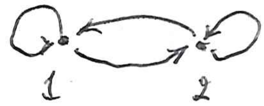
What happens in M_2: 

~~[illegible]~~ Define T by gens p_{ij} and no other relations initially.

$$\Delta : T \longrightarrow M_2 \otimes T$$

$\Delta(p_{ij}) = \tilde{p}_{ij}$ where $\tilde{p}_{ij} = e_{ij} \otimes p_{ij}$.

Better to consider an arb. $\mathbb{C}\Gamma$ where $\Gamma_+ = \Gamma \cup 0$ is semigp with 0 absorbing. Then you want to look at tensor products of Γ graded modules.

<u>Something interesting: path algebra for Γ</u>

Let T be defd by gens p_{ij}, no relations. Get ! alg map $\Delta : T \longrightarrow M_2\mathbb{C} \otimes T$ $\Delta(p_{ij}) = e_{ij} \otimes p_{ij}$ which makes T into a counital comodule for $M_2\mathbb{C} = \Lambda$. ~~Then~~ This structure should be equiv. to an $M_2 \cup 0$ grading of T. But when you come to.

Start again: ~~[illegible]~~ Start with an alg defined by generators, each generator having an assoc. degree in Γ_+. V. Spaces of generators X is Γ_+ graded, so the tensor alg $T(X)$ is also Γ_+ graded.

$$X \xrightarrow{\Delta} \mathbb{C}\Gamma_+ \otimes X$$ ~~get~~

$$\begin{tikzcd}
T(X) \arrow[r, "\Delta"] \arrow[dr, "\Delta'"'] & \mathbb{C}\Gamma_+ \otimes T(X) \arrow[d] \\
& \mathbb{C}\Gamma \otimes T(X)
\end{tikzcd}$$

a

Repeat: $\Lambda = M_2\mathbb{C} = T\otimes T^*$, where $T = \mathbb{C}^2$ with usual left mult by matrices, and T^* = dual of T with contragred. repn.

You know Morita equivalence $V \mapsto T\otimes V$ between vector spaces and (unital) Λ modules. Consider a Λ-module retract of $\Lambda\otimes V$, V a v.s. By Morita eq. this is the same as a vs retract W of $T^*\otimes V = \begin{matrix}V\\ \oplus \\ V\end{matrix}$. Draw it:

$$W \xleftarrow{\beta = (\beta_1\ \beta_2)} \begin{matrix}V\\ \oplus \\ V\end{matrix} \xhookleftarrow{\alpha = \binom{\alpha_1}{\alpha_2}} W \qquad \beta\alpha = 1$$

The retract W is equiv. to the projection $p = \alpha\beta$

$$\begin{matrix}V\\ \oplus \\ V\end{matrix} \xhookleftarrow{\binom{\alpha_1}{\alpha_2}} W \xleftarrow{(\beta_1\ \beta_2)} \begin{matrix}V\\ \oplus \\ V\end{matrix}$$

given by the above composition

$$p = \alpha\beta = \begin{pmatrix}\alpha_1\beta_1 & \alpha_1\beta_2 \\ \alpha_2\beta_1 & \alpha_2\beta_2\end{pmatrix} \qquad p_{ij} = \alpha_i\beta_j$$

$p = p^2$ means $p_{ik} = \sum_j p_{ij}p_{jk}$. Define A by these gens + rels. $A = A^2$. Then a retract W of $\begin{matrix}V\\ \oplus \\ V\end{matrix}$ is equiv. to a homom. $A \to \mathrm{End}(V)$

You know $W = p\left(\begin{matrix}V\\ \oplus \\ V\end{matrix}\right)$ is exact fun of V

so you get an equivalence between reduced A-modules and retracts W of $\begin{smallmatrix} V \\ \oplus \\ V \end{smallmatrix}$

Reduced means $\quad V = \sum_i \alpha_i \sum_j \beta_j V = \sum_i \alpha_i W$

and $\quad {}_A V = \{ v \mid \underbrace{\alpha_i \beta_j v = 0 \quad \forall i, j}_{\Updownarrow \;\; \beta_j v = 0 \quad \forall j} \}$

If V reduced A-module, and $W = p\begin{pmatrix} V \\ \oplus \\ V \end{pmatrix}$
you have

$$\begin{tikzcd} W \oplus W & V \arrow[l, hook', "\binom{\beta_1}{\beta_2}"'] & W \oplus W \arrow[l, two heads, "(\alpha_1 \ \alpha_2)"'] \end{tikzcd}$$

Something seems to be happening here. There is this endom $\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} (\alpha_1 \ \ \alpha_2) = \begin{pmatrix} \beta_1\alpha_1 & \beta_1\alpha_2 \\ \beta_2\alpha_1 & \beta_2\alpha_2 \end{pmatrix}$ of $\begin{smallmatrix} W \\ \oplus \\ W \end{smallmatrix}$

Let's review the graded case.

$$\begin{tikzcd} W & V \oplus V \arrow[l, two heads, "(\beta_1 \ \beta_2)"'] & W \arrow[l, hook', "\binom{\alpha_1}{\alpha_2}"'] \end{tikzcd} \qquad 1 = \beta\alpha = \beta_1\alpha_1 + \beta_2\alpha_2$$

$$\begin{tikzcd} W & W \oplus W \arrow[l, two heads, "(\alpha_1, \ \alpha_2)"'] & W \arrow[l, hook', "\binom{\beta_1}{\beta_2}"'] \end{tikzcd} \qquad \text{for } V \text{ reduced}$$

$$\begin{tikzcd} W \oplus W & V \arrow[l, hook', "\binom{\beta_1}{\beta_2}"'] & W \oplus W \arrow[l, two heads, "(\alpha_1 \ \alpha_2)"'] \end{tikzcd}$$

So you assume ~~$\beta_{jk} = \alpha_j \beta_k = 0 \quad j \neq k$~~

the extra relations $p_{ij}\, p_{kl} = 0 \qquad j \neq k$

equiv. $\alpha_i \beta_j \alpha_k \beta_l = 0 \qquad \forall i, j, k, l$ st $j \neq k$

$\Updownarrow \longleftarrow$ because $\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}$ is inj. $(\beta_1\ \beta_2)$ surj

$\beta_j \alpha_k = 0 \qquad \forall j \neq k$

So in this case you have

$$V = \operatorname{Img}\left\{ \begin{matrix} W \\ \oplus \\ W \end{matrix} \xleftarrow{\begin{pmatrix}\beta_1\alpha_1 & 0 \\ 0 & \beta_2\alpha_2\end{pmatrix}} \begin{matrix} W \\ \oplus \\ W \end{matrix} \right\} = \begin{matrix} \operatorname{Img}(h_1) \\ \oplus \\ \operatorname{Img}(h_2) \end{matrix}.$$

Look at length 1 complexes and 2×2 invertible matrices:

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Repeat. $\Lambda = M_2\mathbb{C} = T \otimes T^*$ is M. eq. to $\mathbb{C}$: Λ-modules (unital) $\simeq$ v.s. via $M \mapsto T^* \otimes_\Lambda M$, $T \otimes V \leftarrow\!\shortmid V$. Hence a Λ-mod retract of $\Lambda \otimes V$ is equiv. to a v.s. retract W of $T^* \otimes V = \begin{matrix} V \\ \oplus \\ V \end{matrix}$, i.e. linear maps

$$W \xleftarrow{\beta = (\beta_1, \beta_2)} \begin{matrix} V \\ \oplus \\ V \end{matrix} \xleftarrow{\alpha = \begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}} W \qquad \sum_i \beta_i \alpha_i = 1_W$$

d W also equivalent to the proj of on $\begin{matrix}V\\ \oplus\\ V\end{matrix}$

$$p = (p_{ij}) = \alpha\beta = \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}(\beta_1\ \beta_2) \qquad p_{ij} = \alpha_i\beta_j.$$

Hence W is equivalent to an A-mod structure, where A is the univ. ring w. gens p_{ij} subject to rels.

$$p_{ik} = \sum_j p_{ij}p_{jk}. \qquad W = p\begin{pmatrix}V\\ \oplus\\ V\end{pmatrix}.$$

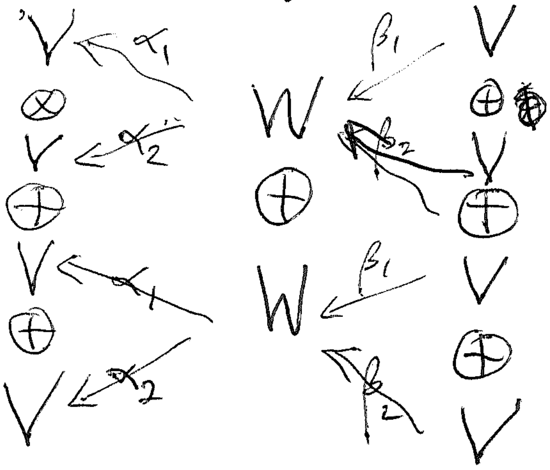
A idemp., exact functor $V \mapsto p\begin{pmatrix}V\\ \oplus\\ V\end{pmatrix}$ from A-mods to vector spaces, killing nil modules. So $W = p\begin{pmatrix}V\\ \oplus\\ V\end{pmatrix}$ is unchanged when the A-module V is replaced by its reduced version. V reduced means $AV = 0$ & ${}_AV = 0$

$$V = \sum_{ij}\alpha_i\beta_j V = \sum_i \alpha_i W, \quad \boxed{\forall ij,\ \alpha_i\beta_j v = 0 \Rightarrow v = 0}$$

$$\Updownarrow$$

$$\forall j\ \beta_j v = 0.$$

$\therefore$ V red $\iff V = \sum_i \alpha_i W$ and $\bigcap_j \operatorname{Ker}\beta_j = 0$



e How to find the reduced version of V

$$\begin{array}{ccccccc} V\oplus V & \xleftarrow{\binom{\alpha_1}{\alpha_2}} & W & \xleftarrow{\beta_1} & & \xleftarrow{\alpha_1} & W & \xleftarrow{(\beta_1\,\beta_2)} & V\oplus V \\ & & \oplus & & V & & \oplus & & \\ V\oplus V & \xleftarrow{\binom{\alpha_1}{\alpha_2}} & W & \xleftarrow{\beta_2} & & \xleftarrow{\alpha_2} & W & \xleftarrow{(\beta_1\,\beta_2)} & V\oplus V \end{array}$$

$$V/_{A}V \longleftarrow V \longleftarrow AV$$

$$\begin{array}{c} V \\ \swarrow \\ \cancel{W}\, V/\mathrm{Ker}(\beta_1,\beta_2) \\ \updownarrow \\ \alpha_1 W + \alpha_2 W/\mathrm{Ker}(\beta_1,\beta_2) \end{array}$$

Thus the reduced version of V namely

$$\mathrm{Img}\left\{A\otimes_A V \longrightarrow \mathrm{Hom}_A(A,V)\right\} = \mathrm{Img}\left\{AV \to V/_{A}V\right\}$$

is

$$V_{red} = \mathrm{Img}\left\{ \begin{matrix} W \\ \oplus \\ W \end{matrix} \xleftarrow{\binom{\beta_1}{\beta_2}} V \xleftarrow{(\alpha_1,\alpha_2)} \begin{matrix} W \\ \oplus \\ W \end{matrix} \right\}$$

Review the situation: You have started with an A-module V and constructed a retract

$$W \xleftarrow{(\beta_1\ \beta_2)} \begin{matrix} V \\ \oplus \\ V \end{matrix} \xleftarrow{\binom{\alpha_1}{\alpha_2}} W$$

of $\begin{matrix} V \\ \oplus \\ V \end{matrix}$. Retract means $\beta_1\alpha_1 + \beta_2\alpha_2 = 1_W$

(so you seem to have

$$W \underset{\beta_1,\ \beta_2}{\overset{\alpha_1,\ \alpha_2}{\rightleftarrows}} V$$

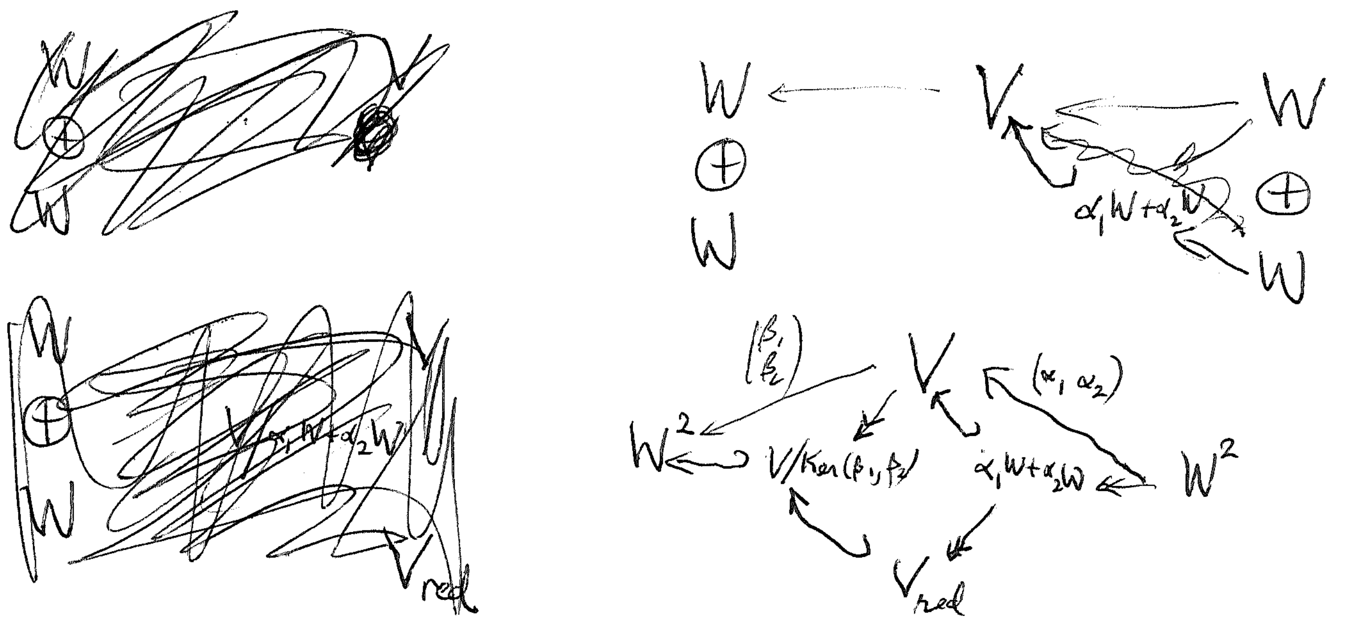
subject to this relation). What's important?

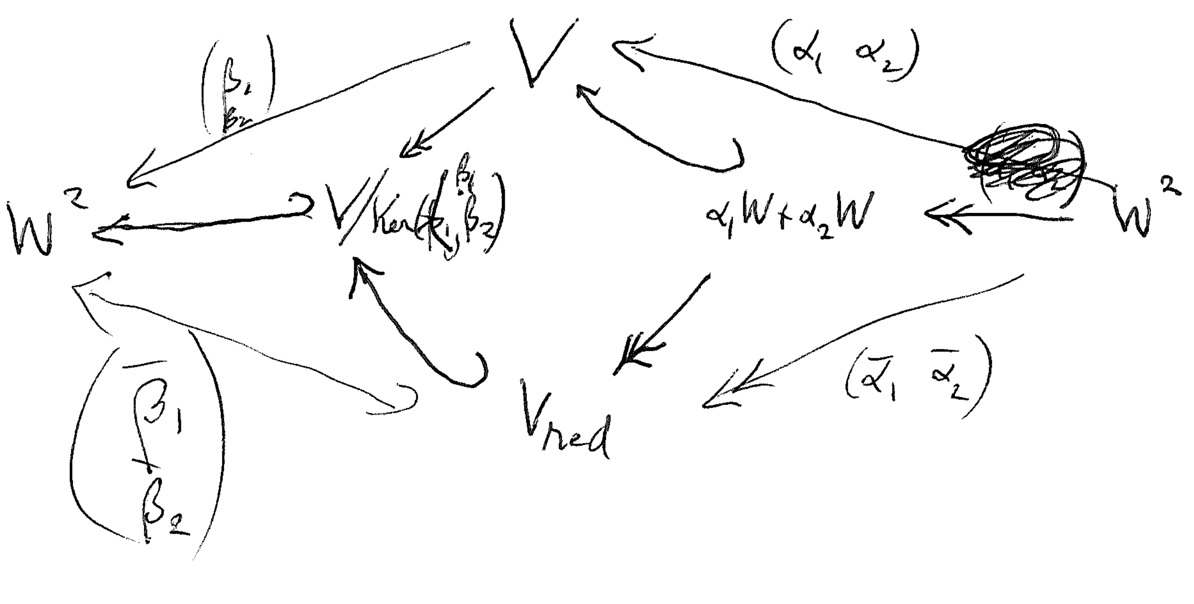
f ~~Next you have ... W~~ Important is that $p = \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}(\beta_1\ \beta_2) = \begin{pmatrix}\alpha_1\beta_1 & \alpha_1\beta_2\\ \alpha_2\beta_1 & \alpha_2\beta_2\end{pmatrix}$ is idemp.

Next without changing W you can replace V by V_{red} which is the image of

$$W \oplus W \xleftarrow{\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}} V \xleftarrow{(\alpha_1\ \alpha_2)} W \oplus W$$

$$\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}(\alpha_1\ \ \alpha_2) = \begin{pmatrix}\beta_1\alpha_1 & \beta_1\alpha_2\\ \beta_2\alpha_1 & \beta_2\alpha_2\end{pmatrix}$$





Aim: To understand the situation where

$\beta_2\alpha_1$ $\beta_1\alpha_2$ are not both zero.

Idea: Explore the link with the Cayley transform picture of a Grassmannian. This is something you should have looked at much earlier. You want V to be a Hilbert space, say fin dim'l, and W ~~to~~ be a closed subspace.

$$W \xleftarrow{(\beta_1\ \beta_2)} \begin{matrix} V \\ \oplus \\ V \end{matrix} \xleftarrow{\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}} W \qquad (\beta_1\ \beta_2) = \begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}^*$$

$$\alpha_1^*\alpha_1 + \beta_1^*\beta_1 = 1.$$

Recall that in the Grassmannian situation $\begin{matrix} V \\ \oplus \\ V \end{matrix}$ is really $\begin{matrix} V_+ \\ \oplus \\ V_- \end{matrix}$, i.e. you have a Hilbert space H together with involution ε. Guess that the graded case $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ you've been studying corresponds to your (ε, F) situation.

Examine then $V = V_1 \oplus V_2 = W \oplus W^\perp$

i.e. a repn of the infinite dihedral group $(\mathbb{Z}/2) * (\mathbb{Z}/2)$. This situation is basically abelian, probably a ~~result~~ consequense of $h_1 + h_2 = 1 \Rightarrow h_1, h_2$ commute.

[redacted] $\Lambda = M_2\mathbb{C} = T \otimes T^*$ is M. eq. to $\mathbb{C}$ via $V \longmapsto T \otimes V$, $T^* \otimes_\Lambda M \longleftarrow\!\shortmid M$. A (unital) Λ-module M splits into $e_{11}M \oplus e_{22}M$, and $e_{21}\cdot$, $e_{12}\cdot$ are inverse isos between $e_{11}M$ and $e_{22}M$, since $e_{21}e_{11} = e_{22}e_{21}$, $e_{12}e_{22} = e_{11}e_{12}$.

A Λ-module retract of $\Lambda \otimes V$

$$\begin{tikzcd} W' & \Lambda \otimes V \arrow[l, "\beta'"'] \arrow[d, equal] & W' \arrow[l, "\alpha'"'] \\ T \otimes W & T \otimes T^* \otimes V & T \otimes W \end{tikzcd} \qquad \beta'\alpha' = 1.$$

$T = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ $\quad T^*$. What is your goal? to describe, parametrize, **understand** [redacted] all retracts of a free Λ module.

Forget free. Take a Λ-module $T \otimes V$ and describe all retracts

$$W' \xleftarrow{\beta'} T \otimes V \xleftarrow{\alpha'} W' \qquad \beta\alpha = 1_{W'}$$

equiv to

$$W \xleftarrow{\beta} V \xleftarrow{\alpha} W \qquad \beta\alpha = 1_W$$

This doesn't lead anywhere. Somehow you are missing the point of a free module. [redacted] Maybe go back to the nuclear picture. E is fin. gen. proj. ~~[illegible]~~ iff the identity map of E is nuclear. This is the ~~basis of~~ background to Serre's thm.

Background to Serre's theorem.

R unital work in $\mathrm{Mod}(R^{op})$.

$$N \otimes_R \mathrm{Hom}_{R^{op}}(M, R) \longrightarrow \mathrm{Hom}_R(M, N)$$

$$n \otimes \lambda \longmapsto (m \longmapsto n\lambda(m))$$

Change letters. Consider a dual pair X, Y over R

$$X \otimes_R Y \to X \otimes_R \mathrm{Hom}_{R^{op}}(X, R) \to \mathrm{Hom}_{R^{op}}(X, X)$$

---

basic idea. Given X right, Y left with

$Y \longrightarrow \mathrm{Hom}_{R^{op}}(X, R)$, $y \longmapsto (x \longmapsto \langle y, x\rangle)$

you get $\quad X \otimes_R Y \longrightarrow \mathrm{Hom}_{R^{op}}(X, X)$

$$x \otimes y \longmapsto (x' \longmapsto x\langle y, x'\rangle)$$

Suppose $\displaystyle\sum_{i=1}^n x_i \otimes y_i \longmapsto id_X$. It means

$$X \xrightarrow{\langle y_i, -\rangle} R^n \xrightarrow{(x_i \cdot)} X$$

$$X \xrightarrow{(y_i)\cdot} R^n \xrightarrow{(x_i)\cdot} X$$

Thus you've embedded as a retract of R^n.

It's clear now what you want to do - make a category

If X, Y are v.s. and $\xi \in X \otimes Y$, choose $\xi = \displaystyle\sum_{i=1}^n x_i \otimes y_i$ with n minimal. Then the x_i are lin. ind. (same for the y_i)

j. ~~...~~ You have the subspaces $V = \sum \mathbb{C} x_i \subset X$ and $W = \sum \mathbb{C} y_i \subset Y$ and ~~...~~ ξ ~~...~~ yields an intrinsic duality between them. To see this change basis from x_i to $x'_j = \sum_i x_i a_{ij}$

Then $\xi = \sum_{i,j} x'_j b_{ji} \otimes y_i$ $\qquad x_i = \sum_j x'_j b_{ji}$

$$= \sum_j x'_j \otimes \sum_i b_{ji} y_i = \sum x'_j \otimes y'_j \qquad y'_j = \sum_i b_{ji} y_i$$

not clear. | start again Let k be a sfield X ~~...~~ a right k vector space, Y a left k vector space, and $\xi \in X \otimes_k Y$. Choose $\xi = \sum_{i=1}^{n} x_i \otimes y_i$ with n least. If $x_i = \sum_{j=1}^{m} x'_j a_{ji}$, then

$$\xi = \sum_i \sum_j x'_j \underbrace{a_{ji} \otimes y_i} = \sum_{j=1}^{m} x'_j \otimes \underbrace{\sum_{i=1}^{n} a_{ji} y_i}_{y'_j}$$

so $m \geq n$. ~~Suppose $m=n$~~ ~~Suppose $m=n$. This means that Then~~ ~~... linear independent,~~ ~~and also the y~~

This implies that the x_i are lin. ind.

Suppose $m = n$, then $\{x'_j\}$ is another basis for $\sum x_i k$ related to $\{x_i\}$ via (a_{ji}). $\therefore$ (a_{ji}) is invertible

$$\begin{array}{ccc} \xi \in X \otimes_k Y & \longrightarrow & \mathrm{Hom}_k(X^*, Y) \\ \cup & & \cup \\ \xi \in V \otimes_k W & \longrightarrow & \mathrm{Hom}_k(V, W) \end{array}$$

K

You've [illegible] looking at Serre's theorem.
Goal: You want to construct "Volodin space" ultimately to solve old problems.
New idea — link between Serre thm. and nuclearity for the id map. Deligne's question (defn of K-gps using the module cat. so that Morita equiv is obvious.)

Retract of a free module idea
Connect this to Cuntz stuff.

Take $\Lambda = \mathbb{C}M_2 = M_2\mathbb{C}$

free module $\Lambda \otimes V$ $\quad V$ a v.s.

E retract of a free Λ-module. Any Λ-module should have this property, but you want to pin things down. I guess this means expressing id_E nuclear form.

$\Lambda = M_2\mathbb{C}$. You are studying retracts of a free Λ-module $\Lambda \otimes V$.
By Morita equivalence these are equivalent to $\mathbb{C}$-module retracts of $T^* \otimes V = V \oplus V$:

$$W \xleftarrow{(\beta_1\ \beta_2)} \begin{matrix} V \\ \oplus \\ V \end{matrix} \xleftarrow{\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}} W \qquad \beta_1\alpha_1 + \beta_2\alpha_2 = 1_W$$

equivalently projections $p = p^2 \in End\begin{pmatrix} V \\ \oplus \\ V \end{pmatrix} = M_2\mathbb{C} \otimes End(V)$

via $p = \begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}(\beta_1\ \beta_2)$, $\qquad p = \sum e_{ij} \otimes \alpha_i \beta_j$

So now you see your mistake I think.

l First of all: a projection $p = p^2 \in \mathrm{End}\begin{pmatrix} V \\ \oplus \\ V \end{pmatrix}$ is equivalent to four operators $p_{ij} \in \mathrm{End}(V)$ satisfying 4 relations $p_{ik} = \sum_j p_{ij} p_{jk}$

$$\begin{tikzcd} W & V \oplus V \arrow[l, "(\beta_1\ \beta_2)"'] & W \arrow[l, "\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}"'] \end{tikzcd} \qquad \begin{aligned} \beta_1\alpha_1 + \beta_2\alpha_2 &= 1 \\ p_{ij} &= \alpha_i \beta_j \end{aligned}$$

Look at the case where V, W are 1-diml say $= \mathbb{C}$.
Recall V is reduced as A-module when

$$\begin{tikzcd} W \oplus W & W \arrow[l, "\begin{pmatrix}\beta_1 \\ \beta_2\end{pmatrix}"'] & W \oplus W \arrow[l, "(\alpha_1\ \alpha_2)"'] \end{tikzcd}$$

surj then inj

is the canonical factorization of $\begin{pmatrix} \beta_1\alpha_1 & \beta_1\alpha_2 \\ \beta_2\alpha_1 & \beta_2\alpha_2 \end{pmatrix}$

$$\begin{vmatrix} \beta_1\alpha_1 & \beta_1\alpha_2 \\ \beta_2\alpha_1 & \beta_2\alpha_2 \end{vmatrix} = \beta_1\alpha_1\beta_2\alpha_2 - \beta_2\alpha_1\beta_1\alpha_2 = 0$$

So this 2×2 matrix has trace 1 and det. 0.

so $\begin{pmatrix} \beta_1\alpha_1 & \beta_1\alpha_2 \\ \beta_2\alpha_1 & \beta_2\alpha_2 \end{pmatrix}$ is idempotent. yes.

$$\begin{pmatrix}\beta_1 \\ \beta_2\end{pmatrix} \underbrace{(\alpha_1\ \alpha_2) \begin{pmatrix}\beta_1 \\ \beta_2\end{pmatrix}}_{\alpha_1\beta_1 + \alpha_2\beta_2 = \beta_1\alpha_1 + \beta_2\alpha_2 = 1.} (\alpha_1\ \alpha_2)$$

m Yesterday you learned something, namely, that retracts of a free Λ-module, $\Lambda = M_2\mathbb{C}$, are different. Let's go over this again, look at Serre thm. (this means writing the identity as a nuclear map, is there a graded version?) and write things up.

$\Lambda = M_2\mathbb{C}$. is Morita equivalent to $\mathbb{C}$

via $V \longmapsto T\otimes V$ $\qquad T = \underbrace{\mathbb{C}^2}_{\text{column vectors}}$ usual left action of $M_2\mathbb{C}$.

$T^*\otimes_\Lambda M \longleftarrow M$

$$\Lambda = T\otimes T^* = \begin{pmatrix}\mathbb{C}\\ \mathbb{C}\end{pmatrix}\otimes\begin{pmatrix}\mathbb{C} & \mathbb{C}\end{pmatrix} \qquad e_i^{\dagger} = e_i\otimes e^{\dagger}$$

a Λ mod retract of $\Lambda\otimes V$ is m.eq. to $\mathbb{C}$-mod retract W of $T^*\otimes V = \begin{matrix}V\\ \oplus\\ V\end{matrix}$:

$$W \xleftarrow{(\beta_1\ \beta_2)} \begin{pmatrix}V\\ V\end{pmatrix} \xleftarrow{\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}} W \qquad \text{sat } \beta_1\alpha_1+\beta_2\alpha_2 = 1$$

proj op

$$p = \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\begin{pmatrix}\beta_1 & \beta_2\end{pmatrix} : \begin{matrix}V\\ \oplus\\ V\end{matrix} \longleftarrow \begin{matrix}U\\ \oplus\\ V\end{matrix}$$

$p_{ij} = \alpha_i\beta_j$.

Equiv. things. 1) Λ-mod retract of $\Lambda\otimes V$, V $\mathbb{C}$-mod

2) $\mathbb{C}$mod retract W of $\begin{pmatrix}V\\ V\end{pmatrix}$

3) $p = p^2 \in \operatorname{End}\begin{pmatrix}V\\ V\end{pmatrix}$

4) $(p_{ij}) \in M_2(\operatorname{End}(V))$ $\qquad p_{ik} = \sum_j p_{ij}p_{jk}$

A-mod. structure on V $\qquad A$ 4 gen 4 rels.

It seems that this type of A-module is not what you want. You want a graded version, namely $V = \begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix}$, consider ($\mathbb{C}$-module) retract

$$W \xleftarrow{(\beta_1\ \beta_2)} \begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix} \xleftarrow{\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}} W \qquad \beta_1\alpha_1 + \beta_2\alpha_2 = 1.$$

equiv to proj. $p = \alpha\beta = \left(\alpha_i \beta_j\right)$

$$\begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix} \xleftarrow{\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}} W \twoheadleftarrow{(\beta_1\ \beta_2)} \begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix}$$

~~Notice that~~

$$p = p^2 \in \mathrm{End}\begin{pmatrix} V_1 \\ \oplus \\ V_2 \end{pmatrix} = \begin{pmatrix} \mathrm{End}(V_1) & \mathrm{Hom}(V_1 \leftarrow V_2) \\ \mathrm{Hom}(V_2 \leftarrow V_1) & \mathrm{End}(V_2) \end{pmatrix}$$

Here $\quad p_{ij} : V_i \leftarrow V_j$

graded version $\quad V = \begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix} \qquad p_{ij} : V_i \leftarrow V_j$

$\sum_j p_{ij} p_{jk} = p_{ik}$. ~~Your alg A is universal~~

~~A gen by p_{ij}~~

before p is a projection in $\mathrm{End}\begin{pmatrix} V \\ \oplus \\ V \end{pmatrix}$

$= M_2\mathbb{C} \otimes \mathrm{End}(V)$ $\qquad p = \sum e_{ij} \otimes p'_{ij}$

an M_2 graded alg. $\qquad$ these are the homog comps of p.

o So you understand the situation now pretty much. What's missing?

~~[illegible]~~ Maybe the aim is to understand $\boxed{\text{projections in an } M_2\text{-graded algebra.}}$ There is a universal such algebra: generators p_{ij} of degree ij relations $p_{ik} = \sum_j p_{ij} p_{jk}$, $p_{ij} p_{k\ell} = 0 \quad j \neq k$.

Let A be the alg with these gens + rels. Define homo

$$\Delta : A \longrightarrow M_2\mathbb{C} \otimes A, \quad \Delta(p_{ij}) = e_{ij} \otimes p_{ij}$$

Better: form t.p. alg $M_2\mathbb{C} \otimes A$, note $p'_{ij} = e_{ij} \otimes p_{ij}$ in this sat. relations, hence $\exists!$ alg map $\Delta \ni \Delta(p_{ij}) = p'_{ij}$

$$(\Delta_A \otimes 1)\Delta(p_{ij}) = (\Delta_A \otimes 1)(e_{ij} \otimes p_{ij}) = e_{ij} \otimes e_{ij} \otimes p_{ij}$$

$$(1 \otimes \Delta_A)\Delta(p_{ij}) = (1 \otimes \Delta_A)(e_{ij} \otimes p_{ij}) = e_{ij} \otimes e_{ij} \otimes p_{ij}.$$

$\therefore$ Δ makes ~~[illegible]~~ A a comodule for ~~the~~ $M_2\mathbb{C}$

What ~~is~~ about the counit cond. $(\eta \otimes 1)\Delta = 1_A$? $\eta : M_2\mathbb{C} \to \mathbb{C}$ is not an alg map. ~~[illegible]~~ Take any word in the generators ~~[illegible]~~ $p_{s_1} \cdots p_{s_n} \ni s_1 \cdots s_n \in M_2$

---

point. ~~a~~ comodule V for $M_2\mathbb{C} = \mathbb{C}[M_2]$ is the same as a counital comodule for $\mathbb{C}[M_{2,+}]$, which is the same as a graded vector space wrt $M_{2,+}$, which is the same as an M_2 graded v.s. ~~[illegible]~~ $\oplus$ Ker Δ

Get act together

P Point: What seems to happen is that you want to replace free by appropriately graded. In the case of a group you consider a module with both Γ action and Γ grading $= \hat{\Gamma}$ action. In the M_2 case you consider a $\Lambda = M_2(\mathbb{C})$ module of the form $T \otimes V$ where $V = \begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix}$ is graded wrt the set of objects.

Let's review in order to clarify things.

M_2 yields the truly $M_2\mathbb{C}$ and M_2 graded algebras which are just Morita contexts $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$. Basic object is a proj. in an M_2 graded alg. $p_{ij} \in A_{ij}$ sat. $p_{ik} = \sum_j p_{ij} p_{jk}$. can form Universal M_2 graded alg A with these gens + rels. A idemp. so any red. mod V has unique graded $V = \begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix}$ s.t. $p_{ij} : V_i \leftarrow V_j$

Is there a Grassmannian around?

The basic object is a projection in a M_2-graded algebra $p = (p_{ij})$. There is a universal M_2-graded alg. A ~~[illegible]~~ with projection p which is gen. by the p_{ij}

Object M_2 graded alg A tog. with a proj $p = (p_{ij})$ in A. $\mathcal{P}_{M_2}$

Start again. You want to construct the universal M_2 graded alg A which represents projections $p = (p_{ij})$ in any M_2 graded algebra. First construct A as an alg: 4 gens p_{ij}, rels $p_{ij}p_{k\ell} = 0 \quad j \neq k$

$$p_{ik} = \sum_j p_{ij} p_{jk}$$

Next ~~construct~~ M_2 grading on A. Note that

$p'_{ij} = e_{ij} \otimes p_{ij} \in M_2\mathbb{C} \otimes A$ sat same relns as p_{ij}, where $\exists!$ alg morph $\Delta : A \longrightarrow M_2\mathbb{C} \otimes A$, $\Delta p_{ij} = p'_{ij}$. Check $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$, Δ is comodule strue on A for coalg $M_2\mathbb{C}$. Get grading of A wrt $M_2 \amalg \{*\}$

$$A = \bigoplus_{ij} A_{ij} \oplus A_* \quad , \quad A_* = \operatorname{Ker} \Delta .$$

You need to show $A_* = 0$. The point is that A_s for $s \in M_2 \amalg \{*\}$ is spanned by all words $p_{s_1} \cdots p_{s_n}$ in the generators with total degree $s_1 \cdots s_n = s$. Here $s_1, \dots, s_n$ are arrows in M_2 and $s_1 \cdots s_n = *$ means that ~~this~~ chain of arrows is not composable, i.e. for some i the source of $s_i \neq$ target of s_{i+1}. In this case you have required $p_{s_i} p_{s_{i+1}} = 0$ in A so $p_{s_1} \cdots p_{s_n} = 0$, showing $A_* = 0$.

Question: Is there a more general assertion? Consider ~~Consider~~ a semigroup with abs. elt $*$, written $\Gamma \amalg \{*\}$, Consider $\Gamma \amalg \{0\}$ a semigroup with abs. element $*$, whence $\mathbb{C}\Gamma$ is a bialgebra, whose counital comodules are Γ-graded vector spaces

$$V = \bigoplus_{s\in\Gamma} V_s \,;$$

~~and whose~~ the product ~~is~~

$\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma$ leads to a tensor product operation on Γ-graded v.s.

$$(V\otimes W)_s = \bigoplus_{s=tu} V_t \otimes W_u$$

A Γ-graded algebra is an alg A whose underlying v.s. is equipped with a Γ-grading: $A = \bigoplus_{s\in\Gamma} A_s$ such that

$$A_s A_t \subset \begin{cases} A_{st} & \text{if } st\in\Gamma \\ 0 & \text{if } st = *. \end{cases}$$

Equivalent (better) to say the comodule map

$\Delta : A \longrightarrow \mathbb{C}\Gamma \otimes A$ $\qquad \Delta(a) = s\otimes a \quad$ if $a\in A_s$

is an alg. map.

$$\Delta(a_s a_t) = (s\otimes a_s)(t\otimes a_t) = st \otimes a_s a_t$$

Consider the universal Γ-graded alg which represents projections in any Γ-graded algebra. ~~[illegible]~~ generators p_s $s\in\Gamma$, rel $p_s = \sum_{s=tu} p_t p_u$, ~~$p_t p_u = 0$ if $tu = *$~~

Better assume Γ finite. Let A have above gen + rel

$$\Delta(p_s) = s\otimes p_s$$

$$A \xrightarrow{\ \Delta\ } \mathbb{C}\Gamma\otimes A$$

Δ makes A into a comodule for $\mathbb{C}\Gamma$, same as $\Gamma \amalg \{*\}$ grading $A = \bigoplus_{s\in\Gamma} A_s \oplus A_*$

$\mathcal{P}$ is the universal M_2 graded alg representing ~~[illegible]~~ projections $p = (p_{ij})$ in any M_2 graded algebra. Try to do ~~construction~~ for general Γ, (at least finite). Let A be a Γ-graded alg,

$$A = \bigoplus_{s\in\Gamma} A_s \quad \text{and coaction map} \quad \Delta : A \longrightarrow \mathbb{C}\Gamma \otimes A, \quad a_s \longmapsto s \otimes a$$

is alg map.

~~Start with~~ The generators + relations are homogeneous. You want to start with

Let $\Gamma_+ = \Gamma \amalg \{*\}$ be a semigroup with abs. elt $*$.
$\mathbb{C}[\Gamma]$ ~~[illegible]~~ $= \mathbb{C}[\Gamma_+]/\mathbb{C}[*]$ the corresp bialg

---

Begin with $\Gamma_+ = \Gamma \amalg \{*\}$ equipped with assoc. product s.t. $*$ is absorbing. $\mathbb{C}[\Gamma] = \mathbb{C}[\Gamma_+]/\mathbb{C}[*]$ is a bialg whose counital comodules are Γ-graded vector spaces $V = \bigoplus_{s\in\Gamma} V_s$, ~~[illegible]~~ the product $\mathbb{C}[\Gamma] \otimes \mathbb{C}[\Gamma] \longrightarrow \mathbb{C}[\Gamma]$ yields $\otimes$ operation on ~~[illegible]~~ comodules

$$(V\otimes W)_u = \bigoplus_{u=st} V_s \otimes W_t$$

Γ graded alg: ~~$A = \bigoplus_{s\in\Gamma} A_s$ i.e. a comodule~~

~~[illegible]~~ is an alg A equipped with Γ grading

$$A = \bigoplus_{s\in\Gamma} A_s \quad \text{equiv. a } \sout{\text{coaction}} \text{ counital comodule st. } \Delta : A \longrightarrow \mathbb{C}[\Gamma] \otimes A, \quad a_s \longmapsto s \otimes a_s$$

Sat Rel. $A_s A_t \subset \begin{cases} A_{st} & \text{if } st \neq * \\ 0 & \text{if } st = * \end{cases}$ equiv. Δ is an alg map

$t \qquad p = \oplus p_s \in \oplus A_s = A$

$$p_u = (p^2)_u = \sum_{u=st} p_s p_t$$

The idea is that Γ-graded v.s. form a tensor category. Given a Γ graded vector s/p X spanned by the generators you can form the tensor alg

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Do the $\Gamma = M_2$ case again. M_2-graded alg = Morita context $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$. P = the universal M_2-graded algebra representing projections in a M_2-graded alg. = the M_2 graded alg with 4 generators p_{ij} and 4 rels $p_{ik} = \sum_j p_{ij} p_{jk}$. In fact this is ~~equal to~~ the alg these 4 generators + 4 relns. and the addendum relns. $p_{ij} p_{k\ell} = 0 \quad j \neq k$.

Why? let A' have the 4 gen + 4 rels as above

Let
$$A \subset M_2\mathbb{C} \otimes A'$$
be subalg gen. by $e_{ij} \otimes p_{ij}$

---

try to get

4 Back to $M_2 = \Gamma$. You consider M_2-graded algebras, i.e M contexts. Want Grassmannian, i.e. universal M_2-graded alg A representing projections in M_2-graded algs. Proj is $p = \sum e_{ij} \otimes p_{ij} \in M_2\mathbb{C} \otimes A$, i.e. 4 elts $p_{ij} \in A_{ij}$ sat. rel. $p_{ik} = \sum_j p_{ij} p_{jk}$, $p_{ij} p_{kl} = 0$ $j \neq k$.

Let $\mathcal{P}$ be the alg gen. by 4 elts p_{ij} subject to ~~theses~~ relations. Claim that $\mathcal{P}$ has a unique M_2 grading s.t. $\deg(p_{ij}) = e_{ij}$. Proof. Observe $p'_{ij} = e_{ij} \otimes p_{ij} \in M_2\mathbb{C} \otimes \mathcal{P}$ satisfy the support + idemp relations, so $\exists!$ alg map Δ

$$\mathcal{P} \xrightarrow{\Delta_{\mathcal{P}}} M_2\mathbb{C} \otimes \mathcal{P} \underset{1 \otimes \Delta_p}{\overset{\Delta_\Gamma \otimes 1}{\rightrightarrows}} M_2\mathbb{C} \otimes M_2\mathbb{C} \otimes \mathcal{P}$$

$$p_{ij} \longmapsto e_{ij} \otimes p_{ij}$$

Δ_p is a comodule structure for the v.s. $\mathcal{P}$, i.e. a grading $\mathcal{P} = \bigoplus_{ij} \mathcal{P}_{ij} \oplus \mathcal{P}_*$ $\quad \mathcal{P}_* = \operatorname{Ker} \Delta_p$

It seems you need to say that for $s \in \Gamma \amalg \{*\}$ $\mathcal{P}_s$ is spanned by words $p_{s_1} \cdots p_{s_n}$ such that $s_1 \cdots s_n = s$. Then point out $s = 0$ occurs $\iff$ s_i, s_{i+1} not composable whence the word is 0 since you've added the relation.

✓ So $P_*=0$ and $P \xrightarrow{\sim} \bigoplus_{ij} e_{ij}\otimes P_{ij}$

What comes next? $P = P^2$. What's important are reduced P-modules.

The point is that because P is M_2-graded, there is a extension of P to an M_2 graded unital ring, semi direct product

$$P \longrightarrow \tilde{P} \longrightarrow \mathbb{C}e_{11} \oplus \mathbb{C}e_{22}$$

$$\tilde{P} = \begin{pmatrix} \tilde{P}_{11} & P_{12} \\ P_{21} & \tilde{P}_{22} \end{pmatrix}$$

M_2 graded unital contains P as ideal.

Any reduced P-module V unique extension to a unital $\tilde{P}$-module. So $V = V$

Question. For gen Γ-grading: $\mathbb{C}\Gamma$ is automatically a Γ graded vector space, Γ graded alg.

$$\Delta : \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma$$
$$s \longmapsto s \otimes s$$

digress to the finite Γ gen case: P_Γ generators p_s $s \in \Gamma$ rels.

Question: When is P_Γ idempotent? usual proof requires $\Gamma = \Gamma\cdot\Gamma$. Note there is a filtration $\Gamma \supset \Gamma\cdot\Gamma \supset \Gamma\cdot\Gamma\cdot\Gamma \supset \cdots$ and it should be true that this linearizes to

$$\mathbb{C}\Gamma \supset (\mathbb{C}\Gamma)^2 \supset \cdots$$

~~The question is whether~~

Begin again with $\mathbb{C}[\Gamma] = \mathbb{C}[\Gamma_+]/\mathbb{C}[*]$

You would like to understand $P_\Gamma = \Gamma$ graded alg gen by p_s of degree s for $s \in \Gamma$ subject to the idemp. relation $p_s = \sum_{s=tu} p_t p_u$.

Let A' be the (ungraded) alg with these gens + rels. Let $\Delta : A' \longrightarrow \mathbb{C}\Gamma \otimes A'$ alg map by $\Delta(p_s) = s \otimes p_s$. Then $A' = \bigoplus_{s\in\Gamma} A'_s \oplus A'_*$ is Γ_+ graded and $P_\Gamma = A'/A'_*$. When is P_Γ idempotent?

When $\Gamma = \Gamma\cdot\Gamma$ for then every s can be written $s = tu$ in at least 1-way so $p_s \in P^2$

So if $A = P_\Gamma$ is idempotent you can consider reduced A-modules. Can you see any multipliers for P_Γ

So you have $\Delta : A \hookrightarrow \mathbb{C}\Gamma \otimes A$

normalizer? If A is a subring of C, then the normalizer should be the largest subring B of C such that $BA \subset A$ and $AB \subset A$, in fact

$B = \{ c \in C \mid cA \subset A,\ Ac \subset A \}$

$$\Delta A \subset \mathbb{C}\Gamma \otimes \tilde{A} = \mathbb{C}\Gamma \oplus_s \mathbb{C}\Gamma \otimes A$$

Question: Can the Γ graded ring $A = P_\Gamma$ be embedded in a Γ-graded unital ring? When can it be? Let $A^\#$ be a Γ-graded ring containing A as Γ-graded ideal such that $A^\#$ is unital. Look at the components of 1, say $1 = \sum_{s\in\Gamma} e_s$. ~~You have~~ 1 is a proj in $A^\#$ which is Γ-graded, hence get $A \longrightarrow A^\#$, $p_s \longmapsto e_s$

× Let $A = P_\Gamma$, let B be a Γ graded ~~ring~~ alg which is unital as an ~~ring~~ alg, & let $1 = \sum_{s\in\Gamma} e_s \in \bigoplus_{s\in\Gamma} B_s = B$ be the identity element of B. 1 is a projection in the Γ-graded alg B, so there is a canon. map of Γ-gr algs $A \longrightarrow B$ sending p_s to e_s $\forall s$. The image of this map is the subalg of B generated by the components e_s, $s\in\Gamma$.

There should be a smallest Γ-graded unital algebra - might be the zero alg. So in the above you are interested in the case $1 \neq 0$. In any case you can take the ~~quotient~~ alg of P_Γ by the relations $p_s \sum_t p_t = p_s = \sum_t p_t p_s$

$$\boxed{\sum_{u=st} p_s p_t}$$

Let B ~~be~~ the universal Γ-graded algebra generated by elements e_s of degree s for $s\in\Gamma$ subj to $\sum_s e_s e_t = e_t = \sum_s e_t e_s$.

How do you ~~construct~~ B?

B' = the univ alg defd by these gens + rels. $\sum_s (s\otimes e_s)(t\otimes e_t)$ $= \sum_s st \otimes e_s e_t$. This doesn't work because the relation $\sum_s e_s e_t = e_t$ is not homog. unless $st = t$ or 0

You are looking at unital Γ-graded algebras, when nonzero ones $\exists$.

Consider $B = \bigoplus_{s\in\Gamma} B_s$ Γ-graded, B unital with $1 = \sum_{s\in\Gamma} e_s \neq 0$ $e_s \in B_s$. Let $A = P_\Gamma$ the universal Γ-graded alg representing projections. Then $1^2 = 1 \Rightarrow \exists!$ Γ-gr alg map $A \longrightarrow B$ sending p_s to e_s $\forall s \in \Gamma$.

Assume now that this map is surjective, i.e. B is generated by the components e_s of 1. You are already assuming A is idempotent equiv $\Gamma = \Gamma\Gamma$. What next?

Go back to the idea of adjoining an identity. Let A be Γ-graded, $\Delta : A \longrightarrow \mathbb{C}\Gamma \otimes A$ the canonical alg map. You can extend $\mathbb{C}\Gamma\otimes A$ to $\mathbb{C}\Gamma \otimes \tilde{A}$, which is semidirect product of $\mathbb{C}\Gamma$ and $\mathbb{C}\Gamma\otimes A$. $\mathbb{C}\Gamma\otimes\tilde{A}$ is Γ-graded in the same way that $\mathbb{C}\Gamma\otimes A$. In fact $\mathbb{C}\Gamma\otimes A$ is an ideal in $\mathbb{C}\Gamma\otimes\tilde{A}$ with quotient alg $\mathbb{C}\Gamma$.

Next look for multipliers.

~~Γ-graded unital ring $B \neq 0$, then~~

$\Gamma = \Gamma^2$ $A = P_\Gamma$ is idempotent, has good module category. Can you say anything about an A-module structure on a vector space V such that $\sum_{s\in\Gamma} p_s = 1$?

2nd Look at the M_2 case again.

$\Gamma = M_2 \quad \mathbb{C}\Gamma = M_2\mathbb{C}$, Γ-graded alg = Morita context. $A = \mathcal{P}_{M_2}$ is a M. cont. so it embeds into a unital M. cont.

$$\begin{pmatrix} \mathbb{C}\oplus A_{11} & A_{12} \\ A_{21} & \mathbb{C}\oplus A_{22} \end{pmatrix} = A \oplus \begin{pmatrix} \mathbb{C}e_1 & 0 \\ 0 & \mathbb{C}e_2 \end{pmatrix}$$

Then any red. A-module structure on V ~~is eq~~ extends uniquely to ~~a unital~~ this unital ring, thus get grading wrt "Objects" $V = V_1 \oplus V_2$ such that $p_{ij} : V_i \leftarrow V_j$ reduced means $\forall i \ V_i = \sum_j p_{ij} V_j$

$(\forall j)(\forall v \in V_j)\big((\forall i)(p_{ij}v_j = 0) \Rightarrow v_j = 0\big)$?

What are the remaining points?

Thing to understand. $\exists$ unital Γ-graded alg $B \neq 0$?

$1 = \sum_{s\in\Gamma} e_s \in \bigoplus_s B_s = B$ is a projection

$\Rightarrow e_s = \sum_{s=tu} e_t e_u \Rightarrow \forall s$ with $e_s \neq 0$, $\exists\, t,u$ with $s = tu$ and $e_t e_u \neq 0$.

This gets too ~~hard~~ hard. Try another approach.

$\Delta A \subset \mathbb{C}\Gamma \otimes \tilde{A}$ In $\mathbb{C}\Gamma\otimes\tilde{A}$ are elements $s \otimes 1$

Do these yield multipliers on ΔA? Should be true for a Γ (a) category. So you first ask for left mult.

$(s\otimes 1)(t\otimes a_t) = st \otimes a_t$

need $st = t$, similarly

for $(t\otimes a_t)(s\otimes 1) = ts\otimes a_t$ to be in ΔA and $a_t \neq 0$ you need $ts = t$. Therefore looks like a cat.

Repeat: $\Gamma = \Gamma \cdot \Gamma$ so that $A = P_{\Gamma}$ is idempotent and Γ-graded. Is there a Mult alg for A involving the Γ-grading?

For example, a Γ-graded multiplier ring.

Look at left mult. $\mathrm{Hom}_{A^{op}}(A, A)$. ~~[illegible]~~

~~[illegible]~~ Mult. alg.

Remember Γ is finite. Let ~~[illegible]~~ $\mu \in \mathrm{Hom}_{A^{op}}(A, A)$

Int. question. A is Γ-graded alg

Let $\mu \in \mathrm{Hom}_{A^{op}}(A, A)$. Question is whether μ has ~~a degree~~ homogeneous components.

Go over ~~[illegible]~~ M_2 case

General situation is: Γ_+ ~~[illegible]~~ $= \Gamma \amalg \{*\}$ is equipped with a semigroup structure (assoc prod) such that $*$ is absorbing ($s* = *s = *$). Get bialg $\mathbb{C}\Gamma = \mathbb{C}\Gamma_+/\mathbb{C}*$

$\Delta(s) = s \otimes s$. Notion of a Γ-graded alg $A = \bigoplus_{s \in \Gamma} A_s$

$$A_s A_t \subset \begin{cases} A_{st} & \text{if } st \neq * \\ 0 & \text{otherwise} \end{cases} \qquad \text{e.g. } \mathbb{C}\Gamma$$

Associated to such an A is a module cat ??

You have left and right Γ-graded modules over A. Look at Γ-graded modules over $\mathbb{C}\Gamma$, i.e.

$V = \bigoplus_{t \in \Gamma} V_t$ a v.s. with Γ grading

and a Γ-action such that $sV_t \subset \begin{cases} V_{st} & st \neq 0 \\ 0 & st = 0. \end{cases}$

If Γ is a group, then V is free as Γ-module

b_1 Review again. $\Gamma = M_2$ $\Lambda = M_2\mathbb{C}$

P_Γ = the universal Γ-graded alg representing projections in any Γ-graded alg. Construction

Let A = the alg generated by elements p_{ij} satis

$$p_{ij}p_{k\ell} = 0 \quad j \neq k, \qquad p_{ik} = \sum_j p_{ij}p_{jk}$$

Let $\Delta : A \longrightarrow \Lambda \otimes A$ be the alg map $\ni$ $\Delta(p_{ij}) = e_{ij} \otimes p_{ij}$. Then $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$ so that A becomes a comodule for Λ, i.e. a graded v.s. wrt Γ_+. So

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Review again. $\Gamma = M_2$, $\Lambda = M_2\mathbb{C}$

P_Γ = universal M_2 graded alg representing projections in any Γ graded alg.

Construction of P_Γ. Let A be the alg with generators p_{ij} subj to rels

$$\left| \begin{array}{l} p_{ij}p_{k\ell} = 0 \quad j \neq k \\ p_{ik} = \sum_j p_{ij}p_{jk} \end{array} \right.$$

Let $\Delta : A \longrightarrow M_2\mathbb{C} \otimes A$ be the alg map s.t. $\Delta(p_{ij}) = e_{ij} \otimes p_{ij}$. Then $(\Delta_\Gamma \otimes 1)\Delta_A = (1 \otimes \Delta_A)\Delta_A$, i.e. A is a $M_2\mathbb{C}$ comodule, which means A is graded wrt $(M_2)_+$:

$$A = \bigoplus_{ij} A_{ij} \oplus A_* \qquad \left| \begin{array}{l} \Delta(a_{ij}) = e_{ij} \otimes a_{ij} \\ \Delta(A_*) = 0 \end{array} \right.$$

Δ alg map $\Longrightarrow$ $A_s A_t \subset A_{st}$ $\quad s,t \in \Gamma_+$

Then $P_\Gamma = A/A_*$

Maybe you want to check that this has the desired property. Let $B = \bigoplus_{ij} B_{ij}$ be a Γ-graded alg, ~~There is the following idea. Take~~ ~~Suf~~ let $\tilde{p} = (\tilde{p}_{ij})$ be a proj. in B. Then there is a unique homom. $A \xrightarrow{\theta} B$ sending p_{ij} in A to $\tilde{p}_{ij}$ in B.

$$A \xrightarrow{\Delta_A} M_2\mathbb{C} \otimes A$$

$$B \xrightarrow{\Delta_B} M_2\mathbb{C} \otimes B$$

You need to show that $A_* \longrightarrow 0$ in B.

Do Γ_+ first. A and B are Γ_+ graded.

A is the ~~$F = M_2$ graded~~ univ algebra with gen p_{ij} subj to relns $\sum_j p_{ij} p_{jk} = p_{ik}$, $p_{ij} p_{kl} = 0$ $j \neq k$

construct $\Delta : A \longrightarrow \mathbb{C}[\Gamma_+] \otimes A$ $\quad \Delta(p_s) = s \otimes p_s$

Δ counital coaction of $\mathbb{C}[\Gamma_+]$ on A $\Rightarrow$ $A = \bigoplus_{s \in \Gamma_+} A_s$

Δ alg map $\Rightarrow$ $A_s A_t \subset A_{st}$ $\quad \forall s, t \in \Gamma_+$

Then $\mathcal{P} = A/A_*$ is a Γ-graded alg.

Let $B = \bigoplus_{s \in \Gamma} B_s$ be an arb. Γ-graded alg

This should be the same as a Γ_+ graded algebra such that $B_* = 0$.

$$\begin{array}{ccccc}
 & & [*] \otimes B & & \\
 & & \uparrow & & \\
B & \longrightarrow & \mathbb{C}[\Gamma_+] \otimes B & \qquad b_s \longmapsto s \otimes b_s & s \in \Gamma_+ \\
\| & & \downarrow & \qquad\qquad \downarrow & \\
B & \longrightarrow & \mathbb{C}[\Gamma] \otimes B & \qquad \begin{cases} s \otimes b_s & s \in \Gamma \\ 0 & s = * \end{cases} &
\end{array}$$

d_1 $\quad A$ univ. alg gens p_{ij}, rels | support, idemp-

let $\Delta : A \longrightarrow \mathbb{C}[\Gamma_+] \otimes A$ be the alg map

~~sat.~~ $\Delta p_{ij} = e_{ij} \otimes p_{ij}$. then $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$

Rep. A univ alg gen by p_{ij} subj to | $p_{ij}p_{k\ell} = 0$ $j \neq k$, $p_{ik} = \sum_j p_{ij}p_{jk}$

$\Delta : A \longrightarrow \mathbb{C}\Gamma_+ \otimes A$ the alg map s.t. $\Delta(p_{ij}) = e_{ij} \otimes p_{ij}$

Check rels $(e_{ij} \otimes p_{ij})(e_{k\ell} \otimes p_{k\ell}) = 0$ $\quad j \neq k$

$$\sum_j (e_{ij} \otimes p_{ij})(e_{jk} \otimes p_{jk}) = \sum_j e_{ik} \otimes p_{ij}p_{jk} = e_{ik} \otimes p_{ik}$$

Then Δ coumital coaction of $\mathbb{C}\Gamma_+$ on the v.s. A.

Start again: You want to prove universal property for your construction of P_Γ.

$\Gamma = M_2 \qquad \mathbb{C}\Gamma = M_2\mathbb{C}$

A univ. alg gen. by p_{ij} rels | supp, idemp

Claim A is Γ_+-graded alg because of alg map

$\Delta : A \longrightarrow \mathbb{C}\Gamma_+ \otimes A \qquad \Delta(p_{ij}) = e_{ij} \otimes p_{ij}$

satisfying $(\Delta_{\Gamma_+} \otimes 1)\Delta_A = (1 \otimes \Delta_A)\Delta_A$, $(\mu \otimes 1)\Delta_A = \mathrm{id}_A$

Try again: $A \xrightarrow{\Delta} \mathbb{C}\Gamma \otimes A$, $\quad A \xrightarrow{\Delta'} \mathbb{C}\Gamma_+ \otimes A$ | both Δ, Δ' send p_{ij} to $e_{ij} \otimes p_{ij}$

~~But the~~ Need to check

relns. $(e_{ij} \otimes p_{ij})(e_{k\ell} \otimes p_{k\ell}) = [*] \otimes p_{ij}p_{k\ell} = 0$ for $j \neq k$

e_1 Start again $\Gamma = M_2$, $\mathbb{C}\Gamma = M_2\mathbb{C}$

$A =$ (universal) alg gen p_{ij} subj to rels $\left| \begin{array}{l} p_{ij}p_{k\ell} = 0 \quad j \neq k \\ p_{ik} = \sum_j p_{ij}p_{jk} \end{array}\right.$

~~Define~~ Let $\Delta_A : A \longrightarrow \mathbb{C}[\Gamma_+] \otimes A$ be the alg map such that $\Delta_A(p_{ij}) = e_{ij} \otimes p_{ij}$ and verify $(\Delta_\Gamma \otimes 1)\Delta_A = (1 \otimes \Delta_A)\Delta_A$. You also need $(\eta \otimes 1)\Delta_A = id$ How does this happen

~~$\mathbb{C}[*]$ is an ideal in the semigroup alg $\mathbb{C}[\Gamma_+]$ and $\mathbb{C}[\Gamma]$?~~

~~Then $\Delta(p_s) = s \otimes p_s$ $s \in \Gamma$~~

~~Point: $\mathbb{C}\Gamma$ is a subalg of $\mathbb{C}\Gamma_+$ NO.~~

~~You have $\mathbb{C}\Gamma_+ / \mathbb{C}[*] = \mathbb{C}\Gamma$. So as far as the alg struct is concerned one has~~

~~$\mathbb{C}\Gamma_+ = \underbrace{\mathbb{C}\Gamma}_{\text{subalg.}} \oplus \underbrace{\mathbb{C}[*]}_{\text{ideal}}$ semi~~

~~Go back to $\Delta : A \longrightarrow \mathbb{C}\Gamma \otimes A$~~

Start again: $\Gamma = M_2$, $\mathbb{C}[M_2] = M_2\mathbb{C}$

A : gens p_{ij} rels $\left| \begin{array}{l} p_{ij}p_{k\ell} = 0 \quad j \neq k \\ p_{ik} = \sum_j p_{ij}p_{jk} \end{array}\right.$

$A \xrightarrow{\Delta} \mathbb{C}\Gamma \otimes A$ $\qquad \Delta p_{ij} = e_{ij} \otimes p_{ij}$

$\Longleftarrow \quad \tilde{\Delta} \searrow \mathbb{C}\Gamma_+ \otimes A$

g_1 $$\Delta : A \longrightarrow \mathbb{C}\Gamma \otimes A \underset{1\otimes\Delta}{\overset{\Delta\otimes 1}{\rightrightarrows}} \mathbb{C}\Gamma\otimes\mathbb{C}\Gamma\otimes A$$

$$\Delta(a) = \sum_{s\in\Gamma} s \otimes e_s a$$ $\{s \mid e_s a \neq 0\}$ finite $\forall a$

$$(\Delta\otimes 1)\Delta(a) = \sum_{s\in\Gamma} s\otimes s\otimes e_s a$$

$$(1\otimes\Delta)\Delta a = \sum_{s\in\Gamma} s\otimes \sum_t t\otimes e_t e_s a$$

$\therefore$ $e_t e_s = \delta_{ts} e_s$ means e_s is ~~a~~ ~~famil~~ family of mutually annihilating projections

Let $A_s = e_s A$. Then $A = \bigoplus_{s\in\Gamma} A_s \oplus A_*$

where $A_* = \bigcap_{s\in\Gamma} \mathrm{Ker}(e_s) = \mathrm{Ker}(\Delta)$ $\quad \Delta = \sum_{s\in\Gamma} s\otimes e_s$

$a_s \in A_s \quad a_t \in A_t$

$$\Delta(a_s)\Delta(a_t) = (s\otimes a_s)(t\otimes a_t)$$

$$\Delta(a_s a_t) = st\otimes a_s a_t$$

$$\therefore \quad st\neq 0 \implies a_s a_t \in A_{st}$$

Try to understand properties of $\sum_{s\in\Gamma} e_s$ i.e. the effect on A of $\mu\in(\mathbb{C}\Gamma)^*$: $\mu(s) = 1 \quad \forall s\in\Gamma$

$$\Delta(a) = \sum_{s\in\Gamma} s\otimes e_s a$$

$$(\mu\otimes 1)\Delta(a) = \sum_{s\in\Gamma} e_s a$$

h_1 $\qquad \sum_{s \in \Gamma} e_s(a'a'')$

$$a'a'' = \left(\sum_s a'_s + a'_*\right)\left(\sum_t a''_t + a''_*\right)$$

$$= \sum_u \left(\sum_{u=st}' a'_s a''_t\right) + \left(\sum_s a'_s\right) a''_* + a'_* \left(\sum_t a''_t\right) + a'_* a''_*$$

$$e_u(a'a'') = \sum_{u=st}' e_s(a')\, e_t(a'')$$

$$\sum_{s \neq *} e_s(a') \sum_{t \neq *} e_t(a'') = \sum_{u \neq *} \overbrace{\sum_{u=st} e_s(a')\, e_t(a'')}^{e_u(a'a'')}$$

$$= \sum_{\substack{s,t \ni \\ st \neq *}} e_s(a')\, e_t(a'')$$

seems OKAY

So it seems that $\sum_{s \neq *}' e_s$ is an alg map from A to itself NO.

$$A = \bigoplus_{s \in \Gamma_+} A_s = \bigoplus_{s \neq *} A_s \oplus A_*$$

Lets begin again. Begin with ① A an alg.

② $\Delta : A \longrightarrow \mathbb{C}\Gamma \otimes A$ a comodule structure on A for $\mathbb{C}\Gamma$: $(\Delta_\Gamma \otimes 1)\Delta = (1 \otimes \Delta)\Delta$

③ Δ alg map.

ℓ_1

$$\Delta a = \sum_{s\neq *} s\otimes e_s a \longrightarrow \sum_{s\neq *} s\otimes s\otimes e_s a$$

$$\parallel$$

$$\longrightarrow \sum_{s\neq *} s\otimes \sum_{t\neq *} t\otimes e_t e_s a$$

$$\Rightarrow \quad e_t e_s a = \begin{cases} 0 & s\neq t \\ e_s a & s=t \end{cases}$$

$$\therefore \Delta = \sum_{s\neq *} s\otimes e_s$$

e_s orth. idemps.

$$\Rightarrow A = \bigoplus_{s\neq *} e_s A \oplus e_* A \qquad e_* = 1-\sum_{s\neq *} e_s$$

$$= \bigoplus_{s\in\Gamma} A_s \oplus A_* = \bigoplus_{s\in\Gamma_+} A_s$$

$$a' = \sum_{s\in\Gamma_+} a'_s \qquad a'' = \sum_{t\in\Gamma_+} a''_t$$

$$a'a'' = \sum_{s,t\in\Gamma_+} a'_s a''_t = \sum_{u\in\Gamma_+}\sum_{u=st} a'_s a''_t$$

$$\boxed{(a'a'')_u = \sum_{u=st} a'_s a''_t}$$

except that in the above you're assuming that A is Γ_+-graded alg. Try to prove this.

$$\Delta a = \bigoplus_{s\in\Gamma} s\otimes e_s a \qquad a = \bigoplus_{s\in\Gamma_+} e_s a$$

$$\tilde{\Delta} a = \bigoplus_{s\in\Gamma_+} s\otimes e_s a = \bigoplus_{s\in\Gamma} s\otimes e_s a + [*]\otimes e_* a$$

j1 Assume A an algebra, $\Delta : A \to \mathbb{C}\Gamma \otimes A$ a comodule structure on the underlying vector space of A for $\mathbb{C}\Gamma$, Δ algebra map.

$$\Delta a = \sum_{s\in\Gamma} s \otimes e_s a \qquad e_s \in \mathcal{L}(A)$$

$$\forall a \left(\{s \mid e_s a \neq 0\} \text{ is fin.}\right)$$

$(\Delta\otimes 1)\Delta a = (1\otimes\Delta)\Delta a$ means

$$e_t e_s a = \begin{cases} e_s a & t=s \\ 0 & t\neq s \end{cases} \qquad \therefore\ e_s \text{ mut ann proj.}$$

$\therefore$ A has v.s. decomp.

$$A = \bigoplus_{s\in\Gamma_+} A_s \qquad \text{where} \quad A_s = e_s A \quad s\in\Gamma, \qquad A_* = e_* A$$

and $e_* = 1 - \sum_{s\in\Gamma} e_s$.

$\cancel{\Delta a = \sum_{s\in\Gamma} s\otimes a_s \quad \text{where } a_s = e_s a}$

Thus $a = \sum'_{s\in\Gamma_+} a_s \qquad a_s = e_s a$

$$\Delta a = \sum_{s\in\Gamma} s\otimes a_s$$

Now use Δ is an alg map $\qquad \Delta(ab) = (\Delta a)(\Delta b)$.

$\cancel{\neq a_s b_t \text{ if } t\neq s}$

$\cancel{\Delta(a_s b_t) = \sum_{u\in\Gamma} u \otimes e_u(a_s b_t)}$

k_1

$$\Delta(ab) = (\Delta a)(\Delta b)$$

$$\Delta(a_s b_t) = (s \otimes a_s)(t \otimes b_t) = st \otimes a_s b_t$$

$$= \begin{cases} 0 & \text{if } st = * \\ st \otimes a_s b_t & \text{if } st \in \Gamma \end{cases}$$

$$\therefore \quad A_s A_t \subset \begin{matrix} 0 & \text{if } st = * \\ A_{st} & \text{if } st \in \Gamma \end{matrix}$$

$st \neq *$

In general.

$$\cancel{\Delta(a_s b_t) = \sum_u a \otimes e_u \left(\sum_{s,t} st \otimes a_s b_t\right)}$$

~~Start~~ again $\quad \Gamma = M_2 \quad \mathbb{C}\Gamma = M_2\mathbb{C}$

A is the (univ) alg w gen p_{ij} rels $\begin{matrix} p_{ij}p_{kl} = 0 \quad j \neq k \\ p_{ik} = \sum_j p_{ij}p_{jk} \end{matrix}$

$\Delta : A \to \mathbb{C}\Gamma \otimes A$ is the alg map ~~with~~ st $\Delta(p_{ij}) = e_{ij} \otimes p_{ij}$. well defined because the right side elts satisfies the relations. Next Δ is a coaction of $(\mathbb{C}\Gamma, \Delta_\Gamma(s) = s \otimes s)$ on A: $(\Delta_\Gamma \otimes 1)\Delta_A = (1 \otimes \Delta_A)\Delta_A$, because these are alg maps agreeing on the gen. But such a comod str. same as a grading $A = \bigoplus_{s \in \Gamma_+} A_s \quad a = \sum_{s \in \Gamma_+} a_s$

$$\Delta a = \sum_{s \in \Gamma_+} s \otimes a_s$$

It might be better to write ~~say that~~

$$A = \bigoplus_{s \in \Gamma} A_s \oplus A_* \Rightarrow \Delta a = \sum_{s \in \Gamma} s \otimes a_s$$

So Δ amounts to a grading of the v.s. A: $A = \bigoplus_{s \in \Gamma} A_s \oplus A_*$

Such that $a = \sum_{s \in \Gamma} a_s + a_* \implies \Delta a = \sum_{s \in \Gamma} s \otimes a_s$

Thus $A_* = \text{Ker}\, \Delta$.

Next, what does it mean for Δ to be an alg map? $\Delta(ab) = (\Delta a)(\Delta b)$.

This ~~is equivalent~~ reduces to the case where a, b homog.

$a = a_s \quad b = b_t \qquad s, t \in \Gamma$

$$\Delta(a_s b_t) = \Delta(a_s)\Delta(b_t) = (s \otimes a_s)(t \otimes b_t)$$
$$= st \otimes a_s b_t$$

Write $a_s b_t = \sum_{u \in \Gamma} c_u + c_* \qquad c_u \in A_u,\ c_* \in A_*$

Then $\Delta(a_s b_t) = \sum_{u \in \Gamma} u \otimes c_u$. ~~and~~ For this to be $st \otimes a_s b_t$ can only happen where

~~$u = st$ and $c_u = a_s b_t$ $u = st$~~

① $st \neq *$, $u = st$, $c_u = a_s b_t$

② $st = *$ all $c_u = 0$, so $a_s b_t = c_*$

Thus ~~if $st \neq *$ in~~ we have

$st \in \Gamma$ i.e. not $* \implies a_s b_t \in A_{st}$

$st = * \implies a_s b_t \in A_*$

m_1 A is an algebra

$\Delta : A \longrightarrow \mathbb{C}\Gamma \otimes A$ is a coaction of $\mathbb{C}\Gamma$ on the underlying vector space of A, which means $(\Delta\otimes 1)\Delta = (1\otimes\Delta)\Delta$

Let $\Delta a = \sum_{s\in\Gamma} s\otimes e_s a$ e_s operators on A

s.t. $(\forall a)$ $\{s \mid e_s a \neq 0\}$ is finite

Then

$$\left.\begin{aligned}(\Delta\otimes 1)\Delta a &= \sum_{s\in\Gamma} s\otimes s\otimes e_s a\\ (1\otimes\Delta)\Delta a &= \sum_{s\in\Gamma}(1\otimes\Delta)(s\otimes e_s a)\\ &= \sum_{s\in\Gamma}\sum_{t\in\Gamma} s\otimes t\otimes e_t e_s a\end{aligned}\right\} \Longrightarrow e_t e_s = \begin{cases} e_s & t=s\\ 0 & t\neq s\end{cases}$$

Thus $\exists$ splitting $A = \bigoplus_{s\in\Gamma} A_s \oplus A_*$ $\quad A_s = e_s A$, $A_* = e_* A$

where $e_* = 1 - \sum_s e_s$. If $a = \sum_{s\in\Gamma} a_s + a_*$, then

$$\Delta a = \sum_{s\in\Gamma} s\otimes a_s$$

Say $\exists$ equiv between coactions Δ and gradings of A indexed by $\Gamma_+ \simeq \Gamma \amalg \{*\}$ given by

$$\Delta = \sum_{s>0} s\otimes e_s$$

Now assume Δ alg hom. $\Delta(a)\,\Delta(a') = \Delta(aa')$ reduces to case a, a' homog.

$$\Delta(a_s a'_t) = (s\otimes a_s)(t\otimes a'_t) = st\otimes a_s a'_t$$

$$\overset{\shortparallel}{\sum_u} u\otimes(a_s a'_t)_u$$

$\mathbf{0}_*$ What does it mean for Δ to be an alg map? Ans $\Delta(aa') = (\Delta a)(\Delta a')$, which reduces to the case $a = a_s$ $a' = a'_t$ $s,t \in \Gamma_+$

$$(\Delta a_s)(\Delta a'_t) = st \otimes a_s a'_t$$

$$\Delta(a_s a'_t) = \sum_{u\in\Gamma} u \otimes e_u(a_s a'_t)$$

If $st = 0$, then $e_u(a_s a'_t) = 0$ $\forall u \in \Gamma \Rightarrow a_s a'_t \in A_*$

$st \neq 0$, $e_u(a_s a'_t) = \begin{cases} 0 & u \neq st \\ a_s a'_t & u = st. \end{cases}$

$$\therefore \quad A_s A_t \subset \begin{cases} A_{\mathbf{0}} & st = 0 \\ A_{st} & st \neq 0. \end{cases}$$

---

back to $\Gamma = M_2$ $\mathbb{C}\Gamma = M_2\mathbb{C}$

A is alg gen by p_{ij} subj to rels. | supp idem

~~[illegible]~~ Let $\Delta : A \longrightarrow \mathbb{C}\Gamma \otimes A$ be alg map st $\Delta(p_{ij}) = e_{ij} \otimes p_{ij}$. Then Δ is a comodule structure on the V.S. A, so $\exists$ splitting $A = \bigoplus_{s\in\Gamma_+} A_s$ char by

$\Delta = \sum_{s\in\Gamma} s \otimes e_s$. Because Δ is an alg map you have $\Delta(a_s a'_t) = (s\otimes a_s)(t\otimes a'_t) =$

$st \otimes a_s a'_t$. Better might be

$$a = \sum_{s\in\Gamma_+} a_s \qquad a' = \sum_{t\in\Gamma_+} a'_t \qquad aa' = \sum_{u\in\Gamma_+} \sum_{u=st} a_s a'_t$$

$$\Delta a = \sum_{s\in\Gamma} s \otimes a_s \qquad \Delta a' = \sum_{t\in\Gamma} t\otimes a'_t,$$

$$\Delta(aa') = \sum_{s\in\Gamma}\sum_{t\in\Gamma} st \otimes a_s a'_t = \sum_{u\in\Gamma} u \otimes \sum_{u=st} a_s a'_t$$

n_1

$$A = \bigoplus_{s\in\Gamma} A_s \oplus A_*$$

$$\Delta a = \sum_{s\in\Gamma} s\otimes a_s + 0$$

$$\Delta a' = \sum_{t\in\Gamma} t\otimes a'_t$$

$$\Delta(aa') = \sum_{s,t\in\Gamma\times\Gamma} st\otimes a_s a'_t$$

$$= \sum_{\substack{s,t\in\Gamma\\ st\neq 0}} st\otimes a_s a'_t$$

$$= \sum_{u\in\Gamma} u\otimes \sum_{u=st} a_s a'_t$$

$$\therefore (aa')_u = \sum_{u=st} a_s a'_t$$

$$\Delta : A \longrightarrow \mathbb{C}\Gamma\otimes A$$

$$\Delta a = \sum_{s\in\Gamma} s\otimes e_s a \qquad \forall a \ \{s \mid e_s a\neq 0\} \text{ is finite}$$

$$\left.\begin{array}{l}(\Delta\otimes 1)\Delta a = \sum_s s\otimes s\otimes e_s a\\ (1\otimes\Delta)\Delta a = \sum_s s\otimes \sum_t t\otimes e_t e_s a\end{array}\right\} \Longrightarrow e_t e_s = \begin{cases}0 & t\neq s\\ e_s & t=s\end{cases}$$

$$\therefore A = \bigoplus_{s\in\Gamma} e_s A \oplus e_* A \qquad e_* = 1 - \sum_{s\in\Gamma} e_s$$

$$\Delta e_t a = t\otimes e_t a \qquad t\in\Gamma$$

$$\Delta e_* a = \sum_{s\in\Gamma} s\otimes e_s(e_* a) = 0$$

P_1 $\Delta : A \longrightarrow \mathbb{C}\Gamma \otimes A$ comult.

yields $A = \bigoplus_{s \in \Gamma_+} A_s$ $\ni$ $\Delta = \sum_{s\in\Gamma_+} s \otimes e_s = \sum_{s \in \Gamma} s \otimes e_s$

$a a' = \sum_{s,t \in \Gamma_+} a_s a'_t$, $\Delta(aa') = \sum_{s,t\in\Gamma_+} (s \otimes a_s)(t \otimes a'_t)$

$= \sum_{u \in \Gamma_+} u \otimes \left(\sum_{\substack{u = st \\ \text{in } \Gamma_+}} a_s a'_t \right)$ $\therefore$

So it seems that $A_s A_t \subset A_{st}$

Repeat. $\Delta : A \longrightarrow \mathbb{C}\Gamma \otimes A$ is a coaction ~~comodu~~ by $\mathbb{C}\Gamma$ on the v.s. A. You know $\Delta = \sum_{s \in \Gamma} s \otimes e_s$

where $e_t e_s = \begin{cases} 0 & t \neq s \\ e_s & t = s \end{cases}$ orth fam. of proj.

$\therefore \exists$ $A = \bigoplus_{s \in \Gamma_+} A_s$ $\quad A_s = e_s A \quad s \in \Gamma$

$A_0 = (1 - \sum_{s \in \Gamma} e_s) A$

unique sp. $\ni$ $\Delta = \sum_{s \in \Gamma_+} s \otimes e_s$.

Relation between $\mathbb{C}\Gamma$ and $\mathbb{C}[\Gamma_+]$ ~~its~~

coalg structure: look dually: $(\mathbb{C}\Gamma)^* =$ fns on Γ

$\mathbb{C}[\Gamma_+]^* =$ fns on Γ_+, get $(\mathbb{C}\Gamma)^* \hookrightarrow (\mathbb{C}\Gamma_+)^*$

as fns. vanishing at $*$. So get ~~com a~~ coalg

map $\mathbb{C}\Gamma_+ \twoheadrightarrow \mathbb{C}\Gamma$ sending ~~$s\in\Gamma$ to s~~

$s \in \Gamma \subset \Gamma_+$ to $s \in \Gamma$

$* \in \Gamma_+$ to 0.

$\S 1$ If C is a coalgebra there is a counital coalg obtained by adjoining a counit η $\tilde{C} = C \times \mathbb{C}$. ~~...~~ Equivalence between comodules for C and counital comodules for $\tilde{C}$.

$$\begin{tikzcd} M \arrow[r, "\Delta"] \arrow[dr, "\tilde{\Delta}"'] & C \otimes M \\ & \tilde{C} \otimes M \arrow[u, "p \otimes 1"'] \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & \mathbb{C}\sigma \arrow[r] & \tilde{C} \arrow[r] \arrow[l, bend right, "\eta"'] & C \arrow[r] & 0 \end{tikzcd}$$

$$\Delta\sigma = \sigma \otimes \sigma$$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & \tilde{A} \arrow[r] & \mathbb{C}1 \arrow[l, bend right] \arrow[r] & 0 \end{tikzcd}$$

What properties should $\tilde{C}$ have.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \arrow[r, "\sigma"] & \tilde{C} \arrow[r, "\text{coalg map}"'] \arrow[l, bend right, "\eta"'] & C \arrow[r] & 0 \end{tikzcd}$$

The problem: You define $\Delta : A \longrightarrow \mathbb{C}\Gamma \otimes A$ a coaction of the coalg $\mathbb{C}\Gamma$ on the underlying vector space of A. noncounital coaction, compat with ~~...~~ alg structures. Δ should automatically extend to a counital coaction $\tilde{\Delta} : A \longrightarrow \widetilde{\mathbb{C}\Gamma} \otimes A$. The problem is to see that $\widetilde{\mathbb{C}\Gamma}$ ~~is an alg and that~~ ~~...~~ has a natural alg structure compatible with the coalg str on $\widetilde{\mathbb{C}\Gamma}$, and ~~the~~ comp. with $\tilde{\Delta}$.

Review the problem. You have a semigroup $\Gamma_+ = \Gamma \amalg \{*\}$ where $*$ is absorbing. Let $\mathbb{C}\Gamma = \mathbb{C}[\Gamma_+]/\mathbb{C}[*]$ be the associated bialg, uses $\mathbb{C}[*]$ is an ideal in $\mathbb{C}[\Gamma_+]$ to get a product on $\mathbb{C}\Gamma$, why is the quotient a coalgebra? ~~doesn't~~ seems obvious

$$\begin{tikzcd} \mathbb{C}[\Gamma_+] \arrow[r, "\Delta"] \arrow[d] & \mathbb{C}[\Gamma_+]\otimes\mathbb{C}[\Gamma_+] \arrow[d] \\ \mathbb{C}[\Gamma_+]/\mathbb{C}[*] \arrow[r, "\bar{\Delta}?"] & (\mathbb{C}[\Gamma_+]/\mathbb{C}[*])\otimes(\mathbb{C}[\Gamma_+]/\mathbb{C}[*]) \end{tikzcd}$$

Look ~~at intri~~ at the tensor category picture.

Γ_+ is a semigroup, $\mathbb{C}\Gamma_+$ is a counital coalg, $\Delta s = s\otimes s$ whose counital comodules are graded vector spaces $V = \bigoplus_{s\in\Gamma_+} V_s$ with resp to Γ_+.

Focus upon set like coalgs. $S \longleftrightarrow \mathbb{C}S$

~~S Γ_+. Formulate.~~

You should understand adjoining a counit to a coalgebra D.

Review the problem, concrete problem. You are given $\mathbb{C}\Gamma$, a set-like coalg, equipped with an assoc. product $\Delta: \mathbb{C}\Gamma\otimes\mathbb{C}\Gamma \to \mathbb{C}\Gamma$ which is a ~~coalg non-counital coalgebra~~ map of coalgs, but does not respect counits. Thus Δ is equivalent to an associative product $\Gamma_+ \wedge \Gamma_+ \to \Gamma_+$

To the coalg $\mathbb{C}\Gamma$ belongs the cat of its counital comodules, which $\cong$ Γ-graded modules $V = \bigoplus_{s\in\Gamma} V_s$

Then Δ corresp to a $\otimes$ prod op.

$$V \otimes W = \bigoplus_{s,t \in \Gamma} V_s \otimes W_t = \bigoplus_{u \in \Gamma} \left(\bigoplus_{u=st} V_s \otimes W_t \right) \oplus \bigoplus_{0=st} V_s \otimes W_t$$

~~[illegible]~~ : Review:

Start with, consider a semigroup S with absorbing element $*$. Let $\Gamma = S - \{*\}$, ~~so~~ $S = \Gamma_+$.

$\mathbb{C}[S]$ is a counital coalg

First step: S set have $\mathbb{C}S$ coalg (coass) cocomm counital, ~~[illegible]~~ whose counital comodules are S-graded vector spaces: $V = \bigoplus_{s \in S} V_s$, whose comodules are ~~[illegible]~~ graded wrt $S_+ = S \amalg \{*\}$: $V = \bigoplus_{s \in S} V_s \oplus V_*$

$$\Delta : V \longrightarrow \mathbb{C}S \otimes V \qquad \Delta = \sum_{s \in S} s \otimes e_s$$

$$S \xrightarrow{f} S' \qquad \mathbb{C}S \longrightarrow \mathbb{C}S'$$

$$V = \bigoplus_{s \in S} V_s \longmapsto V = \bigoplus_{s' \in S'} \left(\bigoplus_{s \in f^{-1}(s')} V_s \right)$$

$$\begin{tikzcd} V \arrow[r, "\Delta"] \arrow[d, equal] & \mathbb{C}S \otimes V \arrow[d, "f \otimes 1"] \\ V \arrow[r, "\Delta'"] & \mathbb{C}S' \otimes V \end{tikzcd}$$

$$\xi \longmapsto \sum_s s \otimes e_s \xi$$
$$\downarrow$$
$$\sum_s f(s) \otimes e_s \xi$$
$$\shortparallel$$
$$\sum_{s'} s' \otimes \sum_{f(s)=s'} e_s \xi$$

Go back to the original problem

$\Gamma = M_2 \qquad \mathbb{C}\Gamma = M_2\mathbb{C} \qquad A$ alg gen by p_{ij}

rel $p_{ij}p_{k\ell} = 0 \quad j \neq k, \quad \sum_j p_{ij}p_{jk} = p_{ik}$.

Define $\Delta : A \longrightarrow \mathbb{C}\Gamma \otimes A$ to be the alg map

$\Delta(p_{ij}) = e_{ij} \otimes p_{ij}$. well-defined since rels satisfied

Check $(\Delta_\Gamma \otimes 1)\Delta = (1 \otimes \Delta)\Delta$ so ~~you have~~

there is a unique splitting $A = \bigoplus_{s \in \Gamma} A_s \oplus A_0$

such that $\Delta a = \sum_{s \in \Gamma} s \otimes e_s a$. Note $A_0 = \mathrm{Ker}(\Delta)$

What is the problem? Use the fact Δ respects mult. to show that A/A_0 is a Γ-graded alg., and that it's the universal Γ-graded alg ~~generated~~ ~~the components of a proj~~ representing projections in any Γ-graded alg.

Let B be a Γ-graded alg equipped with a proj $p = \sum p_{ij} \in B$. One has splitting

$$B = \bigoplus_{s = ij} B_{s=ij} \qquad \Delta_B = \sum_{s \in \Gamma} s \otimes e_s$$

such that $B_s B_t \subset \begin{cases} B_{st} & \text{if } st \in \Gamma \\ 0 & \text{if } st = 0 \end{cases}$

Maybe your point of departure is wrong. namely you should start with B?

Start again with a Γ graded alg B equipped with a proj :

~~$A : B \to \mathbb{C}\Gamma \otimes B$~~

$$B = \bigoplus_{s\in\Gamma} B_s \qquad \Delta = \sum_{s\in\Gamma} s \otimes e_s : B \longrightarrow \mathbb{C}\Gamma \otimes B$$

$$B_s B_t \subset \begin{cases} B_{st} & \text{if } st \neq 0 \text{ i.e. } st \in \Gamma \\ 0 & \text{if } st = 0 \end{cases} \qquad \Delta(bb') = (\Delta b)(\Delta b')$$

also given $\quad p = \sum_{s\in\Gamma} p_s \in B = \bigoplus_{s\in\Gamma} B_s$

Start again bialg $\mathbb{C}\Gamma$ assoc. to semigroup $\Gamma_+ = \Gamma \amalg \{*\}$ such that $*$ is absorbing.

$B = \bigoplus_{s\in\Gamma} B_s$ is a Γ-graded alg : $B_s B_t \subset \begin{cases} B_{st} & st \neq 0 \\ 0 & st = 0 \end{cases}$

coaction associated to the grading

$$\Delta : B \longrightarrow \mathbb{C}\Gamma \otimes B$$

$$\Delta = \sum_{s\in\Gamma} s \otimes e_s$$

let $\quad p = \sum_{s\in\Gamma} p_s \quad$ be a projection in B $\qquad p = p^2$

$$p_s = e_s p$$

$$\Delta p = \sum_{u\in\Gamma} u \otimes p_u = \sum_{s,t\in\Gamma} st \otimes p_s p_t$$

$$p_u = \sum_{u = st} p_s p_t$$

To construct a universal Γ-graded alg $\mathcal{P}_\Gamma$ representing projections in any Γ-graded algebra. Define A by generators $p_s \; s\in\Gamma$ ~~[illegible]~~ relation

The generators and relations are homog wrt Γ

so at first sight A should be Γ graded, Try to do this by defining $\Delta : A \longrightarrow \mathbb{C}\Gamma \otimes A$

$$\Delta(p_s) = \sum_{s\in\Gamma} s \otimes p_s$$

$$\sum_{u=st} \Delta(p_s)\Delta(p_t) = \sum_{u=st} \overset{u}{\cancel{st}} \otimes p_s p_t = \overset{\Delta(p_u)}{u \otimes p_u}$$

so Δ is well-defined. Also $(\Delta\otimes 1)\Delta = (1\otimes\Delta)\Delta$ so that Δ is a coaction. Therefore you know that $\exists!$ Γ_+ grading $A = \bigoplus_{s\in\Gamma} A_s \oplus A_0$ such that

$$\Delta = \sum_{s\in\Gamma} s \otimes \cdot_s \quad \Longrightarrow \quad A_0 = \operatorname{Ker}\Delta$$

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & \mathbb{C}\Gamma \otimes A \\ \downarrow & & \downarrow \\ A/A_0 & \xrightarrow{\bar{\Delta}} & \mathbb{C}\Gamma \otimes A/A_0 \\ A_s & \xrightarrow{\sim} & s \otimes A_s \end{array}$$

Thus P_Γ should be A/A_0

A is spanned by words in the generator p_s $s\in\Gamma$

each word, has a total degree in Γ_+

$$p_{s_1}\cdots p_{s_n} \qquad s_1 \ldots s_n \in \Gamma_+$$

A_s is spanned by all words of total degree s

Let's review the little progress so far. Suppose given a set Γ ~~and semi~~ equipped with a semigroup structure (associative product) on $\Gamma_+ = \Gamma \amalg \{*\}$ such that the basepoint $*$ is absorbing. Then on $\mathbb{C}\Gamma$ one has a bialgebra structure. First description: coproduct $\Delta : \mathbb{C}\Gamma \to \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma$

$\Delta s = s \otimes s$ for $s \in \Gamma$, product: the product on Γ_+ extends linearly to ~~an~~ alg structure on $\mathbb{C}[\Gamma_+]$; $*$ absorbing in $\Gamma_+ \implies \mathbb{C}[*]$ is an ideal in $\mathbb{C}[\Gamma_+]$; so $\mathbb{C}[\Gamma_+]/\mathbb{C}[*]$ is an algebra; identification natural $\mathbb{C}[\Gamma] \simeq \mathbb{C}[\Gamma_+]/\mathbb{C}[*]$.

$$\boxed{[s][t] = \begin{cases} [st] & \text{if } st \in \Gamma \\ 0 & \text{if } st = *. \end{cases} \quad \text{for } s,t \in \Gamma}$$

2nd description

$\mathbb{C}\Gamma$ - comodules

What should you be doing? Introducing the bialgebra $\mathbb{C}\Gamma$ where Γ is a set equipped with an assoc. product on $\Gamma_+ = \Gamma \amalg \{0\}$ such that 0 is absorbing.

$\mathbb{C}\Gamma$-module M same as vector space with ~~action $\Gamma \to \mathrm{End}(M)$, $s \mapsto (m \mapsto sm)$~~ operators $s : m \mapsto sm$ compatible with product: ~~$(st)m$~~ $s(tm) = (st)m$ where st denotes product in $\Gamma_+ = \Gamma \amalg \{0\}$.

~~$\mathbb{C}\Gamma$ comodules same as~~

Simplest is to say that $\mathbb{C}\Gamma$-modules are vector spaces with Γ_+ action

Question: Do $\mathbb{C}\Gamma$ modules have a natural tensor product operation corresponding to $\Delta s = s \otimes s$? This seems clear. Role of counit η of $\mathbb{C}\Gamma$?

Project which Grothendieck must understand, namely, to interpret bialgebras in terms of additive categories with $\otimes$. Galois picture of motives.

Take example $\Gamma = M_2$ so that $\mathbb{C}\Gamma$-modules are equivalent to vector spaces. What is $\otimes$ on $\mathbb{C}\Gamma$-modules arising from Δ?

Recall Γ set equipped with an assoc. product on $\Gamma_+ = \Gamma \amalg \{0\}$ such that $0s = s0 = 0 \quad \forall s \in \Gamma$.

$\mathbb{C}\Gamma$ is naturally a bialg

$$\Delta : \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \qquad \Delta s = s \otimes s$$

$$\mu : \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma \qquad s \otimes t \longmapsto st$$

~~What not Correct way to think is that~~ Good viewpoint is that $\mathbb{C}\Gamma = \mathbb{C}\Gamma_+ / \mathbb{C}[0]$ is more than a functor of the set Γ, is a functor of the pointed set.

~~What best of mathematics~~ ~~of madness~~ Go over stuff you know, perhaps clean up languages. The last idea you had involved tensor category picture for bialgebras. Groupoid version, focus upon, because of Connes's success and your letter to Serre.

~~Better~~ ~~more~~ ~~problems~~ Begin with Grothendieck Galois theory picture. You can identify a group G with the category of G-sets equipped with a fibre functor. This generalizes to ~~groupoids~~ a groupoid Γ as follows. First you have ~~want~~ left C-sets = cat of covariant functors C ⟶ Sets, and right C-sets i.e. C^{op}-sets, ~~tensor~~ pairing $R \times_C L$, gives all ~~ctant~~ funs from C-Sets to Sets. ~~You~~ ~~know~~ ~~topos~~ ~~picture~~ Yoneda

$$h : C^{op} \longrightarrow C\text{-Sets}, \quad X \mapsto h^X = (Y \mapsto \mathrm{Hom}(X,Y)).$$

Gives points in the topos C-Sets, which can be extended to an equivalence between Pro C and Homtop(Sets, C-sets). When C is a groupoid

Pro C = C

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all kinds of things to review

main idea is to generalize to groupoids what has been done for groups. ~~review~~ recover letter to ~~basic~~ ~~tensor~~ ~~picture~~ Serre

group case: here you recover a group G from the category of G-Sets ~~and a~~ together with a fibre functor. Then there's the linearized version where an algebraic group can be recovered from the tensor category of its representations.

Set picture - G group. Then G is equivalent to the categ G-sets tog w a fibre functor. Γ groupoid. Then Γ is equivalent to the groupoid of fibre functors on Γ-sets.

Review the idea - keep it alive until all tax stuff is done. The aim is to generalize "Groth theory for groups" to groupoids. Begin with: a group G is equivalent to the category G-sets together with a fibre functor. Next consider the linearized version

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A group G is equivalent to the category of G-sets together with a fibre functor. ~~Next~~ Next the linearized version. You want to recover G from its representations. Maybe start with the category of all G module (over $\mathbb{C}$), i.e. unital $\mathbb{C}G$ modules. ~~finiteness conditions~~ Things will be confused because of finiteness conditions. Groth envisaged fin. diml reps of G - these the linearized analog ~~version~~ of finite G-sets, so the appropriate group algebra is profinite diml, and best ~~understood~~ handled by passing to the dual. "Tannaka" duality. The finite diml reps of G are comodules for a commutative Hopf algebra.

Look at finite diml reps of G, functions on G whose translates span a fin. diml space, representative fns.

Go over problem to keep the math alive. Begin with Groth picture for ~~Galois theory~~ π_1, namely, a group G is equivalent to the category of G-sets together with a fibre functor. There is also the profinite version with finite G-sets.

Next linearize: the Hopf algebra ~~group ring~~ $\mathbb{C}G$ should be equivalent to the category of G-modules together with a fibre functor, ~~meaning~~ which means a functor to vector spaces respecting $\otimes$. Consider also ~~more~~ the case of finite dimensional reps of G; linear analog of profinite case. Related to ~~~~ proalg groups. Then it's natural to ~~~~ use ~~~~ appropriate dual of $\mathbb{C}G$ - Tannaka alg of rep functions $A(G)$ and representations are comodules: $V \longrightarrow A(G) \otimes V$

G a discrete group. G is equivalent to the topos of G sets ~~~~ equipped with a fibre functor. $\hat{G}$, the profinite completion of G, should be equivalent to the category of finite G-sets equipped with a fibre functor. ~~There's something missing~~

Here you are missing Grothendieck's list of properties which characterize the category of G-sets, or finite G-sets. For example what properties of the category of finite separable extensions of a field k enable you to construct an equivalence with finite $\mathrm{Gal}(\bar{k}/k)$ sets.

Question. How do you know when a category $\mathcal{C}$ is the category of G-sets for some group G?

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where next? Review the ~~[illegible]~~ problem. You want to generalize what Grothendieck did for groups to groupoids. His picture of π_1.

~~[illegible]~~ What is the basic idea? You were trying to describe to Berrick? Thm. that ~~[illegible]~~ any compact ANR has the homotopy type of a finite complex. inverse system of finite complexes.

Recall something ~~[illegible]~~ Wasting too much time

Look at $\Gamma = M_2$ $\quad \mathbb{C}\Gamma = M_2\mathbb{C}$

Consider ~~good~~ reduced $\mathbb{C}\Gamma$ modules, ~~which~~ that is, unital modules over $\mathbb{C}\Gamma$. From the ~~coprod~~ comult $\Delta : \mathbb{C}\Gamma \to \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma$ you get a tensor prod operation on these Δs modules, namely on $M \otimes N$ s operates as $s \otimes s$.

But we have Morita equivalence $M = T \otimes V$ where $V = T^* \otimes_\Lambda M$ $\quad \Lambda = \mathbb{C}\Gamma$. So your $\otimes$ operation ~~is~~ viewed on vector spaces is

$$V, W \longmapsto (T \otimes V) \otimes (T \otimes W) = T \otimes T \otimes V \otimes W$$

~~So the question is what~~ $T \otimes T$ is a Λ-module via Δ, so $T \otimes T = T \otimes \underbrace{\left(T^* \otimes_\Lambda (T \otimes T)\right)}_{\text{2 diml space}}$

Puzzling. $\quad Q = T^* \otimes_\Lambda (T \otimes T)$. ~~What kind~~

V, W, X

$Q \otimes (Q \otimes (V \otimes W))$