Year 2001
Ideas

743-44 ~~[illegible]~~ Wall obstruction to finiteness vanishes for compact ANR $\boxed{\text{Groth Verdier Duality Tool}}$

743 groupoid gens of Groth fib functor thms
your letter to Serre

739 $\mathbb{C}[\Gamma]$ is bialg where $\Gamma \sqcup \{*\}$ is a semigroup with abs. elt.

686, 710 problems with $\Gamma = M_2$ ⟨ understood in the $8\frac{1}{2} \times 11$ pgs mailed from NJ

701 Volodin space ~~of~~ via partitions of 1.
Deligne's question

675 red mods for C are $(V_1, \dots, V_n, W, \; W \overset{\beta_i}{\hookrightarrow} V_i \overset{\alpha_i}{\twoheadleftarrow} W)$
$(\Gamma = M_n ?)$

628 How assembly seems to go beyond Groth's topos picture (for groupoids)

608 $R \times^{e} L$ for R contra, L co

595 Mult for $A \triangleleft C$ are id's

592 new notion $\bigoplus \Lambda e_X \otimes V_X$ of free Λ-module Γ = grpoid

574 Assembly for $\forall\, \Gamma$ groupoid

550 adjoining an identity for Γ-graded algebras?

532 sheaf picture for $\mathcal{G}$ torsor, $\mathcal{G}$ groupoid

512-515 Philosophy discussion

486 $U(n,1)$ action $\mathcal{O}_n$

481 Morita context $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$ with $*$ product

467 $*$ product, can it be used to handle $A\langle D\rangle$?
recall D^2 filtration

466 treating an operator as if it were idempotent

460 strictly reduced M context

438 For $A = A^2$, can $Mult(A)$ be smaller than $Mult(A/K)$ where $AK = KA = 0$?

419 to define semi direct product $Mult(A) \oplus A$ you need $A^2 = A$.

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397 Obstruction to $A = A^2$ being Mor. equiv to a ring with local units?

359 function on ~~[illegible]~~ $\hat{\mathbb{Z}}$ with pos. herm. values is completely positive. Easiest to check when the values are projections

289 * product M context

24~~9~~8-50 brief look at Pedersen-Weibel

204 $\Gamma = \mathbb{Z}/2$ example

174 Ex. C gen. by h, e, $e^2 = e$, $eh = h$

93a Ex. $\Gamma = \mathbb{Z}$, $\Phi = \{-1, 0, 1\}$

85. $\beta\alpha = 1 \Rightarrow p = \alpha\beta$ is idemp. Is converse true? No but describe modules?

72. geometric case
$$\begin{array}{ccc} \mathbb{R} \times_{\mathbb{R}/\mathbb{Z}} \mathbb{R} & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ \mathbb{R} & \longrightarrow & \mathbb{R}/\mathbb{Z} \end{array}$$

64. pos. (continuous) functionals on a nonunital C^* alg

48 ring gen h, k $kh = h$

41 tensor product of Γ-graded algs is not defined in gen.

22 local left units

13 linear eqns criterion for flatness

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636 + earlier work on retracts of $M_2 \otimes V$

701-703 case V, W 1 dim

Jan 16, 2001. Go over the problems ~~...~~ again. Given Γ, Φ say $\Gamma = \mathbb{Z}$ $\Phi = \{-1, 0, 1\}$ to fix the ideas. You have noncomm. rings.

C_Φ: gens h_s for $s \in \Gamma$ rels $h_s h_t = 0$ $s^{-1}t \notin \Phi$

$$\sum_{s \in t\Phi^{-1}} h_s h_t = h_t = \sum_{s \in t\Phi} h_t h_s$$

Observe that $\sum_{s \in \Phi'} h_s$ is an approx. identity. it is a net indexed by finite subsets of Γ such that

$$\lim_{\Phi'} h_s a = \lim_{\Phi'} a h_s = a$$

for any $a \in C_\Phi$. So C_Φ has left + right units.

firm modules: a left C_Φ module N is firm $\iff$

$$N = C_\Phi N \iff N = \sum_{s \in \Gamma} h_s N \iff \sum_{s \in \Gamma} h_s = 1 \text{ on } N$$

~~...~~ Question, better idea that a partition of 1 is a diagonal approximation,

Discuss the circle case $\Gamma = \mathbb{Z}$, $\Phi = \{-1, 0, 1\}$.

Go back to the obstruction. You have constructed a Morita equivalence, but you do not have, have not yet found the dual pair. Let's focus on the circle case. $C = C_c(\mathbb{R})$ and $B = C_c(\mathbb{R}) \rtimes \mathbb{Z}$. Then C is naturally a firm module over the cross product algebra B. Is it possible to use C as firm B, $A = C(\mathbb{R}/\mathbb{Z})$ bimodule to construct a Morita equivalence between A and B?

Think a little about the assembly map. Essentially it is a line bundle for the group ring $\mathbb{C}[\Gamma]$ over $B\Gamma$.

So you have a fibre bundle over $B\Gamma$ with fibre the group ring $\mathbb{C}[\Gamma]$. ~~So you have~~

~~[illegible]~~ linear equations criterion for flatness.

R unital ring, work in $\mathrm{Mod}(R)$

Left unit e in A: $ea = a$ $\forall a$.

Prop. $\exists$ left unit in A $\iff$ $\mathbb{Z}$ is a projective $\tilde{A}^{op}$-mod

A has a left unit

$$0 \longrightarrow A \longrightarrow \tilde{A} \longrightarrow \mathbb{Z} \longrightarrow 0$$

exact seq of $\tilde{A}^{op}$ modules

$$A \subset R \longrightarrow R/A$$

$\exists$ left unit in $A \iff R/A$ proj.

linear equations criterion

$$\begin{tikzcd} R^I \arrow[r, "(x_{ij})"] \arrow[dr, "0"'] & R^J \arrow[r, dashed, "y"] \arrow[d, "(m_j)"] & R^K \arrow[dl, dashed] \\ & M & \end{tikzcd}$$

$$\sum_j x_{ij} m_j = 0$$

any linear relation in M is a consequence of linear relns. in R.

~~[illegible]~~

$$\sum x_{ji} m_i = 0$$

$$\Longrightarrow \exists\ y_{kj},\ m'_k$$

$m = x'm'$

$$\begin{tikzcd} R^I \arrow[r, "x_{ij}"] \arrow[dr, "0"'] & R^J \arrow[r, dashed, "x'_{jk}"] \arrow[d, "m_j"] & R^K \arrow[dl, dashed, "m'_k"] \\ & M & \end{tikzcd}$$

$$(r_i) \longmapsto \Big(\sum_i r_i x_{ij}\Big)_j (r'_j)$$

$$\downarrow$$

$$\sum_j r'_j m_j$$

$$R^I \xrightarrow{(x_{ij})} R^J \longrightarrow C \longrightarrow 0$$

$$0 \to \mathrm{Hom}_{R^{op}}(C,R) \longrightarrow (R^J)^\vee \longrightarrow (R^I)^\vee$$

$$\begin{tikzcd}
0 \arrow[r] & M \otimes_R C^\vee \arrow[r] & M \otimes_R (R^J)^\vee \arrow[r] \arrow[d, equal] & M \otimes_R (R^I)^\vee \arrow[d, equal] \\
& & \mathrm{Hom}_{R^{op}}(R^J, M) \arrow[r, "(x_{ij})^t"] & \mathrm{Hom}_{R^{op}}(R^I, M) \\
& & m_j \arrow[r, mapsto] & \sum x_{ij} m_j = 0
\end{tikzcd}$$

$$F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0$$

$$N^\vee = \mathrm{Hom}_{R^{op}}(N,R)$$

$$M \otimes_R N^\vee \longrightarrow \mathrm{Hom}_{R^{op}}(N,M)$$ nuclear maps $M \dashrightarrow N$

$$n \otimes f \longmapsto (n' \longmapsto n f(n'))$$

$$0 \to C^\vee \longrightarrow F_0^\vee \longrightarrow F_1^\vee$$ exact

$$0 \to M \otimes_R C^\vee \longrightarrow M \otimes_R F_0^\vee \longrightarrow M \otimes_R F_1^\vee$$ exact for M flat

$$\{f : F_0 \to M\} \qquad \{F_1 \to M\}$$

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[dr, bend right, "0"'] & F_0 \arrow[r] \arrow[d] & C \arrow[r] & R^K \arrow[d] \\
& M \arrow[rr, equal] & & M
\end{tikzcd}$$

$$R^I \xrightarrow{(x_i)} R \longrightarrow R/\ell \longrightarrow 0 \qquad \ell = \sum R x_i$$

$$R^J \otimes_R M \longrightarrow M \longrightarrow M/$$

$$0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0$$

$$0 \to \mathrm{Tor}_1^R(\ , \ ) \to M\otimes\mathfrak{a} \to M \to M/M\mathfrak{a} \to 0$$

$$0 \to K \to \overset{\text{free}}{F} \to M \to 0$$

$$0 \to T_1(N) \to K\otimes_R N \to F\otimes_R N \qquad \forall N \in \mathrm{Mod}(R)$$

If $\operatorname{colim}_{i\in I} N_i$, $\quad 0 \to N' \to N \to N'' \to 0$

$$\operatorname{colim} T_1(N_i) = T_1(\operatorname{colim} N_i)$$

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ 0 \to T_1(N') & \to & K\otimes_R N' & \to & K\otimes_R N' & \to & T_0(N') & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & & & \\ 0 \to T_1(N) & \to & K\otimes_R N & \to & F\otimes_R N & \to & T_0(N) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & & & \\ 0 \to T_1(N'') & \to & K\otimes_R N'' & \to & F\otimes_R N'' & \to & T_0(N'') & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

If $T_1(N) = 0$ for $N = R/\mathfrak{a}$ $\qquad \mathfrak{a} = \sum_1^n Rx_i$

then $T_1(N) = 0$ for all N

$$0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0$$

$$\xi \in \mathrm{Ker}(M\otimes_R \mathfrak{a} \to M\otimes_R R = M)$$

$$\xi = \sum_i m_i \otimes x_i \qquad \sum m_i x_i = 0$$

$$m_i = \sum_j m_j' x_{ji}, \quad \sum_i x_{ji} x_i = 0 \qquad \xi = \sum_i \sum_j m_j' x_{ji} \otimes x_i \ {\ne} \atop {=0}.$$

What lin. equ. criterion $\Rightarrow$ flatness

Given $N' \hookrightarrow N$ you want to show that

$M \otimes_R N' \longrightarrow M \otimes_R N$ is injective. First case

to handle is $\mathfrak{a} = \sum_I R a_i \subset R$ $\quad I$ finite

$$R^I \twoheadrightarrow \mathfrak{a} \hookrightarrow R$$

$$M^I \twoheadrightarrow M \otimes_R \mathfrak{a} \xrightarrow{u} M$$

$$k = \sum_I m_i \otimes a_i \longmapsto \sum_I m_i a_i = 0$$

$$\Longrightarrow \quad m_i = \sum_J m'_j x_{ji}, \quad \sum_I x_{ji} a_i = 0$$

$$k = \sum_I \sum_J m'_j x_{ji} \otimes a_i = \sum_J m'_j \overbrace{\sum_I x_{ji} a_i}^{0}$$

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & R^I \arrow[rr, "\cdot(x_{ih})"] \arrow[dr, two heads] & & R^H \arrow[r] & N \arrow[r] & 0 \\ & & & L \arrow[ur, hook] & & & \end{tikzcd}$$

To show $M \otimes_R -$ converts this to an exact sequence.

Try to finish this. First case

$$\begin{tikzcd} R^I \arrow[rr] \arrow[dr, two heads] & & R \\ & \mathfrak{a} \arrow[ur, hook] & \end{tikzcd} \qquad \begin{tikzcd} M^I \arrow[rr] \arrow[dr, two heads] & & M \\ & M \otimes_R \mathfrak{a} \arrow[ur] & \end{tikzcd}$$

$$\mathfrak{a} = \sum_I R x_i$$

First case: $\mathfrak{a}$ ~~left~~ ideal in R. To show

$M \otimes_R \mathfrak{a} \longrightarrow M \otimes_R R = M$ is injective

Let $k = \sum m_i \otimes a_i$ be in the kernel $\quad \sum_I m_i a_i = 0$.

linear reln in M can be factored

$$m_i = \sum_J m'_j x_{ji} \quad , \quad \sum_I x_{ji} a_i = 0 .$$

$$k = \sum_I \sum_J m'_j x_{ji} \otimes a_i = \sum_J m'_j \otimes \overbrace{\sum_I x_{ji} a_i}^{0}$$

Generalization.

$$N' \subset R^H \qquad M \otimes_R N' \xrightarrow{f} M \otimes_R R^H = M^H$$

~~let $k = \sum_I m_i \otimes x_{ih} \in \mathrm{Ker}\, f$~~

$$k \in M \otimes_R N' \qquad k = \sum_I m_i \otimes n'_i = \sum_I m_i \otimes (x_{ih})_{h\in H}$$

you have $\sum_i m_i x_{ih} = 0 \quad \forall h.$

$\Rightarrow \exists$ fact. $m_i = \sum_J m'_j y_{ji}$, $\sum_I y_{ji} x_{ih} = 0 \quad \forall j, h.$

$$k = \sum_I \sum_J m'_j y_{ji} \otimes \underset{\overset{\shortparallel}{(x_{ih})_{h\in H}}}{n'_i} \in M \otimes_R R^H$$

$$= \left(\sum_I \sum_J m'_j y_{ji} \otimes x_{ih} \right)_{h\in H}$$

$$= \left(\sum_J \sum_I m'_j y_{ji} x_{ih} \otimes 1 \right)_{h\in H}$$

$$\boxed{\begin{aligned} k &= \sum_I m_i \otimes (x_{ih})_h \\ &= \sum_J \sum_I m'_j y_{ji} \otimes (x_{ih})_H \\ &= \sum_J m'_j \otimes \sum_I y_{ji} (x_{ih})_H \\ &= 0 \end{aligned}}$$

Try again. $N' \subset R^H \qquad M \otimes_R N' \xrightarrow{\alpha} M \otimes_R R^H \xrightarrow{\sim} M^H$

$k \in \mathrm{Ker}(\alpha)$ $\quad k = \sum_I m_i \otimes (x_{ih})_{h\in H} \longmapsto \underset{\overset{\shortparallel}{0}}{\left(\sum_I m_i x_{ih} \right)_{h\in H}} \in M^H$

$m_i = \sum_J m'_j y_{ji}$, $\sum_I y_{ji} x_{ih} = 0$

There is something you don't understand here. $M \in \mathrm{Mod}(R^{op})$ is flat when $0 \to N' \to N$ exact $\Rightarrow 0 \to M \otimes_R N' \longrightarrow M \otimes_R N$ exact. You want to deduce this from the linear eqns. criterion. There is a reduction to the case of the inclusion $\mathfrak{a} \subset R$ where $\mathfrak{a}$ is a finitely generated left ideal. Let $k \in M \otimes_R \mathfrak{a}$ $\quad k = \sum_I m_i \otimes a_i$ be in the kernel of

$$M \otimes_R \mathfrak{a} \longrightarrow M \otimes_R R = M, \text{ i.e. } \sum_I m_i a_i = 0$$

Thm loc. cit. $\quad m_i = \sum_J m_j' x_{ji}, \quad \sum_I x_{ji} a_i = 0, \forall j$. So

$$k = \sum_I \Big(\sum_J m_j' x_{ji} \Big) \otimes a_i = \sum_J \sum_I m_j' x_{ji} \otimes a_i = \sum_J m_j' \otimes \overbrace{\sum_I x_{ji} a_i}^{0}$$

A similar argument works with a sub R-module $N' \subset R^H$, H finite. Let $k \in \mathrm{Ker}(M \otimes_R N' \to M^H)$ write $k = \sum_I m_i \otimes n_i'$ with $n_i' \in N'$ so that $n_i' = (a_{ih})_{h \in H} \in R^H$. Condition $k = 0$ means

$$\sum_I m_i \otimes (a_{ih})_{h \in H} \longmapsto \underset{\substack{\| \\ 0}}{\sum_I (m_i a_{ih})_{h \in I}} \in M^H$$

fact. $\quad m_i = \sum_J m_j' x_{ji}, \quad (x_{ji})_{J \times I}$ matrix over R

$$\sum_I x_{ji} a_{ih} = 0, \quad \forall j, h$$

Then
$$k = \sum_I m_i \otimes n_i' = \sum_I \sum_J \overbrace{m_j' x_{ji} \otimes (a_{ih})_{h \in H}}^{\in M \otimes_R N'}$$
$$= \sum_J m_j' \otimes \sum_I (x_{ji} a_{ih})_{h \in H} = 0,$$

M flat (i.e. $M \otimes_R -$ exact from $\mathrm{Mod}(R)$ to $\mathcal{Ab}$)

$\Longleftrightarrow$ (i) $\forall$ injections $N' \hookrightarrow N$ one has $M \otimes_R N' \hookrightarrow M \otimes_R N$

(ii) $\forall$ left ideal $\mathfrak{a} \subset R$ —— $M \otimes_R \mathfrak{a} \longrightarrow M \otimes_R R = M$, $m \otimes a \longmapsto ma$

is injective.

$F(N) = M \otimes_R N$. $N =$ union (colim) of the directed system of finitely generated submodules. $N = \bigcup N_\alpha$

N_α f.g. $N'_\alpha = N' \cap N_\alpha$. $N'_\alpha \hookrightarrow N_\alpha$ fin gen

~~can suppose~~

Get back to Γ for the next few hours.
Let's review, get back in the earlier mode.

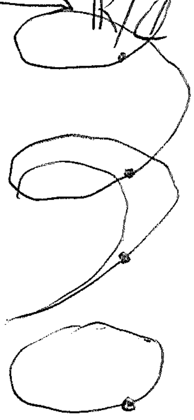
Assembly map. Γ group. $\Gamma \to X \to Y$ compact

$x + \mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$ principal $\mathbb{Z}$-bundle

locally trivial fibre bundle over $\mathbb{R}/\mathbb{Z}$ with fibre $\mathbb{Z}$

$\{x\} \hookrightarrow \mathbb{R}/\mathbb{Z}$ assoc. fibre ______ ______ $\mathbb{C}[\mathbb{Z}]$

$x + \mathbb{Z}$



Try to carry out Serre's thm.

you have a line bundle over $S^1 = \mathbb{R}/\mathbb{Z}$ for the ring $\mathbb{C}[\mathbb{Z}]$

To embed it in a trivial bundle

L $\qquad C(S^1) \otimes \mathbb{C}[\mathbb{Z}]$

end result is a fin. gen. proj module over $\boxed{C(S^1) \otimes \mathbb{C}[\mathbb{Z}]}$

Γ group. Given a principal Γ bundle over Y compact

get $K_0\big(\underbrace{C(Y)}_{C^*} \otimes \mathbb{C}[\Gamma]\big) \longrightarrow K_0\big(C(Y) \overset{*}{\otimes} C^*_r(\Gamma)\big)$

$C(S^1) \overset{*}{\otimes} \mathbb{C}^*_r(\mathbb{Z}) = C(S^1 \times \underbrace{\mathbb{Z}^\vee}_{\chi \text{ on } \mathbb{Z}})$.

$$K\left(C\left(\underset{\substack{\uparrow\\ B\Gamma}}{S^1}\times \underset{\substack{\uparrow\\ C_r(\Gamma)}}{S^1}\right)\right) \qquad KK^0(\mathbb{C}, C(S^1\times S^1))$$

$\otimes$

$x \in H^n(X\times Y)$

$\xi\in H_p(X) \longrightarrow H^{n-p}(Y)$.

<u>slant product.</u> $KK^0(\mathbb{C}, A\otimes B)$

$\times$

$$KK^i(A,\mathbb{C}) \longrightarrow KK^i(\mathbb{C},B)$$

$$KK^i(C(B\Gamma),\mathbb{C}) \longrightarrow KK^i(\mathbb{C}, C_r(\Gamma))$$

K-homology of $B\Gamma$ K-[redacted]theory of $C_r(\Gamma)$.

Try to recall Cuntz's talk.

Γ, $\Phi\subset\Gamma$ finite $\overset{C}{\overset{\shortparallel}{\mathcal{E}\Sigma_\Phi}}$ gens h_s, $s\in\Gamma$

rels. $h_s h_t = 0$ $s^{-1}t\notin\Phi$

has left and right local units $\displaystyle\sum_{s\in t\Phi^{-1}} h_s h_t = h_t = \sum_{s\in t\Phi} h_t h_s$

What do you remember? P_Φ classifies projections supp in Φ in Γ-graded algs $=$ $\hat{\Gamma}$ algs. Formulas:

E module for $C\rtimes\Gamma$.

$$s\otimes\eta \longmapsto s h_1^{1/2}\eta$$

$$E\longrightarrow \mathbb{C}[\Gamma]\otimes E \longrightarrow E$$

$$\xi \longmapsto (s\longmapsto h_1^{1/2}s^{-1}\xi)$$

$$\sum_t h_1^{1/2}t^{-1}\xi \longmapsto \sum_s s h_1^{1/2}h_1^{1/2}s^{-1}\xi = \xi.$$

$$\sum_s s\otimes f_s \longmapsto \sum_t t h_1^{1/2} f_t \longmapsto \left(s\longmapsto \sum_t h_1^{1/2}s^{-1}t h_1^{1/2} f_t\right)$$

so $p_s = h_1^{1/2} s h_1^{1/2} = h_1^{1/2}h_s^{1/2}s$

$$\sum_t p_t p_{t^{-1}s} = h_1^{1/2}\sum_t t h_1^{1/2}h_1^{1/2}t^{-1}s h_1^{1/2} = p_s$$

~~My work~~ How much do you understand.

Contents of earlier notes

p505 n Does a CP map induce a map in K-theory? Seems not.

p505 p. Seems there's a Γ-graded alg morphism

$$P_F \longrightarrow \mathcal{E}_{\Sigma_F} \rtimes \Gamma \qquad p_s \longmapsto h_1^{1/2} h_s^{1/2} s$$

which you've overlooked.

p505 q $\quad K^{top}\Gamma \xrightarrow{\mu} K_*(C_r(\Gamma))$ index map

$$\begin{gathered} \| \\ KK^\Gamma(\underline{E}\Gamma, \mathbb{C}) = \varinjlim_F KK^\Gamma(\mathcal{E}^{ab}_{\Sigma_F}, \mathbb{C}) \end{gathered}$$

p505 t Baaj Skandalis

$$KK^\Gamma(\mathcal{E}_{\Sigma_F}, \mathbb{C}) \simeq KK^\Gamma(P_\Gamma \rtimes \hat{\Gamma}, \mathbb{C})$$

It looks as though you've neglected the Γ gradings on P_F and missed certain things. Go back to

$C = \mathcal{E}_{\Sigma_F}$ (roughly) gens h_s, $s \in \Gamma$ rels $h_s h_t = 0$ $s^{-1}t \in F$, $\sum_s h_s h_t = h_t = \sum_s h_t h_s$.

C has an ~~approx~~ unit (in alg. sense), the net $\sum_{s \in K} h_s$. $\quad B = C \rtimes \Gamma \quad t h_s t^{-1} = h_{ts}$

~~So you see a pro~~ ~~Point is that~~

New point. $B = C \rtimes \Gamma$ and $A = P_F$ are naturally Γ graded, ~~which~~ and you have a projection in B, so you have a Γ graded alg map

$$A = P_F \longrightarrow C \rtimes \Gamma = B$$

Let's review: Outline flatness stuff.

~~fin. pres. module M = cokernel $R^I \to R$~~

$$\begin{tikzcd} M\otimes_R N' \arrow[r] & M\otimes_R N \arrow[r] & M\otimes_R N'' \\ F_\alpha\otimes_R N' \arrow[r] \arrow[u, squiggly] & F_\alpha\otimes_R N \arrow[r] \arrow[u, squiggly] & F_\alpha\otimes_R N'' \arrow[u, squiggly] \end{tikzcd}$$

$M = \varinjlim F_\alpha$ filtered. F_α f.g. free

$N' \longrightarrow N \longrightarrow N''$ exact in $Mod(R)$

$M\otimes_R N' \longrightarrow M\otimes_R N \longrightarrow M\otimes_R N''$ exact?

$k = \sum_I' m_i\otimes n_i \mapsto$

$$\begin{tikzcd} k\in M\otimes_R N \arrow[r] & 0 \quad M\otimes_R N'' \\ F_\alpha\otimes_R N \arrow[r] \arrow[u] & F_\alpha\otimes_R N'' \arrow[u, "F_\beta"'] \end{tikzcd}$$

$k_1 \in$. $\qquad k_1''$

$k_\alpha \quad F_\alpha\otimes_R N \qquad k_\alpha'' \quad F_\alpha\otimes_R N''$

$$F_\beta\otimes_R N' \longrightarrow {}^{k_\beta}\,F_\beta\otimes_R N \longrightarrow {}^{0}\,F_\beta\otimes_R N''$$

$$M\otimes_R N' \longrightarrow {}^{k}\,M\otimes_R N \longrightarrow {}^{0}\,M\otimes_R N''$$

$$\sum m_i\otimes n_i \longrightarrow \sum m_i\otimes p(n_i)$$

$$F \otimes_R N' \hookrightarrow F \otimes_R N$$

$$M \otimes_R 0 \longrightarrow M \otimes_R N' \longrightarrow M \otimes_R N$$

$$k = \sum m_i \otimes n_i' \longmapsto 0$$

This should be straight forward. Let $L \hookrightarrow N$ in $Mod(R)$. Assume M ~~is a right~~ in $Mod(R^{op})$ a filtered colimit of f. free modules. To show $M \otimes_R L \to M \otimes_R N$ injective. Idea: let $\xi = \sum_{i \in I} m_i \otimes l_i$ be in the kernel

means $\sum_I m_i \otimes u(l_i) = 0$ in $M \otimes_R N$ $\qquad u : L \to N$ injection

Have

$$\begin{tikzcd}
(1 \otimes l_i)_I \in R^I \otimes_R L \arrow[r, hook, "1 \otimes u"] \arrow[d, "(m_i \otimes id_L)"'] & R^I \otimes_R N \arrow[d, "(m_i \otimes id_N)"] \\
M \otimes_R L \arrow[r, "1 \otimes u"] & M \otimes_R N
\end{tikzcd}$$

$$\sum m_i \otimes l_i$$

$$\begin{tikzcd}
(l_i) \in \bigoplus_I L \arrow[r, hook] \arrow[d, "(m_i \otimes 1)"] & \bigoplus_I N \ni (u(l_i)) \arrow[d, "(m_i \otimes 1)"] \\
M \otimes_R L \arrow[r] & M \otimes_R N
\end{tikzcd}$$

$$\sum m_i \otimes l_i$$

You have $R^I \to M$ given by the m_i and $u(l_i) \in N$ such that $R^I \otimes_R N \longrightarrow M \otimes_R N$?

Point should be that $M = \varinjlim F_\alpha$

so

$$\begin{tikzcd}
M \otimes_R L \arrow[r] & M \otimes_R N \\
F_\alpha \otimes_R L \arrow[u] \arrow[r, hook] & F_\alpha \otimes_R N \arrow[u]
\end{tikzcd}$$

Given $u : L \hookrightarrow N$ to show $1 \otimes u : M \otimes_R L \hookrightarrow M \otimes_R N$ when M satisfies the linear equations criterion. The point should be that M is a filtered colim of f free modules by the linear equations criterion, hence $M \otimes_R -$ = filtered colim of $F_\alpha \otimes_R -$ which preserves injection.

Do this by hand. Given $\xi = \sum_{i \in I} m_i \otimes l_i \in M \otimes_R L$ such that $(1 \otimes u)\xi = \sum_{i \in I} m_i \otimes u(l_i)$ is zero in $M \otimes_R N$.

Put

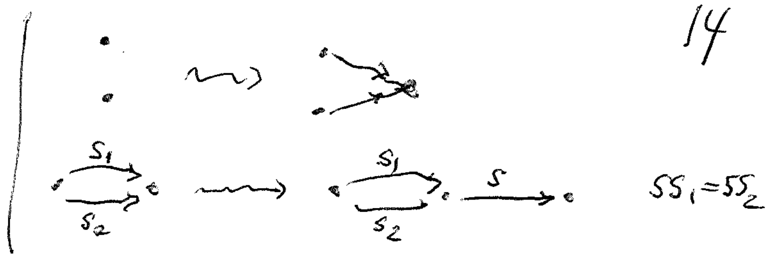
$$\begin{array}{ccc} F_0 & = & R^I \\ \downarrow & & \downarrow (m_i)\cdot \\ M & = & M \end{array} \qquad \begin{array}{ccc} F_0 \otimes_R N & = & N^I \\ \downarrow & & \downarrow (m_i)\otimes - \\ M \otimes_R N & = & M \otimes_R N \end{array}$$

Real problem is how to use $\sum_{i \in I} m_i \otimes u(l_i) = 0$ in $M \otimes_R N$

The linear equations criterion say for any flat R^{op}-module M that the cat of f. free P over M is filtering, but $M = \varinjlim P$ formally, ~~the filt. cat. thing~~ also $\varinjlim P \otimes_R N = M \otimes_R N$ should also be true by adjointness. Steps: To show M flat you need $L \subset N \Rightarrow \underbrace{M \otimes_R L \longrightarrow M \otimes_R N}$ inj. You have by adjointness that $= \varinjlim_\alpha (P_\alpha \otimes_R L) \longrightarrow \varinjlim_\alpha (P_\alpha \otimes_R N)$ and you know that $P_\alpha \otimes_R L \hookrightarrow P_\alpha \otimes_R N$, so the point seems to be that $\varinjlim_\alpha$ respects injections (i.e. kernels).

$$\bigoplus_{\alpha \to \beta} P_\alpha \rightrightarrows \bigoplus_\alpha P_\alpha \longrightarrow \varinjlim P_\alpha$$

$\mathcal{S}$ (small) filtering category.



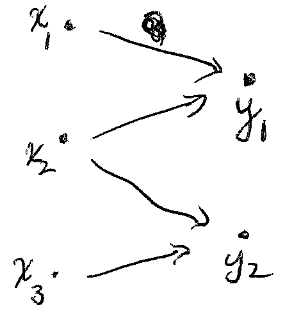
$F : \mathcal{S} \to$ Sets $\qquad \coprod_{x \in \mathrm{Ob}\,\mathcal{S}} F(x) \Big/ \sim \qquad$ where

relation $(x_1, \xi_1 \cancel{\in F(s_1)}) \sim (x_2, \xi_2 \cancel{\in F(s_2)})$

means $\exists$ object x and arrows $x_1 \xrightarrow{s_1} x$, $x_2 \xrightarrow{s_2} x$

such that $F(s_1)\xi_1 = F(s_2)\xi_2$

trans. (x_i, ξ_i) $\quad i = 1, 2, 3.$

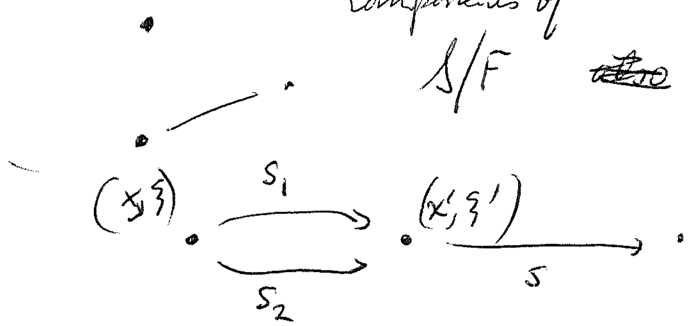


you are forming the cofibred cat $\mathcal{S}/F$ over $\mathcal{S}$ assoc. to F.

Objects $x, \xi \in F(x)$

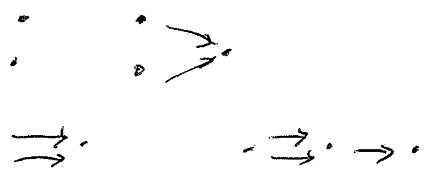
Maps $(x, \xi) \to (x', \xi')$ are map $s : x \to x'$ $\ni$ $s_* \xi = \xi'$.

Components of $\mathcal{S}/F$ ~~also~~ filtering



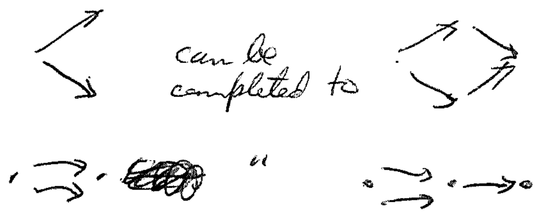
~~equiv.~~

$F : \mathcal{C} \to$ Sets $\qquad \mathcal{C}$ filtering



$\mathcal{C}/F$ cat of (x, ξ) $\quad x$ in $\mathcal{C}$, $\xi \in F_x$

If $\mathcal{C}$ filtering, then $\mathcal{C}/F$ sats.



For C/F get an equivalent relation on objects

$x \sim x' \iff$
$$\begin{tikzcd}[row sep=small] x \arrow[dr] & \\ & y \\ x' \arrow[ur] & \end{tikzcd}$$
ref, symm.

$x \sim x' \sim x''$
$$\begin{tikzcd}[row sep=small] x \arrow[dr] & & \\ & y \arrow[dr] & \\ x' \arrow[ur] \arrow[dr] & & z \\ & y' \arrow[ur] & \\ x'' \arrow[ur] & & \end{tikzcd}$$
clear.

equivalence classes are connected components. ~~This is~~
The ~~space~~ set of components is ~~clearly~~ in general the colim. $C/F\times G \longrightarrow C/F \times C/G$

$(\xi,\eta)\in F_x\times G_x$ ~~x, ξ, η~~

Assume $(x,\xi)\sim(x',\xi')$
$(x,\eta)\sim(x',\eta')$

$(\xi',\eta')\in F_{x'}\times G_{x'}$ ~~Can "enlarge"~~ dominate x, x' by x'' so that $\xi=\xi'$ in $F_{x''}$

Do again. have canonical map $\pi_0(C/F\times G) \longrightarrow \pi_0(C/F)\times\pi_0(C/G)$ First prove surj. Given $\alpha\in\pi_0(C/F)$ $\beta\in\pi_0(C/G)$ rep. α by $\xi\in F(x)$, rep β by $\eta\in G(x')$. Via $\cdot\to\cdot\leftarrow\cdot$ can suppose $x=x'$, whence $(\xi,\eta)\in(F\times G)(x)$ rep. $\gamma\in\pi_0(C/F\times G)$ mapping to (α,β). Next inj. ~~Take two elements of $\pi_0(C/F\times G)$ represent by $(\xi,\eta)\in(F\times G)(x)$ and $(\xi',\eta')\in(F\times G)(x')$. can suppose $x'=x$.~~

Take two elts $\gamma,\gamma'\in\pi_0(C/F\times G)$ such that $pr_1\gamma = pr_1\gamma'$ in $\pi_0(\mathcal{C}/F)$

$pr_2\gamma = pr_2\gamma'$ in $\pi_0(\mathcal{C}/G)$

Rep γ by $(\xi,\eta)\in(F\times G)(x)$

γ' — $(\xi',\eta')\in(F\times G)(x')$ can assume $x=x'$.

Then $\xi\in F(x)$ and $\xi'\in F(x)$ both rep. $pr_1\gamma=pr_1\gamma'$ in $\pi_0(\mathcal{C}/F)$, so by enlarging x, can assume $\xi=\xi'\in F(x)$. ~~State~~ Then η and $\eta'\in G(x)$ both rep. $pr_2\gamma=pr_2\gamma'$, so by enlarging x again can suppose $\eta=\eta'\in G(x)$. Thus $\gamma=\gamma'$.

~~Both rep. by $\xi=\xi'\in F(x)$ and $\eta=\eta'\in G(x)$~~

$$\mathrm{Ker}(u,v)\longrightarrow F \underset{v}{\overset{u}{\rightrightarrows}} G$$

Let $\gamma\in\pi_0\mathcal{C}/F$ sat $u(\gamma)=v(\gamma)$ in $\pi_0(\mathcal{C}/G)$

$$\mathcal{C}/F \underset{v}{\overset{u}{\rightrightarrows}} \mathcal{C}/G$$

Rep γ by $\xi\in F(x)$. Then $u\gamma$, $v\gamma$ rep by $u\xi\in G(x)$, $v\xi\in G(x)$

By enlarging x can suppose $u\xi=v\xi$ in $G(x)$ whence $\xi\in\mathrm{Ker}(u,v)(x)$ rep. $\gamma_1\in\mathcal{C}/\mathrm{Ker}(u,v)$ mapping to γ.

So now what are the steps to go from the filtering category of fin free mods over M to flatness of M.

$\mathcal{C}$ = cat. of finite seq $x=(m_1,\dots,m_n)$ in M

$$F(\vec{m}_\alpha) = \left(R^n \longrightarrow M\right),\quad e_i \mapsto m_i$$

$$\varinjlim F_\alpha = M. \qquad \varinjlim F_\alpha\otimes_R N = M\otimes_R N$$

So what gives? Chain of arguments.

$$N' \hookrightarrow N$$

$$F_\alpha \otimes_R N' \hookrightarrow F_\alpha \otimes_R N \quad \text{as } F_\alpha \text{ free}$$

$$\begin{tikzcd} \varinjlim (F_\alpha \otimes_R N') \arrow[r] \arrow[d, "\cong"'] & \varinjlim (F_\alpha \otimes_R N) \arrow[d, "\cong"'] \\ M \otimes_R N' \arrow[r] & M \otimes_R N \end{tikzcd}$$

$M = \varinjlim F_\alpha$ $\quad \alpha \in$ filtering cat. $\qquad N' \subset N$

because F_α free you have $F_\alpha \otimes_R N' \hookrightarrow F_\alpha \otimes_R N$

$\varinjlim (F_\alpha \otimes_R N) = (\varinjlim F_\alpha) \otimes_R N$ $\quad$ because both $\varinjlim$ and $- \otimes_R N$ are left adjoint funs.

This point (is different from) seems unrelated to the calculation of filtered $\varinjlim$ s

$$\begin{tikzcd} \tilde{\phi}' \in F_\alpha \otimes_R N' \arrow[r, hook] \arrow[d] & F_\alpha \otimes_R N \ni \tilde{\phi} \arrow[d] \\ F_\beta \otimes_R N' \arrow[r, hook] \arrow[d] & F_\beta \otimes_R N \arrow[d] \\ \phi' \in M \otimes_R N' \arrow[r] & 0 \in M \otimes_R N \end{tikzcd}$$

How does the argument go. You take $\phi' \in M \otimes_R N'$ mapping to zero in $M \otimes_R N$. ϕ' lifts to $\tilde{\phi}' \in F_\alpha \otimes_R N'$ for some α, which maps to $\tilde{\phi} \in F_\alpha \otimes_R N$

and then $\tilde{\phi}$ ~~goes~~ maps to zero in $M \otimes_R N$, which means, because of the calculation of $\varinjlim$ for filtering systems, that $\exists\ \alpha \to \beta$ such that $\tilde{\phi}'$ becomes zero in $F_\beta \otimes_R N$, then $\tilde{\phi}'$ becomes zero in $F_\beta \otimes_R N'$ since $F_\beta \otimes_R N' \hookrightarrow F_\beta \otimes_R N$ so $\phi' = 0$.

So ultimately you seem to argue as follows: such that $(1 \otimes u)\phi' = 0$

Suppose given $\phi' \in M^\omega \otimes_R N'$ going to zero in $M \otimes_R N$

Represent ϕ' as $\sum_{i=1}^{\omega} m_i \otimes n_i'$ so that $\sum_1^\omega m_i \otimes u(n_i') = 0$

$$\text{You have}\quad \tilde{\phi} = (1 \otimes u)(\tilde{\phi}') \in F_\alpha \otimes_R N$$
$$= \sum e_i \otimes n_i$$
$$\hookrightarrow \sum m_i \otimes n_i = 0 \in M \otimes_R N$$

Thus you have a sequence $n_1, \dots, n_\omega \in N$ and $m_1, \dots, m_\omega \in M$ such that $\sum_{i=1}^{\omega} m_i \otimes n_i = 0$.

You want to find

m_j', r_{ji} such that $m_i = m_j' r_{ji}$ and $r_{ji} n_i = 0$

~~So you learn something new, I think~~

You should be able to analyze this. Let's review the discussion.

Let M be a right module such that the category of ~~finite free~~ finite free modules over M is filtering. ~~[illegible]~~ or better, let $\alpha \mapsto (F_\alpha \to M)$ be a cofinal functor into this category from a filtering cat into the cat of f.t. free modules over M. ~~Then~~ Suppose given $W' \overset{u}{\subset} W$ left modules. To show $M \otimes_R W' \xrightarrow{1\otimes u} M \otimes_R W$ is injective, using the fact that $\varinjlim F_\alpha \otimes_R W = M \otimes_R W$ Sim. for W'

You take $\xi \in \operatorname{Ker}(1\otimes u)$, ~~[illegible]~~ you write $\xi = \sum m_i \otimes n_i'$ then ~~[illegible]~~ take $F_\alpha \to M$ to be given by the sequence m_i. ~~[illegible]~~ $(1\otimes u)(\xi) = \sum m_i \otimes u(n_i') = 0$ by assumption

i.e.

$$\begin{tikzcd} F_\alpha \otimes_R W' \arrow[r, hook, "1\otimes u"] \arrow[d, "\text{corresp to sequence } w_i"'] & F_\alpha \otimes_R W \arrow[d, "\text{corresp to seq } u(w_i')"] \\ M \otimes_R W' \arrow[r] & M \otimes_R W \end{tikzcd}$$

So what you need to understand seems to be the meaning of a relation $\sum m_i \otimes n_i = 0$

Since $\varinjlim_\alpha F_\alpha \otimes_R N = M \otimes_R N$

α runs over a filtering cat you have exactly I think to prove ~~that~~ the factorization

$$\sum m_i \otimes w_i = 0 \implies m_i = m_j' r_{ji} \;, \quad r_{ji} w_i = 0$$

$$\begin{tikzcd} F_\alpha \otimes_R N \arrow[rr] \arrow[dr] & & M \otimes_R N \\ & F_\beta \otimes_R N \arrow[ur] & \end{tikzcd}$$

$(n_i) \longmapsto \sum m_i \otimes n_i$

$(n_i) \longmapsto (r_{ji} n_i)$

$$\varinjlim_{\{\text{fin fr}\}/M} F \xrightarrow{\sim} M$$

$$\mathrm{Hom}_R\Big(\varinjlim_{\text{fin fr}/M} F, T\Big) = \varprojlim_{\text{fin fr}/M} \mathrm{Hom}_R(F,T)$$

$$\begin{tikzcd} R^k \arrow[d, "{(k,\alpha)}"'] \arrow[dr, "{F(k,\alpha)}"] & \\ M & T \end{tikzcd}$$

w

Here's what I would like to understand precisely

Assume fin fr$/M$ is filtering, let W be any R-module. Suppose $R^I \xrightarrow[R^{op}\text{-map}]{(m_i)\cdot} M$ $\quad R^I \xrightarrow[R\text{-linear}]{\cdot(w_i)} W$

such that $\sum_{i\in I} m_i \otimes w_i = 0$ in $M \otimes_R W$

To find fact. $m_i = \sum_J m'_j r_{ji}$ $\ni$ $\sum_I r_{ji} w_i = 0$

$$\begin{tikzcd} R^I \otimes_R W = W^I \ni (w_i) \arrow[d, "(m_i)\otimes 1"'] \\ M \otimes_R W \end{tikzcd}$$

You have an object $R^I \xrightarrow{(m_i)} M$ of fin fr$/M$ and an element of the functor $-\otimes_R W$ applied to this

Perhaps you can use

$$\begin{tikzcd} \coprod_\alpha \otimes_{\mathbb{C}} R \otimes_{\mathbb{C}} W \arrow[r] & \coprod_\alpha \otimes_{\mathbb{C}} W \arrow[r] & \coprod_\alpha \otimes_R W \arrow[r] \arrow[d] & 0 \\ & & M \otimes_R W & \end{tikzcd}$$

Begin with $\sum m_i \otimes w_i = 0$ in $M \otimes_R W$

You get $\alpha : R^I \xrightarrow{(m_i)} M$ an object of fin fr/M

and an elt $\sum_I e_i \otimes w_i \in F_\alpha \otimes_R W$.

You should begin with $M = \varinjlim_\alpha F_\alpha$ $\alpha \in$ fin fr/M

and $M \otimes_R W = \varinjlim F_\alpha \otimes_R W$.

Start again. Assume M ∋ the cat fin fr/M of maps $\alpha : R^I \longrightarrow M$ (finite sequences) is filtering. Let $F_\alpha = R^I$, i.e. $\alpha \longmapsto F_\alpha$ is obvious fun from fin fr/M ~~⊗~~ → fin free $\subset \mathrm{Mod}(R^{op})$. Then we know

$$\operatorname*{colim}_{\{\alpha\}} F_\alpha \otimes_R W = M \otimes_R W \qquad \text{for any } W.$$

But because fin fr/M is filtering you know for any $\xi \in \mathrm{Ker}\left(F_\alpha \otimes_R W \longrightarrow M \otimes_R W\right)$ $\exists\ \alpha \to \beta$ ∋ $F_\alpha \otimes_R W \longrightarrow F_\beta \otimes_R W$, $\xi \longmapsto 0$

M flat $\Leftrightarrow \forall\, W' \overset{\mu}{\subset} W$, $M \otimes_R W \xrightarrow{1 \otimes \mu} M \otimes_R W'$ injective

Assume not. let $\sum m_i \otimes w_i \in M \otimes_R W$ ∋ $\sum m_i \otimes \mu w_i = 0$ in $M \otimes_R W'$

$\exists$ $m_i = \sum_J m'_j x_{ji}$ $\quad 0 = \sum_I x_{ji} \mu(w_i) = \mu\left(\sum_I x_{ji} w_i\right)$

$$\sum_I m_i \otimes w_i = \sum_{I,J} m_j x_{ji} \otimes w_i$$

A right ideal in R unital.

(i) $\forall a_1 \in A \quad \exists a \quad (1-a)a_1 = 0$

(ii) $\forall a_1, \dots, a_n \in A \quad \exists a \quad (1-a)a_i = 0 \quad \forall i$

(iii) R/A is a flat R^{op}-module

A has local left units.

Recall A has a left unit: $\exists a$ st $(1-a)R = 0$.
$\iff$ R/A is a projective R^{op}-module.

(i) $\Rightarrow$ (ii) Ind $\Rightarrow$ $\exists a'$ $\quad (1-a')a_i = 0 \quad i = 1, \dots, n-1$.

$\exists a'' \quad (1-a'')(1-a')a_n$

then if $1-a = (1-a'')(1-a')$ i.e. $a = a'' + a' - a''a'$

$(1-a)a_i = 0$ for $i = 1, \dots, n$.

(i) $\Rightarrow$ (iii) Let S = monoid $\{(1-a), a \in A\}$ under mult. Claim S filt. cat.

$$\begin{tikzcd} \cdot \arrow[r, shift left, "1-a_0"] \arrow[r, shift right, "1-a_1"'] & \cdot \arrow[r, "1-a"] & \cdot \end{tikzcd}$$

$(1-a)(1-a_0) = (1-a)(1-a_1)$

$(1-a)(a_1 - a_0) = 0$

functor $S \longrightarrow \mathrm{Mod}(R^{op})$

$\cdot \longmapsto R$.

$1-a \longmapsto (1-a)\cdot$

colim $= R/A$

$$\begin{tikzcd}
R \arrow[r, "x_i\cdot"] \arrow[dr, "0"'] & R^I \arrow[r, squiggly, "x_{gi}"] \arrow[d, "(r_i+A)\cdot"] & \cdot \\
(\sum r_i x_i + A)\cdot \arrow[r] & R/A \arrow[r, phantom, "="] & R/A
\end{tikzcd}$$

$$\sum_i r_i x_i \in A$$

$$0 \longrightarrow A \longrightarrow R \longrightarrow R/A \longrightarrow 0$$

~~R/A flat.~~

$$\cancel{\begin{tikzcd}
R \arrow[r, "(x_i)"] \arrow[dr, "0"'] & R^I \arrow[d, "(r_i+A)"] \\
 & R/A
\end{tikzcd}}$$

$$\cancel{\sum_i (r_i+A)\cdot x_i = 0} \qquad \cancel{\sum_i r_i x_i \in a}$$

$$\begin{tikzcd}
R \arrow[r, "\cdot a_i"] \arrow[dr, "0"'] & R \arrow[r, "(x_i)\cdot"] \arrow[d, "(1+A)\cdot"] & R^I \arrow[dl, "(y_i+A)\cdot"] \\
 & R/A &
\end{tikzcd}$$

$$\sum_I (y_i+A)x_i = \sum_I y_i x_i + A = 1+A$$

$$x_i a_i = 0$$

$$\therefore \quad 1 - \sum y_i x_i = a$$

$$1 - a = \sum y_i x_i \quad \text{and} \quad (1-a)a_i = 0$$

$$\begin{tikzcd} R \arrow[r, "a_1\cdot"] \arrow[dr, "0"'] & R \arrow[r, "(x_i)\cdot"] \arrow[d, "(1 \bmod A)\cdot"] & R^I \arrow[dl, "(y_i \bmod A)\cdot"] \\ & R/A & \end{tikzcd}$$

$$1 \equiv \underbrace{\sum y_i x_i}_{1-a} \quad (\bmod A)$$

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A\otimes_R W' \arrow[r] \arrow[d] & A\otimes_R W \arrow[r] \arrow[d] & A\otimes_R W'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & W' \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & W'' \arrow[r] \arrow[d] & 0 \\ & W'/AW' \arrow[r] \arrow[d] & W/AW \arrow[r] \arrow[d] & W''/AW'' \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

$A\otimes_R M \xrightarrow{\sim} AM$ Can you show that $A\otimes_R W \to W$ is injective using local left units. Assume $\sum a_i \otimes w_i \mapsto \sum a_i w_i = 0$

$$R \longrightarrow R^I$$

$$R \longrightarrow R/A$$

N

~~OKAY Now~~ $\sum a_i\otimes w_i \in A\otimes_R W$

$$a\sum a_i \otimes w_i = a\otimes \sum a_i w_i$$

$$\| $$

$$\sum a_i \otimes w_i$$

Point: (i) $(1-a)a_1 = 0 \implies$ (ii) $(1-a_0)a_i = 0 \quad i\in I$

(iii) R/A flat

$$\begin{tikzcd} R^J \arrow[r, "(a_j)\cdot"] \arrow[dr, "0"'] & R \arrow[r, "(x_i)\cdot"] \arrow[d, "(1+A)\cdot"] & R^I \arrow[d, "(y_i+A)"] \\ & R/A \arrow[r, phantom, "="] & R/A \end{tikzcd}$$

$\sum_i y_i x_i \equiv 1 \mod A$

$$1 - \sum y_i x_i = a$$

$$1-a = \sum y_i x_i$$

$$(1-a)a_j = \sum y_i x_i a_j = 0$$

Back to Γ, your ~~most~~ overlooked point about the grading.

$$\mathcal{E}_{\Sigma_\Phi} \quad \text{gens. } h_s,\ s\in\Gamma \ \Big|\ \text{rels.} \quad h_s h_t = 0 \quad s^{-1}t \notin \Phi$$

$$\sum_{s\in t\Phi^{-1}} h_s h_t = t = \sum_{s\in t\Phi} h_t h_s$$

$\mathcal{E}_{\Sigma_\Phi}$ has local left + right units $\sum_{s\in K} h_s$

form $B = \mathcal{E}_{\Sigma_\Phi} \rtimes \Gamma$ an ideal inside $\widetilde{\mathcal{E}_{\Sigma_\Phi}} \rtimes \Gamma$.

~~Algebra~~. B-module E firm when $E = \sum_{s\in\Gamma} h_s E = \sum_{s\in\Gamma} s h_1 E$

Observe $B = \bigoplus_{\Gamma} B_s \qquad B_s = Cs \qquad C = \mathcal{E}_{\Sigma_\Phi}$

B is a Γ-graded algebra. ~~to what~~ Basic Morita equivalence. ~~is clear~~ $E \longmapsto h_1 E = V$

$$\bigoplus s\otimes V$$
$$\|$$
$$E \xrightarrow{\ \alpha\ } \mathbb{C}[\Gamma]\otimes V \xrightarrow{\ \beta\ } E$$
$$\|$$
$$\Big\{ f : \underset{\text{fin supp.}}{\Gamma \to V} \Big\}$$

$$\alpha(\xi)_s = \alpha_1 s^{-1}\xi \qquad \beta f = \sum_s s\beta_1 f_s$$

$$(\beta\alpha)(\xi) = \sum_s s\beta_1\alpha_1 s^{-1}\xi = \sum_s h_s \xi = \xi.$$

~~$(\alpha\beta)f$~~ $$\big(\alpha(\beta f)\big)_s = \sum_t \underbrace{\alpha_1 s^{-1}t\beta_1}_{p_{s^{-1}t}} f_t$$

Γ grading. First, review ungraded theory.
Let E be a $B = C_\Phi \rtimes \Gamma$ - module (finite). E is v.s. with Γ-action and $h_1 : E \circlearrowleft$ such that $h_1(s^{-1}t)h_1 = 0$ for $s^{-1}t \notin \Phi$, ~~and~~ better $h_1 s h_1 = 0$ for $s \notin \Phi$, also $\sum_s s h_1 s^{-1} = 1$. Get.

$$E \xrightarrow{\alpha} \mathbb{C}[\Gamma] \otimes V \xrightarrow{\beta} E$$

$$\mathbb{C}[\Gamma] \otimes V = \bigoplus s \otimes V = \{f : \Gamma \to V \text{ fin supp}\}$$

$$E \xrightarrow{\alpha_1 = \alpha} \overbrace{h_1 E}^{V} \overset{\beta_1}{\hookrightarrow} E$$

anyway ~~the~~ key point is $(\alpha\beta f)_s = \sum_t \alpha_1 s^{-1} t \beta_1 f_t$

So your V for the minimal factorization $E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$ gives $p_s = \alpha_1 s \beta_1 \in \mathrm{End}(V)$ $\quad p_s = 0$ for $s \notin \Phi$

$$\sum_t p_t p_{t^{-1}s} = \sum_t \alpha_1 t \underbrace{\beta_1 \alpha_1}_{h_1} t^{-1} s \beta_1 = \alpha_1 s \beta_1 = p_s$$

Thus you've got a Γ-graded projection in $\mathrm{End}(V)$ finite support ~~in~~ Φ.

Your Morita equivalence links $B = C_\Phi \rtimes \Gamma$ with $A = P_\Phi$ which are both Γ-graded algs. ($\hat{\Gamma}$ algs.)

How do you see that A is Γ-graded? Recall adjoint functors

vector spaces $\qquad$ Γ-graded vector spaces

$$W \qquad\qquad V = \bigoplus_{s \in \Gamma} V_s = (V_s)_{s \in \Gamma}$$

$$\mathrm{Hom}\Big(\bigoplus_{s \in \Gamma} V_s, W\Big) = \prod_s \mathrm{Hom}(V_s, W) = \mathrm{Hom}_{\hat{\Gamma}}\big((V_s)_{s \in \Gamma}, (W)_{s \in \Gamma}\big)$$

Try again

$$\begin{tikzcd} \begin{pmatrix}\text{vector spaces}\\ W\end{pmatrix} \arrow[r, shift left] & \begin{pmatrix}\Gamma\text{-graded v.s.}\\ s \mapsto V_s\end{pmatrix} \arrow[l, bend left, shift left] \end{tikzcd}$$

$$\bigoplus_{s\in\Gamma} V_s \longleftarrow\!\shortmid (s \mapsto V_s)$$

$$W \longmapsto (s \mapsto W)_{s\in\Gamma}$$

$$\mathrm{Hom}\Big(\bigoplus_s V_s, W\Big) = \mathrm{Hom}_{\Gamma}^{a}\big((V_s), (W)\big)$$

Go on to algebras. Γ-graded alg $(B_s)_{s\in\Gamma}$ + $B_s B_t \subset B_{st}$.

$$\mathrm{Hom}_{alg}\Big(\bigoplus_s B_s, C\Big) = \{(f_s : B_s \to C)\}$$

$$\begin{tikzcd} B_s \times B_t \arrow[r] \arrow[d, "f_s \times f_t"] & B_{st} \arrow[d, "f_{st}"] \\ C \times C \arrow[r] & C \end{tikzcd}$$

So what are you trying to understand? The basic point is that a Γ-graded alg $B = \bigoplus_s B_s$ sits inside the ~~constant~~ Γ-graded alg. $\mathbb{C}[\Gamma] \otimes B$, there's a canon. Γ-graded alg. map

$$\begin{array}{ccc} B & \longrightarrow & \mathbb{C}[\Gamma] \otimes B \\ \| & & \\ \bigoplus_s B_s & & \bigoplus s \otimes B_s \end{array}$$

So in the case of $A = P_\Phi$, ~~you defn~~ this is defined by gens + rels.

$$A \longrightarrow \bigoplus_s A = \mathbb{C}[\Gamma] \otimes A$$

$$p_s \qquad\qquad s \otimes p_s$$

I think ~~guess~~ that to make sense of the situation you need a $\boxed{\Gamma\text{-graded}}$ Morita equivalence of some sort. Maybe this means using Γ-graded $\boxed{\text{modules}}$ ✓ over P_Φ. ~~Look~~

Look for an analogue of a projector $p_s = h_1^{1/2} s h_1^{1/2}$

Γ group, Φ finite subset of Γ

Given a Γ-graded alg $B = \bigoplus_s B_s$, a projection $p = \sum p_s \in B$ is a family of $p_s \in B_s$ fin supp sat. $\left\{ p_u = \sum_{u=st} p_s p_t \right.$. Define alg P_Φ by gens p_s $s \in \Gamma$, rels $\left[p_s = 0 \text{ for } s \notin \Phi \right.$.

Then there is a canon alg hom. $P_\Phi \longrightarrow B$ obviously. But P_Φ is Γ-graded.

To clarify. ~~Within~~ You are used to the ring $\mathbb{C} = \mathbb{C}e$ universal for projections. P_Φ is an analog for ~~graded~~ projections $p = \sum p_s$ with Supp $\subset \Phi$ in a <u>Γ-graded</u> alg.

For lecture you need Γ graded v.s., algs., modules.

You want to enrich your Mor. equ. between $A = P_\Phi$ and $B = C_\Phi \rtimes \Gamma$ to include Γ graded modules. So you start with a Γ-graded ($\hat{\Gamma}$-) module V over $A = P_\Phi$ e.g. A itself.

Let $M = \bigoplus_s M_s$ be a Γ-graded $A = \bigoplus_s A_s$ module. Then you have $p_t \in A_t$ acting on M as an operator of left degree t, i.e. $p_t : M_s \longrightarrow M_{ts}$ $\forall s$.

You want to obtain a proj p on $\mathbb{C}[\Gamma] \otimes M$

$$\mathbb{C}[\Gamma] \otimes M = \bigoplus_{s \in \Gamma} M = \{ f : \Gamma \to M \mid f \text{ fin supp} \}$$

want ops respecting left mult by elements of Γ

$$(L_t f)_s = f_{t^{-1}s} \qquad (L_t f)(s) = f(t^{-1}s)$$

$$(pf)(s) = \sum_t p(s^{-1}t) f(t)$$

problem: You are mixing Γ actions and Γ coactions. Maybe not a problem. Philosophy from GNS, pairings Γ spreads things out, but the real action takes place in a "fundamental domain".

What is the meaning of Γ grading?

Γ action together with Γ grading, compatible fashion, yields a rigid structure, get a Morita equivalence with vector spaces.

Ultimate object: free Γ-module

A free Γ-module should be of the form $\mathbb{C}[X]$ where X is a free Γ-set, i.e. (left) Γ-torsor. A section of $X \to X\backslash\Gamma$ then gives a basis for $\mathbb{C}[X]$.

Now you have a picture of $\mathbb{C}[\Gamma] \otimes M$ where M is Γ-graded. You ~~use the~~ declare $s \otimes M_t$ has degree st.

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the idea of 3 hours ago concerned free Γ-graded Γ-modules. What does this mean?

~~Maybe you ask first about Γ:~~ want to split

Start with the ungraded case. You ~~have~~ the free Γ-module $\mathbb{C}[\Gamma] \otimes V$

Start again C_Φ : $h_s, s\in\Gamma$ $h_s h_t = 0$ $s^{-1}t \notin \Phi$

$\sum_s h_s h_t = h_t = \sum_s h_t h_s$

C_Φ is a $B = C_\Phi \rtimes \Gamma$ module Θ

Structure. Let M be a Γ-graded $A = P_\Phi$ module, e.g. $M = A$. Thus $M = \oplus M_s$ $A = \oplus A_t$ where $A_t M_s \subset M_{ts}$. Recall that has gens. p_s for $s \in \Gamma$ subj to rels $p_s = 0$ if $s \notin \Phi$, $p_u = \sum_{st=u} p_s p_t = \sum_s p_s p_{s^{-1}u}$.

~~Method~~ You have ~~construction~~ $E(M) = E(A) \otimes_A M$ where you get a projection on $\mathbb{C}[\Gamma] \otimes M$. Your problem is to work in, incorporate, the Γ grading on M.

One idea maybe worth exploring is to find ~~the~~ a suitable subalgebra of $\mathrm{Hom}_\Gamma(\mathbb{C}[\Gamma]\otimes M, \mathbb{C}[\Gamma]\otimes M)$ containing the canon. ~~proj~~ proj p. $\;\;\| \;\; \mathrm{Hom}(M, \mathbb{C}[\Gamma]\otimes M) \supset \mathbb{C}[\Gamma] \otimes \mathrm{Hom}(M,M)$

The map should send $s \otimes \theta \in \mathbb{C}[\Gamma] \otimes \mathrm{End}(M)$ into $R_s \otimes \theta$ on $\mathbb{C}[\Gamma]\otimes M$, where $R_s t = ts^{-1}$. This

Let $f : \Gamma \longrightarrow M$ have fin. support $\sum_{s\in\Gamma} s \otimes f(s) \in \mathbb{C}[\Gamma]\otimes M$

$$\left(\sum_t R_t \otimes \theta(t)\right)\left(\sum_s s \otimes f(s)\right) = \sum_{t,s} st^{-1} \otimes \theta(t) f(s)$$

$u = st^{-1}$

$$= \sum_u u \otimes \sum_t \theta(t) f(ut)$$

$$= \sum_u u \otimes \sum_s \theta(u^{-1}s) f(s)$$

$$= \sum_s s \otimes \sum_t \theta(s^{-1}t) f(t)$$

$$\mathbb{C}[\Gamma]\otimes M = \bigoplus_{s\in\Gamma} M = \{f:\Gamma\to M \mid \operatorname{Supp} f \text{ is fin}\}$$

$$\mathbb{C}[\Gamma]\otimes M = \bigoplus_{s\in\Gamma} s\otimes M$$

M is a Γ-graded P_Φ-module.

Idea: What is $P_\Phi \rtimes \hat{\Gamma}$?

You need to clarify the relation between graded modules and modules in the case of a graded algebra. You have understood adjoint funs. for Γ-graded vector spaces.

$$V = \bigoplus_s V_s \xrightarrow[\text{grading}]{\text{forget}} V \qquad H \triangleleft G$$

$\hat{\Gamma}$ - modules $\qquad$ $\mathbb{C}$ - modules $\qquad H \rtimes K$

$$\bigoplus_s W \longleftarrow W$$

adjointness:

$$\operatorname{Hom}_{\mathbb{C}}\left(\bigoplus_s V_s, W\right) = \prod_s \operatorname{Hom}(V_s, W)$$
$$= \operatorname{Hom}_{\hat{\Gamma}}((V_s), (W)_s)$$

So next to understand case of a $\hat{\Gamma}$ alg $A = \bigoplus A_s \qquad A_sA_t \subset A_{st}$

Let M be a $\hat{\Gamma}, A$ module: $\quad M = \bigoplus M_t \qquad A_sM_t \subset A_{st}$

Maybe first do Γ, A modules where A is a Γ algebra. Assume A unital and M unitary A-mod

If Γ acts on M compatible: $\quad s*(am) = (s*a)(s*m)$

$$A \rtimes \Gamma. \qquad \operatorname{Mod}(A) \underset{\text{forget}}{\rightleftarrows} \operatorname{Mod}(A\rtimes\Gamma)$$

$$\| $$

$$B \qquad \operatorname{Mod}(A) \underset{}{\overset{B\otimes_A -}{\rightleftarrows}} \operatorname{Mod}(B)$$

$\mathbb{C}$-modules, Γ-modules, $\hat{\Gamma}$-modules.

$$\mathrm{Hom}_{\Gamma}(\mathbb{C}[\Gamma]\otimes V, M) = \mathrm{Hom}_{\mathbb{C}}(V, M)$$

$$\mathrm{Hom}_{E}\Big(\bigoplus_s V_s, W\Big) = \mathrm{Hom}_{\hat{\Gamma}}\big((V_s)_s, (W)_s\big)$$

$$\bigoplus_{s\in\Gamma} W \simeq \mathbb{C}[\Gamma]\otimes W$$

$$\underbrace{\Big(V = \bigoplus_{s\in\Gamma} V_s\Big)}_{\text{a } \hat{\Gamma}\text{-module}} \hookrightarrow \underbrace{\mathbb{C}[\Gamma]\otimes\Big(V = \bigoplus_{s\in\Gamma} V_s\Big)}_{\text{free } \hat{\Gamma}\text{-module on } V}$$

canon map sending V_s to $s\otimes V_s \subset \mathbb{C}[\Gamma]\otimes V$

$$\mathbb{C}[\Gamma]\otimes W \xrightarrow{\eta\otimes 1} W$$
$$s\otimes w \longmapsto w$$

---

fundamental object seems to be a projection of finite support in a Γ-graded (or $\hat{\Gamma}$) alg.

Now before you can discuss $\hat{\Gamma}$-algs $=$ Γ-graded algs. you need $\otimes$ for $\hat{\Gamma}$-modules.

$$\Big(\bigoplus_{s\in\Gamma} V_s\Big)\otimes\Big(\bigoplus_{t\in\Gamma} W_t\Big) = \bigoplus_{s,t\in\Gamma\times\Gamma} V_s\otimes W_t$$

$$= \bigoplus_{u\in\Gamma}\Big(\bigoplus_{t} V_{ut^{-1}}\otimes W_t\Big)$$

Situation arising. Γ-action $+$ Γ-grading w. compatibility is very strong condition.

What you can do. Given a vector space M you can form the Γ-module $\mathbb{C}[\Gamma]\otimes M$. You want to do

~~when~~ more generally when M is Γ-graded
What's important seems to be to get a Γ-inv. projection.

Start again. Your aim? Γ-invariant projection on $\mathbb{C}[\Gamma]\otimes M = \bigoplus_{s\in\Gamma}(s\otimes M) \simeq \bigoplus_{s\in\Gamma} M$.

What does such a thing look like?

An operator

$$\begin{array}{ccc} \bigoplus_t M & \xrightarrow{T} & \prod_s M \\ \| & & \| \\ \left\{ \begin{array}{c} f:\Gamma\to M \\ \text{supp } f \text{ fin} \end{array} \right\} & & \{g:\Gamma\to M\} \end{array}$$

$$(Tf)(s) = \sum_t T(s,t)\,f(t)$$

$$(TL_u f)(s) = \sum_t T(s,t)\,f(u^{-1}t) = \sum_t T(s,ut)\,f(t)$$

$$(L_u Tf)(s) = Tf(u^{-1}s) = \sum_t T(u^{-1}s,t)\,f(t)$$

$$T(s,ut) = T(u^{-1}s,t) \iff \underline{T(us,ut) = T(s,t)}$$

$$\Rightarrow T(1,ut) = T(u^{-1},t) \qquad \qquad \| \;\; T(1,s^{-1}t)$$

$$\Rightarrow T(1,s^{-1}t) = T(s,t).$$

$$(Tf)(s) = \sum_t T(s^{-1}t)\,f(t).$$ and if you want T to map fin. supp into finite support you need $T(\;)$ to have finite support Φ

NO $\quad T(\;)m$ ______ $\forall m$.

Prop: Description of operators $T : \mathbb{C}[\Gamma] \otimes M \circlearrowleft$ commuting with left translations: $t(s \otimes m) = ts \otimes m$ in terms of left kernels $(Tf)(s) = \sum_t \theta(s^{-1}t) f(t)$ where $\theta : \Gamma \longrightarrow \mathcal{L}(M)$ has finite support.

Composition
$$(T_1 T_2 f)(s) = \sum_u \theta_1(s^{-1}u)(T_2 f)(u)$$
$$= \sum_u \theta_1(s^{-1}u) \sum_t \theta_2(u^{-1}t) f(t)$$
$$= \sum_t \left(\sum_u \theta_1(s^{-1}u)\theta_2(u^{-1}t) \right) f(t).$$

So a proj left-inv. on $\mathbb{C}[\Gamma] \otimes M$ is a $p : \Gamma \longrightarrow \mathcal{L}(M)$ finite support such that
$$\sum_u p(s^{-1}u) p(u^{-1}t) = p(s^{-1}t)$$
or
$$\sum_{\{(x,y) \mid xy = z\}} p(x) p(y) = p(z)$$

---

digress to ask what things look like when you keep the left Γ-action but ~~use right~~ make your left inv. operators act on the right.

$$(fT)(s) = \sum_t f(t)\theta(s^{-1}t)$$

$$(f T_1 T_2)(s) = \sum_u (fT_1)(u)\, \theta_2(s^{-1}u)$$
$$= \sum_u \sum_t f(t)\theta_1(u^{-1}t)\, \theta_2(s^{-1}u)$$

If you combine this with inversion of the variable in $\theta(?)$, then you get

$$(fT_1)(s) = \sum_t f(t)\,\theta_1(t^{-1}s)$$

$$(fT_1T_2)(s) = \sum_u (fT_1)(u)\,\theta_2(u^{-1}s)$$

$$= \sum_t f(t)\,\theta_1(t^{-1}u)\,\theta_2(u^{-1}s)$$

so things look much nicer. Can you check this independently? ~~And the answer is clear.~~

$$\mathrm{Hom}_\Gamma(\mathbb{C}[\Gamma]\otimes M, \mathbb{C}[\Gamma]\otimes M)$$

$$= \mathrm{Hom}(M, \mathbb{C}[\Gamma]\otimes M) \supset \mathbb{C}[\Gamma]\otimes \mathcal{L}(M)$$

$$(t\otimes\theta)(s\otimes m) = st^{-1}\otimes \theta m$$

$$(t\otimes\theta)\cdot = R_t\otimes\theta$$

$$\overbrace{\sum_t t\otimes\theta(t)}^{T}\ \overbrace{\sum_s s\otimes f(s)}^{f} = \sum_{t,s} st^{-1}\otimes\theta(t)f(s)$$

$$= \sum_u u \otimes \sum_{u=st^{-1}}' \theta(t)f(s)$$

$$(Tf)(u) = \sum_{\substack{u=st^{-1}\\ \Downarrow \\ t^{-1}=s^{-1}u \Rightarrow t=u^{-1}s}} \theta(t)f(s) = \sum_s \theta(u^{-1}s)\,f(s)$$

$$(Tf)(s) = \sum_t \theta(s^{-1}t)\,f(t)$$

Still not really clear. Review what is clear. $\mathrm{End}_\Gamma(\mathbb{C}[\Gamma]\otimes M) \simeq \{\theta : \Gamma \to \mathrm{End}(M) \mid \theta \text{ fin supp}\}$

How $(\theta f)(s) = \sum_t \theta(s^{-1}t) f(t)$. You've made the same mistake as before, namely, there is a difference between having $s \mapsto \theta(s)$ of finite support and having $\forall m,\ s \mapsto \theta(s)m$ of fin. supp.

Now how to make real progress? Think! It's possible some notational simplification might arise ~~occur~~ from using $A \rtimes \Gamma$ with Γ on the right.

Leave this for now.

Return to M which has this finite support $p : \Gamma \longrightarrow \mathcal{L}(M)$ satisf

$$p(s) = \sum_t p(t)\,p(t^{-1}s) = \sum_t p(st^{-1})\,p(t)$$

~~What else do you need? Next~~ Suppose M is Γ-graded: $M = \bigoplus_{s\in\Gamma} M_s$ such that left mult by $p(t)$ has degree t. i.e. $p(t)M_s \subset M_{ts}$.

Begin again. Describe the situation. You have a Morita equivalence between $B = C_\Phi \rtimes \Gamma$ (whose fin modules are Γ-modules with a suitable equivariant partition of unity $1 = \sum_\Gamma h_s$) and $A = P_\Phi$ (which describes projections in a Γ-graded (alt: $\hat{\Gamma}$) algebra.

P_Φ represents projections in a Γ-graded alg with Supp $\subset \Phi$.

$B = \bigoplus_{s\in\Gamma} B_s$ is a Γ-graded alg: means ~~such that~~ $B_s B_t = B_{st}$

~~and p~~ projection in B ~~means $p = \sum p_s$~~
with Supp $\subset \Phi$ means $p = \sum_s p_s \in B = \bigoplus_s B_s$
satisfies $p_s \neq 0 \Rightarrow s \in \Phi$, $p^2 = p$.

$\exists$ Γ-graded algebra P_Φ $\ni$

$$\mathrm{Hom}_{\Gamma\text{-gr algs}}(P_\Phi, B) = \left\{ p \in B \;\middle|\; \begin{matrix} p^2 = p \\ p_s \neq 0 \Rightarrow s \in \Phi \end{matrix} \right\}$$

Proof: Define $\underbrace{P_\Phi}_{=\text{ noncomutal alg}}$ gens p_s $s \in \Gamma$ rels $\begin{cases} p_s = \sum_t p_t p_{t^{-1}s} \\ p_s = 0 \quad s \notin \Phi \end{cases}$

Claim P_Φ has ! Γ-grading $\ni$ p_s degree s.

$$T(V) = V \oplus V\otimes V \oplus V \otimes V \otimes V \oplus \cdots$$

obvious Γ-graded algebra when V is Γ-graded

J ideal generated by relations which are homogeneous.

Trickier method. Form alg $\mathbb{C}[\Gamma] \otimes P_\Phi$ with Γ grading $s \otimes P_\Phi$ degree s. Thus P_Φ here has constant degree 1. ~~$s \otimes p_s = \sum_t t(t^{-1}s) \otimes p_t p_{t^{-1}s}$~~

Family $\tilde{p}_s = s \otimes p_s$. $\tilde{p}_s = 0$ $s \notin \Phi$

$$\sum_t \tilde{p}_t \tilde{p}_{t^{-1}s} = \sum_t (t \otimes p_t)(t^{-1}s \otimes p_{t^{-1}s}) = \sum_t (s \otimes p_t p_{t^{-1}s}) = s \otimes p_s = \tilde{p}_s$$

$\therefore \exists !$ alg map

$$P_\Phi \longrightarrow \mathbb{C}[\Gamma] \otimes P_\Phi$$

$$p_s \longmapsto s \otimes p_s$$