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May 24, 2001 (Alice 39 yesterday)

Topic: Γ-graded vector spaces and algebras

Background: You have already encountered these when Γ is a group (more generally a monoid, a semi group even). A Γ-graded algebra is ~~an~~ algebra B equipped with a splitting into subspaces $B = \bigoplus_s B_s$ indexed by Γ such that $B_s B_t \subset B_{st}$. Example: If A is a Γ algebra, the crossproduct $B = \Gamma \ltimes A$ $= \bigoplus_s sA = \bigoplus_s As$ is a Γ-graded algebra.

Example: The algebra $A = P_{\Gamma,\Phi}$ is defined by generators $p(s)$ for $s \in \Gamma$ subject to the relations $p(s) = 0$ for $s \notin \Phi$, $p(s) = \sum_t p(t)p(t^{-1}s)$. These are homogeneous (both generators + relations) if $p(s)$ is assigned the degree s, so A should be a Γ-graded algebra, with the Γ-grading specified in this way. One way to show this would be to define the Γ grading on the free algebra with the generators $p(s)$ - this uses the tensor product operation on Γ-graded vector spaces defined by

$$(V \otimes W)_s = \bigoplus_t V_t \otimes W_{t^{-1}s}.$$

Then you check that any element in the ideal generated by the relations is a sum of homogeneous elements lying in this ideal. Hence the ideal is a Γ-graded subspace of the free algebra, and b the quotient A inherits a Γ-grading making it a Γ-graded alg.

Here is a cleaner way to show that A is a Γ-graded algebra. The tensor product algebra $\mathbb{C}\Gamma \otimes A$ is a Γ-graded algebra with $(\mathbb{C}\Gamma \otimes A)_s = s \otimes A$. Using the generators + relations defining A one has a unique alg homomorphism

$$\Delta : A \longrightarrow \mathbb{C}\Gamma \otimes A \qquad \Delta p(s) = s \otimes p(s)$$

Let A_s be the subspace of A spanned by monomials $p(s_1)\, p(s_2) \cdots p(s_n)$ in the generators of total degree $s_1 s_2 \cdots s_n$ equal to s. Clearly $\Delta(a_s) = s \otimes a_s$ for any $a_s \in A_s$.

Also $A = \sum_{s \in \Gamma} A_s$. This must be a direct sum since if we have $\sum a_s = 0$ with $a_s \in A_s$, then applying Δ yields $\sum_s s \otimes a_s = 0$ in $\mathbb{C}\Gamma \otimes A$, which is possible only if $a_s = 0$ for all s. Thus we have a Γ-grading on A making A a Γ-graded algebra.

Problem: A Morita context is a ring equipped with a grading $A = \bigoplus_{i,j=1,2} A_{ij}$ such that if we write the four components of an element a of A as a 2×2 matrix:

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \simeq A$$

the multiplication in A is given by matrix multiplication that is

$$\begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{pmatrix}\begin{pmatrix} a''_{11} & a''_{12} \\ a''_{21} & a''_{22} \end{pmatrix} = \begin{pmatrix} a'_{11}a''_{11} + a'_{12}a''_{21} & a'_{11}a''_{12} + a'_{12}a''_{22} \\ a'_{21}a''_{11} + a'_{22}a''_{21} & a'_{21}a''_{12} + a'_{22}a''_{22} \end{pmatrix}$$

Here we have a grading indexed by the set Γ of ordered pairs ij with $i, j = 1$ or 2. The product $A_{ij}A_{kl}$ is zero when $j \neq k$ and contained in A_{il} when $j = k$. Thus Γ is the set of arrows in the groupoid having objects in $\{1, 2\}$ and with exactly one map from one object to another. The analog of $\mathbb{C}\Gamma$ in this situation is the path algebra of the groupoid, which is $M_2(\mathbb{C})$.

Our aim now is to find a good generalization of the two types of $\mathbb{C}\Gamma$ discussed above.

The coalgebra $\mathbb{C}\Gamma$. If Γ is any set, then $\mathbb{C}\Gamma$ equipped with the coproduct

$$\Delta : \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \qquad \Delta s = s \otimes s$$

is a coassociative, cocommutative, and counital coalgebra. The counit is $\eta : \mathbb{C}\Gamma \longrightarrow \mathbb{C}$, $\eta s = 1$.

Ex: If Γ is a point, then $\mathbb{C}\Gamma = \mathbb{C}$ with $\Delta(1) = 1 \otimes 1 \in \mathbb{C} \otimes \mathbb{C}$.

By a point of a coalgebra C we mean a coalgebra morphism $\phi : \mathbb{C} \longrightarrow C$, equivalently an element $\underbrace{\xi}_{\phi(1)=} \in C$ satisfying $\Delta \xi = \xi \otimes \xi$. If C has counit $\eta : C \to \mathbb{C}$, then the point ϕ will be called unital when ϕ respects counits: $\eta\phi = \mathrm{id}$, equivalently $\eta(\xi) = 1$. Note that the identity and zero are the only coalg morphisms from $\mathbb{C}$ to $\mathbb{C}$. Thus a point ϕ is not unital iff it is zero.

Calculation of the points of $\mathbb{C}\Gamma$. Let $\xi = \sum_{s \in \Gamma} \lambda_s s$ in $\mathbb{C}\Gamma$ satisfy $\Delta(\xi) = \xi \otimes \xi$, that is

$$\sum_s \lambda_s (s \otimes s) = \sum_{s,t} \lambda_s \lambda_t (s \otimes t).$$

Then $\lambda_s \lambda_t = 0$ for $s \neq t$ and $\lambda_s^2 = \lambda_s$. Thus either all $\lambda_s = 0$ and we have the zero point, or there is exactly one $\lambda_s = 1$ and the rest are zero. Therefore

$$\boxed{\begin{array}{lcl} \text{Points}(\mathbb{C}\Gamma) & = & \Gamma \cup \{0\} \\ \text{Unital Points}(\mathbb{C}\Gamma) & = & \Gamma \end{array}}$$

Clearly one has functors $C \longmapsto \text{Points}(C)$ from coalgebras to sets with basepoint, and $C \longmapsto \text{Unital Points}(C)$ from counital coalgebras to sets. In the unital case one can recover Γ from $\mathbb{C}\Gamma$ as $\Gamma = \text{Unital Points}(\mathbb{C}\Gamma)$. Moreover one has an equivalence between the categories of sets and the category of counital coalgebras which are set-like, that is, spanned by their points.

In the nonunital case one gets an equivalence of categories between sets with basepoint and set-like coalgebras as follows. Note first that there is a 1-1 correspondence (essentially) between sets and sets with basepoint given by $\Gamma \longmapsto \Gamma_+ = \Gamma \cup \{0\}$. However there are more morphisms in the category of sets with base point. A map $\Gamma_+ \xrightarrow{f} \Gamma'_+$

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amounts to a partially defined map from Γ to Γ', that is, a map from a subset of Γ, the domain $D(f)$, to Γ'.

Next observe that the coalgebra $\mathbb{C}\Gamma$ can be expressed

$$\mathbb{C}\Gamma = \mathbb{C}\Gamma_+/\mathbb{C}\{0\}$$

showing that $\Gamma_+ \longmapsto \mathbb{C}\Gamma$ is a well-defined functor from sets with basepoint to set-like coalgebras. This functor has the quasi-inverse $C \longmapsto \mathrm{Points}(C)$, yielding the desired equivalence of categories.

Since $\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma' = \mathbb{C}[\Gamma\times\Gamma']$, one has

$$\mathrm{Points}(\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma') = (\Gamma\times\Gamma')_+ = \Gamma_+ \wedge \Gamma'_+ .$$

In other words the tensor product operation on set-like coalgebras corresponds to the smash product on sets with basepoint.

Perhaps you should be more careful, namely, check that $\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma' \xrightarrow{\sim} \mathbb{C}[\Gamma\times\Gamma']$, $s\otimes t \longmapsto (s,t)$ is an isomorphism of coalgebras. The tensor product coalgebra has coproduct

$$(*) \qquad \begin{array}{ccc} \mathbb{C}\Gamma\otimes\mathbb{C}\Gamma' & \xrightarrow{\ \Delta\ } & \mathbb{C}\Gamma\otimes\mathbb{C}\Gamma'\otimes\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma' \\ s\otimes t & & s\otimes t\otimes s\otimes t \end{array}$$

which corresponds dually to $(a\otimes b)(a'\otimes b') = aa'\otimes bb'$ for algebra tensor product. $(*)$ under $\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma' = \mathbb{C}[\Gamma\times\Gamma']$ becomes

$$\begin{array}{ccc} \mathbb{C}[\Gamma\times\Gamma'] & \xrightarrow{\ \Delta\ } & \mathbb{C}[\Gamma\times\Gamma']\otimes\mathbb{C}[\Gamma\times\Gamma'] \\ (s,t) & \longmapsto & (s,t)\otimes(s,t) \end{array}$$

so it's clear.

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May 28, 2001

Continue to identify sets and pointed sets via $\Gamma \longmapsto \Gamma_+ = \Gamma \amalg \{0\}$. Recall the equivalence of categories between pointed sets and set-like coalgebras given by $\Gamma_+ \longmapsto \mathbb{C}\Gamma$, $C \longmapsto$ points of C.

TFAE: (1) a product $\mu : \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma$ which respects the coalgebra structures.

(2) A binary operation $\Gamma_+ \times \Gamma_+ \longrightarrow \Gamma_+$ such that 0 is absorbing: $0x = 0 = x0$.

(3) A pointed set map $\Gamma_+ \wedge \Gamma_+ \longrightarrow \Gamma_+$

Proof. Γ_+ is the subset of points in $\mathbb{C}\Gamma$. Because μ is a coalgebra map it restricts to a binary operation $(\xi,\eta) \mapsto \mu(\xi\otimes\eta)$ on Γ_+. In effect given points ξ, η in C, then

$$\begin{tikzcd}
\xi\otimes\eta \quad \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \arrow[r, "\mu"] \arrow[d, "\Delta_{\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma}"'] & \mathbb{C}\Gamma \quad \mu(\xi\otimes\eta) \arrow[d, "\Delta_{\mathbb{C}\Gamma}"] \\
(\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma)\otimes(\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma) \arrow[r, "\mu\otimes\mu"] & \mathbb{C}\Gamma\otimes\mathbb{C}\Gamma
\end{tikzcd}$$

$$\xi\otimes\eta\otimes\xi\otimes\eta \qquad\qquad \mu(\xi\otimes\eta)\otimes\mu(\xi\otimes\eta)$$

Thus 1) yields 2). Next the absorbing property of 0 means that the μ operation on Γ_+ descends to the smash product $\Gamma_+\times\Gamma_+/\Gamma_+\vee\Gamma_+ = \Gamma_+\wedge\Gamma_+$, so 2) yields 3). Finally $\Gamma_+\wedge\Gamma_+ = (\Gamma\times\Gamma)_+ =$ Points of $\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma = \mathbb{C}[\Gamma\times\Gamma]$, so that a pointed set map $\Gamma_+\wedge\Gamma_+ \longrightarrow \Gamma_+$ is equivalent to a coalg map $\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma \to \mathbb{C}\Gamma$.

g Next associativity

TFAE: (1) $\mu : \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma$ is associative

(2) the induced product $\Gamma_+ \times \Gamma_+ \longrightarrow \Gamma_+$ is associative

(3) the map $\bar{\mu} : \Gamma_+ \wedge \Gamma_+ \to \Gamma_+$ on points satisfies

$$\bar{\mu}(\bar{\mu} \wedge 1) = \bar{\mu}(1 \wedge \bar{\mu}) \quad \text{from } \Gamma_+ \wedge \Gamma_+ \wedge \Gamma_+ \longrightarrow \Gamma_+$$

Proof. (1) $\Rightarrow$ (2) because you are restricting the product in $\mathbb{C}\Gamma$ to the subset Γ_+. (2) $\Rightarrow$ (3) because the product on $\Gamma_+ \times \Gamma_+$ descends to $\Gamma_+ \wedge \Gamma_+$. In other words the two maps $\Gamma_+ \times \Gamma_+ \times \Gamma_+ \rightrightarrows \Gamma_+$ giving associative descend to $\bar{\mu}(\bar{\mu} \wedge 1)$ and $\bar{\mu}(1 \wedge \bar{\mu})$. Finally (3) $\Rightarrow$ (1) by the equivalence between coalgebras $\mathbb{C}\Gamma$ and ptd sets Γ_+.

At this point one has described bialgebras with set-like coalgebra structure in terms of semigroups $\Gamma_+ = \Gamma \sqcup \{0\}$ with absorbing basepoint 0. Note that any subset of a ring closed under product and containing zero yields such a Γ_+, and that $\mathbb{C}\Gamma$ is the largest ring generated by Γ_+. $\mathbb{C}\Gamma$ is an obvious generalization of the group ring of a group.

Next discuss Γ-graded vector spaces and algebras.

Prop. Equivalence between a comodule structure on V for the coalgebra $\mathbb{C}\Gamma$, where Γ is a set, and a grading of $V = \bigoplus_{s \in \Gamma} V_s \oplus V_0$ with respect to Γ_+. The comodule V is counital $\Leftrightarrow V_0 = 0$, so that V is graded wrt Γ.

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<u>Proof</u>: Given a coproduct $\Delta_V : V \to \mathbb{C}\Gamma \otimes V$ which is coassociative:

$$V \xrightarrow{\Delta_V} \mathbb{C}\Gamma\otimes V \underset{1\otimes\Delta_V}{\overset{\Delta_{\mathbb{C}\Gamma}\otimes V}{\rightrightarrows}} \mathbb{C}\Gamma\otimes\mathbb{C}\Gamma\otimes V$$

Δ_V has the form $\Delta_V v = \sum_{s\in\Gamma} s\otimes e_s(v)$ where the $e_s \in \mathrm{End}(V)$ satisfy the finiteness condition $\forall v$ $e_s(v)=0$ for a$\forall s$. Then equality of

$$(\Delta_{\mathbb{C}\Gamma}\otimes 1)\Delta_V v = \sum_s s\otimes s\otimes e_s(v)$$
$$(1\otimes\Delta_V)\Delta_V v = \sum_{s,t} s\otimes t\otimes e_te_s(v)$$

for all v is equivalent to $e_te_s=0$ for $s\neq t$ and $e_s^2=e_s$. The e_s are annihilating projections on V such that $\sum' e_s$ is defined by the finiteness condition and it a projection. Then we have the ~~splitting~~

$$V=\bigoplus_{s\in\Gamma} e_sV \oplus \left(1-\sum e_s\right)V$$

which yields the Γ_+-grading with $V_s=e_sV$, $V_0=(1-\sum e_s)V$. Also $\sum e_s=(\eta\otimes 1)\Delta_V$, so that Δ_V is a counital coproduct $\iff \sum e_s = 1$.

Next let $\mathbb{C}\Gamma$ be the bialgebra arising from a semi-group Γ_+ with absorbing basepoint 0. Define a <u>Γ-graded algebra</u> A to be an algebra equipped with a Γ-grading

$$A=\bigoplus_{s\in\Gamma}A_s \quad \text{s.t.} \quad A_sA_t\begin{cases}\subset A_{st} & \text{if } st\in\Gamma\\ = 0 & \text{if } st=0.\end{cases}$$

The Γ-grading is equivalent to a counital comodule structure on A for the coalgebra $\mathbb{C}\Gamma$, i.e. a coproduct

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$$\Delta : A \longrightarrow \mathbb{C}\Gamma \otimes A \quad , \quad \Delta a = \sum_{s \in \Gamma} s \otimes e_s a \ , \ \sum_{s \in \Gamma} e_s = 1$$

In other words $\Delta(a_s) = s \otimes a_s$ for $a_s \in A_s$.

The compatibility condition between grading + product can be expressed as saying that Δ is an algebra homomorphism. In effect, $\Delta(a_s a_t) = (s \otimes a_s)(t \otimes a_t) = st \otimes a_s a_t$ which implies that $a_s a_t \in A_{st}$ for $st \neq 0$, and $a_s a_t = 0$ if $st = 0$ (since Δ is injective because of the counit η).

June 10, 2001

Review the multiplier algebra Mult(A) for an algebra A. A <u>multiplier</u> on A is defined to be a pair of operators on A

$$\mu = (a \mapsto \mu a, \quad a \mapsto a\mu)$$

satisfying

$$\mu(a_1 a_2) = (\mu a_1) a_2$$
$$a_1(\mu a_2) = (a_1 \mu) a_2$$
$$(a_1 a_2)\mu = a_1(a_2 \mu)$$

The product $\mu\nu$ of two multipliers is defined by

$$(\mu\nu)a = \mu(\nu a) \qquad a(\mu\nu) = (a\mu)\nu$$

and it makes Mult(A) into a sub-algebra:

$$\mathrm{Mult}(A) = \Big\{ \mu \in \underbrace{\mathrm{Hom}_{A^{op}}(A,A)}_{\text{left multipliers}} \times \underbrace{\mathrm{Hom}_A(A,A)^{op}}_{\text{right multipliers}} \;\Big|\; (a_1\mu)a_2 = a_1(\mu a_2) \Big\}$$

More generally if $(X, Y, \langle y, x\rangle)$ is a <u>dual pair</u> over A one can define its multiplier algebra to be

$$\mathrm{Mult}(X, Y, \langle y,x\rangle) = \Big\{ \underbrace{}_{(x \mapsto \mu x,\; y \mapsto y\mu)} \in \mathrm{Hom}_{A^{op}}(X,X) \times \mathrm{Hom}_A(Y,Y)^{op} \;\Big|\; \langle y\mu, x\rangle = \langle y, \mu x\rangle \Big\}$$

Mult(A) is the special case with the A^{op}-module $X = A$, the A module $Y = A$, and the pairing $\langle y, x\rangle = yx$

Let A be an ideal in the algebra R. Then each $r \in R$ yields a multiplier

$$\mu_r = (a \mapsto ra, \quad a \mapsto ar)$$

whence one has an alg homomorphism $\mu : R \longrightarrow \mathrm{Mult}(A)$. Restricting to A (in other words taking $R = A$) one gets a canonical algebra map

$$A \xrightarrow{\phi} Mult(A), \qquad (\phi_a a' = aa', \; a'\phi_a = a'a)$$

with the following properties:

1) $\operatorname{Ker}\phi = \{ a \in A \mid Aa = aA = 0 \}$.

2) $\mu\phi_a = \phi_{\mu a}$ and $\phi_a \mu = \phi_{a\mu}$ $\forall a \in A, \mu \in Mult(A)$,

hence $A/\operatorname{Ker}\phi = \phi_A$ is an ideal in $Mult(A)$.

Check 2). $(\mu\phi_a)a' = \mu(\phi_a a') = \mu(aa') = (\mu a)a' = \phi_{\mu a}a'$

$a'(\mu\phi_a) = (a'\mu)\phi_a = (a'\mu)a = a'(\mu a) = a'\phi_{\mu a}$

$\therefore$ $\mu\phi_a = \phi_{\mu a}$, and similarly for the other order.

Next look at semi-direct products for algebras which are analogous to such products for groups, where to form $Q \ltimes K$ one needs a homomorphism from Q to $Aut(K)$. For algebras the analog is an alg map $R \xrightarrow{\varphi} Mult(A)$ and the product on $R \ltimes A$ is defined by $(r + a)(r' + a') = rr' + (\varphi_r a' + a\varphi_{r'} + aa')$.

There is a slight problem with associativity as follows. It's enough to consider $R = Mult(A)$. There are 8 associativities to check: $a_1 a_2 a_3$; three involving one μ: $\mu a_1 a_2$, $a_1 \mu a_2$, $a_1 a_2 \mu$ OK by defn. of multiplier; three involving one a: $\mu\nu a$, $\mu a \nu$, $a\mu\nu$, where the first + third OK by def of product of multipliers; One involving three multipliers which is OK.

So there is a problem with $\boxed{(\mu a)\nu \overset{?}{=} \mu(a\nu)}$, and

There are two ways to proceed. If $A = A^2$ is assumed then OK because

$$(\mu(a_1 a_2))\nu = ((\mu a_1) a_2)\nu = (\mu a_1)(a_2 \nu)$$
$$\mu((a_1 a_2)\nu) = \mu(a_1(a_2 \nu)) = (\mu a_1)(a_2 \nu)$$

Thus no problem with $\mathrm{Mult}(A) \ltimes A$ when $A = A^2$.

On the other hand, ~~the~~ applying ϕ takes μ, a, ν into μ, ϕ_a, ν which satisfies associativity as $\mathrm{Mult}(A)$ is a ring.

June 18, 2001.

Recall that if $e^2 = e$ in a ring B, then one has a Morita context: $\begin{pmatrix} eBe & eB \\ Be & B \end{pmatrix} < M_2 B$ which is associated to the dual pair over B given by eB, Be and the pairing $\langle b_1 e, e b_2 \rangle = b_1 e b_2$. (Note: $eB \otimes_B Be = eBe$)
This Morita context yields a Morita equivalence between the unital ring eBe and the ideal BeB which is idempotent. One has a canon. surjective ring morphism $Be \otimes_{eBe} eB \twoheadrightarrow BeB$ whose kernel is killed by B (hence by BeB) on both left and right.

We now generalize this construction [redacted] to any element h of B. Consider the dual pair over B given by the right ideal hB, the left ideal Bh, and the pairing

$$b_1 h * h b_2 = b_1 h b_2$$

which is well-defined since $b_1 h = 0$ or $h b_2 = 0$ $\Longrightarrow b_1 h b_2 = 0$. This yields the Morita context

$$\begin{pmatrix} hB \otimes_B Bh & hB \\ Bh & B \end{pmatrix}$$

where the product in the ring $hB \otimes_B Bh$ is

$$(hb_1 \otimes b_2 h) * (hb_3 \otimes b_4 h) = hb_1 \otimes b_2 h b_3 b_4 h$$

Define the $*$ product on hBh by

$$hbh * hb'h = hbhb'h$$

Then the canonical map $hb_1 \otimes b_2 h \mapsto hb_1 b_2 h$ from $hB \otimes_B Bh$ to hBh respects $*$ products, showing that $*$ product on hBh is associative.*

Similarly
$$(hb_1 \otimes b_2 h) * hb_3 = hb_1 b_2 h b_3$$
$$b_0 h * (hb_1 \otimes b_2 h) = b_0 h b_1 b_2 h,$$
the actions of $hB \otimes_B Bh$ on hB and Bh respectively descend to* actions of hBh given by $*$ product:
$$hbh * hb_3 = hbhb_3$$
$$b_0 h * hbh = b_0 hbh$$

(*These statements are not accurate unless $B = B^2$, which is the case when $BhB = B$. Thus it would have been better to proceed as follows.)

Consider the M_2-graded abelian group
$$\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$$
and define the $*$ product on it, using the formulas which hold when h^2. More precisely, there are 8 products associated to this Morita context, 4 of which lead to expressions containing h^2; these give the $*$ products
$$hbh * hb'h = hbhb'h \qquad hbh * hb' = hbhb'$$
$$bh * hb'h = bhb'h \qquad bh * hb' = bhb'$$

Here's a way to understand better the Morita context $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$. This context is essentially determined by the dual pair over B given by the B^{op}-module $X^* = hB$, the B module $X = Bh$, and the pairing $b'h * hb = b'hb$.

So $(hB, Bh, \langle b'h, hb\rangle = b'hb)$ is a quotient of the dual pair $(B, B, \langle b', b\rangle = b'hb)$. You have eliminated from the latter Morita context the obvious ~~[illegible]~~ degeneracies arising from the annihilators ${}_hB$ and B_h.

Relation of $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$ to $\begin{pmatrix} jBi & jB \\ Bi & B \end{pmatrix}$

where the latter Mor. context is supposed to correspond to quadruples $(V, W, \overbrace{W \overset{j}{\twoheadrightarrow} V \overset{i}{\hookrightarrow} W}^{h=ij})$. Now $Bi = B/\{b \mid bi = 0\}$, but $bi = 0 \iff bij = 0$ when j is surjective; so $Bi = B/B_h$. Similarly $jB = B/\{b \mid jb = 0\}$ and $jb = 0 \iff ijb = 0$ when i is injective; so $jB = B/{}_hB$.

Conclude that the dual pairs

$$\left\{\begin{array}{l} (B, B, \langle b', b\rangle = b'hb) \\ (B/{}_hB, B/B_h, \text{same}) \\ (jB, Bi, \langle b'i, jb\rangle = b'hb) \end{array}\right.$$

are essentially equivalent.

Let's consider the Morita context (here $h \in B$)

$$\begin{pmatrix} A & Y \\ X & B \end{pmatrix} = \begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$$

the product is the $*$ product, i.e. as if $h^2 = h$.

Then this context is strictly idempotent assuming $BhB = B$.

$B = BhB \subset BB$, $XY = Bh * hB = BhB = B$

$YX = hBBh = hBh = A$, $A^2 = hBhBh = hBh = A$

$YB = hB^2 = hB = Y$, $AY = YXY = YB = Y$

$BX = B^2h = Bh = X$, $XA = XYX = BX = X$.

Let's describe the Morita equivalence associated to this Morita context. We use the reduced module picture, i.e. $\mathcal{M}_r(A)$ is the category of A-modules V such that $AV = V$ and ${}_AV = 0$. You know that the functor of the equivalence from $\mathcal{M}_r(B)$ to $\mathcal{M}_r(A)$ is given by

$$W \longmapsto \mathrm{Im}\{Y \otimes_B W \xrightarrow{\alpha} \mathrm{Hom}_B(X, W)\}$$

$$y \otimes w \longmapsto (x \mapsto (xy)w).$$

(This should be true quite generally, certainly for a strictly idempotent Morita context)

The map α factors

$$hB \otimes_B W \twoheadrightarrow hW \hookrightarrow \mathrm{Hom}_B(Bh, W)$$

$$hb \otimes w \longmapsto hbw, \quad hw' \longmapsto (b'h \mapsto \underbrace{b'h * hw'}_{b'hw'})$$

The second map is injective, since $b'hw' = 0$ for all $b' \in B$ implies $hw' \in {}_BW$ which is zero assuming W reduced. The first map is surjective as $hBW = hW$ since $BW = W$ for W reduced.

Thus the functor giving the equivalence $M_r(B) \longrightarrow M_r(A)$ is $W \longmapsto hW$. As a check note that $A(hW) = hBhW = hBhBW = hBW = hW$; also $A(hw) = 0 \iff hBhw = 0 \Rightarrow Bhw = BhBhw = 0 \Rightarrow hw \in$ [redacted]${}_B W = 0$.

Next look at the inverse functor from $M_r(A)$ to $M_r(B)$ which sends V to $W = \mathrm{Im}\{Bh\otimes_A V \longrightarrow \mathrm{Hom}_A(hB, V)\}$. Thus

$$\underbrace{Bh\otimes_A V}_{\substack{B \text{ conil free} \\ \text{as } B = B^2}} \twoheadrightarrow W \hookrightarrow \underbrace{\mathrm{Hom}_A(hB, V)}_{B \text{ nil free as}}$$

$$\begin{array}{ccc} \mathrm{Hom}_B(B, \mathrm{Hom}_A(hB, V)) = & \mathrm{Hom}_A(hB\otimes_B B, V) \\ & \uparrow\wr \\ & \mathrm{Hom}_A(hB, V) \end{array}$$

since $hB\otimes_B B \longrightarrow hB$ is surjective with A-nil kernel and ${}_A V = 0$
[redacted] Thus W is A-reduced.

Now assume B satisfies ${}_B B = 0$ and $B_B = 0$. Then hB is A [redacted]-reduced, and Bh is A^{op} reduced. Also since [redacted] $B(Bh) = Bh$ and ${}_B Bh \subset {}_B B = 0$, one sees that $A = hBh$ is A-reduced, and similarly A is A^{op}-reduced.

Motivation: At the conference nearly one month ago Joachim told me that he could extend the Morita equivalence, which arises in the assembly map for a group Γ (+ finite support condition), to the case of the groupoid M_n ($Ob = \{1, \dots, n\}$, $Ar = Ob \times Ob$). On one side one has the algebra A universally generated by the the components of a projection in an M_n graded algebras. On the other side one has the algebra B, which is a sort of crossproduct with the noncommutative n-simplex.

In order to reconstruct Joachim's result it seems worthwhile to look more generally at assembly for a groupoid Γ. Geometrically assembly for a group involves constructing a K class starting from a principal bundle for the group. So you want to understand principal bundles (or torsors) for a groupoid.

Let's approach this problem from Groth's topos viewpoint, which gives an elegant category picture of classifying topoi for groupoids (without homotopies, partitions of unity, etc.)

Let C be a small category, let C-sets be the category $\mathrm{Fun}(C, \mathrm{sets})$ of covariant functors L and C^{op}-sets the category of contravariant functors R. Because we use left functional notation: $(fg)(x) = f(g(x))$, it is convenient to write a chain of composable arrows in $\mathcal{C}$ with the arrows pointed to the left, so that the composition of

$$Z \xleftarrow{f} Y \xleftarrow{g} X \quad \text{is} \quad Z \xleftarrow{fg} X.$$

We write Ob for the set of objects, and Ar for the set of arrows:

$$Ar = \coprod_{X,Y\in\mathcal{O}} Ar(X,Y) \qquad Ar(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$$

composition: $$Ar(X,Y)\times Ar(Y,Z) \longrightarrow Ar(X,Z)$$
$$(X\xleftarrow{f}Y),\ (Y\xleftarrow{g}Z) \longmapsto (X\xrightarrow{fg}Z).$$

A $\mathcal{C}$-set L is a set over Ob: $$L = \coprod_{X\in Ob} L(X)$$ together with a left action by Ar: $$\underset{\xrightarrow{\text{source}}}{Ar} \times_{Ob} L \longrightarrow L,$$ i.e.

$$\coprod_{(Y,X)\in\mathcal{O}} Ar(Y,X)\times L(X) \longrightarrow \coprod_{Y\in Ob} L(Y),$$ satisfying appropriate identity and associativity conditions. Similar a $\mathcal{C}^{op}$-set R is a set over Ob with right action by Ar: $$R\times_{Ob} Ar \longrightarrow R$$

With this notation understood we can define the "tensor product" $R\times_{\mathcal{C}} L$ to be the set

$$R\times_{\mathcal{C}} L = \mathrm{Coker}\left\{R\times_{Ob}L \leftleftarrows R\times_{Ob} Ar\times_{Ob} L\right\}$$
$$(f^*\rho,\lambda) \longleftarrow (\rho, f, \lambda)$$
$$(\rho, f_*\lambda) \longleftarrow (\rho, f, \lambda)$$

In other words a set map $R \times_{\mathcal{C}} L \to S$ is the same as a family of maps

$$\phi_X : R(X) \times L(X) \longrightarrow S \qquad \forall X \in Ob$$

such that

$$\begin{tikzcd} R(Y) \times Ar(Y,X) \times L(X) \arrow[r] \arrow[d] & R(X) \times L(X) \arrow[d, "\phi_X"] \\ R(Y) \times L(Y) \arrow[r, "\phi_Y"] & S \end{tikzcd}$$

commutes $\forall X, Y \in Ob$.

It is easy to establish the following bilinearity property

$$\mathrm{Hom}_{sets}(R \times_{\mathcal{C}} L, S) = \mathrm{Hom}_{\mathcal{C}^{op}\text{sets}}(R, \mathrm{Hom}_{sets}(L,S))$$
$$= \mathrm{Hom}_{\mathcal{C}\text{-sets}}(L, \mathrm{Hom}_{sets}(R,S))$$

from which it follows that $R \times_{\mathcal{C}} L$ respects arb. ind limits. If you take $R = h_X = \mathrm{Hom}_{\mathcal{C}}(-,X)$, then

$$\mathrm{Hom}_{sets}(h_X \times_{\mathcal{C}} L, S) = \mathrm{Hom}_{sets}(L(X), S)$$

by Yoneda's lemma, whence

$$\boxed{h_X \times_{\mathcal{C}} L = L(X)}, \text{ sim } \boxed{R \times_{\mathcal{C}} h^X = R(X)}$$

Therefore $\boxed{h_X \times_{\mathcal{C}} h^Y = \mathrm{Hom}_{\mathcal{C}}(Y,X) = Ar(X,Y)}$, and then using $\varinjlim_{X/R} h_X = R$ etc, yields the general $R \times_{\mathcal{C}} L$ by right continuity.

The category $\mathcal{C}$-sets is a topos. In the Grothendieck theory it is natural to define a $\mathcal{C}$-torsor over a space B to be a topos map from Sh_B, sheaves ~~of~~ sets over B, to $\mathcal{C}$-sets. Such a map is given by the inverse image functor $f^*: \mathcal{C}\text{-sets} \longrightarrow Sh_B$ which is required to be rtcont and left exact (respects finite proj. lim's).

Consider $B = pt$. A right continuous functor $F: \mathcal{C}\text{-sets} \longrightarrow \text{sets}$ has the form (up to a canonical isomorphism) $F(L) = R \times_{\mathcal{C}} L$, where R is the $\mathcal{C}^{op}$-set: $\mathcal{C}^{op} \underset{\text{Yoneda}}{\hookrightarrow} \mathcal{C}\text{-sets} \xrightarrow{F} \text{sets}$,

i.e. $R(X) = F(h^X)$. When is F left exact? I think this happens iff $\mathcal{C}/R$ is filtering, equivalently R is prorepresentable. Assuming this, it follows that $\text{Pro}\,\mathcal{C}$ is the category of points in $\mathcal{C}$-sets.

For a space B a rtcont $F: \mathcal{C}\text{-sets} \longrightarrow Sh_B$ should have the form $F(L) = R \times_{\mathcal{C}} L$, where R is a $\mathcal{C}$-sheaf over B, i.e. the functor $\mathcal{C}^{op} \underset{\text{Yoneda}}{\hookrightarrow} \mathcal{C}\text{-sets} \xrightarrow{F} Sh_B$. The left exactness of F should be equivalent to each stalk of R being pro-representable.

Next simplify to a groupoid Γ where ~~pro~~ representable functors are representable. A Γ torsor over B is a Γ^{op}-sheaf R such that each stalk is representable.

Example: $\Gamma = M_2$. A Γ^{op}-set is the same thing as an ordered pair of sets together with an isomorphism between them. It is representable iff both sets are points. Clearly there is a unique torsor up to canonical ism.