So where are you? You have a space B a $\mathcal{G}^{op}$-sheaf R over B, which means a sheaf over B equipped with a map $R \xrightarrow{s}$ constant sheaf $\mathcal{G}_0$ ~~and a~~ (in terms of etale spaces/B, $R \longrightarrow B \times \mathcal{G}_0$, and a map $R \underset{s \searrow \mathcal{G}_0 \swarrow t}{\times} \mathcal{G}_1 \longrightarrow R$ assoc. + identity stuff

$$R_X \times \mathcal{G}_{1,X,Y} \longrightarrow R_Y$$

$$\mathcal{G}_{1,X,Y} = \mathrm{Hom}_{\mathcal{G}}(Y,X)$$

$$R_X \times \mathcal{G}_{X,Y} \times \mathcal{G}_{Y,Z} \longrightarrow R_Z$$

$$R(X) \times \mathrm{Hom}(Y,X) \times \mathrm{Hom}(Z,Y) \longrightarrow R(Z)$$

$$\underbrace{(\xi, \quad f}, \quad g)$$

$$Z \xrightarrow{g} Y \xrightarrow{f} X$$

$$(f^*\xi, g)$$

$$\underbrace{R(Y) \times \mathrm{Hom}(Z,Y)}$$

$$Z \xrightarrow{fg} X$$

$$\downarrow$$

$$R(Z)$$

$$g^*(f^*\xi) = (fg)^*\xi,$$

Still very confused.

Where are you? You are looking at a fun. $\mathcal{G}^{op} \longrightarrow Sh_B$, ~~such~~ i.e. a $\mathcal{G}^{op}$-sheaf R over B. Thus R is a sheaf over B equipped with $R \underset{\mathcal{G}_0}{\times} \mathcal{G}_1$

What do you have? You have a functor $R : \mathcal{G}^{op} \longrightarrow Sh_B$ (locally representable?)

What precisely is R? R consists of [redacted] sheaves R_X $\forall X \in \mathcal{G}_0$ and maps

$$\mu_{XY} : R_X \times \mathcal{G}_{XY} \longrightarrow R_Y \qquad \mathcal{G}_{XY} = \mathrm{Hom}_{\mathcal{G}}(Y, X)$$

satisfying id and assoc. conditions.

Locally representable? First do for a point. ~~You want a point~~ Fix a pt $b \in B$, then R_X becomes a set $\forall X$, $R_X . \mu_{XY}$ define a fun $\mathcal{G}_b^{op} \longrightarrow$ sets. Rep. means. $\exists X \in \mathcal{G}_0$, and $\xi \in R_X$ such that $\forall Y$

$$\begin{aligned} \mathrm{Hom}(Y, X) &\xrightarrow{\sim} R_Y \\ f &\longmapsto f^*\xi \end{aligned}$$

$$\begin{array}{ccccc} \mathrm{Hom}(Y, X) & \longrightarrow & \mathrm{Hom}(R_X, R_Y) & \longrightarrow & R_Y \\ f & & f^* & & f^*\xi \end{array}$$

I think the good way to proceed is to form ~~the~~ $\mathcal{G}/R$ and to worry about this having a ~~final~~ final object locally.

Review notation. $R : \mathcal{G}^{op} \longrightarrow Sh_B$. Picture of $\mathcal{G}$

$$\mathcal{G}_2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{G}_1 \;\substack{\longrightarrow\\ \longrightarrow}\; \mathcal{G}_0$$

~~Simp~~ nerve of a category $\mathcal{C}$

recall $\mathcal{C}$ consists of Ob, Ar, $id, s, t, \circ$

What you need is a notation that fit well with modules.

$$Y \qquad X \qquad X \times_Y X \qquad X \times_Y X \times_Y X$$

You want a composition notation, product, which

$$R \times^{\mathcal{C}} L \qquad\qquad Ar = \coprod_{(x_0, x_1)} \mathcal{G}'(x_0, x_1)$$

~~Notation of $\mathcal{G}(x)$~~

$$\mathcal{O} \leftleftarrows \mathcal{A} \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \mathcal{A} \times_{\mathcal{O}} \mathcal{A}$$

or

$$\begin{tikzcd} \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathcal{A} \times_{\mathcal{O}} \mathcal{A} \arrow[r, "pr_1"] \arrow[r, shift left] \arrow[r, shift right, "pr_2"'] & \mathcal{A} \arrow[r, shift left] \arrow[r, shift right] & \mathcal{O} \end{tikzcd}$$

you write down sets. A set $\mathcal{O}$ of objects a set $\mathcal{A}$ of arrows. Each arrow has source and target. You have to decide conventions about composable arrows. Try $\mathcal{A} \times_{\mathcal{O}} \mathcal{A} =$

$$\{(f, g) \in \mathcal{A} \times \mathcal{A} \mid s(f) = t(g)\} = \{ \xleftarrow{f} \xleftarrow{g} \}$$

When you have $\mathcal{A} \rightrightarrows \mathcal{O}$

$$d_0(x' \longleftarrow x) = x$$

$$d_1(x' \longleftarrow x) = x'$$

$$\begin{tikzcd} & (x'' \xleftarrow{f} x' \xleftarrow{g} x) \arrow[dl, "d_0"'] \arrow[d, "d_1"] \arrow[dr, "d_2"] & \\ (x' \xleftarrow{g} x) & (x'' \xleftarrow{fg} x) & (x'' \xleftarrow{f} x') \end{tikzcd}$$

$$d_0 = s$$

$$d_1 = t$$

You ~~...~~ want a notation that will enable transition from categories to rings.

R right module over A, L left module

$$R\otimes A\otimes A\otimes L \begin{array}{c}\xrightarrow{d_0}\\ \xrightarrow{d_1}\\ \xrightarrow{d_2}\end{array} R\otimes A\otimes L \begin{array}{c}\xrightarrow{d_0}\\ \xrightarrow{d_1}\end{array} R\otimes L$$

and the faces d_i replace $\otimes$ by $\cdot$

$$d_0 d_2 (r\otimes a\otimes a'\otimes l) = d_0(r\otimes a\otimes a'l) = ra\otimes a'l$$

$$d_1 d_0 (\text{———}) = d_1(ra\otimes a'\otimes l) = ra\otimes a'l$$

What's different ~~...~~ involves the set of objects. There seems to be something interesting here. When you pass from a category $\mathcal{C}$ to its arrow ring $\mathbb{Z}[C]$ the partially defined composition in the category is extended by zero.

$$\mathcal{G}^{op} \overset{h}{\hookrightarrow} \mathcal{G}^\wedge \xrightarrow{f^*} Sh_B \qquad f^*h : \mathcal{G}^{op} \longrightarrow Sh_B$$

$$X \mapsto h^X \longmapsto f^*(h^X) \qquad \overset{\shortparallel}{R}$$

R is $\mathcal{G}^{op}$-sheaf (gen. of $\mathcal{G}^{op}$-set)

What does a $\mathcal{G}^{op}$-set look like?

$$R \leftleftarrows R\times_{\mathcal{O}} \mathcal{A} \leftleftarrows R\times_{\mathcal{O}}\mathcal{A}\times_{\mathcal{O}}\mathcal{A} \qquad \text{nerve of } \mathcal{G}/R$$

$$\mathcal{O} \leftleftarrows \mathcal{A} \leftleftarrows \mathcal{A}\times_{\mathcal{O}}\mathcal{A} \qquad \text{nerve of } \mathcal{G}$$

$\mathcal{G}$ groupoid. $\mathcal{C}$ category. Ob Ar

You have to decide on source + target in some sense. Given an ordered pair (X, Y) of Objects you have a set $\mathcal{C}(X, Y)$ of arrows, you have to say the direction of the arrows – usually X is the source and Y is the target. This is relevant for composition.

$$\begin{array}{ccccc} \mathcal{C}(X,Y) & \times & \mathcal{C}(Y,Z) & \longrightarrow & \mathcal{C}(X,Z) \\ f & & g & & gf \end{array}$$

$$\boxed{\mathcal{C}^{op}(X,Y) = \mathcal{C}(Y,X)}$$

$$\begin{array}{ccccc} \mathcal{C}^{op}(X,Y) & \times & \mathcal{C}^{op}(Y,Z) & \longrightarrow & \mathcal{C}^{op}(X,Z) \\ \parallel & & \parallel & & \parallel \\ \mathcal{C}(Y,X) & \times & \mathcal{C}(Z,Y) & & \mathcal{C}(Z,X) \end{array}$$

$$\begin{array}{cccccc} f & & g & & & gf \\ \mathcal{C}(X,Y) & \times & \mathcal{C}(Y,Z) & \longrightarrow & \mathcal{C}(X,Z) & \big\downarrow \\ & & & & & (gf)^t \\ \mathcal{C}^{op}(Y,X) & \times & \mathcal{C}^{op}(Z,Y) & \longrightarrow & \mathcal{C}(Z,X) & \\ f^t & & g^t & & f^t g^t & f^t g^t \end{array}$$

~~[illegible]~~ problem then arising from the notation used for the composition

It won't make any difference for a groupoid because $\mathcal{G}$ and $\mathcal{G}^{op}$ are isom. categories.

$$\begin{tikzcd}
(f, g) \arrow[r, mapsto] & gf \\
\mathcal{G}^{op}(X,Y) \times \mathcal{G}^{op}(Y,Z) \arrow[r] & \mathcal{G}^{op}(X,Z) \arrow[d, "\cong"] \\
\mathcal{G}(Y,X) \times \mathcal{G}(Z,Y) \arrow[r] & \mathcal{G}(Z,X) \\
(f^{-1}, g^{-1}) &
\end{tikzcd}
\qquad
\begin{array}{c} gf \\ \downarrow \\ (gf)^{-1} \\ \| \\ f^{-1}g^{-1} \end{array}$$

decide on simplest notation

What happens in a category is that given a triple of objects (X, Y, Z) and maps $f: X \to Y$ and $g: Y \to Z$, there is a composite map from X to Z. The problem is whether to denote the composition of f first ~~then~~ followed by g as gf or fg. Functions lead to $g(f(x)) = (gf)(x)$.

~~The way to proceed might be~~

Where to start? Suppose given $R: \mathcal{G}^{op} \longrightarrow Sh_B$ ie a $\mathcal{G}^{op}$-sheaf you form $\mathcal{G}/R$ which is a category (groupoid) object in Sh_B. ~~So what to do Picture~~ Picture, draw nerves.

~~In the case of a point~~ ~~R is a sheaf over~~

What is R? R consists ~~of~~ a family $R(X)$, $X \in \mathcal{O}$ of sheaves

---

Outline. ~~(Nerve of small)~~

C small cat, nerve of C

C cons. of C^{ob}, C^{ar} two sets and 4 maps

$$C^{ob} \underset{\longleftarrow}{\overset{id}{\rightleftarrows}} C^{ar} \xleftarrow{\circ} C^{ar} \times_{C^{ob}} C^{ar}$$

Yesterday you learned that ~~arrows~~ composition of arrows can be ~~written in~~ denoted two ways:

left, corresp. to functions $(fg)(x) = f(g(x))$

right ~~left~~ ops. $x(gf) = (xg)f$

This means that ~~you use two conventions you want~~ ~~two~~ the nerve will depend on your choice. Use left operators.

$$C^{ar} \longleftarrow C^{ar} \times_{\{s,t\}} C^{ar}$$

$$g \xleftarrow{d_0} (f, g) = (x \xleftarrow{f} y \xleftarrow{g} z)$$

$$fg \xleftarrow{d_1}$$

$$f \xleftarrow{d_2}$$

$$C^{ob} \rightleftarrows C^{ar} \rightleftarrows C^{ar} \times_{s,t} C^{ar} \quad \cdots$$

$R : \mathcal{G}^{op} \longrightarrow Sh_B$ means family of sheaves $R(X)$, $X \in \mathcal{G}^{ob}$, and family of maps

$$\underbrace{\mathcal{G}^{ar}(\cancel{Y}, \cancel{X})} \longrightarrow \mathrm{Hom}_{Sh_B}\big(R(Y), R(X)\big)$$

source X
target Y

$$\left(Y \xleftarrow{f} X\right) \longmapsto \cancel{R(X) \xrightarrow{f^*} R(Y)}$$

$$R(Y) \xrightarrow{f^*} R(X)$$

assemble the $R(X)$ into $\quad R^{ob} = \coprod_{X \in \mathcal{G}^{ob}} R(X) \qquad p : R^{ob} \to \mathcal{G}^{ob}$

and the f^* into

$$R^{ar} = \coprod_{(Y \leftarrow X) \in \mathcal{G}^{ar}} R(\cancel{X}) \simeq R^{ob} \times_{(s,t)} \mathcal{G}^{ar}$$

~~vg 1a 9602 5002 9377~~
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~~8733 7000 8524~~
~~9440 260 9219~~

Spend next $\frac{1}{2}$ hour on maths

A = universal alg gen. by the components p_{ij}, $1 \le i, j \le n$, of a proj in a M_2-graded alg

relns. $\quad p_{ik} = \sum_j p_{ij} p_{jk}$, $\quad p_{ij} p_{kl} = 0 \quad j \ne k$.

satisfied by $\quad e_{ij} \in M_n$. $\quad$ Why $\quad e_{ij} = |i\rangle\langle j|$

No

$$e_{ij} e_{kl} = \begin{cases} 0 & j \ne k \\ e_{il} & j = k \end{cases}$$

Define $\quad \Delta : A \longrightarrow \mathbb{C}M_2 \otimes A$

$$\Delta(p_{ij}) = e_{ij} \otimes p_{ij}$$

Define $\Delta : A \longrightarrow \mathbb{C}M_{A} \otimes A$

$$\Delta(p_{ik}) = e_{ij} \otimes p_{ij}$$

$$\Delta(p_{ik}) = e_{ik} \otimes p_{ik}$$

$$\Delta\Big(\sum_j p_{ij} p_{jk}\Big) = \sum_j \underbrace{(e_{ij} \otimes p_{ij})(e_{jk} \otimes p_{jk})}$$

$$= \sum_j e_{ik} \otimes p_{ij} p_{jk} \quad = e_{ik} \otimes p_{ik}$$

So you have this alg. A, which is M_A graded. ~~The next point~~ Recall in the case of group Γ,

$$\Delta : A \longrightarrow \mathbb{C}\Gamma \otimes A$$

$$\Delta(p_s) = s \otimes p_s$$

~~You got~~ ~~You~~ what you did. You made p act internally somehow, Suppose you take an A-module V, i.e. you have operators $p_{ij} \in \mathcal{L}(V)$ satisfying the relations

One thing you have is $\sum_{ij} e_{ij} \otimes p_{ij} = \big(\; p_{ij} \;\big)$

Think: You are so slow. What you are trying to do is to use the universal projection to construct a retraction. So have this A module V, you form $\mathbb{C}M_2 \otimes V$ ~~but it has~~

then apply $\mathbf{p} = \begin{pmatrix} & | & \\ - & p_{ij} & - \\ & | & \end{pmatrix}$ to $M_2 \times V = \begin{pmatrix} & | & \\ - & V & - \\ & | & \end{pmatrix}$

to get something interesting.

review: $A = \overset{\text{univ}}{\text{alg}}$ gen. by components p_{ij} of a proj in a M_2 graded algebra.

$$\Delta : A \longrightarrow \mathbb{C}M_n \otimes A$$
$$p_{ij} \qquad\qquad e_{ij} \otimes p_{ij}$$

$$\sum_j (e_{ij} \otimes p_{ij})(e_{jk} \otimes p_{jk})$$

$$= \sum_j \underbrace{e_{ij} e_{jk}}_{e_{ik}} \otimes p_{ij} p_{jk} = e_{ik} \otimes p_{ik}$$

$\mathbb{C}M_n \otimes A = M_n A$ has a ~~can~~ canonical projection ~~[illegible]~~ namely $\sum e_{ij} p_{ij} = \begin{pmatrix} p_{11} & & p_{1n} \\ & & \\ p_{n1} & & p_{nn} \end{pmatrix}$

Now how can you use this? The first thing the occurs to me is that $M_n \otimes A$ acts on $\mathbb{C}^n \otimes A$ ~~as right~~ commuting with A^{op} action. Column vectors. Q: A unital? ~~The unital~~

$\tilde{A} =$ the $\overset{\text{unital}}{\text{alg.}}$ gen. gen. by $\tilde{p}_{ij}$ + above rels.

$$\Delta : \tilde{A} \longrightarrow \mathbb{C}M_n \otimes \tilde{A}$$

What happens? $\mathbb{C}M_n \otimes \tilde{A} = M_n\mathbb{C} \times M_n A$

$$A \xrightarrow{\Delta} M_n A \hookrightarrow M_n \tilde{A}$$ unital algebra

$$p_{ij} \longmapsto e_{ij} \otimes p_{ij} = e_{ij} \otimes p_{ij}$$

You have a proj $p = \sum_{ij} e_{ij} \otimes p_{ij}$. Let's try using Greek letters for the maps (ij) in the groupoid. $p = \sum e_\alpha \otimes p_\alpha \in M_n A$. You can split $M_n \tilde{A}$ using p.

What might be interesting is what it means to adjoin an identity to the M_n-graded algebra A

Recall that Δ is compatible with the M_n grading on A and the M_n grading on $M_n A = M_n \mathbb{C} \otimes A$ where A has 1 grading $?$

$\Gamma_+ \times \Gamma_+ \longrightarrow \Gamma_+$

Go over your $\Gamma \ni \Gamma_+$ is a semigroup with absorbing element 0.

$\mathbb{C}\Gamma$ is a bialg with mult.

$$\begin{tikzcd}
\Gamma_+ \times \Gamma_+ \times \Gamma_+ \arrow[r, shift left, "\mu \times 1"] \arrow[r, shift right, "1 \times \mu"'] \arrow[d, "\pi \times 1"] & \Gamma_+ \times \Gamma_+ \arrow[r, "\mu"] \arrow[d, "\pi"] & \Gamma_+ \\
\Gamma_+ \wedge \Gamma_+ \times \Gamma_+ \arrow[r, "\bar{\mu} \times 1"] \arrow[d] & \Gamma_+ \wedge \Gamma_+ \arrow[r, "\bar{\mu}"] & \Gamma_+ \\
\Gamma_+ \wedge \Gamma_+ \wedge \Gamma_+ & &
\end{tikzcd}$$

So go back to $\Gamma = M_2$ ~~which~~

$$\mathbb{C}M_2 = M_2\mathbb{C}$$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$\Delta : A \longrightarrow \mathbb{C}\Gamma \otimes A \underset{1 \otimes \Delta}{\overset{\Delta \otimes 1}{\rightrightarrows}} \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \otimes A$$

$$\Delta(a_\alpha) = \alpha \otimes a_\alpha \quad \begin{array}{l} \longmapsto \alpha \otimes \alpha \otimes a_\alpha \\ \longmapsto \alpha \otimes (\alpha \otimes a_\alpha) . \end{array}$$

$$\Delta(a_\alpha a_\beta) = (\alpha \otimes a_\alpha) \cdot (\beta \otimes a_\beta) = \alpha\beta \otimes a_\alpha a_\beta$$

$$A_\alpha A_\beta \subset \begin{cases} 0 & \alpha\beta = 0 \\ A_{\alpha\beta} & \alpha\beta \neq 0 . \end{cases}$$

question: Is $\mathbb{C}\Gamma \otimes A$ a Γ-graded alg?

Review: If Γ_+ is a semigroup with $*$ absorbing then $\mathbb{C}\Gamma = \mathbb{C}\Gamma_+ / \mathbb{C}\{*\}$ is a bialgebra with coproduct $\Delta s = s \otimes s$, product

$$\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma = \mathbb{C}(\Gamma_+ \wedge \Gamma_+)/\mathbb{C}\{*\} \longrightarrow \mathbb{C}\Gamma_+ / \mathbb{C}\{*\} = \mathbb{C}\Gamma$$

$\therefore$ point is that $s, t \in \Gamma$ then $\mu : \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma$ given by $\mu(s \otimes t) = \begin{cases} st & \text{if } st \neq * \\ 0 & \text{if } st = * . \end{cases}$

Observe you can adjoin an identity to any semigroup to make it a monoid.

Now you understand ~~the~~ the Γ grading on $\mathbb{C}\Gamma \otimes A$ every elt of A has degree 1

Look briefly at adjoining identity to a Γ-graded alg. A

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[dr, "id"'] & \mathbb{C}\Gamma \otimes A \arrow[r, shift left, "\Delta\otimes 1"] \arrow[r, shift right, "1\otimes\Delta"'] \arrow[d, "\varepsilon\otimes 1"] & \mathbb{C}\Gamma\otimes\mathbb{C}\Gamma\otimes A \\ & A & \end{tikzcd}$$

$$A = \bigoplus_{s\in\Gamma} A_s$$

$$\Delta(a_s) = s \otimes a_s \in \mathbb{C}\Gamma \otimes A.$$

$$\therefore \Delta(a_s a_t) = st \otimes a_s a_t \Longrightarrow \begin{array}{ll} a_s a_t = 0 & \text{when } st = 0 \\ a_s a_t \in A_{st} & \text{when } st \neq 0. \end{array}$$

Back to $\Gamma = M_2$. Γ is a

A is M_2-graded i.e. a Morita context

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad \begin{array}{ccc} A & \xrightarrow{\Delta} & M_2\mathbb{C} \otimes A \\ a_\alpha & \longmapsto & e_\alpha \otimes a_\alpha \end{array}$$

Now you want to understand a unital Morita context. This should mean that the Mor. cont. $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ is a unital ring

Let $\mathbb{1} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{21} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix}$ $\qquad \begin{pmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{pmatrix}$

$$\sum_j \varepsilon_{ij} a_{jk} = a_{ik}$$

$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} \begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \overset{a_{11}}{\overset{\shortparallel}{\varepsilon_{11} a_{11}}} & 0 \\ \underset{\underset{0}{\shortparallel}}{\varepsilon_{21} a_{11}} & 0 \end{pmatrix}$$

$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon & \end{pmatrix}$$

$$\sum_j \varepsilon_{ij}\, a_{jk} = a_{ik}$$

$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} \begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \varepsilon_{11} a_{11} & 0 \\ \varepsilon_{21} a_{11} & 0 \end{pmatrix}$$

$$\varepsilon_{11} a_{11} = a_{11}$$
$$\varepsilon_{21} a_{11} = 0$$

$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ a_{21} & 0 \end{pmatrix} = \begin{pmatrix} \varepsilon_{12} a_{21} & 0 \\ \varepsilon_{22} a_{21} & 0 \end{pmatrix}$$

$$\varepsilon_{12} a_{21} = 0$$
$$\varepsilon_{22} a_{21} = a_{21}$$

$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \overset{a_{11}}{\varepsilon_{11} a_{11} + \varepsilon_{12} a_{21}} & \overset{a_{12}}{\varepsilon_{11} a_{12} + \cancel{\varepsilon_{12} a_{22}}} \\ \underset{a_{21}}{\varepsilon_{21} a_{11} + \varepsilon_{22} a_{21}} & \underset{a_{22}}{\cancel{\varepsilon_{21} a_{12}} + \varepsilon_{22} a_{22}} \end{pmatrix}$$

$$0 = \varepsilon_{12} A_{21} = \varepsilon_{12} A_{22} \qquad 0 = \varepsilon_{21} A_{11} \neq \varepsilon_{21} A_{12}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} = \begin{pmatrix} \overset{a_{11}}{a_{11} \varepsilon_{11} + a_{12} \varepsilon_{21}} & \overset{a_{12}}{a_{11} \varepsilon_{12} + a_{12} \varepsilon_{22}} \\ \underset{a_{21}}{a_{21} \varepsilon_{11} + a_{22} \varepsilon_{21}} & \underset{a_{22}}{a_{21} \varepsilon_{11} + a_{22} \varepsilon_{22}} \end{pmatrix}$$

$$\cancel{a_{12} \varepsilon_{22} = 0}$$

$$\varepsilon_{12} A_{21} = \varepsilon_{21} A_{22} \qquad \cancel{\varepsilon_{11} a_{11} = a_{11}}$$

To understand a unital Morita context.

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$ when is such a graded ring unital?

Use multipliers. The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is a

Consider a Γ-graded algebra $\quad A \xrightarrow{\Delta} \mathbb{C}\Gamma \otimes A$

$$A_s \ni a \longmapsto s \otimes a$$

can you see obvious multipliers on such an. For example suppose Γ is a group.

$$A_s A_t \subset A_{st}$$

Consider $\Gamma = M_2 \qquad \mathbb{C}\Gamma = M_2\mathbb{C}$ arrow ring of the groupoid M_2

$$A = \bigoplus_\alpha A_\alpha$$

So it is probably important to emphasize the special case of a groupoid. What structure does the arrow ring of a groupoid have?

So look at $\mathbb{C}\Gamma$ the arrow ring. Note that

Let Γ be the set of arrows in a cat. Point. For each object X you get an idemp. 1_X and $\sum_{X \in Ob} 1_X$ should be a local left + right unit.

$$f \in Ar(Z, Y), \quad g \in Ar(Y, X)$$

$$fg \in Ar(Z, X)$$

Then it should be clear that reduced left $\mathbb{C}\Gamma$ modules are the same as covariant functors from the category to $Mod(\mathbb{C})$ and right ones are contrav. funs.

The arrow ring for M_n is $M_n\mathbb{C}$. The category picture gives the unit $\sum_{i=1}^{n} e_{ii}$ What does adjoining an identity mean? Take a semigroup. ?

What you have at this point an understanding the arrow ring for a category, groupoid. Where next?

In the group case you have

$$A \xrightarrow{\Delta} \mathbb{C}\Gamma \otimes A$$

$$\|$$

$$\bigoplus_\alpha A_\alpha$$

$$\Delta(a) = \alpha \otimes a \qquad \text{for } a \in A_\alpha$$

$$A \longrightarrow M_2 \otimes A$$

$$a_\alpha \longmapsto \alpha \otimes a_\alpha$$

$$p = \sum_\alpha p_\alpha \xrightarrow{\Delta} \sum \alpha \otimes p_\alpha$$

$$p^2 = \sum_{\beta\gamma} p_\beta p_\gamma \longmapsto \sum_{\beta\gamma} \beta\gamma \otimes p_\beta p_\gamma$$

$$p_\alpha = \sum_{\alpha = \beta\gamma} p_\beta p_\gamma$$

In the end you have the following problem:

$p \in M_2\mathbb{C} \otimes A$ $\qquad$ $p \in \mathbb{C}\Gamma \otimes A$ acts on $\mathbb{C}\Gamma \otimes V$

$p = \sum_{\alpha \in M_2} e_\alpha \otimes p_\alpha$ $\qquad$ $p = \sum_{s \in \Gamma} s \otimes p_s$

There is something which involves s^{-1}. instead of ~~keeping~~ left mult by s on $\mathbb{C}\Gamma$, you use right mult by s^{-1} which commutes left Γ-mult. You might try the same thing for $p = \sum e_\alpha \otimes p_\alpha$ on $M_2\mathbb{C} \otimes V$

maybe $\sum_{\alpha \in M_2} e_{\alpha^{-1}} \otimes \rho_\alpha$ internally on $M_2\mathbb{C} \otimes V$

this commutes with left $M_2\mathbb{C}$ operates.

This looks like it ought to work.

So start with Γ a groupoid, let A be the Γ-graded algebra (means $A = \bigoplus A_\alpha$ where α ranges over the arrows of the groupoid and product is like the $\boxed{\text{arrow ring } \mathbb{C}\Gamma}$ i.e.

$$A_\alpha A_\beta = \begin{cases} 0 & \text{if } \alpha\beta = 0 \\ A_{\alpha\beta} & \text{if } \alpha\beta \neq 0. \end{cases}$$) you want

NO at this point you don't care about the grading.

You ~~say~~ want an A-module V, which means operators ρ_α on V $\quad \rho_\alpha = \sum_{\alpha = \beta\gamma} \rho_\beta \rho_\gamma$

Then you want a proj op p

$$W \xrightarrow{\alpha} \mathbb{C}\Gamma \otimes V \xrightarrow{\beta} W$$

$$V$$

Let's see if you can guess the ~~alg~~ theoretic picture of a G-torsor. Take $G = M_2$ first.

A G-torsor over B is a functor $G^{op} \xrightarrow{R} Sh_B$ which is locally representable. When $G = M_2$, a functor R without the last condition is equivalent to two sheaves F_1 F_2 over B and an isom. between them. Representable stalkwise means F_1, F_2 are final ~~sheaves~~. Linearize to see what happens. R should become a $\mathbb{C}[M_2^{op}]$-sheaf over B

Look at M_2, groupoid, $\mathbb{C}M_2 = M_2\mathbb{C}$ ring of 2x2 matrices is a bialg. What was the last idea yesterday. You tried Groth's version of an M_2 torsor, and it doesn't lead anywhere. Now to linearize. Before: $\overbrace{M_2^{op}}^{\text{functor}} \longrightarrow Sh_B$ which is an ordered pair F_1, F_2 of sheaves over B and an isom $F_1 \xrightarrow{\sim} F_2$. You can control the situation via stalks. When you linearize you replace sheaves of M_2^{op} sets by sheaves of $(M_2\mathbb{C})^{op}$ modules. Morita equivalence still holds. Another point is you want continuous functions.

You've reached the following situation: There seems to be a version of "assembly" for $\Gamma = M_2$, which

---

Groupoid M_2 object 1, 2 unique map for each ordered pair of objects.

$\mathbb{C}[M_2] = M_2\mathbb{C}$ arrow alg of the groupoid M_2

functors $M_2 \longrightarrow$ vector spaces are left $M_2\mathbb{C}$-modules

$\mathbb{C}[C]$ = arrow ring of C, basis given by the set of arrows in C $\cdot \xleftarrow{f} \cdot \xleftarrow{g} \cdot$. So it should be true that a left reduced $\mathbb{C}[C]$ module is a covariant functor.

Your idea now is to treat $\mathbb{C}\mathcal{G}$ in analogy with $\mathbb{C}\Gamma$. So life goes on.

Let $\mathcal{G}$ be the groupoid M_n i.e. n distinct objects say $1, 2, \dots, n$ and $Ar \simeq Ob \times Ob$
$f \longmapsto (\text{target}(f), \text{source}(f))$

~~...~~ You want to start with a retract of a free $\mathbb{C}\mathcal{G}$ module

$$W \xrightarrow{\alpha} \mathbb{C}\mathcal{G} \otimes V \xrightarrow{\beta} W \qquad \beta\alpha = 1$$

In our situation $\mathbb{C}\mathcal{G} = M_n\mathbb{C} = \mathbb{C}^n \otimes (\mathbb{C}^n)^*$

Now use the Morita equivalent of $M_n\mathbb{C}$ with $\mathbb{C}$.
above retract equivalent to $\qquad \mathbb{C}\mathcal{G} = \mathbb{C}^n \otimes (\mathbb{C}^n)^*$

$$\mathbb{C}^n \otimes_{\mathcal{G}} W \longrightarrow (\mathbb{C}^n)^* \otimes V \longrightarrow \mathbb{C}^n \otimes_{\mathcal{G}} W$$

So if you start with V then the possible W's are retracts of ~~...~~ $V^{\oplus n}$ to a projection of an $V^{\oplus n}$ i.e. a $p \in M_n\mathbb{C} \otimes \mathrm{End}(V)$

Take $n = 2$. Want proj.

Repeat. basic object is a retract of the free $M_n\mathbb{C}$-module generated by V:

$$* \qquad W \xrightarrow{\alpha} M_n\mathbb{C} \otimes V \xrightarrow{\beta} W \qquad \beta\alpha = 1.$$

~~...~~ Use the Mor eq ~~...~~ between $M_n\mathbb{C}$ and $\mathbb{C}$ given by $\begin{pmatrix} \mathbb{C} & E^* \\ E & E \otimes E^* = M_n\mathbb{C} \end{pmatrix}$ $\quad E = \mathbb{C}^n$ column vect

$*$ equiv. to $\qquad E^* \otimes_{M_n\mathbb{C}} W \longrightarrow E^* \otimes V \longrightarrow E^* \otimes_{M_n\mathbb{C}} W$

call this $\qquad \overline{W} \longrightarrow \mathbb{C}^n \otimes V \longrightarrow \overline{W}$

So the point is that a retract W of the $M_n\mathbb{C}$ mod $M_n\mathbb{C} \otimes V$ is equiv. to a retract of ~~...~~ $\mathbb{C}^n \otimes V$

so what is a retract of $V^{\oplus n} = \mathbb{C}^n \otimes V$

~~eq~~ It's equivalent to ~~a projection on of~~

$p = p^2$ in End$(\mathbb{C}^n \otimes V) = M_n\mathbb{C} \otimes \mathrm{End}(V)$

i.e. ~~to~~ $p = \sum_{ij} e_{ij} \otimes p_{ij}$ where the p_{ij} satisfy

$$p = \Big(\sum_{ij} e_{ij} \otimes p_{ij}\Big)\Big(\sum_{kl} e_{kl} \otimes p_{kl}\Big)$$

$$= \sum_{i,j=k,l}{}' e_{il} \otimes p_{ij}p_{kl} = \sum_{i,k} e_{il} \otimes \sum_j p_{ij}p_{jl}$$

i.e. $p_{il} = \sum_j p_{ij} p_{jl}$.

So one has the following equivalence:

~~[illegible]~~

A-module structure on V

~~vect~~ $\mathbb{C}$-module retract W of $\mathbb{C}^n \otimes V$

$M_n\mathbb{C}$ ——— W of $M_n\mathbb{C} \otimes V$

Remaining step

$$\begin{tikzcd} W \arrow[r, hook, "\alpha"] \arrow[dr, "j"'] & M_n\mathbb{C}\otimes V \arrow[r, "\beta"] \arrow[d, shift right, "\eta"'] & W \\ & V \arrow[u, shift right, "\varepsilon"'] \arrow[ur, "\iota"'] & \end{tikzcd}$$

$\varepsilon : \mathbb{C} \to \big(\mathbb{C}e_{11}$

Here α is coinduced by j

β is induced by ι

Review. Retract of a free $M_n\mathbb{C} = \Lambda$ module

$\Lambda = T \otimes T^*$

$T = \mathbb{C}^n$ col. v.

$\beta\alpha = 1_W$

$$W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W$$

$$\alpha\beta = p = p^2 \qquad p \in \Lambda \otimes \mathrm{End}(V)$$

$$T^* \otimes_\Lambda W \longrightarrow T^* \otimes V \longrightarrow T^* \otimes_\Lambda W$$

You are trying to set up an equivalent between between A module V and B modules W. But you don't know yet what B is.

$$W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W$$

because Λ is a unital ring, β equivalent to a $\mathbb{C}$ linear $i : V \longrightarrow W$

Dually α should be equivalent to $j : W \longrightarrow V$ $\mathbb{C}$ linear, but a choice has to be made

$$\mathrm{Hom}_\Lambda(W, \mathrm{Hom}_\mathbb{C}(\Lambda, V)) = \mathrm{Hom}(W, V)$$

so you need an isom

$$\Lambda \otimes V \longrightarrow \mathrm{Hom}_\mathbb{C}(\Lambda, V) = \mathrm{Hom}_\mathbb{C}(\Lambda, \mathbb{C}) \otimes V$$

$$\mathrm{Hom}_\Lambda(\Lambda \otimes V, \mathrm{Hom}_\mathbb{C}(\Lambda, V)) = \mathrm{Hom}(\Lambda \otimes_\Lambda \Lambda \otimes V, V)$$

$$= \mathrm{Hom}_\mathbb{C}(\Lambda \otimes V, V)$$ So you need a linear functional on Λ. Trace

basic object is retract of a free Λ-module

$$W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W \qquad \Lambda = M_n \mathbb{C}$$

same as projection in $\mathrm{End}_\Lambda(\Lambda \otimes V) = \Lambda \otimes \mathrm{End}(V)$ same as an A-module structure on V. Problem to find alg B operating on any such retract W such that you have a Morita equivalence between A and B. You propose to use the identity $1 \in \Lambda$ and the trace $tr : \Lambda \to \mathbb{C}$, to define

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "j"'] & \Lambda \otimes V \arrow[r, "\beta"] \arrow[d, shift right, "tr \otimes 1"'] & W \\ & V \arrow[u, shift right, "\varepsilon = 1 \otimes -"'] \arrow[ur, "i"'] & \end{tikzcd}$$

doesn't look right since $tr(1) = 2$.

You want an "equivariant" splitting of $\Lambda \otimes V$. Partition of 1

In the case of $\Lambda = M_n\mathbb{C}$ you have the partition $\sum_{i=1}^{n} e_{ii}$. sum of identity maps corresp to objects.

So what are you trying to say? Take

$$\sum \beta e_{ii} \alpha$$

So on W, besides the Λ operation you have operator $h_i = \beta e_{ii} \alpha$ $\quad i = 1, \dots, n$ adding up to 1 on W

$$\underbrace{W \xhookrightarrow{\alpha} \Lambda \otimes V}\xrightarrow{\beta} W$$

on here have e_{ij}
with relations $e_{ij}\, e_{kl} = \begin{cases} 0 & j \neq k \\ e_{il} & j = k \end{cases}$

translates to $h_{ij} = \beta e_{ij} \alpha =$

$$h_{ij} h_{kl} = \beta e_{ij} \underbrace{\alpha\beta}_{p} e_{kl} \alpha = \beta e_{ij} e_{kl} \alpha\beta\cancel{\alpha}$$

$$= \begin{cases} 0 & j \neq k \\ \underbrace{\beta e_{il} \alpha}_{h_{il}} & j = k \end{cases} \qquad ??$$

$\Lambda = M_n \mathbb{C}$ basis is e_{ij}

You want the actual projection $\pi_{ii} : \Lambda \to \mathbb{C} e_{ii}$

$$\begin{tikzcd} \bar{W} \arrow[r] \arrow[rr, bend left, "1"] & V \oplus V \arrow[r] & \bar{W} \end{tikzcd}$$

$\Lambda = M_A \mathbb{C}$ $\quad M_A$-graded

$W \xhookrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W$ $\quad$ retract of the free Λ-mod gen. by V.

Problem: ~~[illegible]~~ find natural operators on any such W. Λ itself operates on the left.

Perhaps you have made the same mistake as before namely thinking that $\mathbb{C}\Gamma \otimes \mathrm{End}(V)$ is the endo alg of the free Γ-module $\mathbb{C}\Gamma \otimes V$ (say Γ finite)

Yes. $\Lambda \otimes V \qquad \Lambda = M_2\mathbb{C}$

$$\mathrm{End}_\Lambda(\Lambda \otimes V) = \Lambda^{op} \otimes \mathrm{End}(V)$$

namely $\quad (s \otimes \varphi)(t \otimes v) = ts \otimes \varphi v$

$$(s_1 \otimes \varphi_1)\left[(s_2 \otimes \varphi_2)(t \otimes v)\right] = (s_1 \otimes \varphi_1)\left[ts_2 \otimes \varphi_2 v\right]$$
$$= ts_2 s_1 \otimes \varphi_1 \varphi_2 v$$

$$(s_1 \otimes \varphi_1)(s_2 \otimes \varphi_2) = s_2 s_1 \otimes \varphi_1 \varphi_2$$

Therefore

$$T^* \otimes_\Lambda W \longrightarrow T^* \otimes V \longrightarrow T^* \otimes_\Lambda W$$

$$\underset{A}{p = \sum_{ij} p_{ij}} \longmapsto \underset{\Lambda \otimes A}{\sum_{ij} e_{ij} \otimes p_{ij}} \qquad \text{want this to act}$$

"internally" on $\Lambda \otimes V$.

$$p(\lambda \otimes v) = \sum_{ij} \lambda e_{ji} \otimes p_{ij} v$$

this should also hold for p on $T^* \otimes V$

$$p(p(\lambda \otimes v)) = \sum_{kl} \sum_{ij} \lambda \underbrace{e_{ji} e_{lk}}_{\begin{cases} 0 & i \neq l \\ e_{jk} & i = l \end{cases}} \otimes p_{kl} p_{ij} v$$

$$= \sum_{kj} \lambda e_{jk} \otimes \sum_l p_{kl} p_{lj} v$$

$$= \sum_{kj} \lambda e_{jk} \otimes p_{kj} v = p(\lambda \otimes v)$$

Note: there's some resemblance between this and the way a Γ action and Γ grading combine to yield a crossproduct.

So now you understand p on $T^* \otimes V$. Still a way to go.

$$W \xhookrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W \qquad \beta\alpha = 1.$$

You have Λ acting on the left.

$$T^* \otimes_\Lambda W \qquad T^* \otimes V \qquad T^* \otimes_\Lambda W$$

Basically want

~~[illegible]~~ Repeat. Λ arrow ring ~~of~~ a groupoid

$$W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W$$

guess that there should be an operator on W, an h operator. $\beta\alpha = 1_W$

$$\frac{T \otimes T^* \otimes V}{}$$

$$M_2 \mathbb{C} \cdot \otimes V \longrightarrow W$$

$$\begin{array}{c} \cup\!\!\cup \\ V \end{array}$$

maybe the idea is that there are two projections $\mathbb{C}$ linear on $\Lambda \otimes V$

latest idea Λ = arrow alg of the groupoid is a left Λ-module (also a right module but you have used this when applying p to $\Lambda \otimes V$).

Λ is also graded wrt G. This gives ~~various~~ a partition of Λ.

Repeat: $$W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W$$

You want to find the ~~natural~~ ring B which operates on all W's arising from A-modules V. B contains left mult by elts of Λ and also operators $h_s = \beta e_s \alpha$, where e_s is the projection of Λ onto $\mathbb{C}s$, arising from the G grading of Λ. Then $\sum_{s \in G} h_s = 1$ on W.

B should be generated by Λ and the h_s, ideally a kind of cross product. Is there a crossproduct algebra using a groupoid?

$G \ltimes D$ should a G graded alg

Can you define what it means for G to act on D?? D may not be an algebra.

Repeat: ~~Start with~~ Start with an A-mod structure on V, whence a projection
$$p(\lambda \otimes v) = \sum_{s \in G} \lambda s^{-1} \otimes p_s v$$
on the free Λ-module $\Lambda \otimes V$, hence a

Λ-module retract

$$W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W \qquad \beta\alpha = 1.$$

The problem now is to define the ~~right~~ appropriate alg B which acts on any such W, and which leads to Morita equiv.

Examples of operators to go in B, ~~left~~ mult by Λ. $h_s = \beta e_s \alpha$ where e_s is proj onto $s \otimes V$ defined by the $\mathcal{G}$ grading. This family of h_s satisfies $\sum_{s\in\mathcal{G}} h_s = 1_W$, but Cuntz has a partition indexed by the objects of the groupoid.

So go back to first idea, using $1 \in \Lambda$ and the trace $tr : \Lambda \to \mathbb{C}$

$$\begin{tikzcd}
W \arrow[r, "\alpha"] \arrow[dr, "j"'] & \Lambda \otimes V \arrow[r, "\beta"] \arrow[d, shift right] & W \\
& V \arrow[u, shift right] \arrow[ur, "\iota"'] &
\end{tikzcd}$$

~~If you get it ...~~ Apply Morita equiv. $\Lambda = T \otimes T^*$

$$W^\# \xrightarrow{\alpha^\#} T^* \otimes V \xrightarrow{\beta^\#} W^\# \qquad \text{where } W^\# = T^* \otimes_\Lambda W$$

You know $T^* \otimes V \longrightarrow W^\# \longrightarrow T^* \otimes V$

$$p(\lambda \otimes v) = \sum \lambda e_{ji} \otimes p_{ij} v$$

$$p(\lambda\otimes v)=\sum_{i,j}\lambda e_{ji}\otimes p_{ij}v$$

$$p(p(\lambda\otimes v))=\sum_{k,l,i,j}\lambda e_{ji}e_{lk}\otimes p_{kl}p_{ij}v$$

$$=\sum\lambda e_{jk}\otimes\underbrace{\sum_{l}p_{kl}p_{lj}}_{p_{kj}}v$$

Review ↗

$$W\xrightarrow{\alpha}\Lambda\otimes V\xrightarrow{\beta}W \qquad \beta\alpha=1_W$$

list all the operators on W you get and any relations between them.

$h_{ij}=\beta\pi_{ij}\alpha$ $\quad$ π_{ij} projects onto $\mathbb{C}e_{ij}$

How to handle M_n $\quad$ $\Lambda=M_n\mathbb{C}$. basic object is a retract of a free Λ module

$$W\xrightarrow{\alpha}\Lambda\otimes V\xrightarrow{\beta}W \qquad \begin{matrix}\beta\alpha=1\\ \alpha\beta=p\end{matrix}$$

p is equivalent to an Λ-module structure on V

Now want to find the ring B whose modules are such W. Idea: You have Λ left mult on W. Other operators are $\pi_{ij}:\Lambda\otimes V\longrightarrow\mathbb{C}e_{ij}\otimes V$

$\pi_{ij}(\lambda)=e_{ii}\lambda e_{jj}$ ← right mult by e_{jj}

$\sum_{ij}\pi_{ij}=\mathrm{id}$ on Λ $\quad$ so you have a partition of unity.

$$W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W$$

partition of unity on Λ is $\sum \pi_{ij} = id$

where $\pi_{ij}(\lambda) = e_{ii}\lambda e_{jj}$

$$\pi_{ij}(e_{kl}) = e_{ii} e_{kl} e_{jj} = \begin{cases} 1 & \text{if } k=i \text{ and } l=j \\ 0 & \text{if not.} \end{cases}$$

$$\pi_{ij}(\lambda \otimes v) = e_{ii}\lambda e_{jj} \otimes v$$

These ops sum to id.

$$\overset{w}{\overset{\shortparallel}{\beta\alpha w}} = \sum_{ij} \beta\, \pi_{ij}\, \alpha w = \sum_{ij} e_{ii} \beta \lambda \,?$$

Let $\alpha w = \sum_{ij} e_{kl} \otimes v(kl)$. Then

$$\pi_{ij}(\alpha w) = e_{ij} \otimes v(ij)$$

$$\pi_{ij}(\alpha w) = e_{ii}\left(\sum_{k,l} e_{kl} \otimes v_{kl}\right) e_{jj} = e_{ij} \otimes v_{ij}$$

Let $\xi \in \Lambda \otimes V$, then $\xi = \sum_{ij} e_{ij} \otimes v(ij)$

and

$$e_{kk}\, \xi\, e_{ll} = \sum_{ij} e_{kk} e_{ij} e_{ll} \otimes v(ij)$$

$$= e_{kl} \otimes v(kl)$$

so $\alpha w = \sum_{k,l} e_{kk}\, \alpha(w)\, e_{ll} = \sum_{l} \alpha$

$$W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W$$

$$\alpha\beta = p \qquad p(\lambda \otimes v) = \sum_{ij} \lambda e_{ji} \otimes p_{ij} v$$

typical element of $\Lambda \otimes V$ has the form

$$\sum e_{ij} \otimes v(ij)$$

$$p\Big(\sum_{ij} e_{ij} \otimes v(ij)\Big) = \sum_{ij} \sum_{kl} e_{ij} e_{lk} \otimes p_{kl}\, v(ij)$$

Try instead $$p\Big(\sum_{ij} e_{ji} \otimes v(ij)\Big) = \sum_{ij} \sum_{kl} \overbrace{e_{ji} e_{lk}}^{\begin{cases} 0 & \text{if not} \\ e_{jk} & \text{if } i=l \end{cases}} \otimes p_{kl}\, v(ij)$$

$$= \sum_{ijk} e_{jk} \otimes p_{ki}\, v(ij)$$

~~again~~ $\cancel{p\Big(\sum e_{ij} \otimes v(ij)\Big)}$

$$p\Big(\sum_{ij} e_{ji} \otimes v(ij)\Big) = \sum_{ijk} e_{ji} e_{ik} \otimes p_{ki}\, v(ij)$$

$$= \sum_{jk} e_{jk} \otimes \sum_{i} p_{ki}\, v(ij)$$

$$= \sum_{jk} e_{kj} \otimes \sum_{i} p_{ji}\, v(ik)$$

p

check it again

$$p\Big(\sum_{ij} e_{ji}\otimes v(ij)\Big) = \sum_{ij}\sum_{k\ell} e_{ji}e_{\underset{\overset{\shortparallel}{i}}{\ell}k}\otimes p_{k\underset{\overset{\shortparallel}{i}}{\ell}}\, v(ij)$$

$$= \sum_{kj} e_{jk}\otimes \sum_i p_{ki}\, v(ij)$$

$$p\Big(\sum_s s^{-1}\otimes v(s)\Big) = \sum_s\sum_t s^{-1}t^{-1}\otimes p_t\, v(s)$$

$$= \sum_u u^{-1}\otimes \sum_{u=ts} p_t\, v(s)$$

$$\cancel{s \longmapsto v(s)}$$

$$(pv)(u) = \sum_t p_t\, v(t^{-1}u)$$

$$= \sum_s p_{us^{-1}}\, v(s)$$

Λ groupoid ring, arrow ring,

$$W \hookrightarrow \Lambda\otimes V \longrightarrow W$$

$$\cancel{p\Big(\sum_s s\otimes f(s)\Big) = \sum_{s,u} su^{-1}\otimes p_u f(s)}$$

$$= \sum$$

$s = tu^{-1}$

$su = t \quad u = s^{-1}t$

$$p\Big(\sum_t t\otimes f(t)\Big) = \sum_{t,u} tu^{-1}\otimes p_u f(t)$$

$$= \sum_s s\otimes \sum_t p(s^{-1}t) f(t)$$

Discuss situation

$$W \xleftrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W$$

May think of W as a <u>cov. functor</u> from the groupoid to vector spaces.

$$W = \bigoplus_{x \in Ob} 1_x W$$

$$\Lambda = \bigoplus_{x \in Ob} 1_x \Lambda$$

Λ spanned by the arrows $x \leftarrow y$

What do you know about a category?

$Ar(X, Y)$ representable functors.

So $\Lambda = \bigoplus_X \Lambda 1_X$ Λ as a left Λ-module splits into left ideals corresp to the representable functors

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So far you begin with the category of A-modules V, i.e. vector space tog. with ops $p(s)$ for each $s \in \Gamma$ satisfying the idempotence condition $p(u) = \sum_{u=st} p(s)p(t)$. Put another way

$$p = \sum_{s \in \Gamma} s \otimes p(s) \in \Lambda \otimes End(V)$$

$$p^2 = \sum_{s,t \in \Gamma} st \otimes p(s)p(t) = \sum_u u \otimes \sum_{u=st} p(s)p(t)$$

Review: You are trying to extend from a group to a groupoid, using essentially the same formulas. You begin with an Λ-module structure V, that is a vector sp tog with operators $p(s) \in \mathrm{End}(V)$, $s \in \Gamma$ such that $p = \sum_{s \in \Gamma} s \otimes p(s) \in \Lambda \otimes \mathrm{End}(V)$ is idempotent: $p^2 = \sum_{s,t} st \otimes p(s)p(t) = \sum_u u \otimes \sum_{u=st} p(s)p(t)$

whence $\boxed{\sum_{u=st} p(s)p(t) = p(u).}$

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Given such a family of operators $p(s)$ define p on $\Lambda \otimes V$ by

* might be better notation.

$$p\Big(\sum_t t \otimes f(t)\Big) = \sum_{u,t} tu^{-1} \otimes p(u) f(t)$$

rough
$s = tu^{-1}$
$s^{-1} = ut^{-1}$
$s^{-1}t = u$

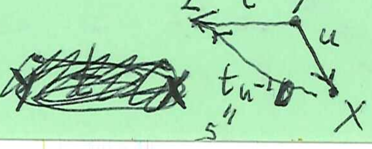
$$= \sum_s s \otimes \sum_t p(s^{-1}t) f(t)$$

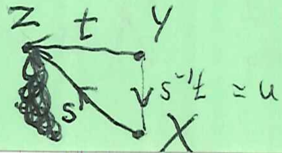
Better might be

$$p\Big(\sum_t t \otimes f(t)\Big) = \sum_t \underbrace{\Big(\sum_u tu^{-1} \otimes p(u) f(t)\Big)}_{\sum_s s \otimes p(s^{-1}t) f(t)}$$

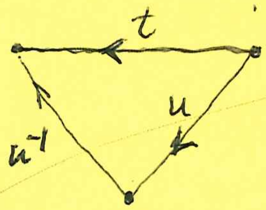
t fixed then we have

$\{u \mid tu^{-1} \text{ defd}\}$ $\qquad\qquad$ $\{s \mid s^{-1}t \text{ defined}\}$





Repeat: $p\left(\sum_t t\otimes f(t)\right) = \sum_t \left(\sum_u tu^{-1}\otimes p(u)f(t)\right)$

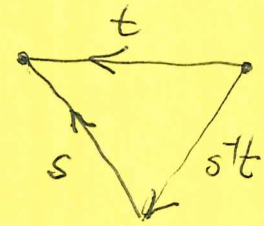


In $\sum_u tu^{-1}\otimes p(u)f(t)$ think t is fixed and u runs over all arrows with same source as t.

$$= \sum_t \sum_s s\otimes p(s^{-1}t)f(t).$$

So you have the formula

$$(pf)(s) = \sum_t p(s^{-1}t)f(t)$$



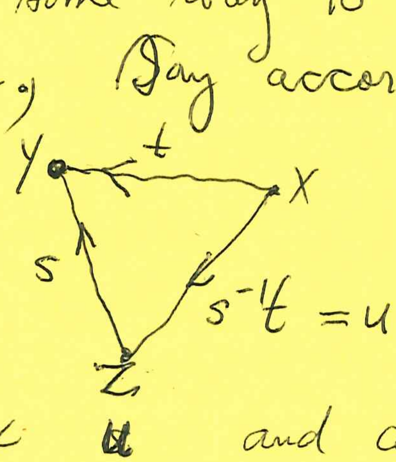
~~[illegible]~~ sum takes place over all t with same target ~~as~~ s

You want to factor p appropriately:

$$\alpha\beta\left(\sum_t t\otimes f(t)\right) = \sum_s s\otimes js^{-1}\sum_t t\, i\, f(t)$$
$$= \sum_s s\otimes \sum_t p(s^{-1}t)\, f(t)$$

$js^{-1}ti$

Is there some way to factor $p(s^{-1}t)$ into $js^{-1}\circ ti$, say according to intermediate object



Halifax ~~[illegible]~~

~~[illegible]~~ fix u and ask for solutions of $u = s^{-1}t$

$$\{(s,t) \mid u = s^{-1}t\} = \coprod_Y Ar(Y,Z)\times Ar(Y,X)$$

and ~~this should~~ for each ~~ones~~ Y, the piece should be an orbit under the isot. gp $Ar(Y,Y)$

Now go back to M_n

$$W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W \longrightarrow \Lambda \otimes V$$

What you have is p.

$$(pf)(s) = \sum_t p(s^{-1}t) f(t)$$

Repeat earlier idea. You have a rtcont functor from A-modules V to vector spaces W and you want to find an algebra B operating naturally on this functor so that the functor is a Morita equivalence.

Go back to earlier idea that the projection operators π_s on Λ as vector space yield to "compressed" operators on W. $h_s = \beta \pi_s \alpha$.

$$W = p(\Lambda \otimes A) \otimes_A V$$

$$\begin{tikzcd} p(\Lambda\otimes A) \arrow[r, "\alpha"] & \Lambda\otimes A \arrow[loop below, "\pi_s\otimes 1"] \arrow[r, "\beta"] & p(\Lambda\otimes A) \end{tikzcd}$$

these are all A^{op}-maps.

$$W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W$$

Can you recover V from W?

In any case it is clear that you have a partition of unity W: $\sum_{s\in\Gamma} h_s = 1$.

Γ groupoid, $\Lambda = \mathbb{C}\Gamma$, does Γ act on the alg $\mathbb{C}\Gamma$ allowing one to form a cross product alg $\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma$ as in the group case, thereby getting a Morita equiv. In the group case the ~~basic~~ Morita equivalence arises from left mult by Γ on the Γ-graded vector space $\mathbb{C}[\Gamma]$. Better to say that a Γ-module M with Γ grading ~~equivariant partition indexed by Γ such that Γ respects~~

$$M = \bigoplus_{s\in\Gamma} M_s \quad \text{such that} \quad tM_s \subset M_{ts}$$

is canonically isom to $\mathbb{C}\Gamma \otimes M_1$.

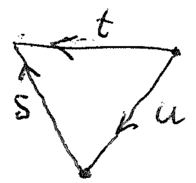
Is there a corresp ~~statement~~ in the groupoid case? Consider then M a Γ-graded module, where Γ is a groupoid

$$M = \bigoplus_{s\in\Gamma} M_s$$

Γ finite set only identity maps.

$\mathbb{C}\Gamma$ groupoid alg, suppose finitely many objects so that $\mathbb{C}\Gamma$ is unital. A unital $\mathbb{C}\Gamma$-mod is the same as a functor from Γ to Vect.

Repeat. If V is A-module, get ρ on $A\otimes V$ ~~defin~~ given by



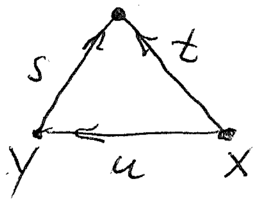
$$\rho\Big(\sum_t t\otimes f(t)\Big) = \sum_{\cancel{t}} \sum_{\cancel{u}\, s} \overset{s}{\cancel{tu^{-1}}} \otimes \rho(\overset{s^{-1}t}{\cancel{u}}) f(t)$$

$$= \sum_s s \otimes \sum_t \rho(s^{-1}t) f(t)$$

$s = tu^{-1}$
$u = s^{-1}t$

You ultimately want to expect $p(u)$ as a sum over factorizations of u as $s^{-1}t$.

$$(pf)(s) = \sum_t \underbrace{p(s^{-1}t)}\, f(t).$$

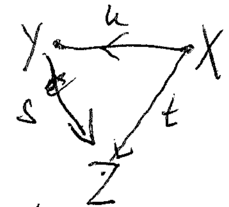


this to be written as a sum over possibly intermediate objects occurring in a factorization of u.

What would you like to happen? Recall that s^{-1} should be viewed, or written, as s^*. At some point you should explore this. The groupoid ring should be a $*$ algebra in an obvious way.

What would you like to happen.

$$p(u) = \sum_Z q(s)^* q(t)$$



$u = s^{-1}t$

You know that

$$p(u) = \sum_{u = s^{-1}t} p(s^{-1})p(t)$$

Suppose you have the M_n case. Then

$$(pf)(s) = \sum_t \underbrace{p(s^*t)\, f(t)}$$
$$= \sum_{s^*t = s_1^* t_1} p(s_1^*)p(t_1)\, f(t)$$

$$W \overset{\alpha}{\hookrightarrow} \Lambda\otimes V \overset{\beta}{\twoheadrightarrow} W$$

On $\Lambda\otimes V$ you have the projection operator 22

π_s onto $\mathbb{C}s \subset \Lambda$ $\forall s \in \Gamma$

$\sum \pi_s = 1$ $h_s = \beta\pi_s\alpha$, $\sum h_s = 1$

So the h_s operator on W as well as the s.

Look at $t h_s u = \beta(t\pi_s u)\alpha$

$s = ij$ then $\pi_s(\lambda) = e_{ii}\lambda e_{jj}$

Repeat. $W \overset{\alpha}{\hookrightarrow} \Lambda\otimes V \xrightarrow{\beta} W$ $\Lambda\otimes V$

$\cup$

On $\Lambda\otimes V$ have projections $\pi_s : \Lambda\otimes V \twoheadrightarrow s\otimes V$

Also have left mult by s operators $\Lambda\otimes V$.

So the Γ action and the Γ-grading.

How are they related? Take the M_n case

Two kinds of operators namely left mult by s on $\Lambda\otimes V$ and projection operator $\pi_t : \Lambda\otimes V \circlearrowleft$ image $t\otimes V$.

Can suppose $V = \mathbb{C}$. $\Lambda = \mathbb{C}\Gamma$ where Γ is a conn groupoid trivial isotropy groups. Let x, y, z be objects s, t, u arrows.

Question: Assuming Ob finite, What is the alg generated by the s. and π_t

$M_n\mathbb{C} = \Lambda$ has basis $s \in M_n$ $\quad s \in Ob \times Ob$

L_s for left mult by s.

π_s proj onto $\mathbb{C}s$

You are working in the adjoint rep not the standard rep.

$$\begin{array}{cc} & j \\ i & \begin{pmatrix} \cdot & 1 & \end{pmatrix} \end{array} \qquad \pi_{ij}(\lambda) = e_{ii}\lambda e_{jj}$$

find the relations

π_{ij}

$M_2\mathbb{C}$. $\quad (e_{ij}\,\pi_{kl})(\lambda) = e_{ij}e_{kk}\lambda e_{ll} = \begin{cases} 0 & j \neq k \\ e_{ij}\lambda e_{ll} & j = k \end{cases}$

$$(e_{ij}\pi_{kl})(\lambda) = e_{ij}\,\pi_{kl}(\lambda) = e_{ij}\,e_{kk}\lambda e_{ll} = \begin{cases} 0 & \text{if } j \neq k \\ e_{ij}\lambda e_{ll} & \text{if } j = k \end{cases}$$

$$(\pi_{kl}\,e_{ij})(\lambda) = \pi_{kl}(e_{ij}\lambda)$$

$$= \underline{e_{kk}\,e_{ij}\lambda e_{ll}} = \begin{cases} 0 & \text{if } k \neq i \\ e_{ij}\lambda e_{ll} & \text{if } k = i \end{cases}$$

$$e_{kk}e_{ij}\lambda e_{ll} = \begin{cases} e_{kj}\lambda e_{ll} & \text{if } k = i \\ 0 & k \neq i \end{cases}$$

$$\pi_{k\ell}(\lambda) = e_{kk}\lambda e_{\ell\ell}$$

$$(e_{ij} \circ \pi_{k\ell})(\lambda) = e_{ij}\, e_{kk}\lambda e_{\ell\ell} = \begin{cases} e_{ik}\,\pi_{k\ell}(\lambda) & \text{if } j=k \\ 0 & \text{if } j\neq k \end{cases}$$

$$(\pi_{k\ell} \circ e_{ij})(\lambda) = e_{kk}\, e_{ij}\lambda\, e_{\ell\ell} = \begin{cases} e_{kj}\lambda e_{\ell\ell} & \text{if } k=i \\ 0 & \text{if } k\neq i \end{cases}$$

$$\pi_{k\ell}(e_{ij}\lambda) = e_{kk}\, e_{ij}\lambda\, e_{\ell\ell} = \begin{cases} 0 & \text{if } k\neq i \\ e_{ij}\lambda e_{\ell\ell} \\ e_{kk}e_{kj}\lambda e_{\ell\ell} \end{cases} \Big\} \text{ if } k=i$$

$$\boxed{\pi_{k\ell}\circ e_{ij} = \begin{cases} e_{ij}\circ \pi_{j\ell} & \text{if } k=i \\ 0 & \text{if } k\neq i \end{cases}}$$

Repeat this calculation You are looking at operators on $\Lambda = \mathbb{C}\Gamma$ in particular π_s proj ops assoc. to the grading and ~~op.~~ op. e_s left mult by $s\in\Gamma$.

$$\pi_{k\ell}\left(\sum \lambda_{ij} e_{ij}\right) = \lambda_{k\ell}e_{k\ell}$$

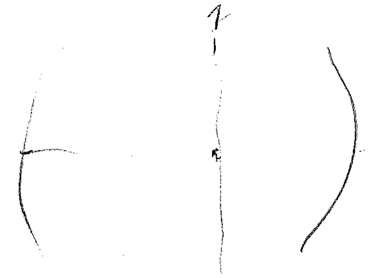
$$\pi_{k\ell}(\lambda) = e_{kk}\lambda e_{\ell\ell}.$$

$$(\pi_{k\ell}\circ e_{ij})(\lambda) = e_{kk}\, e_{ij}\lambda\, e_{\ell\ell} = \begin{cases} 0 & k\neq i \\ e_{ij}\lambda e_{\ell\ell} & k=i \end{cases}$$

$$e_{ij}\lambda e_{\ell\ell} = e_{ij}\, e_{jj}\lambda e_{\ell\ell} = (e_{ij}\circ\pi_{j\ell})(\lambda)$$

Repeat the calculation

$$\pi_{k}^{l}(\lambda) = e_{kk}\lambda e_{ll}$$



$$(e_{ij}\circ\pi_k^l)(\lambda) = e_{ij}e_{kk}\lambda e_{ll} = \begin{cases} 0 & j\neq k \\ e_{ik}\,\pi_{k}^{l}(\lambda) & j = k\end{cases}$$

$$e_{ij}\,\pi_k^l = \begin{cases} 0 & j\neq k \\ e_i \end{cases}$$

$$(e_{ij}\circ\pi_k^l)(\lambda) = e_{ij}\,e_{kk}\lambda e_{ll} = \begin{cases} 0 & j\neq k \\ e_{ik}\,\pi_k^l & j=k\end{cases}$$

$$(\pi_k^l\circ e_{ij})(\lambda) = e_{kk}(e_{ij}\lambda)e_{ll} = \begin{cases} 0 & k\neq i \\ e_{ij}\lambda e_{ll} \\ \quad {}^{\shortparallel} \end{cases}$$

$$(e_{ij}\cdot\pi_j^l)(\lambda) = e_{ij}\,e_{jj}\lambda e_{ll}$$

$$\pi_k^l(e_{ij}\lambda) = e_{kk}e_{ij}\lambda e_{ll} = \begin{cases} 0 & k\neq i \\ e_{ij}(\pi_j^l\lambda) \end{cases}$$

So it looks like there is a kind of normal form. Note that the l doesn't change. This is the effect of right mult by e_{ll}

Repeat again. $\Gamma = M_n$ $\Lambda = M_n\mathbb{C}$

$\pi_{k\ell}$ grading projection onto $\mathbb{C}e_{k\ell}$

$$\pi_{k\ell}(\lambda) = e_{kk}\lambda e_{\ell\ell} \quad \text{in } M_n\mathbb{C}$$

$$(\pi_{k\ell}\circ\hat{e}_{ij})(\lambda) = e_{kk}(\hat{e}_{ij}\lambda)e_{\ell\ell} = \delta_{ki}\, e_{ij}\, e_{jj}\lambda e_{\ell\ell}$$

$$\boxed{\pi_{k\ell}\bullet e_{ij} = \delta_{ki}\, e_{ij}\cdot\pi_{j\ell}}$$

So look at the operators $\beta\pi_{k\ell}\alpha = h_{k\ell}$

Try $\pi_\ell(\lambda) = \lambda e_{\ell\ell}$

$$(\pi_\ell\circ e_{ij})(\lambda) = \pi_\ell(e_{ij}\lambda) = e_{ij}\lambda e_{\ell\ell} = e_{ij}\,\pi_\ell(\lambda)$$

$$\beta\pi_{k\ell}\alpha\, e_{ij} = \delta_{ki}\, e_{ij}\beta\pi_{j\ell}\alpha$$

$$h_{k\ell}\, e_{ij} = \delta_{ki}\, e_{ij}\, h_{j\ell}$$

Repeat. $\Gamma = M_n$ $\Lambda = M_n\mathbb{C}.$

$$W \overset{\alpha}{\hookrightarrow} \Lambda\otimes V \overset{\beta}{\twoheadrightarrow} W$$

~~Each object determines a pro~~ Right mult on Λ by the identity maps of the groupoid yields partition of 1

Look then at $\Lambda \otimes V = M_n V$

~~Given~~ $\cancel{\sum e_{ij} \otimes v_{ij}}$ ~~define~~ $\cancel{\pi_k}$

Define π_k on $\Lambda \otimes V$ by

Start again V is a A-module ie vector space with ~~[illegible]~~ operators $p(s)$ $s \in \Gamma$.

$$\begin{tikzcd} W \arrow[r, hook, "\alpha"] & \Lambda \otimes V \arrow[r, two heads, "\beta"] \arrow[d, shift right, "\eta_1"'] & W \\ & V \arrow[u, shift right, "\varepsilon_1"'] & \end{tikzcd}$$

If the case of a group alg you have projections e_s of $\Lambda \otimes V$ onto $s \otimes V$.

Last night tried to review topos idea

$$\mathcal{C} \qquad \mathcal{C}^\wedge = \mathrm{Fun}(\mathcal{C}, \mathcal{S}ets)$$

~~Then~~ a topos map $\mathcal{T} \xrightarrow{f} \mathcal{C}^\wedge$ is given by $f^*: \mathcal{C}^\wedge \longrightarrow \mathcal{T}$ f^* rtcont, left exact

~~If~~ ~~then~~ $\mathcal{T} = \mathcal{S}ets$ (the pt topos). then f^* rtcont means f^* given by "twisting" wrt $R \in (\mathcal{C}^{op})^\wedge$

$$f^*(L) = R \times^{\mathcal{C}} L = \varinjlim_{X \text{ in } \mathcal{C}/R} h_X \times^{\mathcal{C}} L = \varinjlim_{X \text{ in } \mathcal{C}/R} L(X)$$

$$\mathcal{C}^{op} \hookrightarrow \mathcal{C}^\wedge \qquad \text{Yoneda}$$
$$Y \mapsto h^Y$$

~~In the case of a groupoid~~

IDEA: Groth has all these nice category ideas which should be linearized

In the case of a groupoid prorepresentable is the same as representable.

$$\mathcal{G}^{op} \hookrightarrow \mathrm{Fun}(\mathcal{G}, \text{sets}) \xrightarrow{f^*} Sh_B$$

$$\mathcal{C}^{op} \subset (\mathrm{Pro}\,\mathcal{C})^{op} \hookrightarrow \mathrm{Fun}(\mathcal{C}, \text{sets})$$

$$X \qquad X_\alpha \qquad \varinjlim h^{X_\alpha}(Y) = \varinjlim \mathrm{Hom}(X_\alpha, Y)$$

So now take

$$W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W$$

Λ = arrow ring of the groupoid Γ.

you want a partition of $\Lambda \otimes V$, really of Λ

What you want is to see if there is a relation, link between the category situation:

$$\mathcal{G}^{op} \hookrightarrow \mathrm{Fun}(\mathcal{G}, \text{sets}) = \mathcal{G}^\wedge$$

and the assembly stuff you are studying.

$$\Lambda = \mathbb{C}[Ar] = \bigoplus_{Y, X \in Ob} \mathbb{C}[Ar(Y,X)]$$

$$Y \xleftarrow{f} X$$

Λ is the arrow ring, reduced Λ-modules same as cov funs, etc. $R \otimes_\Lambda L$. Yoneda? You want a category inside of Λ-modules

Λ = arrow ring of Γ

$\{\text{red } \Lambda\text{-modules}\} = \mathrm{Fun}(\Gamma, a_{\mathbb{C}})$

You want Yoneda. For each obj X you want a cov. fun. $\Gamma \longrightarrow a_{\mathbb{C}}$ i.e. a left Λ-module

$$\Lambda e_X = \bigoplus_Y \mathbb{C}[\mathrm{Ar}(Y,X)] = \mathbb{C}[h^X]$$

$$\mathrm{Hom}_\Lambda(\mathbb{C}[h^X], M) = L(X).$$

$$\Lambda = \bigoplus_X \underbrace{\mathbb{C}[h^X]}_{\Lambda e_X}$$

$$\Lambda = \mathbb{C}[\Gamma] = \bigoplus_X \underbrace{\mathbb{C}[\mathrm{Ar}(Y,X)]}_{\mathbb{C}[\{Y \xleftarrow{f} X\}]} = \bigoplus_X \underbrace{\mathbb{C}[h^X]}_{\Lambda e_X}$$

So you have this splitting of Λ as a left Λ module.

$$\begin{tikzcd} W \arrow[r, hook, "\alpha"] & \Lambda \otimes V \arrow[r, "\beta"] \arrow[d] & W \\ & \mathbb{C}[h^X] \otimes V & \end{tikzcd}$$

Aim toward reconstructing V from W.

$$\Lambda e_X = \sum \mathbb{C} e_{YX}$$

It seems that you have some type of inducing taking place

First see about the M.eq situation

$$\begin{tikzcd}
T^* \otimes_A W \arrow[r, hook] & T^* \otimes V \arrow[r, two heads] \arrow[d, shift right, "\partial X"'] & T^* \otimes_A W \\
& V \arrow[u, shift right, "\iota_X"'] &
\end{tikzcd}$$

~~Point to make is that for each object X there seems to be a map W → V~~

Try to understand in the M_n situation how V might be recovered, assuming it is reduced. V is reduced when $V = \sum p(s) V$ and $\bigcap_s \operatorname{Ker}(p(s) \text{ on } V) = 0$

$$\bar{W} \hookrightarrow \mathbb{C}^n \otimes V \longrightarrow \bar{W}$$

It seems time for ~~better~~ details. Begin with the A for M_2.
$$p = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \in M_2 \otimes \operatorname{End}(V) = \operatorname{End}(\mathbb{C}^2 \otimes V) \qquad p^2 = p$$

$$p = \sum_{ij} e_{ij} \otimes p_{ij}$$

typical element of $\mathbb{C}^2 \otimes V$ is $\sum_k e_k \otimes f_k = \cancel{f}$

$$p \sum_k e_k \otimes f_k = \sum_{ij\,k} \overbrace{e_{ij} e_k}^{\delta_{jk} e_i} \otimes p_{ij} f_k = \sum_{i,j} e_i \otimes p_{ij} f_j$$

$$= \sum_i e_i \otimes \sum p_{ij} f_j \qquad (pv)(i) = \sum p(ij) \cancel{f}(j)$$

So given $p = p^2$ on $\mathbb{C}^n \otimes V = V^{\oplus n}$.

$$(p_{ij}) \in M_n(\mathrm{End}(V)), \quad \sum_j p_{ij} p_{jk} = p_{ik}$$

What does it mean for V to be reduced.

$$W \xleftarrow{(\beta_1\ \beta_2)} V \oplus V \xleftarrow{\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}} W \xleftarrow{(\beta_1\ \beta_2)} V \oplus V$$

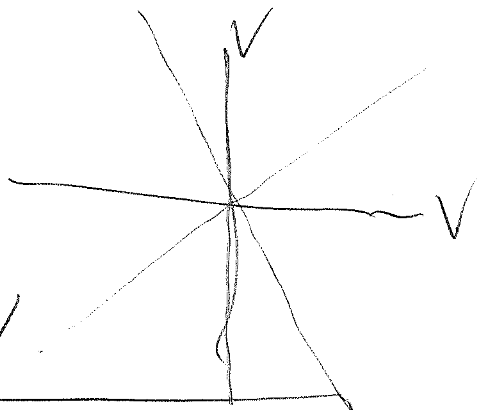
$$\beta_1\alpha_1 + \beta_2\alpha_2 = 1 \implies \begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}(\beta_1\ \beta_2) \text{ idemp.}$$

$$p_{ij} = \alpha_i \beta_j \qquad \sum_{ij} \alpha_i \beta_j V$$

~~If you want to~~

$$\sum_j p_{ij} p_{jk} = \sum_j \alpha_i \beta_j \alpha_j \beta_k = \alpha_i \beta_k = p_{ik}$$

$$W = \beta_1 V + \beta_2 V$$



$$\alpha_1 W = \alpha_1\beta_1 V + \alpha_1 \beta_2 V$$
$$\alpha_2 W = \alpha_2\beta_1 V + \alpha_2 \beta_2 V.$$

$$\therefore \boxed{V = \sum p_{ij} V \iff V = \alpha_1 W + \alpha_2 W}$$

$$\boxed{\bigcap \mathrm{Ker}(p_{ij} \text{ on } V) = 0 \iff 0 = \bigcap_j \mathrm{Ker}(\beta_j \text{ on } V)}$$

$$0 = \alpha_i \beta_j v \quad i = 1, 2 \iff \beta_j v = 0 \quad j = 1, 2$$

So what do we learn? An A module structure on V consists of $p_{ij} \in \mathrm{End}(V)$ $1 \leq i, j \leq n$ sat $\sum_j p_{ij} p_{jk} = p_{ik}$

whence you have a retract

$$W \xleftarrow{(\beta_1 \cdots \beta_n)} V^{\oplus n} \xhookleftarrow{\begin{pmatrix}\alpha_1 \\ \vdots \\ \alpha_n\end{pmatrix}} W \qquad \sum_{i=1}^{n} \beta_i \alpha_i = 1_W$$

with $p_{ij} = \alpha_i \beta_j$. V is reduced A-mod iff

$$V = \sum_1^n \alpha_i W \quad \text{and} \quad \bigcap_{j=1}^{n} \mathrm{Ker}(\beta_j : V \to W) = 0$$

Assuming V is reduced, you should be able to recover V from the retract W.

$$W \xhookrightarrow{\alpha} \mathbb{C}^n \otimes V \xrightarrow{\beta} W$$

$$\alpha_1 W + \alpha_2 W = V$$

$$W \xleftarrow{(\beta_1 \ \beta_2)} \begin{matrix} V \\ \oplus \\ V \end{matrix} \xhookleftarrow{\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}} W$$

$p_{ij} = \alpha_i \beta_j$ Assume $V = \alpha_1 W + \alpha_2 W$, let $v \in V$

write $v = \alpha_1 w_1 + \alpha_2 w_2$, write $w_1 = \beta_1 v_1 + \beta_2 v_2$
$w_2 = \beta_1 v_1' + \beta_2 v_2'$

then
$$\begin{aligned} v = {} & \alpha_1 \beta_1 v_1 + \alpha_1 \beta_2 v_2 \\ & + \alpha_2 \beta_1 v_1' + \alpha_2 \beta_2 v_2' \end{aligned} \in \sum_{ij} p_{ij} V$$

To understand M_2 case completely. A has generators p_{ij} $\quad i,j=1,2$ subject to relations $\sum_j p_{ij}p_{jk} = p_{ik}$, i.e. $\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$ is idempotent. Let V be an A-module. Then

$$V \oplus V \xleftarrow{\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}} V \oplus V \quad \text{is idempotent}$$

$$p\left(\sum_k e_k \otimes f(k)\right) = \sum_{ijk} \overbrace{e_{ij}\, e_k}^{e_i \delta_{jk}} \otimes p_{ij} f(k)$$

$$= \sum_{ij} e_i \otimes p_{ij} f(j) = \sum_i e_i \otimes \sum_j p_{ij} f(j)$$

$$W \xleftarrow{(\beta_1 \;\; \beta_2)} V \oplus V \xleftarrow{\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}} W \qquad \sum \beta_i \alpha_i = 1_W$$

$$p = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} (\beta_1 \;\; \beta_2) = \begin{pmatrix} \alpha_i \beta_j \end{pmatrix}$$

What seems to happen is that by introducing W the image of p on $V \oplus V$, the retract of $V \oplus V$ corresp to p, you actually get a factorization of p into $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} (\beta_1 \cdots \beta_n)$

Mayer Vietoris, the simplest partition situation

$$W \xleftarrow{(\beta_1 \ \ \beta_2)} \begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix} \xleftarrow{\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}} W$$

It appears your mistake was ~~using~~ trying to use $\Lambda \otimes V$ instead of allowing V to depend on the source object. You want a free Λ module to be a direct sum of representable functors.

$$\bigoplus_x \Lambda e_x \otimes V_x . \quad \mathbb{C}[h^x]$$

Let's work this out in the simplest case M_2: two objects. V_1, V_2.

First digress to understand the ring Λ, which should be slightly different from what you expected.

$M_2\mathbb{C} = \Lambda$ this is the arrow ring of the groupoid M_2. ~~Now you consider~~ $\Lambda = T \otimes T^*$

You are after a retract a "free" Λ-module

$$W \xleftarrow{\beta} \begin{pmatrix} \overbrace{\Lambda e_{11}}^{T} \otimes V_1 \\ \underbrace{\Lambda e_{22}}_{T} \otimes V_2 \end{pmatrix} \xleftarrow{\alpha} W$$

What is new is the meaning of free Λ module.
So you do get $T \otimes \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ for your free module

So by M. eq you get back to

$$W^{\#} \xleftarrow{(\beta_1\ \beta_2)} \begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix} \xleftarrow{\binom{\alpha_1}{\alpha_2}} W^{\#}$$

~~So the point~~

Start again. You have ~~the~~ a new notion of free Λ-module for $\Lambda = \mathbb{C}M_2 = M_2\mathbb{C}$, which leads to retracts of the form

$$W \xleftarrow{\beta} T \otimes \begin{pmatrix} V_1 \\ \oplus \\ V_2 \end{pmatrix} \xleftarrow{\alpha} W$$

then by M. eq. to retracts

$$W^{\#} \xleftarrow{(\beta_1\ \beta_2)} \begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix} \xleftarrow{\binom{\alpha_1}{\alpha_2}} W^{\#}$$

You propose now to study the latter ~~problem~~ to understand A. ~~[illegible]~~ You need to look at arb projections on $\begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix}$

$$p = p^2 \text{ in } \mathrm{End}\begin{pmatrix} V_1 \\ \oplus \\ V_2 \end{pmatrix}$$ ~~naturally~~ an M_2 graded alg

Your $p = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \in \begin{pmatrix} \mathrm{End}(V_1) & \mathrm{Hom}(V_2,V_1) \\ \mathrm{Hom}(V_1,V_2) & \mathrm{End}(V_2) \end{pmatrix}$

so as before you ~~find~~ get $p = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} (\beta_1\ \ \beta_2)$

I'm confused. You have the notion of free module where you are given V_X for each object X. Thus it should be clear that the modules to consider ~~seem to~~ are graded

---

Back to M_2. new notion of free module which involves representable functors. Looks good because of topos background. If $\mathcal{G}$ is a groupoid then a topos map $Sh_B \xleftarrow{f^*} \mathcal{G}^\wedge$ is described by a functor $R : \mathcal{G}^{op} \longrightarrow Sh_B$, i.e. a right $\mathcal{G}$ sheaf over B, whose stalks are representable. In other words R is a sheaf over B with right $\mathcal{G}$ action which means you are given $R \xrightarrow{\text{source}} Ob\,\mathcal{G}$ and $R \times_{Ob\,\mathcal{G}} Ar\,\mathcal{G} \longrightarrow R$ making a contravariant functor.

$\mathcal{G}$ groupoid, $\mathcal{G}^\wedge = \mathrm{Fun}(\mathcal{G}, \text{sets}) = \{\mathcal{G}\text{-sets}\}$ is a topos

a topos map from Sh_B to $\mathcal{G}^\wedge$ is given by

a fun $Sh_B \xleftarrow{f^*} \mathcal{G}^\wedge$ $\quad f^*$ rt cont + left exact

f^* rt cont implies $\exists$ canon $f^*L \simeq R \times_{\mathcal{G}} L$ where R is the $\mathcal{G}^{op}$-sheaf over B given by

$$\mathcal{G}^{op} \xrightarrow[\text{Yoneda}]{} \mathcal{G}^\wedge \xrightarrow{f^*} Sh_B$$

Finall f^* left exact means that $\mathcal{G}^0/R$ (this should be the crossproduct of the $\mathcal{G}^{op}$ action on R) has a final object locally over B.

Repeat: $\mathcal{G}$ groupoid, $\mathcal{G}^\wedge = \mathrm{Fun}(\mathcal{G}, \text{sets})$

topos map $Sh_B \longrightarrow \mathcal{G}^\wedge$ given by $f^*: \mathcal{G}^\wedge \longrightarrow Sh_B$ rt cont + left exact. f^* rt cont implies f^* has the form $f^*(L) = R \times_{\mathcal{G}} L$ where R is the $\mathcal{G}^{op}$-sheaf: $\mathcal{G}^{op} \xhookrightarrow[\text{Yoneda}]{} \mathcal{G}^\wedge \xrightarrow{f^*} Sh_B$

f^* left exact means ~~the functo~~ that at each $b \in B$ the contrav functor given by R_b with $\mathcal{G}^{op}$ acting is representable, better to say the $\mathcal{G}^{op}$-set ~~given by~~ R_b is representable.

~~[illegible] groupoid ring. You have not got all the details. Look at [illegible]~~

---

where to start. Λ = arrow ring of $\mathcal{G}$. Yesterday you learned that there might be a new notion of free Λ-module, namely $\bigoplus_X \Lambda e_X \otimes V_X$, a direct sum of representable functors $\Lambda e_X = \mathbb{C}[h^X]$ $= \bigoplus_Y \mathbb{C}[\mathrm{Ar}(Y,X)]$. For a connected groupoid the functors $\mathbb{C}[h^X] = \Lambda e_X$ are all isomorphic, so it is not really a new notion. Only in so far that the old ~~[illegible]~~ version of free module with one generator Λ is replaced by Λe_X which is smaller. ~~Now you need~~

Repeat situation. Look at $G = M_2$

An M_2 graded ring is a ~~Morita context~~

Let's review what you know. ~~Consider $\Lambda \otimes V$~~

Because Λ is graded wrt M_2 ?

Non comm Mayer-Vietoris.

First understand groupoid consisting of 2 elts

1, 2 only the identity maps allowed.

$$W \xleftarrow{(\beta_1\ \beta_2)} \begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix} \xleftarrow{\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}} W \qquad \beta_1\alpha_1 + \beta_2\alpha_2 = 1_W$$

$$(p_{ij}) = \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\begin{pmatrix}\beta_1 & \beta_2\end{pmatrix} = \begin{pmatrix}\alpha_1\beta_1 & \alpha_1\beta_2 \\ \alpha_2\beta_1 & \alpha_2\beta_2\end{pmatrix}$$

$$\Lambda = \mathbb{C}e_{11} \oplus \mathbb{C}e_{22}$$

$p = p^2$ in a Γ graded ring.

$$\Gamma = \{e_1, e_2\} \qquad \boxed{\begin{matrix} e_1^2 = e_1, \quad e_2^2 = e_2 \\ e_1e_2 \quad e_2e_1 \quad \text{undefined} \end{matrix}}$$

$A = A_1 \oplus A_2$

$A_1A_1 \subset A_1, \quad A_2A_2 \subset A_2$

$A_1A_2 = A_2A_1 = 0$

$$A \longrightarrow \mathbb{C}\Gamma \otimes A$$

$$a_1 \longmapsto e_1 \otimes a_1$$

$$a_2 \longmapsto e_2 \otimes a_2$$

$p_1^2 = p_1, \quad p_2^2 = p_2$

$p_1p_2 = p_2p_1 = 0$

Next $\Gamma = M_2$. $\quad A = \bigoplus_{i,j=1,2} A_{ij}$

$$A_{ij} A_{kl} = \begin{cases} 0 & j \neq k \\ A_{il} & j = k \end{cases}$$

IDEA that you should be careful about $e_X^2 = e_X$

M_2-graded alg $=$ Morita context.

Interested in $p = p^2$ in an M_2-graded alg, get an M_2 ~~ring~~ graded ring A $\quad$ A idemp. and naturally M_2 graded.

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & M_2\mathbb{C} \otimes A \\ \cup & & \\ a_{ij} & \xrightarrow{\Delta} & e_{ij} \otimes a_{ij} \end{array}$$

But the units e_{ii}, e_{jj} are multipliers on A so that reduced modules naturally split

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Argument: A universal alg gen. by components p_{ij} of a projection in an M_n-graded alg. Then A is naturally M_n graded also idempotent. The diagonal units e_{ii} can be adjoined to A

$$\underbrace{e_{kk}}(a_{ij})(b_{mn}) = (a_{kj})(b_{mn}) = a_{kj} b_{jn}$$

Let A be graded wrt a groupoid Γ

~~[illegible]~~ Can you show that the units e_X for X any object. ~~A is graded wrt~~ A is Γ graded with Γ a groupoid. For each object X of Γ define ~~[illegible] left mult by μ_X~~ a multiplier μ_X of A by:

$$\mu_X\cdot(Z\xleftarrow{f}Y)=\begin{cases}0 & X\neq Z\\ (X\xleftarrow{f}Y) & X=Z\end{cases}$$

$$(Z\xleftarrow{f}Y)\cdot\mu_X=\begin{cases}0 & Y\neq X\\ (Z\leftarrow X) & Y=X.\end{cases}$$

$$\mu_X(Z\xleftarrow{f}Y)(Y\xleftarrow{g}U)=\mu_X$$

$$(\mu_X a_f)\,a_g \overset{?}{=} \mu_X(a_f a_g)$$ both $=0$ if $X\neq \text{target}(f)$; $=$ if $X=\text{targ}(f)$

$$\underbrace{(a_f\mu_X)\,a_g}_{\neq 0\ \Rightarrow\ \text{source}(f)=X} = a_f(\mu_X a_g)$$

~~source (f) = target g~~

$X=$ target(g)

~~left mult [illegible] μ_X $\mu_X f$~~

You have a Γ-graded ring A: $A_f A_g \subseteq \begin{cases}0 & fg \text{ not defd}\\ A_{fg} & \text{oth.}\end{cases}$

X object let e_X be the operator on A defined by $e_X a_f = \begin{cases} a_f & \text{if } \text{targ}(f)=X\\ 0 & \text{if not}\end{cases}$

Maybe take care of the cases by using the graded.

$$A \xrightarrow{\Delta} \mathbb{C}\Gamma \otimes A \subset \mathbb{C}\Gamma \otimes \tilde{A}$$

$$a_f \overset{\Delta}{\longmapsto} f \otimes a_f$$

what elements of $\mathbb{C}\Gamma \otimes \tilde{A}$ yield multipliers on A.

$$([X] \otimes 1)(f \otimes a_f) = [X]f \otimes a_f = \begin{cases} f \otimes a_f & X = t(f) \\ 0 & X \neq t(f) \end{cases}$$

$$(f \otimes a_f)([X] \otimes 1) = f[X] \otimes a_f = \begin{cases} f \otimes a_f & \text{if } X = \text{source } f \\ 0 & X \neq \text{source } f \end{cases}$$

So it seems clear that you can adjoin units belonging to objects. Back to M_2. Now your A is M_2-graded and idempotent. ~~Idempotent implies~~

So a red. A-module V splits into $e_{11}V \oplus e_{22}V$. ~~At this point~~ So maybe what this means is

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Repeat: Given the groupoid M_2, you have the notion of a M_2-graded algebra (= Mor. context), and can form A the univ. alg gen. by the components of a proj in a M_2-graded alg. A is M_2-graded and we can adjoin e_{11}, e_{22} to A to get a unital Morita context. So next consider an A-module V

$$W \overset{\beta}{\twoheadleftarrow} M_2\mathbb{C} \otimes V \overset{\alpha}{\hookleftarrow} W$$

A = univ. alg gen by components p_{ij} of a proj in a M_2-graded alg.

A is idempotent, M_2-graded, and can be enlarged to a unital M_2-graded alg.

$$\cancel{A \xrightarrow{\Delta} M_2\mathbb{C}\otimes A} \qquad \cancel{\bigoplus_{ij} A_{ij} \ni a_{ij} \longmapsto e_{ij}\otimes a_{ij}}$$

Why A is M_2-graded.
Define $\Delta : A \longrightarrow M_2\mathbb{C}\otimes A$ to be the alg map such that $\Delta(p_{ij}) = e_{ij}\otimes p_{ij}$

check relations $(e_{ij}\otimes p_{ij})(e_{kl}\otimes p_{kl}) = \begin{cases} 0 & \text{for } j\neq k \\ e_{il}\otimes p_{ij}p_{jk} & j=k \end{cases}$

$$\sum_j \Delta(p_{ij})\Delta(p_{jk}) = \sum_j (e_{ij}\otimes p_{ij})(e_{jk}\otimes p_{jk})$$

$$= e_{ik}\otimes \sum_j p_{ij}p_{jk} = e_{ik}\otimes p_{ik} = \Delta(p_{ik})$$

$\Delta : A \longrightarrow M_2\mathbb{C}\otimes A$ is an alg map

check

$$\begin{tikzcd} A \arrow[r, "\Delta"] & M_2\mathbb{C}\otimes A \arrow[r, shift left, "\Delta\otimes 1"] \arrow[r, shift right, "1\otimes\Delta"'] \arrow[d, "\eta"] & M_2\mathbb{C}\otimes M_2\mathbb{C}\otimes A \\ & A & \end{tikzcd}$$

$$p_s \longmapsto s\otimes p_s \rightrightarrows \begin{matrix} s\otimes s\otimes p_s \\ s\otimes s\otimes p_s \end{matrix}$$

Next point $\quad A \xrightarrow{\Delta} M_2\mathbb{C}\otimes A \subset M_2\mathbb{C}\otimes\tilde{A}$

claim that $e_{11}\otimes 1$, $e_{22}\otimes 1$ in $M_2\mathbb{C}\otimes\tilde{A}$ such that left or right mult by these elts preserves $\Delta A = \bigoplus e_{ij}\otimes A_{ij} \subset M_2\mathbb{C}\otimes A$

$$(e_{11}\otimes 1)(e_{ij}\otimes a_{ij}) = \delta_{1i}\, e_{ij}\otimes a_{ij} = \begin{cases} 0 & i\neq 1 \\ e_{ij}\otimes a_{ij} & \end{cases}$$

Better ~~[illegible]~~ $\Delta : A \hookrightarrow \Lambda \otimes A$

$$\begin{array}{ccc} A & \hookrightarrow & \Lambda \otimes A \\ \cup & & \cup \\ A_s & \xrightarrow{\sim} & s \otimes A_s \end{array}$$

look at $e_X \otimes 1 \in \Lambda \otimes \tilde{A}$ contains $\Lambda \otimes A$ as ideal

$$(e_X \otimes 1)\Delta(a_s) = (e_X \otimes 1)(s \otimes a_s) = e_X s \otimes a_s$$

$$= \begin{cases} s \otimes a_s & \text{if } X = \text{target}(s) \\ 0 & \text{if } X \neq \text{target}(s) \end{cases}$$

$$\in \Delta(A_s)$$

Therefore you find that e_{11}

~~[illegible]~~ Review M_2

$A \xrightarrow{\Delta} \Lambda \otimes A$, $A = \oplus A_s$, $\Delta(a_s) = s \otimes a_s$

$$\begin{array}{c} \Lambda \otimes A \\ \cap \\ \Lambda \otimes \tilde{A} \end{array}$$

~~A_s~~

Inside $\Lambda \otimes \tilde{A}$ you have ~~[illegible]~~ subalg ~~[illegible]~~

$\Delta A \oplus$ ~~[illegible]~~ $\bigoplus_X \mathbb{C}(e_X \otimes 1)$, you can adjoin ~~[illegible] the idempoten~~ the idempotents belong to objects.

$$(e_X \otimes 1)\Delta(a_s) = (e_X \otimes 1)(s \otimes a_s) = e_X s \otimes a_s$$

So let's see how this works for M_2. Let V be an A-module, ρ_s

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A-module V has operator ρ_{ij}

You want to understand clearly the M_n situation. ~~Let~~ But it should be simpler to treat a connected groupoid Γ. ~~Assembly~~ arrow ring $\mathbb{C}\Gamma$, basis $\coprod_{Y,X} \underbrace{a_\Gamma(Y,X)}_{\mathrm{Hom}_\Gamma(X,Y)} = a_\Gamma$

notion of Γ-graded alg. $\quad (Y|\Gamma|X)$

$$A \longrightarrow \mathbb{C}\Gamma \otimes A$$

Γ-graded alg A is alg with ~~splitting~~ $A = \bigoplus_{s\in\Gamma} A_s$

$\ni \quad A \xrightarrow{\Delta} \mathbb{C}\Gamma \otimes A$ is ~~a ring~~ alg map

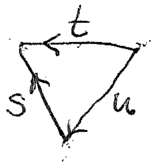
$$\begin{array}{c} A \\ \parallel \\ \bigoplus_s A_s \ni a_s \end{array} \quad \Delta(a_s) = s \otimes a_s \quad \Bigg| \quad \Delta(a_s a_t) = st \otimes a_s a_t$$

a Γ graded alg

$p = p^2$ in A means $p_s = \sum_{s=tu} p_t p_u$

Define A_Γ by gens + rels. $\quad V$ an A-module

$\Lambda \otimes V \ni \sum t \otimes f(t) \qquad tu^{-1} = s$



$$p\Big(\sum_t t \otimes f(t)\Big) = \sum_t \sum_u tu^{-1} \otimes p(u) f(t)$$

$$= \sum_s s \otimes \sum_t p(s^{-1}t) f(t)$$

What do you want to do? Settle question of V being graded with respect to objects.

$$\Lambda = \mathbb{C}\Gamma = \bigoplus_{X} \mathbb{C}[h^X] \quad \Lambda e_X$$

basis $\left(?\,|\Gamma|\,X\right)$

$\Gamma = M_2$

$$W \xleftarrow{\beta} \Lambda \otimes V \xhookleftarrow{\alpha} W$$

$$W^\# \xleftarrow{(\beta_1\ \beta_2)} \begin{matrix} V \\ \oplus \\ V \end{matrix} \xleftarrow{\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}} W^\#$$

$$\begin{matrix} V \\ \oplus \\ V \end{matrix} \xhookleftarrow{\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}} W^\# \twoheadleftarrow{(\beta_1,\ \beta_2)} \begin{matrix} V \\ \oplus \\ V \end{matrix}$$

so you should start maybe with

$$\begin{matrix} V \\ \oplus \\ V \end{matrix} \xleftarrow{\begin{pmatrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{pmatrix}} \begin{matrix} V \\ \oplus \\ V \end{matrix}$$

and introduce the image

$$\begin{matrix} V \\ \oplus \\ V \end{matrix} \xhookleftarrow{\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}} W^\# \twoheadleftarrow{(\beta_1,\ \beta_2)} \begin{matrix} V \\ \oplus \\ V \end{matrix}$$

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other picture

$$\begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix} \xhookleftarrow{\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}} W^\# \twoheadleftarrow{(\beta_1\ \beta_2)} \begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix}$$

There are two situations

V ungraded

$$\begin{tikzcd} V \oplus V \arrow[r, tail, "\binom{\alpha_1}{\alpha_2}"] & W^{\#} & V \oplus V \arrow[l, two heads, "(\beta_1 \beta_2)"'] \end{tikzcd}$$

V graded wrt objects

$$\begin{tikzcd} V_1 \oplus V_2 \arrow[r, tail, "\binom{\alpha_1}{\alpha_2}"] & W^{\#} & V_1 \oplus V_2 \arrow[l, two heads, "(\beta_1 \beta_2)"'] \end{tikzcd}$$

In the graded case $(p_{ij}) \in \begin{pmatrix} \mathrm{Hom}(V_1, V_1) & \mathrm{Hom}(V_2, V_1) \\ \mathrm{Hom}(V_1, V_2) & \mathrm{Hom}(V_2, V_2) \end{pmatrix}$

Start with ungraded case, ~~and~~ go through the process of making V reduced. You ~~feel~~ think that then V will split

In the ungraded case when is V reduced.

$$V = \sum_{ij} p_{ij} V = \sum_{ij} \alpha_i \beta_j V \subset \sum_i \alpha_i W^{\#}$$

$$\sum_{ij} \alpha_i \beta_j v_{ij} \qquad \subset \sum_i \alpha_i \sum_j \beta_j V$$

Repeat

$$\sum_{ij} p_{ij} V = \sum_{ij} \alpha_i \beta_j V$$

$$\subset \sum_i \alpha_i W^{\#}$$

$$\sum_{ij} p_{ij} V = \sum_{ij} \alpha_i \beta_j V = \begin{matrix} \alpha_1\beta_1 V + \alpha_1\beta_2 V \\ + \alpha_2\beta_1 V + \alpha_2\beta_2 V \end{matrix}$$

$$\simeq \begin{matrix} \alpha_1 W^\# \\ + \\ \alpha_2 W^\# \end{matrix} = V \iff V = \alpha_1 W^\# + \alpha_2 W^\#$$

$$\bigcap \operatorname{Ker}(p_{ij} \text{ on } V) = \bigcap \operatorname{Ker}(\alpha_i \beta_j \text{ on } V)$$

$$\alpha_i \beta_j v = 0 \quad \forall i j \implies \beta_j v = 0 \quad \forall j.$$

$$\therefore \bigcap_{ij} \operatorname{Ker}(p_{ij} \text{ on } V) = \bigcap_j \operatorname{Ker}(\beta_j \text{ on } V)$$

So you seem to understand what a reduced ungraded A module is. Question: What is reduced graded module.

Assume V reduced A-module

$$\begin{tikzcd} A \otimes_A V \arrow[rr, "\text{canon}"] \arrow[dr, two heads] & & \operatorname{Hom}_A(A, V) \\ & V \arrow[ur, hook] & \end{tikzcd}$$

therefore you get $e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ which ~~splits V into~~ gives a $\mathbb{Z}/2$ grading

suppose $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ is idemp. does this imply that $A_s A_t = A_{st}$ $\forall s, t$. A Γ graded and $A = A^2$, does this imply A Γ-idemp?

Consider a Morita context $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ which is idempotent as a ring. Let V be a reduced A-module. You know that you can embed A as ideal in a unital Morita context.

$$R = \begin{pmatrix} \tilde{A}_{11} & A_{12} \\ A_{21} & \tilde{A}_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix} \oplus \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

~~It's clear~~ You know the A action on V extends uniquely to a unital R-action, hence it should be clear that $V = \begin{pmatrix} e_{11}V \\ e_{22}V \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$

with [redacted] $A_{ij} V_k \subseteq \begin{cases} 0 & j \neq k \\ V_i & j = k \end{cases}$

$$\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$$

is idempotent when A_{11}, A_{22} idem. but not comp. idemp: $A_{21}A_{12} = 0$

Return to $A = \left\langle p_{ij}\text{'s} \quad \begin{array}{l} p_{ij}p_{kl} = 0 \quad j \neq k \\ \sum_j p_{ij}p_{jk} = p_{ik} \end{array} \right\rangle$

$$A \xrightarrow{\Delta} M_n A$$

$$p_{ij} \overset{\Delta}{\longmapsto} e_{ij} \otimes p_{ij}$$

~~So what?~~ So have A with its <u>universal M_2 graded proj</u> (p_{ij}). Reduced

You need to formulate things clearly in order to believe them. Given A gen by p_{ij} rels

$$\left| \begin{array}{ll} p_{ij}p_{kl} = 0 & j \neq l \\ p_{ik} = \sum_j p_{ij}p_{jk} & \end{array} \right.$$

and a reduced A-module V. You know V is graded by object projections. Thus in $V = \begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix}$ with $p_{ij} V_k \subseteq \begin{cases} 0 & j \neq k \\ V_i & j = k \end{cases}$

In End(V) there are besides the p_{ij}, [redacted] the object units e_{ii}

$$\begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix} \xleftarrow{\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}} \begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix}$$

$$\sum_{ij} p_{ij} V = \sum p_{ij} V_j = \begin{pmatrix} p_{11}V_1 + p_{12}V_2 \\ \\ p_{21}V_1 + p_{22}V_2 \end{pmatrix}$$