ts^{-1} \otimes p(s)\, a(t)$$

$$\mathbb{C}[\Gamma]\otimes V \xrightarrow{\ p\ } \mathbb{C}[\Gamma]\otimes V$$

$$p\Big(\sum_t t\otimes f(t)\Big) = \sum_s s\otimes \sum_t p(s^{-1}t)f(t).$$

~~that~~ want this to appear as "internal" action of $\sum s\otimes p_s$ Put $u = ts^{-1}$ $s = u^{-1}t$

$$p\Big(\sum_t t\otimes f(t)\Big) = \sum_{s,t} ts^{-1}\otimes p(s)f(t)$$

$$= \sum_{t,u} u\otimes p(u^{-1}t)f(t)$$

$$= \sum_{t,s} s\otimes p(s^{-1}t)\,f(t).$$

Maybe you can now explain the formulas, ~~and~~ avoiding ~~the~~ question about whether elements of $\mathbb{C}[\Gamma]\otimes V$ should be best described as $\sum s\otimes f(s)$ or $\sum s\otimes f(s^{-1})$. The principle which you have missed until now is that you want to give an intrinsic definition of $p = \sum s\otimes p_s$ on the $\mathbb{C}[\Gamma]$ left, A right bimodule $\mathbb{C}[\Gamma]\otimes A$.

Go over again. ~~Note~~ A is Γ-graded with ~~def~~ $p_s\in A_s$.

Try to use the universal map. property of A wrt Γ-graded algebras.

$\mathbb{C}[\Gamma]\otimes V$ is a Γ-graded vector space with degree ? Maybe better to look at $\mathbb{C}[\Gamma]\otimes A$ as left Γ, right A-bimodule. Look at the tensor prod algebra $\mathbb{C}[\Gamma]\otimes A$ as a Γ-graded algebra with canonical proj $\sum s\otimes p_s$. confused. Try again

Look at $\begin{cases} \mathbb{C}[\Gamma]\otimes A & \text{as } \mathbb{C}[\Gamma], A\text{-bimodule} \\ \mathbb{C}[\Gamma]\otimes V & \text{as } \mathbb{C}[\Gamma]\text{-module} \end{cases}$

$$(\mathbb{C}[\Gamma]\otimes A)\otimes_A V = \mathbb{C}[\Gamma]\otimes V$$

now you can do other things, namely

~~lots of concentration~~

Let's review what you learned. You found long ago the projection.

$$p : \mathbb{C}[\Gamma]\otimes V \longrightarrow \mathbb{C}[\Gamma]\otimes V$$

$$p\sum_{t\in\Gamma} t\otimes f(t) = \sum_{s\in\Gamma} s\otimes \sum_t p(s^{-1}t)f(t)$$

The claim is this projection is the operator

$\sum(\cdot s^{-1})\otimes(p(s)\cdot)$ on $\mathbb{C}[\Gamma]\otimes V$

because

$$\left(\sum_s(\cdot s^{-1})\otimes(p(s)\cdot)\right)\left(\sum_t t\otimes f(t)\right) = \sum_{s,t\in\Gamma} ts^{-1}\otimes p(s)f(t)$$

$u = ts^{-1}$ $\qquad s = u^{-1}(ts^{-1})s = u^{-1}t$

$$= \sum_{u,t} u\otimes p(u^{-1}t)f(t) = \sum_{s,t} s\otimes p(s^{-1}t)f(t)$$

What do you want next? You have p operating on the bimodule $\mathbb{C}[\Gamma]\otimes A$. You want to see ~~that~~ whether $\underline{p(\mathbb{C}[\Gamma]\otimes A)}$ is naturally Γ-graded.

You want the Γ-grading to be compatible with left Γ mult and A^{op}-mult.

Take $W = B$ which is a Γ-graded alg. $h \in B_1$, so $V = Bh$?? You expect

$$p(\mathbb{C}[\Gamma]\otimes A) = E = Bh$$

---

p is the operator $\sum_s (\cdot s^{-1}) \otimes (p(s)\cdot)$ on $\mathbb{C}[\Gamma]\otimes A$

take $u\otimes a$ $\quad u \in \Gamma \quad a \in A_t$

$$p(u\otimes a) = \sum_s \underbrace{us^{-1}}_{\in \Gamma} \otimes \underbrace{p(s)a}_{\in A_{st}} \qquad us^{-1}st = ut.$$

So p has degree 1 for the natural grading on the Γ, A bimodule $\mathbb{C}[\Gamma]\otimes A$ namely where $u\otimes A_t$ has degree ut.

$B = \mathbb{C}[\Gamma]\otimes \mathcal{E}$

$Bh = \mathbb{C}[\Gamma]\otimes \mathcal{E}h$

Look $\mathbb{C}[\Gamma]\otimes A$ as a $\mathbb{C}[\Gamma], A$ bimodule

Then it ~~has a ring of~~ endos of the form $(\cdot s)\otimes(a\cdot)$

~~$(\cdot s^{-1})\otimes(a\cdot)$ acting on $t\otimes b$ is $ts^{-1}\otimes ab$~~

Assume A graded and grade the bimodule $\mathbb{C}[\Gamma]\otimes A$ by $|s\otimes a| = s|a|$ Then

$\cdot s\otimes a\cdot$ applied to $t\otimes b$ is $ts\otimes ab$
$ts|a||b|$

Start again with $(\cdot s^{-1}\otimes(a_t\cdot))$ applied to $u\otimes a_v$ is $us^{-1}\otimes a_t a_v$ which has degree $us^{-1}tv$

Seems only to work for $s=t$

Review. $\mathbb{C}[\Gamma]\otimes\tilde{A}$ left Γ, right A bimodule,

~~Same as a~~ $R\otimes S$ R, S bimodule

Make it clearer. An B-A bimodule is the same as a $B\otimes A^{op}$-module. In the unital case ~~this~~ ~~the endos~~ one has ~~an isom.~~ a bijection

$B\otimes A^{op} \longrightarrow B\otimes A$

so the B-A free bimodule with the gen[101] is equiv. to the free $B\otimes A^{op}$ mod with 1 gen. Its endos is the ring $(B\otimes A^{op})^{op} = B^{op}\otimes A$

$\mathbb{C}[\Gamma]\otimes\tilde{A}$ left Γ right A bimodule

endo ring is $\mathbb{C}[\Gamma]^{op}\otimes\tilde{A}$ working on the inside which is isom to $\mathbb{C}[\Gamma]\otimes\tilde{A}$ acting ~~$\sigma(s\otimes a)$~~

$s(\sigma\otimes\alpha)a = s\sigma^{-1}\otimes\alpha a$

next comes the grading question

So you have the <u>bimodule</u> $\mathbb{C}[\Gamma]\otimes A$ with endos $s\otimes a \longmapsto s\sigma^{-1}\otimes \alpha a$ for $\sigma\otimes\alpha$ in the <u>alg</u> $\mathbb{C}[\Gamma]\otimes A$. This allows you to define ρ on the <u>bimodule</u>. degree of $s\otimes a$ is $s|a|$
degree of $s\sigma^{-1}\otimes\alpha a$ is $s\sigma^{-1}|\alpha||a|$

It seems that there is no way to obtain the degree of the action of $\sigma\otimes\alpha$ upon $s\otimes a$, namely $s\sigma^{-1}|\alpha||a|$, from the degree of $\sigma\otimes\alpha$, namely $\sigma|\alpha|$, and the degree of $s\otimes a$, namely $s|a|$. Not well expressed

Better is to consider the action of $\sigma\otimes\alpha$ on <u>bimod</u> namely $s\otimes a \longmapsto s\sigma^{-1}\otimes\alpha a$. This preserves the total degree on the bimodule $|s\otimes a| = s|a|$ when $\sigma^{-1}\otimes\alpha$ has degree $\cancel{1}$ i.e $|\alpha| = \sigma$

Summary. have bimodule $\mathbb{C}[\Gamma]\otimes A$ with Γ-grading $\deg(s\otimes A_t) = st$ and an "internal" action of the tens. prod alg $\mathbb{C}[\Gamma]\otimes A$, where $(\sigma\otimes\alpha)*(s\otimes a) = s\sigma^{-1}\otimes\alpha a$. The restriction of this action to the subalg $\bigoplus \sigma\otimes A_\sigma$ of $\mathbb{C}[\Gamma]\otimes A$ preserves the Γ grading on the bimodule.

Point. Consider $\mathcal{A} = \mathbb{C}[\Gamma]\otimes A$ $\quad (s\otimes a)(s'\otimes a') = ss'\otimes aa'$
This algebra $\mathcal{A}$ is not Γ-graded for the total degree $\deg(s\otimes A_t) = st$

Repeat. $E^b = \mathbb{C}[\Gamma]\otimes\tilde{A}$ considered as left Γ, right A bimodule | E' is the free left $R = \mathbb{C}[\Gamma]\otimes\tilde{A}^{op}$ module with one generator $1\otimes 1$. $\mathrm{Hom}_R(R,R) = R^{op} = \mathbb{C}[\Gamma]^{op}\otimes\tilde{A}$ $\simeq \mathbb{C}[\Gamma]\otimes\tilde{A}$. So the tensor prod. alg acts on the bimodule E^b via $(s\otimes a)*(t\otimes b) = ts^{-1}\otimes ab$

Consider $E^b = \mathbb{C}[\Gamma]\otimes\tilde{A}$ as a left Γ, right A bimodule. Then $\mathcal{a} = \mathbb{C}[\Gamma]\otimes A$ acts on E^b via $(s\otimes a)*(t\otimes b) = ts^{-1}\otimes ab$

$$(s\otimes a)*\big((s'\otimes a')*(t\otimes b)\big)$$

$$= (s\otimes a)*(ts'^{-1}\otimes a'b) = (ts'^{-1}s^{-1}\otimes aa'b) = (t(ss')^{-1}\otimes (aa')b)$$

$$= (ss'\otimes aa')*(t\otimes b).$$

Not true that $\mathcal{a}$ is a Γ-graded algebra.

$$(s\otimes a_t)(s'\otimes a'_{t'}) = ss'\otimes a_t a'_{t'}$$

$st \quad s't' \qquad\qquad ss' \; tt'$

$$E^b_s = \bigoplus_{s=tu} t\otimes A_u$$

Point E^b has total degree

$$\underbrace{(s\otimes A_t)}_{\mathcal{a}_{st}} * \underbrace{(s'\otimes A_{t'})}_{E^b_{s't'}} \subset s's^{-1}\otimes A_tA_{t'} \subset E^b_{s's^{-1}tt'}$$

Thus the Γ-grading on E^b is preserved by $\bigoplus_{s\in\Gamma} s\otimes A_s \subset \mathbb{C}[\Gamma]\otimes A = \mathcal{a}$

$E_f = \mathbb{C}[\Gamma] \otimes \tilde{A}$ ~~with~~ with obvious structure of left Γ, right A bimodule.

Define action $(s \otimes a) * (t \otimes a') = ts^{-1} \otimes aa'$ of $\mathcal{A} = \mathbb{C}[\Gamma] \otimes A$, alg with $(s \otimes a)(s' \otimes a') = ss' \otimes aa'$, on E_f. ~~Claim $\mathcal{A}$ acts via bimodule~~ Claim this define a left $\mathcal{A}$ module structure on E_f which respects the bimodule structure.

Obvious Γ-grading on E_f

$t \otimes \tilde{A}_t \subset (E_f)_{st}$ $\qquad \deg(t \otimes a) = t \deg(a)$

$p = \sum s \otimes p_s \in \underset{\text{alg}}{\mathcal{A}} = \mathbb{C}[\Gamma] \otimes A$ satisfies $p^2 = p$

$E = pE_f$ $\qquad$ Cleaner to use "internal" action

$$p(t \otimes a) = \sum_s ts^{-1} \otimes p_s a \qquad u = ts^{-1}, \quad s = u^{-1}t$$

$$p\left(\sum_t t \otimes f(t)\right) = \sum_{t,s} ts^{-1} \otimes p_s f(t)$$

$$= \sum_{t,u} u \otimes p_{u^{-1}t} f(t)$$

p is a projection on the free Γ, A bimodule E_f. It preserves the Γ grading on E_f. $\therefore$ $E = pE_f$ is a Γ-graded, (Γ, A) bimodule

Repeat. You seek the Morita context $\begin{pmatrix} B & E \\ F & A \end{pmatrix}$ 271

E will be a summand of the bimodule

$\boxed{E_f = \mathbb{C}[\Gamma]\otimes\tilde{A}}$ obvious ~~left~~ Γ action
right A action

Let $\mathcal{a} = \mathbb{C}[\Gamma]\otimes\tilde{A}$.

Consider $\mathbb{C}[\Gamma]\otimes\tilde{A}$

~~Define product on $\mathbb{C}[\Gamma]\otimes A$~~

Let $\mathcal{a}$ be the unital alg given by $\mathbb{C}[\Gamma]\otimes\tilde{A}$ with prod. $(s\otimes a)(s'\otimes a') = ss'\otimes aa'$. Let E_f be the left $\mathbb{C}[\Gamma]$, right $\tilde{A}$ bimodule given by $\mathbb{C}[\Gamma]\otimes\tilde{A}$ with multiplication given by $s(t\otimes a)a' = st\otimes aa'$. ~~Define~~ ~~by an act~~ an action of $\mathcal{a}$ on E_f by

$$(s\otimes a)*(t\otimes a') = ts^{-1}\otimes aa'$$

This makes $\mathcal{a}$ operate on E_f which respects the bimodule structure.

$$\mathcal{a} \xrightarrow{\sim} \mathrm{End}_{\mathbb{C}[\Gamma]\otimes\tilde{A}^{op}}(E_f) \qquad \text{OKAY}$$

$$\text{So } (s\otimes a)*(t\otimes a') = ts^{-1}\otimes aa'$$

E_f is Γ graded $\quad E_f = \bigoplus_{s,t} s\otimes\tilde{A}_t$

Subalg of $\mathcal{a}$ gen. linearly by $\bigoplus_s s\otimes A_s$

Study $\mathbb{C}[\Gamma] \otimes A$ | alg: Γ commutes with A.
bimod: Γ left, A right.

$$A = \bigoplus_{s \in \Gamma} A_s$$

Claim: A is naturally a Γ-graded alg; unique Γ-grading resp. product: $A_s A_t \subset A_{st}$

and $p_s \in A_s$ $\qquad A = \mathcal{P}_{\Gamma, \Phi}$ $\qquad p_s \ s \in \Gamma$, $\ p_s = 0 \ s \notin \Phi$, $\ p_s = \sum_{s=tu} p_t p_u$

$$\begin{array}{ccccc} A & \xrightarrow{\Delta} & \mathbb{C}[\Gamma] \otimes A & \overset{\Delta \otimes 1_A}{\underset{1_{\mathbb{C}[\Gamma]} \otimes \Delta}{\rightrightarrows}} & \mathbb{C}[\Gamma] \otimes \mathbb{C}[\Gamma] \otimes A \\ p_s & \longmapsto & s \otimes p_s & \longmapsto & s \otimes s \otimes p_s \\ & & & & s \otimes s \otimes p_s \end{array}$$

So where are you? The bimodule $E^f_{\mathcal{A}}$ left Γ right A.

module is Γ-graded having $s \otimes A_t$ of degree st

$$p(s \otimes a) = \sum_{t \in \Gamma} st^{-1} \otimes p_t a$$

$\deg(s \otimes a) = s \deg(a)$

$\deg(st^{-1} \otimes p_t a) = st^{-1} \cdot t \cdot \deg(a)$

$\therefore$ p respects the Γ-grading, and so

$E = p(E^f_{\mathcal{A}})$ is a Γ-graded, left Γ, right A bimod

What is the relation between the Γ grading and Γ action?

$$E^f_{\mathcal{A}} = \mathbb{C}[\Gamma] \otimes A \qquad (E^f_{\mathcal{A}})_u = \bigoplus_{u = st} s \otimes A_t$$

$$E^f_s = \bigoplus_t st^{-1} \otimes A_t \qquad E^f_t = \bigoplus_{u \in \Gamma} u \otimes A_{u^{-1}t}$$

$$s E^f_t = \bigoplus_{u \in \Gamma} su \otimes A_{u^{-1}t} = (E^f)_{st}$$

It seems that E^f has ~~is a Γ-graded Γ-mod~~ both Γ-grading and Γ action, which means that it should have the form

$$E^f = \mathbb{C}[\Gamma] \otimes (E^f)_1 \qquad \text{where } (E^f)_1 = \sum_{t\in\Gamma} t^{-1} \otimes A_t$$

Check this. $E^f = (\mathbb{C}\Gamma \otimes A) = \bigoplus_u \bigoplus_t t \otimes A_{t^{-1}u}$

$$ts^{-1} \otimes p_s A_{t^{-1}u} \subset ts^{-1} \otimes A_{st^{-1}u}$$

$E^f = \mathbb{C}\Gamma \otimes A$ has Γ-grading where $\deg(s \otimes A_u) = su$

$$p(s \otimes a_u) = \sum_t st^{-1} \otimes p_t a_u \quad \text{degree } su$$

What seems important then is the image of p on

$$\bigoplus_s s \otimes A_{s^{-1}} = \bigoplus t^{-1} \otimes A_t$$

$D = \mathbb{C}[\Gamma] \otimes A$ Γ left, A right bimodule

w Γ grading: $D_u = \bigoplus_{u=st} s \otimes A_t$. Then

$s' E^f_u \subset E^f_{s'u}$, $E^f_u A_{t'} \subset E^f_{ut'}$. Also have

operator $p\, E^f_u = \sum_{u=st}$

$E^\# = \mathbb{C}[\Gamma] \otimes A$ considered as left Γ, right A bimodule

outside action

inside action $\quad t : s \otimes a \longmapsto st^{-1} \otimes a$

$\qquad\qquad a' : s \otimes a \longmapsto s \otimes a'a$

$$E^\#_s = \bigoplus_{s=s_1s_2} s_1 \otimes A_{s_2} = \bigoplus_{s_1} s_1 \otimes A_{s_1^{-1}s}$$

$E^\#_0 = \mathbb{C}[\Gamma]\otimes A$ considered as a left Γ, right A bimod

structures on $\mathbb{C}[\Gamma]\otimes A$

algebra $(s\otimes a)(s'\otimes a') = ss'\otimes aa'$

Γ grading $(\mathbb{C}[\Gamma]\otimes A)_s = \bigoplus_{s_1\in\Gamma} s_1\otimes A_{s_1^{-1}s}$

note: ~~alg~~ Γ-grading is not comp. with alg structure.

outside left Γ, right A bimod structure

$t(s\otimes a) = ts\otimes a$, $(s\otimes a)a' = s\otimes aa'$

respects the Γ grading

inside left A, right Γ bimodule structure

$a'(s\otimes a)t = st\otimes a'a$

define $E^\#$ to be $\mathbb{C}[\Gamma]\otimes A$ with left Γ, right A bimodule, respects grading.

Also have ~~right~~ inside action

$s\otimes a \longmapsto st^{-1}\otimes a'a$ $(t\otimes a')*(s\otimes a) = st^{-1}\otimes a'a$

giving endom. of $E^\#$ as left Γ, right A bimodule

~~spect~~ representing the t.p. alg $\mathbb{C}[\Gamma]\otimes A$ as endom. of $E^\#$

$\sum s\otimes p_s$

observe that $a'\in A_t$ $t\otimes a'\in t\otimes A_t$

~~Sym~~ Γ grading on $E^\#$ is preserved by $\bigoplus t\otimes A_t \overset{alg}{\subset} \mathbb{C}[\Gamma]\otimes A$

~~If~~ $E = pE^\#$, $\bigoplus t\otimes A_t$ preserves Γ-grading on $E^\#$

Let $E = pE^{\#}$. $E^{\#} = \mathbb{C}[\Gamma] \otimes E_1^{\#}$

$$E_1^{\#} = \bigoplus_s s^{-1} \otimes A_s \qquad p(s^{-1}\otimes a_s) = \sum_t \overbrace{s^{-1}t^{-1} \otimes p(t) a_s}^{\in (ts)^{-1} \otimes A_t A_s}$$

So it should follow ~~that~~ $E = \mathbb{C}[\Gamma] \otimes pE_1^{\#}$ E_1

Something is funny. You have a

$$(\mathbb{C}[\Gamma] \otimes A)_1$$

$$E^{\#} = \mathbb{C}[\Gamma] \otimes A = \cancel{\bigoplus s \otimes \bigoplus_{s'} s' \otimes A}$$

$$= \bigoplus_u u \otimes \bigoplus_s s \otimes A_{s^{-1}u}$$

$$E_1^{\#} = \bigoplus_s s \otimes A_{s^{-1}} = \bigoplus_s s^{-1} \otimes A_s \qquad \sum_s p(us^{-1}) a_s$$

$$p\Big(\sum_s s^{-1} \otimes a_s\Big) = \sum_{s,t} \underbrace{s^{-1}t^{-1}}_{(ts)^{-1}} \otimes p(t) a_s = \sum_u u^{-1} \otimes \overline{\sum_{u=ts} p(t) a_s}$$

$E^{\#} = \mathbb{C}[\Gamma] \otimes A$ left Γ, right A bimodule

Γ grading: $s \otimes A_{s'} \subset E^{\#}_{ss'}$, $E_s^{\#} = \sum_t t \otimes A_{t^{-1}s}$

$$E_1^{\#} = \sum_{t\in\Gamma} t^{-1} \otimes A_t \qquad p(s \otimes a_{s'}) = \sum_t st^{-1} \otimes \underbrace{p(t) a_{s'}}_{A_{ts'}}$$

$u = ts^{-1}$, $us = t$, $s = u^{-1}t$

$$\sum_s \sum_t ts^{-1} \otimes p(s) f(t)$$

$$= \sum_u u \otimes \sum_t p(u^{-1}t) f(t)$$

$E^\# = \mathbb{C}[\Gamma] \otimes A$ left $\mathbb{C}[\Gamma]$, right A bimodule

Γ-grading $E^\#_s = \bigoplus_{s=tu} t \otimes A_u$, compatible with left $\mathbb{C}[\Gamma]$, right A multiplication

Interested in endos of this bimodule

$$s \otimes a \longmapsto st^{-1} \otimes a$$ define Γ action on $E^\#$

$$s \otimes a \longmapsto s \otimes a'a$$ defines A action on $E^\#$

$$\boxed{p(s \otimes a) = \sum_t st^{-1} \otimes p(t)a}$$

$ut = s \to t = u^{-1}s$, $u = st^{-1}$

$$p\Big(\sum_s s \otimes f(s)\Big) = \sum_{s,t} st^{-1} \otimes p(t) f(s)$$

$$= \sum_u u \otimes \sum_s p(u^{-1}s) f(s).$$

$$(pf)(s) = \sum_t p(s^{-1}t) f(t)$$

$$E^\#_s = \sum_{s=tu} t \otimes A_u$$

$$p(t \otimes a) = \sum_{s \in \Gamma} ts^{-1} \otimes p(s) a$$

$$p\Big(\sum_t t \otimes f(t)\Big) = \sum_{t,u} \overbrace{tu^{-1}}^{s} \otimes p(u) f(t)$$

$s = tu^{-1}$, $u = s^{-1}t$.

$$p\Big(\sum_t t \otimes f(t)\Big) = \sum s \otimes p(s^{-1}t) f(t)$$

$$\boxed{(pf)(s) = \sum_t p(s^{-1}t) f(t)}$$

$$E^{\#}_s = \sum_{s=s's''} s'\otimes A_{s''} = \sum_t t\otimes A_{t^{-1}s}$$

$$p(t\otimes a) = \sum_{s\in\Gamma} ts^{-1}\otimes p(s)a$$

$|t\otimes a| = t|a|$

$|ts^{-1}\otimes p(s)a| = ts^{-1}|p(s)a|$
$= ts^{-1}\cancel{s}|a|$

$$pp(t\otimes a) = \sum_{s,u\in\Gamma} ts^{-1}u^{-1}\otimes p(u)p(s)a$$

$$= \sum_{s,u} t(us)^{-1}\otimes p(u)p(s)a$$

$$= \sum_{t,v} tv^{-1}\otimes \underbrace{\sum_{v=us} p(u)p(s)a}_{p(v)} = p(t\otimes a)$$

$$E^{\#}_1 = \sum_t t\otimes A_{t^{-1}}$$

$$p(t\otimes a(t)) = \sum_s s\otimes\sum_t p(s^{-1}t)\,a(t)$$

You have to live with this form.

$$\begin{pmatrix} B=\Gamma\ltimes\mathcal{E} & E=\mathbb{C}\Gamma\otimes E_1 \\ F=F_1\rtimes\mathbb{C}\Gamma & A \end{pmatrix}$$

What do you have? Idea is that E yields $V\longmapsto E\otimes_A V$. You want to make ~~clear~~ this Morita context. Dual pair?

$$E = p(E^{\#}) \qquad F = (F^{\#})p$$

$$\begin{pmatrix} B = \Gamma \ltimes B_1 & \begin{array}{c}\mathbb{C}\Gamma\otimes E_1 \\ \| \\ E = p(\mathbb{C}\Gamma\otimes A)\end{array} \\ F = (A\otimes\mathbb{C}\Gamma)p & A \end{pmatrix}$$

$$\begin{array}{c} \| \\ F = F_1 \otimes \mathbb{C}\Gamma \end{array}$$

Is there a chance $E\otimes_A F = p(\mathbb{C}\Gamma\otimes A\otimes\mathbb{C}\Gamma)p$
$= (\mathbb{C}\Gamma\otimes E_1)\otimes_A(F_1\otimes\mathbb{C}\Gamma)$

Studying $E = p(E^\#)$, $E^\# = \mathbb{C}\Gamma\otimes A$ Γ, A bimod

First you need to understand ~~the pairing~~ things before applying p.

$\mathbb{C}\Gamma\otimes\tilde{A} = E^\#$ $\qquad E^\#\otimes_A F^\# = \mathbb{C}\Gamma\otimes\tilde{A}\otimes\mathbb{C}\Gamma$

$F^\# = \tilde{A}\otimes\mathbb{C}\Gamma \qquad A$

Problem: to link B to $E = p(E^\#)$

$$\begin{tikzcd} E \arrow[r, "\tilde{j}"] \arrow[dr, "j"'] & \mathbb{C}\Gamma\otimes A \arrow[r, "\tilde{i}"] \arrow[d, shift left, "\eta_1"] & E \arrow[r, "\tilde{j}"] & \mathbb{C}\Gamma\otimes A \\ & A \arrow[u, shift left, "\varepsilon_1"] \arrow[ur, "i"'] & & \end{tikzcd}$$

$\tilde{j}\xi = \sum_{s\in\Gamma} s\otimes js^{-1}\xi$, $\quad \tilde{i}\tilde{j}\xi = \sum_s s\,i\,js^{-1}\xi$, $\quad \tilde{j}\tilde{i}\left(\sum_t t\otimes f(t)\right)$
$= \tilde{j}\sum_t t\,i\,f(t)$

$\tilde{i}\left(\sum_s s\otimes f(s)\right) = \sum_s s\,i\,f(s)$ $\qquad = \sum_s s\otimes\sum_t js^{-1}t\,i\,f(t)$

So what happens must be as follows. You know that B is a firm ring by local units. You have explicit Morita equivalence given by $V \longmapsto E \otimes_A V$, $W \longmapsto hW$

so you should have isom. $hB \otimes_B W \longrightarrow hW$

$$V \mapsto E \otimes_A V = p(\mathbb{C}\Gamma \otimes V) = W$$

$$\| $$

$$(p(\mathbb{C}\Gamma \otimes \tilde{A})) \otimes_A V$$

$$\underbrace{p(\mathbb{C}\Gamma \otimes \tilde{A})}_{E} \otimes_A V$$

E is the image of p on $E^{\#} = \mathbb{C}\Gamma \otimes \tilde{A} = \{\sum t \otimes f(t) \mid f : \Gamma \to \tilde{A}$ f.supp$\}$

$$(pf)(s) = \sum_{t \in \Gamma} p(s^{-1}t) f(t)$$

$E^{\#}$ is Γ-graded

$$E^{\#} = \bigoplus_{u \in \Gamma} E^{\#}_u \qquad E^{\#}_u = \bigoplus_{u = st} s \otimes \tilde{A}_t$$

$p = \boxed{\cancel{\sum s \otimes p_s}}$ on $E^{\#}$ $\qquad p(s \otimes a_{(t)}) = \sum su^{-1} \otimes ua_{(t)}$.

preserves the grading, But $\cdot A$ does not.

$$E = pE^{\#} = \bigoplus_{x \in \Gamma} pE^{\#}_x$$

$$\begin{pmatrix} B & E \\ F & A \end{pmatrix}$$

it should be true that $E \xrightarrow{\sim} Bh$ up to nil modules, also $\exists\, h_K \ni h_K h = h$ so you might be able to lift h into Bh and maybe into E.

Problem: Can you produce an element in E which somehow corresponds to h in B.

$$E = \left\{ \sum t \otimes f(t) \in \mathbb{C}\Gamma \otimes \tilde{A} \;\middle|\; pf = f \right\}.$$

better $E = p(\mathbb{C}\Gamma \otimes \tilde{A})$

There is an obvious element of E, namely

$$p(1 \otimes 1) = \sum_s s^{-1} \otimes p(s) \qquad \text{better } \sum s \otimes p(s^{-1})$$

Check $f(t) = p(t^{-1})$. $\qquad \displaystyle\sum_t p(s^{-1}t)\,p(t^{-1}) = p(s^{-1})$.

~~So how do~~ Now you have check things.

$$\begin{pmatrix} B & E \\ F & A \end{pmatrix}$$

~~So you have~~

~~You need to define~~

$W \longmapsto hW = V \qquad W \xrightarrow{f} hW \overset{i}{\hookrightarrow} W$

$$E = p(\mathbb{C}\Gamma \otimes \tilde{A})$$

$$\begin{tikzcd}
E \arrow[r, leftrightarrow, "\tilde{j}"] \arrow[dr, "j"'] & \mathbb{C}\Gamma \otimes \tilde{A} \arrow[r, "\tilde{i}"] \arrow[d, shift right] & E \arrow[r, "\tilde{j}"] & \mathbb{C}\Gamma \otimes \tilde{A} \\
& \tilde{A} \arrow[u, shift right] \arrow[ur, "i"'] & &
\end{tikzcd}$$

If W is a ~~fin~~ B^{op}-module, the corresp reduced A^{op}-module is Wh. Take $W = B$, Bh is a left B right A bimodule, $Bh \ni h$

so you have a left ideal.

~~[illegible]~~ Work out today's lecture

finish the Morita equivalence

B-fin $\otimes W \longmapsto hW$ A-reduced.

$h = ij : W \overset{j}{\twoheadrightarrow} V \overset{i}{\hookrightarrow} W$

$$W \xrightarrow{\tilde{j}} \mathbb{C}\Gamma \otimes V \xrightarrow{\tilde{i}} W \xrightarrow{\tilde{j}} \mathbb{C}\Gamma \otimes V$$

$$\tilde{j}w = \sum_{s \in \Gamma} s \otimes js^{-1}w \qquad \tilde{i}\Big(\sum_s s \otimes f(s)\Big) = \sum_s s\,i\,f(s)$$

$$\tilde{i}\tilde{j}w = \sum_s \underbrace{sijs^{-1}}_{shs^{-1}} w = w.$$

Check $s \longmapsto js^{-1}w$ has fin. support

$$\sum_s shs^{-1}w = w$$

$\{s \mid s i j s^{-1} w = 0\}$ finite.

$\hat{\Gamma}$-alg $\quad A = \bigoplus_s A_s \;, \quad A_s A_t \subset A_{st}$

example: $\quad \mathbb{C}\Gamma \otimes B \qquad B$ alg

$$\bigoplus_s sB$$

Claim $\quad \mathrm{Hom}_{\hat{\Gamma}\text{-algs}}\Big(\bigoplus_s A_s, \mathbb{C}\Gamma \otimes B\Big) = \mathrm{Hom}_{\text{algs}}\Big(\bigoplus_s A_s, B\Big)$

$\psi = (\psi_s : A_s \to sB) \qquad\qquad \phi = (\phi_s : A_s \to B)$

$\ni$

$$\begin{array}{ccc} A_s \otimes A_t & \longrightarrow & A_{st} \\ \downarrow \psi_s \otimes \psi_t & & \downarrow \psi_{st} \\ sB \otimes tB & \longrightarrow & stB \end{array}$$

$\ni$

$A = P_{\Gamma, \Phi}$

$$A \xrightarrow{\Delta_A} \mathbb{C}\Gamma \otimes A \overset{\Delta \otimes 1}{\underset{1 \otimes \Delta}{\rightrightarrows}} \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \otimes A$$

$$p_s \qquad s \otimes p_s \longrightarrow s \otimes s \otimes p_s$$

Return to $A = \mathcal{P}_{\Gamma,\Phi}$, $B = \Gamma \ltimes \mathcal{E}_{\Gamma,\Phi}$

firm B module $=$ Γ-module W with operator h

$$\sum_s{}' shs^{-1}w = w \qquad \forall w \in W$$

$$hsh = 0 \qquad s \notin \Phi$$

$$h = ij : W \overset{j}{\twoheadrightarrow} V \overset{i}{\hookrightarrow} W \qquad \Rightarrow \qquad jsi = 0 \quad s \notin \Phi.$$

particular of 1 says $W = \sum_s{}' siV$ so that any $w =$ finite sum of siv

injective

$$\sum_s (si) js^{-1}w = w \qquad \Rightarrow \{s \mid js^{-1}w \neq 0\} \text{ finite}$$

~~So now take~~ Start with B firm ring

B $\quad Bh$ $\quad$ Bh is the reduced A^{op}-module corresp to B as firm B-mod

hB $\quad$ hB is the reduced A-module corresp to B as firm B^{op}-module

hB is A, B bimodule

$$B \xrightarrow{j = h\cdot} hB \overset{i}{\hookrightarrow} B$$

$$p(s) = jsi = hs \quad \text{on } hB. \qquad p(s)(hB) = hshB = 0 \quad s \notin \Phi$$

$$\sum_t p(t)p(t^{-1}s)(hb) = \sum_t ht\,ht^{-1}s\,hb = hshb = p(s)(hb).$$

So what do you need to prove?

You now ~~have~~ should have enough to ~~abstract~~ understand the Morita context.

B-form modules $\qquad h = ij : W \xrightarrow{j} V \xrightarrow{i} W$

$B = \Gamma \ltimes \mathcal{E}$ $\qquad$ A form B-mod is a Γ-mod W tog. with an op $h \ni$ $\quad hsh = 0 \quad s \notin \Phi$

$\sum_s shs^{-1}w = w$ $\qquad$ this sum is assumed finite so

$\{s \mid sijs^{-1}w \neq 0\} = \{s \mid js^{-1}w \neq 0\}$ finite.

Go through the steps.

Given W you factor: $h = ij : W \xrightarrow{j} V \xrightarrow{i} W$

$\therefore V = hW$, i inc, $j = h\cdot$

Then let $p(s) = jsi$ on V $\qquad$ i.e. $p(s) = hs$ on hW

Clear $p(s) = 0$ for $s \notin \Phi$

$$\sum_t p(t)p(t^{-1}s)(hw) = \sum_t hth t^{-1}(shw) = hshw = p(s)(hw)$$

So hW is an A-module.

~~Another point you ge~~

reduced? $\qquad \sum s\underbrace{h}_{ij}s^{-1}w = w$

(Yesterday you thought about Γ graded v.s. and encountered the idea that $\mathbb{C}\Gamma \otimes V$ is the $\hat{\Gamma}$-module cogenerated by the v.s. V.

$$\mathrm{Hom}_{\mathbb{C}}\Big(\bigoplus_s W_s, V\Big) = \prod_s \mathrm{Hom}(W_s, V)$$

$$= \mathrm{Hom}_{\hat{\Gamma}}(W, \mathbb{C}\Gamma \otimes V)$$

Note Γ is a set here.

~~[illegible]~~ This reminds me of the fine topology, direct sum of lines ~~[illegible]~~ in Groth.s TVS theory, used by Ulrike in the context of assembly maps.)

reduced:

$$w = \sum_s si\, js^{-1}w \;\Rightarrow\; W = \sum siV$$

$$\Rightarrow V = jW = \sum p(s)V = AV$$

$$0 = \bigcap_s \mathrm{Ker}(js^{-1}: W \to V) \overset{i}{\supset} \bigcap_s \mathrm{Ker}(js^{-1}i : V \to V)$$

$$\bigcap_s \mathrm{Ker}(js^{-1}i : V \to V) = A^{V}$$

$$W \xrightarrow{\tilde{j}} \mathbb{C}\Gamma \otimes V \xrightarrow{\tilde{i}} W$$

$$\tilde{j}(w) = \sum_s s \otimes js^{-1}w, \quad \tilde{i}\Big(\sum_s s \otimes f(s)\Big) = \sum_s si\, f(s)$$

$$\tilde{j}\,\tilde{i}\Big(\sum_t t \otimes f(t)\Big) = \sum_s s \otimes \sum_t \underbrace{js^{-1}ti}_{p(s^{-1}t)} f(t)$$

This should ~~[illegible] produce~~ provided an isomorphism of $W \longmapsto hW \longmapsto p(\mathbb{C}\Gamma \otimes hW)$ with the identity.

9:08 This is clear, the isomorphism is given by $\tilde{j}$

Question: Is it possible to define a category of triples (W, V, some canonical maps) as in GNS, except that W and V should determine each other?

Review the formulas again. Given

$W \in Mod(\Gamma)$, $V \in Mod(\mathbb{C})$, $\quad W \xrightarrow{j} V \xhookrightarrow{i} W$

such that $\quad jsi = 0 \quad$ for $\quad s \notin \Phi$

$$\boxed{\begin{array}{ll} js^{-1}ti = 0 & \text{for } s^{-1}t \notin \Phi \\ \sum_s sijs^{-1} = 1_W & \text{on } W. \end{array}}$$

these form a category $\mathcal{C}$

firm B-modules $\longrightarrow \mathcal{C} \longleftarrow$ reduced A-modules

$$W \longmapsto hW, \quad W \xrightarrow{j=h} hW \xhookrightarrow[\text{inc.}]{i} W$$

($\mathcal{C}$ is the category whose objects are (W, V, i, j) where W is a Γ-module, V a v.s., $j: W \to V$, $i: V \hookrightarrow W$ linear maps $\ni$ $\left(\begin{array}{ll} jsi = 0 & \text{for } s \notin \Phi \\ \sum sijs^{-1}w = w & \text{all } w \in W. \end{array} \right)$

$$hshW = 0 \iff ijsij = 0.$$

$p(s)v = jsiv \quad \rightsquigarrow \quad V$ reduced A-module

Given an A-module: ops $p(s)$ on V for $s \in \Gamma$ $\ni$ $\quad p(s) = 0 \quad s \notin \Phi$, $\quad \sum_t p(t)p(t^{-1}s) = p(s)$

left Γ-module, right A module $\quad \mathbb{C}\Gamma \otimes \tilde{A}$

Define operator p on $\mathbb{C}\Gamma \otimes \tilde{A}$ by

$$p(t \otimes a) = \sum_s ts^{-1} \otimes p(s)a \qquad u = ts^{-1} \quad us = t \quad s = u^{-1}t$$

$$p\Big(\sum_t t \otimes a(t)\Big) = \sum_{t,s} ts^{-1} \otimes p(s)a(t) = \sum_s s \otimes \Big(\sum_t p(s^{-1}t)a(t)\Big)$$

Observe $\left| \begin{array}{l} p \text{ ~~is~~ an endo of } \mathbb{C}\Gamma \otimes \tilde{A} \text{ as } \Gamma, A \text{ bimod} \\ p^2 = p \text{ on } \mathbb{C}\Gamma \otimes \tilde{A} \\ p \text{ preserves total } \Gamma \text{ degree on } \mathbb{C}\Gamma \otimes \tilde{A} \end{array} \right.$

$$\deg(s \otimes a) = s \deg(a)$$

Set $E = p(\mathbb{C}\Gamma \otimes \tilde{A})$.

Maybe you want to define $W(V) = p(\mathbb{C}\Gamma \otimes V)$

observe

$$\begin{array}{ccc} (\mathbb{C}\Gamma \otimes \tilde{A}) \otimes_A V & = & \mathbb{C}\Gamma \otimes V \\ \downarrow\uparrow & & \downarrow\uparrow \\ p(\mathbb{C}\Gamma \otimes \tilde{A}) \otimes_A V & & p(\mathbb{C}\Gamma \otimes V) \end{array}$$

$V \mapsto W(V)$ is exact and it kills nil A-modules.

$\therefore$ $W(V) = E \otimes_A V$ $\qquad$ E is A^{op}-~~flat~~ projective firm

Problem: to identify E with Bh

First show E is a firm B-module

---

need simpler viewpoint. $E = p(\mathbb{C}\Gamma \otimes \tilde{A})$, $F = (\tilde{A} \otimes \mathbb{C}\Gamma)p$

There should be an obvious ~~pairing~~ dual pair

$(\tilde{A} \otimes \mathbb{C}\Gamma, \mathbb{C}\Gamma \otimes \tilde{A})$.

10:11 Let's try identifying Bh with $E = p(\mathbb{C}\Gamma \otimes A)$

$$B \xrightarrow{g = \cdot h} Bh \xrightarrow{\iota = \text{incl.}} B$$

General situation: $\begin{array}{c} W \underset{\iota}{\overset{f}{\rightleftarrows}} V \\ \Gamma \qquad \mathbb{C} \end{array}$ $\qquad \begin{pmatrix} B & E \\ F & A \end{pmatrix}$

Start with $E = p(\mathbb{C}[\Gamma] \otimes A)$

define B action, show it's a finin B module

$$\tilde{j}(\xi) = \sum s \otimes j s^{-1} \xi$$

$$\tilde{\imath}\tilde{j}(\xi) = \sum s \iota j s^{-1} \xi = \xi$$

$$\begin{tikzcd}
E \arrow[r, "\tilde{J}"] \arrow[dr, "j"'] & \mathbb{C}\Gamma \otimes \tilde{A} \arrow[r, "\tilde{\imath}"] \arrow[d, shift right, "\eta_1"'] & E \\
& \tilde{A} \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "\iota"'] &
\end{tikzcd}$$

$$p(1 \otimes 1) = \sum_s t^{-1} \otimes p(t) \qquad jt\iota$$

$$\sum_t p(s^{-1}t)\, p(t^{-1}) = p(s^{-1})$$

The element in E corresp to h in Bh might turn out to be $p(1 \otimes 1) = \sum_t t \otimes p(t^{-1}) \in E \subset \mathbb{C}\Gamma \otimes A$

Still puzzled.

$$\overset{W}{\overset{\shortparallel}{B}} \in \mathcal{M}(B) \qquad \overset{V}{\overset{\shortparallel}{Bh}} \in R(\tilde{A})$$

You need to understand this isomorphism

$$B = \mathbb{C}\Gamma \overset{p}{\otimes} hB$$

$$W = \mathbb{C}\Gamma \overset{p}{\otimes} hW$$

$$W \xhookrightarrow{\tilde{J}} \mathbb{C}\Gamma \otimes hW \xrightarrow{\tilde{\imath}} W$$

You seem stuck showing $Bh = \mathbb{C}\Gamma \otimes_P A$
I think you can see the element of $\bar{E}$ corresp to

~~the~~ Let's set up the situation carefully. Stick to left modules. Get a function $W \longmapsto hW$ from firm B modules to reduced A-modules, and an exact functor killing nil modules $V \longmapsto \mathbb{C}\Gamma \otimes_P V$ from A-modules to firm B-modules. I think you have a proof that these are inverse. If so then you get

$$B = \mathbb{C}\Gamma \otimes_P hB = (\mathbb{C}\Gamma \otimes_P A) \otimes_A hB$$

You know that $E = \mathbb{C}\Gamma \otimes_P A$ is A^{op} flat and B-firm

hB is the image of $h\cdot : B \longrightarrow B$, $b \longmapsto hb$

h is an ~~operator~~ element of B, hence an operator on B-modules

What should be true is that the Morita context is $\begin{pmatrix} B & Bh \\ hB & A \end{pmatrix}$

where $Bh = \mathbb{C}\Gamma \otimes_P A$, $hB = A \otimes_P \mathbb{C}\Gamma$ are flat firm over A, firm over B. There is a pairing $Bh \times hB \longrightarrow B$ to be understood, but should involve removing one factor of h.
Also there should be a ring isom. ~~surjection~~ ~~$hB \times B$~~ $Bh \otimes_A hB \xrightarrow{\sim} B$ and a ~~ring~~ surjection $hB \otimes_B Bh \longrightarrow A$

Problem: Construct a Morita context $\begin{pmatrix} B & Bh \\ hB & hBh \end{pmatrix}$

Check that $\begin{pmatrix} B & Bh \\ hB & hBh \end{pmatrix}$ is a Morita context

with 6 obvious products: $b_1 b_2$, $b_1(b_2 h)$, $(hb_1)b_2$

$hb_1(hb_2h)$, $hb_1h(b_2h)$, and 2 products

$(hb_1)(b_2h) = hb_1b_2h$

involving h^{-1}:

$(b_1h)(hb_2) = b_1hb_2$

?

$(hb_1h)(hb_2) = hb_1hb_2$

$\begin{pmatrix} B & P \\ Q & A \end{pmatrix}$ BB, BP, QB, QP

AA, AQ, QB, PQ

$\begin{pmatrix} B & Bh \\ hB & hBh \end{pmatrix}$

BB	BP	QB	QP obvious
b_1b_2	b_1b_2h	hb_1b_2	hb_1b_2h

AA

$(hb_1h)(hb_2h) \stackrel{\text{df}}{=} hb_1hb_2h$

PA

$(b_1h)(hb_2h) = b_1hb_2h$

AQ

$(hb_1h)(hb_2) = hb_1hb_2$

PQ

$(b_1h)(hb_2) = b_1hb_2$

Other method: exhibit Bh, hB as a dual pair over B with pairing $\langle b_1 h, hb_2\rangle = b_1 h b_2$

Then you get a Morita context $\begin{pmatrix} B & Bh \\ hB & hB\otimes_B Bh \end{pmatrix}$

$$hB \otimes_B Bh \downarrow hBh$$

$$(hb_1 \otimes b_2 h) hb_3 = hb_1 b_2 h b_3$$

$$b_1 h (hb_2 \otimes b_3 h) = b_1 h b_2 b_3 h$$

$$b_1 h (hbh) = b_1 hbh$$

$$(hb_1 \otimes b_2 h)(hb_3 \otimes b_4 h) = hb_1 \otimes b_2 h b_3 b_4 h$$

$$(hbh)(hb'h) = hbhb'h$$

Next step. ~~Bh Relate~~ Compare

$$hB \otimes_B W \longrightarrow hW \quad \text{onto should}$$

follow from the partition.

$$w = \sum s\, \iota_y\, s^{-1} w$$

$$hw = \sum \underbrace{hsh}_{hB}\, \underbrace{s^{-1}w}_{W}$$

$$p(s)h = hsh$$

$$\sum hb_\alpha \otimes w_\alpha \longmapsto \sum hb_\alpha w_\alpha = 0$$

$$\downarrow p(s)$$

$$\sum hs\underbrace{hb_\alpha}_{\in B} \otimes_B w_\alpha$$

$$B \otimes_B W = W$$

A firm (when Γ finite)

$$A \otimes_A A \longrightarrow A$$

$$\tilde{p}_u = \sum_{u=st} p_s \otimes p_t \longleftarrow\!\shortmid\; p_u$$

Do the $\tilde{p}_u$ satisfy the relations

$$\sum_{u=st} \tilde{p}_s \tilde{p}_t = \sum_{u=st}\left(\sum_{s=s_1s_2} p_{s_1} \otimes p_{s_2}\right)\left(\sum_{t=t_1t_2} p_{t_1} \otimes p_{t_2}\right)$$

$$= \sum_{u=st} \sum_{s=s_1s_2} \sum_{t=t_1t_2} p_{s_1}p_{s_2} \otimes p_{t_1}p_{t_2}$$

$$= \sum_{u=st} p_s \otimes p_t = \tilde{p}_u$$

---

what next? It should be true that the A,B bimodule hB in the Morita context is B^{op} firm, hence also A-firm, since $AhB = hB$. $\therefore$ hB, Bh should be firm on either side. Critical thing is whether

$hB \otimes_B Bh \longrightarrow A$ is an isom.

Still need $\quad Bh \simeq p(\mathbb{C}\Gamma \otimes A)$

$\qquad\qquad\quad hB \simeq (A \otimes \mathbb{C}\Gamma)p$

At the moment you should have a Morita context $\begin{pmatrix} B & Bh \\ hB & A \end{pmatrix}$ yielding your Morita equivalence between B and A. But there are many points to be checked.

Review the Morita equivalence. Let W be a ferin B-module, in other word, W is a Γ-module equipped with $\mathbb{C}$-linear $h: W \to W$ satisfying $hsh = 0$ for $s \in \Phi$, $\sum shs^{-1}w = w$ $\forall w$.

~~Let $V = hW$, let $p(s) = hs$ for hW to itself~~

Let $p(s) = hs$ from hW to hW. Then $p(s) = 0$ $s \notin \Phi$. $\sum_t p(t)p(t^{-1}s)hw = \sum_t hthtt^{-1}shw = hshw = p(s)hw$

So hW is a A-modules. Claim reduced.

$w = \sum_s shs^{-1}w$ $\therefore$ $W = \sum_s shW$

$hW = \sum_s hshW = \sum_s p(s)hW$ $\therefore$ $hW = A \cdot hW$

Also let $v \in hW$, and suppose $\forall s$ $0 = p(s)v = hsv$

Then $0 = \sum_s s^{-1}hsv = v$ so $v = 0$.

So far you've constructed

$$\{\text{ferin } B\text{-mods}\} \longrightarrow \{\text{red } A\text{-modules}\}$$
$$W \longmapsto hW \text{ with } p(s) = hs \text{ on } hW$$

IDEA walking home - make clear the Γ-graded aspects of $\begin{pmatrix} B & Bh \\ hB & A \end{pmatrix}$

~~Given W, V ⊂ W as above form~~

unmotivated version. Given V with A-module st. given by $p(s)$ on V. Form the Γ-module

$$\mathbb{C}\Gamma \otimes V \qquad u(s\otimes v) = us\otimes v$$

Let p be the operator on $\mathbb{C}\Gamma\otimes V$ given by

$$p(s\otimes v) = \sum_t st^{-1}\otimes p(t)v$$

Properties $p(u(s\otimes v)) = u\,p(s\otimes v)$, $p^2 = p$

$$p^2(s\otimes v) = \sum_t st^{-1}u^{-1}\otimes p(u)p(t)v$$

$$= \sum_g s(g)^{-1}\otimes \underbrace{\sum_{g=ut} p(u)p(t)v}_{p(g)} = p(s\otimes v)$$

Let $W = \underbrace{p(\mathbb{C}\Gamma\otimes V)}$, W is Γ-module

$$W = \Big\{ \sum s\otimes f(s) \;\Big|\; f:\Gamma\to V \text{ fin. supp } \sum_u u\otimes f(u) = \sum_{s,t} \overset{u \,=\, ts^{-1}}{ts^{-1}}\otimes p(s)f(t) = \sum_{u,t} u\otimes p(u^{-1}t)f(t) \Big\} \qquad s = u^{-1}t$$

$W = \{\sum s\otimes f(s) \mid f(s) = \sum_t p(s^{-1}t)f(t)\}$ Now you have a precise formula for W ~~[illegible]~~ as certain $f:\Gamma\to V$

The Γ operations are easy $\sum_s us\otimes f(s) = \sum_s s\otimes f(u^{-1}s)$

what is the operator $h = ij = \tilde{\imath}\, \varepsilon_1 \eta_1 \tilde{j}$

$$\begin{tikzcd} W \arrow[r, "\tilde{j}"] \arrow[dr, "j"'] & \mathbb{C}\Gamma \otimes V \arrow[r, "\tilde{\imath}"] \arrow[d, shift right, "\eta_1"'] & W \\ & V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, hook, "i"'] & \end{tikzcd}$$

$\cancel{P_0}f \quad \mathbb{C}\Gamma \otimes V \longrightarrow W \hookrightarrow \mathbb{C}\Gamma \otimes V$

$\sum \cancel{s} \otimes f(s) \qquad\qquad \sum_s \cancel{s} \otimes \sum_t p(s^{-1}t) f(t)$

There are two ways to ~~think~~ think of an elt of W, namely an $f(t)$ satisfying $\sum_t p(s^{-1}t) f(t) = f(s)$ and $f(t)$ of the form $f(t) = \sum_t p(s^{-1}t)\, g(t)$ some g

$$\begin{tikzcd} \overset{g}{\mathbb{C}\Gamma \otimes V} \arrow[r, "p"] & \mathbb{C}\Gamma \otimes V \arrow[r, "p"] \arrow[d, shift right, "\eta_1"'] & \mathbb{C}\Gamma \otimes V \\ & {} \arrow[u, shift right, "\varepsilon_1"'] & \end{tikzcd}$$

$$\sum_s s \otimes g(s) \longmapsto \boxed{\cancel{p(s^{-1}t)g(t)}}\ \sum_s s \otimes \overbrace{\sum_t p(s^{-1}t)\, g(t)}^{f(s)}$$

$$\begin{tikzcd} \sum_s s \otimes \sum_t p(s^{-1}t)\, g(t) \arrow[d, mapsto, "\eta_1"'] & 1 \otimes \sum p(t) g(t) \\ \sum_t p(t) g(t) \arrow[ur, mapsto, "\varepsilon_1"'] & \end{tikzcd}$$

$$p\left(1 \otimes \sum_t p(t) g(t)\right) = \sum s \otimes \sum_t p(s) p(t)\, g(t)$$

apparently it sends $f = pg \qquad f(s) = \sum_t p(s^{-1}t)\, g(t)$

to $\quad p(1 \otimes f(1))$

Have to describe $p(\mathbb{C}\Gamma \otimes V)$ as B-module.

$$\cancel{p\Big(\sum_t t \otimes f(t)\Big) = \sum_{s,t} ts^{-1} \otimes p(s) f(t)}$$

$$\cancel{p\Big(\sum_s s \otimes f(s)\Big) = \sum_u \sum_t ut \otimes p(t) f(u)}$$

$\cancel{B = ut^{-1}}$

$$p\Big(\sum_t t \otimes f(t)\Big) = \sum_t \sum_u tu^{-1} \otimes p(u) f(t)$$

$s = tu^{-1}$
$su = t$
$u = s^{-1}t$

$$= \sum_s s \otimes \sum_t p(s^{-1}t) f(t)$$

So $\mathbb{C}\Gamma \otimes V$ is identified w. $\{ f : \Gamma \to V \mid f \text{ fin. supp.}\}$

Γ action $(tf)(s) = f(t^{-1}s)$. Why

$$t \sum_s s \otimes f(s) = \sum_s ts \otimes f(s) = \sum_s s \otimes f(t^{-1}s)$$

What is the operator h on $p(\mathbb{C}\Gamma \otimes V)$

$$\begin{tikzcd}
\mathbb{C}\Gamma \otimes V \arrow[r] & p(\mathbb{C}\Gamma \otimes V) \arrow[r, hook, "\tilde{j} = \mathrm{inc}"] \arrow[dr, "j"'] & \mathbb{C}\Gamma \otimes V \arrow[r, "\tilde{i} = p"] \arrow[d, shift right, "\eta_1"'] & p(\mathbb{C}\Gamma \otimes V) \\
 & & V \arrow[u, shift right, "\varepsilon_1"'] \arrow[ur, "i"'] &
\end{tikzcd}$$

$h = ij$

Let $f \in p(\mathbb{C}\Gamma \otimes V)$, then $jf = f(1)$, $\varepsilon_1 jf = 1 \otimes f(1)$

$$hf = ijf = p\varepsilon_1 jf = p(1 \otimes f(1)) = \sum_s s \otimes p(s^{-1}) f(1)$$

Thus h on $f \in p(\mathbb{C}\Gamma \otimes V)$ is the lin. fnl. $f \mapsto f(1)$ followed by the function $p(s^{-1})$.

$$(hf)(s) = p(s^{-1}) f(1)$$

rank 1 operator

In other words you have $W = p(\mathbb{C}\Gamma \otimes V)$
$= \{ \sum_s s \otimes f(s) \mid f(s) = \sum_t p(s^{-1}t) f(t) \}$.

with Γ-action $\boxed{(tf)(s) = f(t^{-1}s)}$

and $W \xrightarrow{j} V \xrightarrow{i} W$

$f \longmapsto f(1) \longmapsto p(1 \otimes f(1)) = \left(\sum_s s \otimes p(s^{-1}) \right) f(1)$.

So $\boxed{(hf)(s) = p(s^{-1}) f(1)}$. Can you see that

W is a [redacted] firm B-module?

$$(tht^{-1}f)(s) = (ht^{-1}f)(t^{-1}s) = p(s^{-1}t)\,(t^{-1}f)(1)$$

$$\sum_t (tht^{-1}f)(s) = \sum_t p(s^{-1}t)\, f(t) = f(s).$$

$$(hthf)(s) = p(s^{-1}) \cdot (thf)(1)$$

$$= p(s^{-1})(hf)(t^{-1})$$

$$\boxed{(hthf)(s) = p(s^{-1})\, p(t) f(1)}$$

?

$\cancel{(hthf)(s) = p(s^{-1})(thf)(1)}$

$\cancel{(hf)(t^{-1}) \neq p(t) f(1)}$

So $p(t) = 0$ for $t \notin \Phi$

$\Rightarrow hth = 0$

~~Naturally~~ Next you want to take $V = A$ and to identify $p(\mathbb{C}\Gamma \otimes A)$ with Bh. It is clear that you have such an isom.

V is an A-module

identify $\mathbb{C}\Gamma \otimes V$ with $\{f : \Gamma \to V,\ \text{fin. supp}\}$.

~~[illegible]~~ $\sum_s s \otimes f(s) \longleftrightarrow f$

Γ-action $t\left(\sum_s s \otimes f(s)\right) = \sum_s ts \otimes f(s) = \sum_s s \otimes f(t^{-1}s)$

$(tf)(s) = f(t^{-1}s)$.

p = the op on $\mathbb{C}\Gamma \otimes V$ defd by $\qquad u = ts^{-1},\ us = t,\ s = u^{-1}t$

$$p\left(\sum_t{}' t \otimes f(t)\right) = \sum_{s,t} ts^{-1} \otimes p(s) f(t)$$

$$p(t \otimes v) = \sum_{s \in \Phi} ts^{-1} \otimes p(s) v \qquad = \sum_u u \otimes \sum_t p(u^{-1}t) f(t)$$

$\therefore$

$$(pf)(u) = \sum_t p(u^{-1}t) f(t)$$

p is a Γ-module endo, $p^2 = p$.

$W = p(\mathbb{C}\Gamma \otimes V)$ is a summand of the Γ-module $\mathbb{C}\Gamma \otimes V$.

Define $W \xrightarrow{j} V \xrightarrow{\iota} W$ by

$$jf = f(1), \quad \iota v = p(1 \otimes v) = \sum s^{-1} \otimes p(s) v$$

$$(\iota v)(s) = p(s^{-1}) v$$

$h = \iota j \qquad (hf)(s) = p(s^{-1}) f(1)$

$$\sum_t (tht^{-1}f)(s) = \sum_t (ht^{-1}f)(t^{-1}s) = \sum_t p(s^{-1}t) \underbrace{(t^{-1}f)(1)}_{f(t)}$$

$$= (pf)(s) = f(s)$$

$$(hthf)(s) = p(s^{-1})(thf)(1) = p(s^{-1})(hf)(t) = p(s^{-1}) \overset{0 \text{ for } t \in \Phi}{p(t)} f(1)$$

You want to clarify the situation. Given an A-module V can form $\mathbb{C}\Gamma\otimes V$, a precursor for W, has Γ action and a corresponding (equivariant) partition of unity $\sum_{s\in\Gamma} e_s = 1$

At the moment you have an opaque way ~~method~~ to see that $p(\mathbb{C}\Gamma\otimes V)$ is a firm B-module. The Γ action is clear but the operator h is ~~not so so clear~~ obscure

You have $$p(\mathbb{C}\Gamma\otimes V) \simeq \left\{ \sum_s s\otimes f(s) \;\middle|\; \begin{array}{l} f \text{ finite supp} \\ f = pf \end{array} \right\}$$

then $(hf)(s) = p(s^{-1})f(1)$. Not clear enough.

Instead try ~~to~~ putting $W = p(\mathbb{C}\Gamma\otimes V)$ introduce Γ maps.

$$W \xrightarrow{\tilde{j}=\mathrm{inc}} \mathbb{C}\Gamma\otimes V \xrightarrow{\tilde{i}=p} W$$

Go back to GNS, Hilbert space version

~~[illegible]~~ Basic idea of GNS is illustrated by following: ~~[illegible]~~ unitary rep of Γ on $\mathcal{H}$

$V \overset{i}{\subset} \mathcal{H}$ closed subspace $\ni$ $\overline{\Gamma V} = \sum_{s\in\Gamma} sV = \mathcal{H}$

then have $p(s) = i^* s i$ positive def. function on Γ values in $\mathcal{L}(V)$, from which you can reconstruct the repn of Γ on $\mathcal{H}$. Take $\Gamma = \mathbb{Z}$, ~~say~~ then the pos. def fn. $p(s)$ is equivalent to a measure on S^1, operator-valued. Suppose $p(s)$

finite suppose – then the measure is ≥ 0 matrix Laur poly $\rho(z) = \sum \rho_n z^n$. Recall not easy to see when $\rho(z) \geq 0$ on S^1, however easy if you ask that $\rho(z)^2 = \rho(z)$.

Yesterday you used the description of $W = p(\mathbb{C}\Gamma \otimes V)$ as functions $f : \Gamma \to V$ of finite supp satisfying $f(s) = (pf)(s) = \sum_t p(s^{-1}t) f(t)$ to show that W is a firm B-module. However the operator h is awkward, not pretty. There should be a better picture. The idea is that $\mathbb{C}\Gamma \otimes V$ is a Γ-module with a special kind of partition of unity. A grading is a special kind of partition of 1, namely disjoint. Cech idea

In the geometric situation you form

$$\overset{\textstyle\rightarrow}{\underset{\textstyle\rightarrow}{\rightarrow}} X \times_Y X \rightrightarrows X \longrightarrow Y \qquad X = \coprod U_\alpha$$

It seems clear that you end up with some noncommutative version related to the b' complex.

$$A \otimes A \otimes A \rightrightarrows A \otimes A \longrightarrow A$$

For the moment you should focus on

$$\mathbb{C}\Gamma \otimes V$$