Review. You are making a calculation carefully so as to handle the left-right choices. You start with a representation of Γ on H tog. with op. $h_1 \geq 0$ such that $(h_s = s h_1 s^{-1})_{s\in\Gamma}$ is a partition of unity: $\sum h_s = 1$

Put $V_s = \overline{h_s^{1/2} H} = s V_1$. canonical maps

$$H \xrightarrow{\alpha} \bigoplus_{s\in\Gamma}^{(2)} V_s \xrightarrow{\alpha^*} H \qquad \alpha^*\alpha = 1$$

$$\xi \longmapsto (s \mapsto h_s^{1/2}\xi)$$

$$(s \mapsto \eta_s) \longmapsto \sum_s h_s^{1/2}\eta_s$$

This much could be done for an arb. partition of unity on a Hilbert space. Now bring in the group action of Γ on $\bigoplus_s^{(2)} V_s$. Here there is this system of imprimitivity which is simply transitive under Γ. So you get an isom.

$$\bigoplus_{s\in\Gamma}^{(2)} V_1 \xrightarrow[\sim]{\theta} \bigoplus_{s\in\Gamma}^{(2)} V_s$$

$$(s \mapsto \xi_s) \longmapsto (s \mapsto \overset{\eta_s}{s\xi_s})$$

what is t on $(s \mapsto \eta_s)$, it should be $(ts \mapsto t\eta_s)$ equiv. replacing s by $t^{-1}s$: $s \longmapsto t\eta_{t^{-1}s}$

$$\therefore (t\eta)_s = t\eta_{t^{-1}s} \quad \text{for } \bigoplus V_s$$

$$\begin{array}{ccc} \bigoplus_s V_1 & \xrightarrow{\theta} & \bigoplus V_s \\ & & \Big\downarrow t \\ \bigoplus_s V_1 & \xrightarrow{\theta} & \bigoplus V_s \end{array}$$

$$(s \mapsto \xi_s) \xrightarrow{\theta} (s \mapsto s\xi_s)$$

$$\downarrow$$

$$(s \mapsto t\,t^{-1}s\,\xi_{t^{-1}s})$$

$$(s \mapsto \xi_{t^{-1}s}) \qquad (s \mapsto s\,\xi_{t^{-1}s})$$

$$\begin{array}{ccccc} H & \xrightarrow{\alpha} & \bigoplus_s V_s & \xrightarrow{\theta^{-1}} & \bigoplus_s V_1 \\ \xi & \longmapsto & (\alpha\xi)_s = h_s^{1/2}\xi & \longmapsto & (\theta^{-1}\alpha\xi)_s = s^{-1}h_s^{1/2}\xi = h_1^{1/2}s^{-1}\xi \\ & & \uparrow \theta & & \\ & & \bigoplus_s V_1 & & \end{array}$$

$$(t\alpha'\xi)_s = (\alpha'\xi)_{t^{-1}s} = h_1^{1/2}(t^{-1}s)^{-1}\xi = h_1^{1/2}s^{-1}t\xi$$

So replace α by $\alpha' = \theta^{-1}\alpha$

$$H \xrightarrow{\alpha'} \bigoplus_s^{(2)} V_1 \xrightarrow{\alpha'^*} H$$

$$\xi \qquad \alpha'(\xi)_s = h_1^{1/2}s^{-1}\xi$$

Formulas: On $\bigoplus_{s\in\Gamma}^{(2)} V_1$ the action of $t\in\Gamma$ is

$$\boxed{(t\eta)_s = \eta_{t^{-1}s}} \qquad \text{where } \eta : \Gamma \longrightarrow V_1$$

check $(t_1(t_2\eta))_s = (t_2\eta)_{t_1^{-1}s} = \eta_{t_2^{-1}t_1^{-1}s} = \eta_{(t_1t_2)^{-1}s} = ((t_1t_2)\eta)_s$

$$H \xrightarrow{\alpha'} \bigoplus_s^{(2)} V_1 \qquad \boxed{(\alpha'\xi)_s = h_1^{1/2}s^{-1}\xi}$$

Check $(t(\alpha'\xi))_s = (\alpha'\xi)_{t^{-1}s} = h_1^{1/2}s^{-1}t\xi \qquad (\alpha'(t\xi))_s$

$$(\alpha'^*\eta, \xi) = (\eta, \alpha'\xi) = \sum_s (\eta_s, h_1^{1/2}s^{-1}\xi)$$

$$= \sum_s (s h_1^{1/2}\eta_s, \xi) \qquad \boxed{\alpha'^*\eta = \sum_s s h_1^{1/2}\eta_s}$$

check $\alpha'^*\alpha'\xi = \sum_s s h_1^{1/2} h_1^{1/2} s^{-1}\xi = \sum_s h_s\xi = \xi$

$$\alpha'^*(t\eta) = \sum_s sh_1^{1/2}(t\eta)_s = \sum_s sh_1^{1/2}\eta_{t^{-1}s} = \sum_{ts} tsh_1^{1/2}\eta_s = t\alpha'^*\eta$$

what remains is to "do the descent". I am not being clear, but I mean to do the GNS business: H with Γ action & equivariant partition of unity can be reconstructed from V_1 and the function $s \longmapsto h_1^{1/2} s h_1^{1/2}$ from Γ to $\mathcal{L}(V_1)$

why: $p = \alpha'\alpha'^* : \bigoplus_{s\in\Gamma}^{(2)} V_1 \circlearrowleft$ is a projection Γ-equivariant whose image is H.

Look at $\bigoplus_{s\in\Gamma}^{(2)} V_1 = \ell^2(\Gamma, V_1) \ni \eta = (s \mapsto \eta_s)$

where $(t\eta)_s = \eta_{t^{-1}s}$. OKAY can you write this as $V_1 \otimes \mathbb{C}[\Gamma]$? Get left + right straight.

~~[illegible]~~

Stick to the Hilb. space picture. The key ~~situation~~ is a unitary repn H of Γ ~~such that~~ where there exists a closed subspace $j: V \hookrightarrow H$ such that $H = \bigoplus_{s\in\Gamma}^{(2)} sjV$

Thus H is completion of $\mathbb{C}[\Gamma]\otimes V = \bigoplus_{s\in\Gamma} sjV$. Description elements of H ~~[illegible]~~ equiv. to functions on Γ to V.

$$\sum_{s\in\Gamma} sj\eta_s \in \bigoplus_{s\in\Gamma}^{(2)} sjV$$

If you describe elements of H as $\sum_{s\in\Gamma} sj\eta_s$ what is action of $t\in\Gamma$: $t\left(\sum_s sj\eta_s\right) = \sum_s tsj\eta_s = \sum_{t^{-1}s\in\Gamma} sj\eta_{t^{-1}s}$

so you get $(t\eta)_s = \eta_{t^{-1}s}$ action of $t\in\Gamma$ on $\ell^2(\Gamma, V)$

Note that you are using the left regular representation consistent with Γ left acting on $\mathbb{C}[\Gamma]\otimes V$.

Next to understand projection ~~[illegible]~~ ~~then~~

~~[illegible]~~

Return to replacing V by $h_1^{1/2}V$

$$H \xrightarrow{\alpha} \bigoplus_{s\in\Gamma}^{(2)} sV$$

$$\xi \longmapsto h_1^{1/2} s\xi \quad ?$$

$$\overline{h_s^{\frac12}H} = s\overline{h_1^{1/2}H} = sV$$

$$H \longrightarrow \bigoplus_{s\in\Gamma} V_s$$

$$\xi \longmapsto \sum_s h_s^{1/2}\xi = \sum_s s\,\underbrace{h_1^{1/2}s^{-1}\xi}_{\alpha'_s\xi}$$

$$\left\|\sum_s h_s^{1/2}\xi\right\|^2 = \sum_s \|h_s^{1/2}\xi\|^2 = \sum_s (\xi, h_s\xi) = \|\xi\|^2$$

$$\alpha'(\xi) = \sum_{s\in\Gamma} s\underbrace{\left(h_1^{1/2}s^{-1}\xi\right)}_{\eta_s} \in \bigoplus_{\Gamma}^{(2)} sV_1$$

~~[illegible]~~

At the moment you have Γ acting on H, $h_1 \geq 0$, $h_s = sh_1s^{-1}$, $\sum h_s = 1$, $V = V_1 = \overline{h_1H}$

$$H \xrightarrow{\alpha} \bigoplus_{s\in\Gamma}^{(2)} sV \xrightarrow{\alpha^*} H$$

$$\xi \xmapsto{\alpha} \sum_s s\,h_1^{1/2}s^{-1}\xi$$

$$\sum_s s\,\eta_s \ (\in V) \xmapsto{\alpha^*} \sum_s s h_1^{1/2}\eta_s$$

$$h_s^{1/2}\xi = sh_1^{1/2}s^{-1}\xi$$

$$\left(\sum s\eta_s, \sum sh_1^{1/2}s^{-1}\xi\right) = \sum_s (\eta_s, h_1^{1/2}s^{-1}\xi) = (sh_1^{1/2}\eta_s, \xi)$$

check $\alpha^*\alpha = \sum_s sh_1^{\frac12}h_1^{1/2}s^{-1}\xi = \sum_s h_s\xi = \xi$

$$t\alpha(\xi) = \sum_{t^{-1}s} s\,h_1^{1/2}s^{-1}t\xi = \alpha(t\xi)$$

$$(t\eta)_s = \eta_{t^{-1}s}$$

$$t\sum_s s\eta_s = \sum_s ts\eta_s = \sum_{t^{-1}s} \eta_{t^{-1}s}$$

So now you have the Γ-module picture pretty clear. Next ~~find~~ find the data needed to reconstruct H from V. Recall $\underline{V = \overline{h_1^{1/2} H}} = \overline{h_1 H}$.

The point – any projector $p=p^2=p^*$ on $\bigoplus^{(2)}_{s\in\Gamma} sV$ commuting with Γ determines $H = \mathrm{Im}(p)$ which is a unitary repn. of Γ, and an operator h_1 which is

$$H \xhookrightarrow{\alpha} \bigoplus^{(2)} sV \xrightarrow{j_1} V \xrightarrow{i_1} \bigoplus^{(2)} sV \xrightarrow{\alpha^*} H$$

$h_1 = \alpha^* i_1 i_1^* \alpha \geq 0$, should satisfy $\sum_s s h_1 s^{-1} = 1$ on H since $\sum s i_1 i_1^* s^{-1} = 1$ on $\bigoplus sV$. It might not be true that $H \xrightarrow{j_1 \alpha} V$ is surjective

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Problem. You have $p = p^* = p^2$ on $\bigoplus^{(2)}_{s\in\Gamma} sV$
p commutes with left Γ multiplication

$$\begin{tikzcd} H \arrow[r, hook, "\alpha"] & \bigoplus^{(2)}_{s\in\Gamma} sV \arrow[r, "\alpha^*"] \arrow[d, shift right, "i_1^* = j_1"'] & H \\ & V \arrow[u, shift right, "i_1"'] & \end{tikzcd}$$

$$p = \alpha\alpha^* : \bigoplus_{s\in\Gamma} sV \longrightarrow \bigoplus_{s\in\Gamma} sV$$

p commutes with left mult by Γ

You know that $\sum_{s\in\Gamma} i_s j_s = \mathrm{id}$ on $\bigoplus sV$

so that ?

What is an equivariant map from $\bigoplus^{(2)} sV$ to itself. Same as a linear map $V \longrightarrow \bigoplus_s sV$.

$$\mathrm{Hom}_\Gamma\left(\mathbb{C}[\Gamma]\otimes V, \mathbb{C}[\Gamma]\otimes V\right) = \mathrm{Hom}(V, \mathbb{C}[\Gamma]\otimes V)$$

$$\mathbb{C}[\Gamma]\otimes V = \bigoplus_s sV$$ typical element is

$\sum_{s\in\Gamma} s\eta_s$ with $\eta : \Gamma \longrightarrow V$. $\quad t\sum_s s\eta_s = \sum ts\eta_s$
$= \sum s\eta_{t^{-1}s}$.

You want to understand how an endo of $\mathbb{C}[\Gamma]\otimes V$ looks. It amounts to a linear map $V \longrightarrow \mathbb{C}[\Gamma]\otimes V$, thus it has the form of a function on Γ with values in $\mathrm{Hom}(V,V)$, call this function $s \longmapsto \rho_s$. No

You want to understand operators

$$T = \bigoplus_\Gamma sV \longrightarrow \bigoplus_\Gamma tW$$

which are Γ-module maps. ~~Assume $\dim(V) < \infty$~~

T is equivalent to a linear map $V \longrightarrow \bigoplus_\Gamma tW$

$$\sum t\rho_t : v \longmapsto \sum t\rho_t v \in \bigoplus tW$$

Then $\qquad sv \longmapsto \sum_t st\rho_t v = \sum_t s\rho_{s^{-1}t} v$

maybe you should look at $T : \bigoplus sV \longrightarrow \bigoplus tW$

focus on $t\rho_t T\iota_s s^{-1} : V \longrightarrow W$

$$V \xhookrightarrow{\iota_s s^{-1}} \bigoplus_s sV \qquad \bigoplus_s sV \xrightarrow{T} \bigoplus_t tW$$

given $V \longrightarrow \bigoplus_t tW \qquad \sum_t t\rho_t \qquad \rho_t \in \mathrm{Hom}(V,W)$

Conclude that $\mathrm{Hom}_\Gamma\left(\bigoplus_s sV, \bigoplus_t tW\right) = \mathrm{Hom}(V, \bigoplus tW)$

$$\doteq \{\rho = \sum t\rho_t \mid \rho_t \in \mathrm{Hom}(V,W)\}$$

$$T(sv) = s\sum t\rho_t v \qquad T$$

So what is going on? $\quad sV = \overline{h_s H} = \overline{s \underbrace{h_1 H}_{V}}$

$$H \xrightarrow{\alpha} \bigoplus sV \xrightarrow{\alpha^*} H$$

$$\xi \longmapsto \sum_s s h_1^{\frac{1}{2}} s^{-1} \xi$$

$$\sum_s s\eta_s \longmapsto \sum s h_1^{1/2} \eta_s$$

$$\|\sum h_s^{\frac{1}{2}} \xi\|^2 = \sum_s \underbrace{\|h_s^{1/2}\xi\|^2}_{(\xi, h_s \xi)} = \|\xi\|^2$$

$$(\alpha\xi, \sum s\eta_s) = \sum_s (h_s^{\frac{1}{2}}\xi, \eta_s)$$
$$= \sum_s (\xi, h_s^{\frac{1}{2}} \eta_s)$$

$$\alpha \cdot \alpha^* \sum_t t\eta_t = \alpha \sum_t t h_1^{\frac{1}{2}} \eta_t$$

$$= \sum_s s h_1^{\frac{1}{2}} s^{-1} \sum_t t h_1^{\frac{1}{2}} \eta_t = \sum_t \sum_s s h_1^{\frac{1}{2}} s^{-1} t h_1^{\frac{1}{2}} \eta_t$$

$$H \xrightarrow{\alpha} \bigoplus sV \xrightarrow{\alpha^*} H \qquad h_s^{\frac{1}{2}}\xi = s h_1^{\frac{1}{2}} s^{-1}\xi$$

$$\xi \longmapsto \sum_s h_s^{\frac{1}{2}}\xi = \sum_s s h_1^{\frac{1}{2}} s^{-1}\xi$$

$$\left(\sum_s s h_1^{\frac{1}{2}}\xi, \sum_t t\eta_t\right) = \sum_s (h_1^{\frac{1}{2}}\xi, \eta_s$$

Try again.

$$H \xrightarrow{\alpha} \bigoplus_{s\in\Gamma} sV \xrightarrow{\alpha^*} H$$

$$\boxed{\alpha(\xi) = \sum h_s^{\frac{1}{2}}\xi} = \sum s h_1^{\frac{1}{2}} s^{-1}\xi$$

$$\boxed{\alpha^*\left(\sum_s s\eta_s\right) = \sum_s s h_1^{\frac{1}{2}} \eta_s}$$

$$(\alpha(\xi), \sum_t t\eta_t) = \sum_s (h_1^{\frac{1}{2}} s^{-1}\xi, \eta_s)$$
$$= \sum_s (\xi, s h_1^{\frac{1}{2}} \eta_s)$$

$$\alpha^*\alpha\xi = \sum_s s h_1^{\frac{1}{2}} h_1^{\frac{1}{2}} s^{-1}\xi = \xi$$

$$\alpha\alpha^* \sum s\eta_s = \alpha\left(\sum_s s h_1^{\frac{1}{2}} \eta_s\right) = \sum_t h_t^{\frac{1}{2}} \sum_s s h_1^{\frac{1}{2}} \eta_s$$

$V = \overline{h_1^{1/2} H}$ $\qquad sV = \overline{h_s^{1/2} H}$ $\qquad \alpha(\xi) = \sum_s h_s^{1/2}\xi \in \overset{(2)}{\bigoplus_s} sV$

$\|\alpha\xi\|^2 = \sum_s (\xi, h_s\xi) = \|\xi\|^2$ $\qquad = \sum_s s h_1^{1/2} s^{-1}\xi$

$$\begin{tikzcd} H \arrow[r, "\alpha"] & \overset{(2)}{\bigoplus_s} sV \arrow[r, "\alpha^*"] \arrow[d, shift right, "j_1 s^{-1} = j_s"'] & H \\ & V \arrow[u, shift right, "\iota_s = s\iota_1"'] & \end{tikzcd}$$

$\left(\alpha^* \sum_s s\eta_s, \xi\right) = \left(\sum_s s\eta_s, \sum h_s^{1/2}\xi\right)$
$= \sum_s (\eta_s, h_1^{1/2} s^{-1}\xi)$

$\sum_s \iota_s j_s = \sum_s s\iota_1 j_1 s^{-1} = id$

$\sum_s \alpha^*\iota_s j_s \alpha = \sum_s s\underbrace{(\alpha^*\iota_1 j_1 \alpha)}_{h_1} s^{-1}$

$j_1\alpha\xi = j_1 \sum_s h_s^{1/2}\xi$
$= h_1^{1/2}\xi$

$\boxed{j_1\alpha = h_1^{1/2}}$ $\boxed{\alpha^*\iota_1 = h_1^{1/2}}$

$$\boxed{\alpha^*\left(\sum_s s\eta_s\right) = \sum_s s h_1^{1/2}\eta_s}$$

$s\eta_s = h_s^{\frac{1}{2}}\xi$
$\eta_s = h_1^{1/2} s^{-1}\xi$ $\qquad \sum_s s h_1 s^{-1}\xi = \xi.$
check.

$\alpha^*\iota_1 = h_1^{1/2}$

Remaining point $p = \alpha\alpha^*$ is a Γ-projector on $\overset{(2)}{\bigoplus_s} sV$ which is Γ-graded. This should mean p is a projector is Γ-graded algebra.

$$\bigoplus_s sV \xrightarrow{\alpha\alpha^*} \bigoplus_t tV$$

$$j_t\alpha\alpha^*\iota_s = j_1 t^{-1}\alpha\alpha^* s\iota_1 = j_1\alpha t^{-1}s\alpha^*\iota_1$$

So here you have $V \xrightarrow[\text{inclusion}]{\alpha^*\iota_1 = h_1^{1/2}} H$ and $H \xrightarrow{j_1\alpha} V$

$j_t p\, i_s = h_1^{1/2} t^{-1} s h_1^{1/2}$ $\qquad$ group elts compressed to V.

Summary: From H, Γ, h_1 you get $V = \overline{h_1^{1/2}H}$ and $p = (p_t)$
$p_t =$ ~~[illegible]~~ $h^{1/2} t^{-1} h^{1/2} \in \mathcal{L}(V)$. Conversely GNS should allow you to reverse the

Review: $H,\ \Gamma,\ h_s = s h_1 s^{-1} \geq 0,\ \sum h_s = 1$

$V = \overline{h_1^{1/2} H} \qquad sV = \overline{h_s^{1/2} H} \qquad \alpha(\xi) = \bigoplus h_s^{1/2}\xi \in \bigoplus_s^{(2)} sV$

$\left(\alpha^*(\bigoplus s\eta_s), \xi\right) = \left(\bigoplus s\eta_s, \bigoplus h_s^{1/2}\xi\right) = \sum (\eta_s, h_1^{1/2}s^{-1}\xi) = \sum (s h_1^{1/2}\eta_s, \xi)$

$\alpha^*(\bigoplus s\eta_s) = \sum_s s h_1^{1/2}\eta_s = \sum_s h_s^{1/2} s\eta_s$

$$\boxed{t(\bigoplus s\eta_s) = \bigoplus s\eta_{t^{-1}s}}$$

$$\begin{tikzcd} H \arrow[r, "\alpha"] & \bigoplus_s sV \arrow[r, "\alpha^*"] \arrow[d, shift right, "j_1 s^{-1} = j_s"'] & H \\ & V \arrow[u, shift right, "\iota_s = s\iota_1"'] & \end{tikzcd}$$

$j_s\alpha\xi = j_1 s^{-1}(\bigoplus h_t^{1/2}\xi) = h_1^{1/2} s^{-1}\xi$

$\boxed{j_1\alpha(\xi) = h_1^{1/2}\xi}$ $\qquad j_s\alpha(\xi) = j_1 s^{-1}\alpha(\xi) = j_1\alpha(s^{-1}\xi) = h_1^{1/2}s^{-1}\xi,$

$\boxed{\alpha^*\iota_1\eta = h_1^{1/2}\eta}$

$$\boxed{\begin{aligned} j_1\alpha &= h_1^{1/2} : H \longrightarrow V \\ \alpha^*\iota_1 &= h_1^{1/2} : V \longrightarrow H \\ j_s\alpha &= h_1^{1/2}s^{-1} : H \longrightarrow V \\ \alpha^*\iota_s &= s h_1^{1/2} : V \longrightarrow H \end{aligned}}$$

check

$\alpha^*\alpha = \alpha^*\left(\sum \iota_s j_s\right)\alpha = \sum_s s\,(\overbrace{\alpha^*\iota_1 j_1\alpha}^{h_1})\,s^{-1} = id_H$

$\alpha\alpha^* : \bigoplus_t tV \longrightarrow \bigoplus_s sV$

$j_s\alpha\alpha^*\iota_t = j_1\alpha\, s^{-1}t\, \alpha^*\iota_1 = h_1^{1/2}s^{-1}t\,h_1^{1/2}$

is the compression of the group element $s^{-1}t$ to an operator in V.

Given $H, \Gamma, h_1 \ni \sum_{s\in\Gamma} h_s = 1$ get $V = \overline{h_1^{1/2}H}$

and $H \xrightarrow{j_1\alpha = h_1^{1/2}} V \xrightarrow{\alpha^*\iota_1 = h_1^{1/2}} H$

~~[illegible]~~

Summary: Given $H, \Gamma, h_1, \sum h_s = 1$, you get

$V = h_1^{1/2} H$ and maps $H \xrightarrow{j_1 \alpha = h_1^{1/2}} V \qquad V \xrightarrow{\alpha^* \iota_1 = h_1^{1/2}} H$

such that $p_s = j_1 \alpha s \alpha^* \iota_1 = h_1^{1/2} s h_1^{1/2} \in \mathcal{L}(V)$

has a complete positivity property. ~~[illegible]~~
idempotence

$$p_s p_t = h_1^{1/2} s h_1 t h_1^{1/2} = h_1^{1/2} h_s st h_1^{1/2}$$

$$\sum_s p_s p_{s^{-1}t} = \sum_s h_1^{1/2} h_s t h_1^{1/2} = h_1^{1/2} t h_1^{1/2} = p_t$$

Go backwards. Suppose given V a Hilbert space and a family of operators $p_s \in \mathcal{L}(V)$ satisfying

$p_s^* = p_{s^{-1}} \qquad p_s = \sum_{\substack{t,u \\ tu = s}} p_t p_u$. Then can you define a

[illegible] p on $\overset{(2)}{\bigoplus_{s \in \Gamma}} sV$. ~~[illegible]~~

what is $pv = \bigoplus_s$? $\qquad p_s = j_1 \alpha s \alpha^* \iota_1 = h_1^{1/2} s h_1^{1/2}$

$$p = \sum_s \iota_s j_s \, p \sum_t \iota_t j_t = \sum_s s \iota_1 j_1 s^{-1} p \sum_t t \iota_1 j_1 t^{-1}$$

$$= \sum_{s,t} s \iota_1 \underline{(j_1 s^{-1} t p \iota_1)} j_1 t^{-1}$$

$$j_1 s^{-1} t \alpha \alpha^* \iota_1 = \underline{(j_1 \alpha)(s^{-1} t)(\alpha^* \iota_1)}$$

~~So you need to know something~~

You have to get the data of the Morita equiv. under control. $(H, \Gamma, h_1) \rightsquigarrow (V, p_s) \qquad V = h_1^{1/2} H, \quad p_s = h_1^{1/2} s h_1^{1/2}$

Conversely given V, p_s $\qquad \begin{cases} p_u = \sum_{\substack{s,t \\ u = st}} p_s p_t . \\ p_t^* = p_{t^{-1}} \end{cases}$

you construct p on $\overset{(2)}{\bigoplus_s} sV$ $\qquad p = \alpha\alpha^*$

$$p\iota_1 = \sum_{s} \iota_s j_s p \iota_1 = \sum_{s} s\, \underbrace{\iota_1 j_1 s^{-1} p\, \iota_1}_{\iota_1 \underbrace{j_1 \alpha s^{-1} \alpha^* \iota_1}_{p_{s^{-1}}}}$$

$$p\iota_1 = \sum_{s} s\, \iota_1 p_{s^{-1}}$$

Again.

$$\begin{tikzcd} H \arrow[r, "\alpha"] & \bigoplus tV \arrow[r, "\alpha^*"] \arrow[rr, bend left, "p"] & H \arrow[r, "\alpha"] & \bigoplus sV \end{tikzcd}$$

$$j_s p \iota_t = j_1 s^{-1} \alpha\alpha^* t\, \iota_1 = (j_1\alpha)\, s^{-1}t\, (\alpha^*\iota_1)$$

$$p\iota_t = \sum_s \iota_s j_s p \iota_t = \sum_s s\iota_1 (j_1\alpha) s^{-1}t\, (\alpha^*\iota_1)$$

The problem ~~[illegible]~~? What should happen?

Given V, Γ you get $\overset{(2)}{\bigoplus_s} sV$ unitary rep. of Γ

$$t\left(\bigoplus s\eta_s\right) = \bigoplus s\, \eta_{t^{-1}s}$$

This is forced because you want Γ to left act and the translates sV ~~[illegible]~~ to yield a Γ-grading. Next you can consider any projector p on $\overset{(2)}{\bigoplus} sV$ commuting with Γ. The image of p will give an H with unitary Γ action. ~~Moreover you have~~

$$\mathrm{Hom}_\Gamma\left(\bigoplus_t tV, \bigoplus_s sV\right) = \mathrm{Hom}\left(V, \overset{(2)}{\bigoplus} sV\right)$$

Given an $T : V \longrightarrow \bigoplus sV$ it extends to a Γ map $\bigoplus tV \longrightarrow \bigoplus sV$

$$j_s T \iota_t = j_1 (s^{-1}t) T \iota_1$$

$$T = \sum_{s,t} \iota_s j_s T \iota_t j_t$$

Something should be very simple

Let $T_s = j_s T \iota_1 : V \hookrightarrow \oplus tV \xrightarrow{T} \oplus sV \xrightarrow{j_s} V$

$= j_1 s^{-1} T \iota_1$

Then $T \iota_t = \bigoplus_s \iota_s j_s T \iota_t = \oplus s \iota_1 j_1 s^{-1} t T \iota_1$

What's going on? It should be simple to describe. You want to decribe $T : \oplus tV \longrightarrow \oplus sV$ commuting with Γ. For $\dim V < \infty$ you know Γ-linear T is equiv to $T\iota_1 : V \longrightarrow \bigoplus_s sV$. So T splits into components. $T = \sum_s T_s$ T_s unique Γ-linear op $\ni T_s \iota_1 = j_s T \iota_1$

Here the problem. Starting point (algebraic version) is a Γ-module M ~~such that~~ containing a subspace V such that the canon. map $\bigoplus_{s \in \Gamma} sV \longrightarrow M$ is an isom. Equivalently ~~the~~ the vector space M is given a grading $M = \bigoplus_{s \in \Gamma} M_s$ indexed by the set Γ ~~and~~ which is compatible with the left Γ action on Γ and M: $tM_s \subset M_{ts}$.

Now consider operators ~~between~~ on such a "free" Γ-module. A Γ-linear operator $T : M = \mathbb{C}[\Gamma] \otimes V \longrightarrow M$ is the same as a linear map $V \longrightarrow M = \bigoplus_s sV$, so it splits ~~into~~ $T = \sum_s T_s$ where $T_s V \subset sV$. Say T is homogeneous of degree t when $TV \subset tV$. Let U be homogeneous of degree u: U is Γ-linear and $UV \subset uV$. Then $T(U(V)) \subset T(uV) = uTV \subset utV$. So the degree of TU is backwards: ut. ~~So the should~~

Back to $\mathbb{C}[\Gamma]\otimes V = \bigoplus_s sV$ $\quad\Gamma$ acts by left mult

$$\operatorname{Hom}_\Gamma(\mathbb{C}[\Gamma]\otimes V, \mathbb{C}[\Gamma]\otimes V) = \operatorname{Hom}(V, \mathbb{C}[\Gamma]\otimes V)$$

$$\varphi : V \longrightarrow \mathbb{C}[\Gamma]\otimes V \qquad \varphi = \boxed{\bigoplus s\varphi_s}\,\textstyle\sum s\otimes\varphi_s \qquad \varphi_s\in\mathcal{L}(V)$$

Let $\hat\varphi : \mathbb{C}[\Gamma]\otimes V \longrightarrow \mathbb{C}[\Gamma]\otimes V$ be the Γ linear extension

$$\hat\varphi(t\otimes v) = t\varphi(1\otimes v) = t\sum_s s\otimes\varphi_s(v) = \sum_s ts\otimes\varphi_s(v)$$

$$\hat\varphi\Big(\sum_t t\otimes\eta_t\Big) = \sum_t t\,\hat\varphi(1\otimes\eta_t) = \sum_t t\sum_s s\varphi_s\eta_t$$

$$= \sum_{t,s} ts\,\varphi_s\eta_t$$

$$\hat\varphi\Big(\bigoplus_t t\eta_t\Big) = \sum_t t\,\hat\varphi\,\eta_t = \sum_t ts\varphi_s\eta_t$$

Thus if $\varphi = \bigoplus_s s\varphi_s \in \bigoplus s\mathcal{L}(V)$

then $\hat\varphi\Big(\bigoplus_t t\eta_t\Big) = \bigoplus_{t,s} ts\varphi_s\eta_t \in \mathbb{C}[\Gamma]\otimes V$

$$= \bigoplus_{t,s} s\,\varphi_{t^{-1}s}\eta_t = \bigoplus_{t,s} t\,\varphi_s\eta_{ts^{-1}}$$

Note wrong order

Another notation maybe better for the analysis.

$$\mathbb{C}[\Gamma]\otimes V = \{\eta : \Gamma\longrightarrow V \mid \text{finite support}\}$$

left action of Γ is $(t\eta)(s) = \eta(t^{-1}s)$

$$t\sum_s s\otimes\eta_s = \sum_s ts\otimes\eta_s = \sum s\otimes\eta_{t^{-1}s}$$

$$\operatorname{End}_\Gamma(\mathbb{C}[\Gamma]\otimes V) = \mathbb{C}[\Gamma]\otimes\mathcal{L}(V) \qquad \text{at least for } \dim V<\infty$$

this order forced so as to commute with left Γ mult.

$$\Big(\sum_s s\otimes\varphi_s\Big)\Big(\sum_t t\otimes\eta_t\Big) = \sum_{s,t} ts\otimes\varphi_s\eta_t$$

composition is $\Big(\sum_s s\otimes\varphi_s\Big)\Big(\sum_t t\otimes\psi_t\Big) = \sum_{s,t} ts\otimes\varphi_s\psi_t = \sum_u u\otimes\sum_{ts=u}\varphi_s\psi_t$

~~Yes / Another variation~~ Let $p \in \mathrm{End}_\Gamma(\mathbb{C}[\Gamma]\otimes V)$

Then $p = \sum s\otimes p_s \in \mathbb{C}[\Gamma]\otimes \mathcal{L}(V)$, assume $p^2=p$.

$$p^2 = \sum_{s,t} ts\otimes p_s p_t = \sum_{u\in\Gamma} u\otimes \sum_{ts=u} p_s p_t$$

So a Γ projection on $\mathbb{C}[\Gamma]\otimes V$ is the same as a function $s\mapsto p_s \in \mathcal{L}(V)$ satisfying

$$p_u = \sum_{ts=u} p_s p_t$$

Go back to the Hilb. space situation

$$H,\ \Gamma,\ h_1,\ h_s = s h_1 s^{-1} \qquad V = \overline{h_1^{1/2}H} \xleftarrow{h_1^{1/2}} H$$

$$H \xrightarrow[\xi\mapsto \oplus_s h_s^{1/2}\xi]{\alpha} \bigoplus_s sV \xrightarrow[\sum s\otimes\eta_s \mapsto \sum s h_1^{1/2}\eta_s = \sum h_s^{1/2} s\eta_s]{\alpha^*} H$$

$$(\alpha^*\alpha)(\xi) = \sum_s h_s^{1/2}h_s^{1/2}\xi = \sum_s h_s\xi = \xi.$$

Since $\alpha^*\alpha = id_H$, $p=\alpha\alpha^*$ is a projector commuting with Γ so it should have the form $\sum s\otimes p_s$: $\sum t\otimes\eta_t \mapsto \sum_{ts} ts\otimes p_s\eta_t$

$$\underbrace{\left(\sum_s s\otimes p_s\right)(1\otimes v)} = \sum s\otimes p_s v$$

$$p(1\otimes v) = \alpha\alpha^*(1\otimes v) = \alpha h_1^{1/2} v = \bigoplus_s h_s^{\frac12}h_1^{\frac12}v$$

$$= \sum_s s\underbrace{h_1^{\frac12}s^{-1}h_1^{\frac12}}_{p_s}v$$

$p_s = h_1^{1/2}s^{-1}h_1^{1/2}$. Check this

$$\sum_{ts=u} p_s p_t = \sum_{\substack{ts=u\\ t}} h_1^{1/2}\overset{u^{-1}t}{(s^{-1})}h_1 t^{-1}h_1^{1/2} = \sum_t h_1^{1/2}u^{-1}h_t h_1^{1/2} = p_u$$

You seem to be missing the good notation for a cross product. Once this is straightened out things should go smoothly.

~~Almost there!!!!~~ OKAY

Method: use inverses. H Hilbert space with Γ action, V closed subspace $\ni$ sV are orthogonal and sum dense in H. Then $\overset{(2)}{\bigoplus_s} sV \xrightarrow{\sim} H$ however you want to assign $\deg(sV) = s^{-1}$. So you write vs to mean $s^{-1}v$??

---

what was the point yesterday? Homogeneous components of an operator on a Γ graded space. Consider

$$\mathbb{C}[\Gamma] \otimes V = \bigoplus_{s \in \Gamma} s \otimes V$$

Γ action $\quad t(s \otimes v) = ts \otimes v$

Γ grading $\quad \underset{M}{\underset{\shortparallel}{(\mathbb{C}[\Gamma] \otimes V)}} = s \otimes v \qquad M_s = s \otimes V$

$tM_s \subset M_{ts}$

$$\mathrm{Hom}_\Gamma(R \otimes V, R \otimes V) = \mathrm{Hom}(V, R \otimes V) \supset R \otimes \mathrm{Hom}(V, V)$$

Start with a "free" Γ ~~representation~~ module

$M = R \otimes V = \bigoplus_{s \in \Gamma} s \otimes V$ An operator Φ on M commuting with Γ-action is the same as a linear map $\varphi : V \longrightarrow M$ via $\Phi(t \otimes v) = t\varphi(v)$. Among the $\varphi \in \mathrm{Hom}(V, R \otimes V)$ are those of the form $\sum_t t \otimes \varphi_t \in R \otimes \mathrm{Hom}(V, V)$ where the sum is finite

One ~~has~~

$$\boxed{\left(\sum_t t\otimes \varphi_t\right)\left(\sum_s s\otimes v_s\right) = \sum_{t,s} ts\otimes \varphi_t v_s}$$

and the composition of $\Psi\Phi$ of operators assoc. to ψ, ϕ is given by the same formula.

$$\left(\sum_t t\otimes\psi_t\right)\left(\sum s\otimes\phi_s\right) = \sum ts\otimes\psi_t\phi_s$$

You want to focus on the grading, what grading?

~~namely given a graded vector space~~

$M = R\otimes V$ has a natural grading indexed by the set Γ, namely $M = \bigoplus_{s\in\Gamma} M_s$ where $M_s = s\otimes V$ the subspace of $R\otimes V$.

An operator T on M is said to be homog of degree $t\in\Gamma$ when $TM_s \subset M_{ts}$ $\forall s$. If T_i has deg t_i for $i=1,2$, then $T_1T_2M_s \subset T_1M_{t_2s} \subset M_{t_1t_2s}$ so T_1T_2 has deg t_1t_2.

The general framework should be comodules over the coalgebra $\mathbb{C}[\Gamma]$, $\Delta s = s\otimes s$ for a grading wrt the set Γ. Perhaps a tensor product defined for comodules over a Hopf alg.

$$M\otimes N = \bigoplus_{s,t} M_s\otimes N_t = \bigoplus_u\left(\bigoplus_{st=u} M_sN_t\right)$$

This tensor product is appropriate for $R_sM_t \subset M_{st}$, $R_sR_t\subset R_{st}$, $W_sR_t\subset W_{st}$ for a Γ-graded alg R Γ-graded left + right modules over R

Keep things simple. Γ set, then you have notion of Γ-graded vector space $M = \bigoplus_{s\in\Gamma} M_s$, same as a comodule for $\mathbb{C}[\Gamma]$, $\Delta s = s\otimes s$

If Γ set, you have notion of Γ-graded v.s. $M = \bigoplus_{s\in\Gamma} M_s$

Given two Γ-graded v.s. $M = \bigoplus_{s\in\Gamma} M_s$, $N = \bigoplus_{s\in S} N_s$

then $M\otimes N = \bigoplus_{s,t\in\Gamma\times\Gamma} M_s\otimes N_t$ is $\Gamma\times\Gamma$ graded. Now

if Γ is a group you can push forward wrt $\underset{(s,t)}{\Gamma\times\Gamma} \longrightarrow \underset{st}{\Gamma}$

to get a Γ graded v.s.

$$M\otimes N = \bigoplus_u \Big(\bigoplus_{st=u} M_s\otimes N_t\Big)$$

This is a kind of convolution type tensor product:

$$(M\otimes N)_u = \bigoplus_{u=st} M_s\otimes N_t = \bigoplus_s M_s\otimes N_{s^{-1}u}$$

So Γ-graded ~~modules~~ vector spaces form ~~a~~ $\otimes$-category allowing one to define Γ-graded algebras R and Γ-graded R-modules: $R_s R_t \subset R_{st}$, $R_s M_t \subset M_{st}$, also left mods. Note that Γ-graded vector spaces are the same as comodules for the comm. coalg $\mathbb{C}[\Gamma]$, $\Delta s = s\otimes s$.

You feel that the important question concerns the behavior of operators on a Γ-graded module. Specifically look at the group ring $R = \bigoplus_{s\in\Gamma} \mathbb{C}s$ which is both a Γ-graded left (resp. right) R-module.

$$R_s R_t \subset R_{st} \qquad\qquad R_t R_s \subset R_{ts}$$

and these two actions of Γ? commute.

Look at things as follows. Let $M = \bigoplus_{s\in\Gamma} M_s$ be Γ graded, let Γ left act on M, $t: M\longrightarrow M$ $\ni$ $tM_s \subset M_{ts}$

Then $t: M_1 \xrightarrow{\sim} M_s$ so $M = \bigoplus sM_1$

Back to grading. Starting point is a Γ-module M equipped with a subspace V whose translates sV $s \in \Gamma$ are indep + span M:

$$\bigoplus_{s\in\Gamma} sV \xrightarrow{\sim} M \qquad \mathbb{C}[\Gamma]\otimes V \xrightarrow{\sim} M.$$

In this way M acquires a grading indexed by the set Γ. Alt: a comodule wrt $\mathbb{C}[\Gamma]$, $\Delta s = s\otimes s$

Maybe a better starting point would be ~~...~~ to consider ~~...~~ graded v.s. $\bigoplus_{x\in X} M_x$ with respect to a set X. Given $M = \bigoplus_{x\in X} M_x$, $N = \bigoplus_{y\in Y} M_y$ you have a tensor prod. $\bigoplus_{(x,y)\in X\times Y} M_x \otimes N_y$. Also given $f: X\to Y$, $M = \bigoplus_x M_x$ get $f_!(M) = \bigoplus_y \left(\bigoplus_{x\in f^{-1}(y)} M_x\right)$ pushforward of the system M_x under $f: X\longrightarrow Y$.

Then given $M = \bigoplus_{s\in\Gamma} M_s$, $N = \bigoplus_{t\in\Gamma} N_t$ you can form $M\otimes N = \bigoplus_{(s,t)\in\Gamma\times\Gamma} M_s\otimes N_t$ and push forward with $(s,t)\mapsto st$ to get a Γ graded v.s. $(M\otimes N) = \bigoplus_{u\in\Gamma} \bigoplus_{st=u} M_s\otimes N_t$

This construction should corresp. to $\otimes$ of comodules under the Hopf algebra $\mathbb{C}[\Gamma]$. ~~...~~

~~...~~ Look at the group ring $\mathbb{C}[\Gamma] = \bigoplus_{s\in\Gamma} \mathbb{C}s$ two actions of Γ: $t * s = \begin{cases} ts \\ st^{-1}\end{cases}$ which commute.

How do you ~~...~~ clarify things. Start with operators on a graded vector space. Fix look at ~~...~~ $\mathbb{Z}$ gradings. $V = \bigoplus_{n\in\mathbb{Z}} V_n$. The operators on a graded vector space should be a graded ring,

at least under appropriate finiteness. ~~What about~~
If use the obvious Γ-grading: $\mathbb{C}[\Gamma] = \bigoplus_{s\in\Gamma} \mathbb{C}s$. Put $R = \mathbb{C}[\Gamma]$, $R_s = \mathbb{C}s$. ?

Start again with a Hilbert space repn H of Γ ~~such that~~ and a closed $V < H$ such that $H = \hat{\bigoplus}_{s\in\Gamma} sV$. Consider a Γ-invariant operator T on H.

$$\begin{array}{rl} \mathrm{Hom}_\Gamma(R\otimes V, R\otimes V) = & \mathrm{Hom}(V, R\otimes V) \\ & \quad\cup \\ & R\otimes \mathrm{Hom}(V,V) \end{array}$$

Point is that with this way $(r\otimes\varphi)(r'\otimes v) = (r'r\otimes \varphi v)$. So as a ring one has $R^{op}\otimes \mathrm{End}(V)$. $\mathrm{Hom}_R(R,R) = R^{op}$. For $R=\mathbb{C}[\Gamma]$ and r is ~~sum of~~ lin. comb. of s, homogeneous op

maybe you want simply to use the ring $R\otimes \mathrm{End}(V)$
Let's begin again with $H, \Gamma, V \ni \bigoplus^{(2)}_{s\in\Gamma} sV \xrightarrow{\sim} H$

What's important? ~~Start again with~~ Answer – a grading on an algebra A. This means a splitting $A = \bigoplus_\mu A_\mu$ of A as vector space such that $\forall \mu,\nu$ $A_\mu A_\nu \subset A_\lambda$ for some $\lambda = \mu * \nu$. Then the set of indices has some sort of product. ~~which should~~ There's a problem with $A_\lambda = 0$, which should be sorted out.

Morita contexts.

Question: Is there a way to interpret a Morita context as a graded algebra. ~~[illegible]~~ ~~[illegible]~~ A Morita context $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ is a ring with a splitting into 4 abelian subgroups such that 8 of the possible 16 products are zero. Perhaps you have a grading wrt. a category with two objects and $id \circlearrowleft \bullet \rightleftarrows \bullet \circlearrowright id$ for arrows.

Question: When is a ~~[illegible]~~ Morita context unital?

This should be exactly the case of a unital ring R equipped with idempotent e : $R = \begin{pmatrix} eRe & eRe^{\perp} \\ e^{\perp}Re & e^{\perp}Re^{\perp} \end{pmatrix}$

Clear: $R = (e+e^{\perp})R(e+e^{\perp})$

~~[illegible]~~ You start with the ~~[illegible]~~ ~~[illegible]~~ notion of grading, i.e. ~~[illegible]~~ vector space equipped with a splitting indexed by a set S: $V = \bigoplus_{s\in S} V_s$. ~~[illegible]~~ To define graded algebra what you need is to assign to an ordered pair of elements (s,s') of S a third element $s*s'$ ~~[illegible]~~ partially defined operation $S\times S \supset T \longrightarrow S$, want

$$A_s A_{s'} \subset \begin{cases} A_{s*s'} & s*s' \text{ defd.} \\ 0 & \text{otherwise} \end{cases}$$

Let skip this + focus on $S =$ group Γ. Γ-graded alg A is one with sp. $A = \bigoplus_{s\in\Gamma} A_s$ + $A_s A_t \subset A_{st}$)

have notion of graded left module $A_s M_t \subseteq A_{st}$

right —— $N_t A_s \subset N_{ts}$

bimodule is Γ-graded is OK. $\left(\begin{array}{l} A_s M_t \subset M_{st} \\ M_t B_u \subset M_{tu} \end{array} \right.$

Suppose $V = \bigoplus_{s \in \Gamma} V_s$ is Γ graded.

$\boxed{\text{wallflowers}}$

to get past the obstruction

Let Γ be a group. Notion of Γ-graded vector space $V = \bigoplus_{s \in \Gamma} V_s$, tensor product of these:

$$(V \otimes W)_s = \bigoplus_{tu = s} V_t \otimes W_u,$$

Γ-graded algebra: $A = \bigoplus_{s \in \Gamma} A_s$, $A_s A_t \subset A_{st}$,

left and right Γ-graded modules over a Γ-graded alg.

Question: Given a Γ-graded vector space V, is there a natural Γ-graded alg of endomorphisms? More generally you want $\mathrm{Hom}(V, W)$ Γ-graded v.s.

universal:

$$\mathrm{Hom}_{\hat\Gamma\text{-mod}}(U, \mathrm{Hom}(V,W)) = \mathrm{Hom}_{\hat\Gamma}(U \otimes V, W)$$

It looks as if there are two Hom's corresp. to choice $U \otimes V$ and $V \otimes U$.

Take $U = \mathbb{C}u$ $\deg(u) = u$. Then degree u part of $\mathrm{Hom}(V, W)$, denoted $\mathrm{Hom}^{(u)}(V, W)$, should be

$$\mathrm{Hom}_{\hat\Gamma}(\underbrace{\mathbb{C}u \otimes V}_{\Sigma_u}, W) \qquad \mathbb{C}u \otimes \bigoplus_s V_s = \bigoplus_s uV_s$$

$$\Sigma^u V = \Sigma^u_\otimes V = \bigoplus_s V_{u^{-1}s}$$

$$(\Sigma_u^\otimes V)_s = \bigoplus_{s = ut} \Sigma_u^\otimes V_t$$

Review: Γ group, consider Γ-graded vector space

$$V = \bigoplus_{s\in\Gamma} V_s . \quad \text{tensor prod.} \quad (V\otimes W)_s = \bigoplus_{s=tu} V_t \otimes W_u$$

These are the same as comodules over coalg $\mathbb{C}[\Gamma]$, $\Delta s = s\otimes s$.

So you have a tensor category. ~~Look at internal~~ ~~Hom~~ Notion of Γ-graded alg and Γ-graded left and right modules, $A = \bigoplus_s A_s$, $M = \bigoplus_s M_s$

$A_s M_t \subset M_{st}$ resp $N_s A_t \subset N_{st}$.

You have this tensor product operation on Γ-modules, assoc. but not commutative. Question about internal Hom:

$$\mathrm{Hom}_{\hat\Gamma}(U, \underline{\mathrm{Hom}}(V,W)) = \mathrm{Hom}_{\hat\Gamma}(U\otimes V, W)$$

~~take $U = \mathbb{C}[t]$ $\quad (U\otimes V)_s = \mathbb{C}[t]_s \otimes$~~

take $U = \Sigma^u \mathbb{C}$ $\qquad (\Sigma^u \mathbb{C})_s = \begin{cases} 0 & s\neq u \\ \mathbb{C} & s = u. \end{cases}$

Assume $\underline{\mathrm{Hom}}(V,W)$ satisfies formula above get

$$\underline{\mathrm{Hom}}(V,W)_u = \mathrm{Hom}(\Sigma^u\mathbb{C}, \underline{\mathrm{Hom}}(V,W))$$

$$= \mathrm{Hom}(\Sigma^u \mathbb{C}\otimes V, W)$$

$$(\Sigma^u\mathbb{C}\otimes V)_s = \bigoplus_{tt'=s} \underbrace{(\Sigma^u\mathbb{C})_t}_{=\begin{cases}\mathbb{C} & \text{if } u=t \\ 0 & \text{if not}\end{cases}} \otimes V_{t'}$$

$$= \bigoplus_{\substack{t' \\ \ni ut'=s}} V_{t'} = V_{u^{-1}s}$$

$$\therefore \underline{\mathrm{Hom}}(V,W)_u = \bigoplus_{s\in\Gamma} \mathrm{Hom}(V_{u^{-1}s}, W_s)$$

Check. Is there a way to compose homogeneous operators.
$$V \xrightarrow{\deg a} W \xrightarrow{\deg b} X$$

$$\Sigma^a \longrightarrow \underline{\mathrm{Hom}}(V, W)$$
$$\Sigma^b \longrightarrow \underline{\mathrm{Hom}}(W, X)$$

i.e. $\Sigma^a \otimes V \longrightarrow W$, $\Sigma^b \otimes W \longrightarrow X$

$$\Sigma^b \otimes \Sigma^a \otimes V \longrightarrow \Sigma^b \otimes W \longrightarrow X$$

$$(\Sigma^b \otimes \Sigma^a)_s = \bigoplus_{\substack{t,u \\ s = tu}} \underbrace{(\Sigma^b_t \otimes \Sigma^a_u)}_{\begin{cases} 0 \text{ unless } t=b, \ u=a \\ \text{other } \mathbb{C}. \end{cases}} = \begin{cases} \mathbb{C} & s = ba \\ 0 & s \neq ba \end{cases}$$

$$= \Sigma^{ba}$$

$$\begin{cases} \mathbb{C} \text{ if } t=b \\ 0 \text{ if not} \end{cases} \qquad \begin{cases} \mathbb{C} \text{ if } u=a \\ 0 \text{ if not} \end{cases}$$

$$(\Sigma^b \otimes \Sigma^a)_s = \bigoplus_{\{(t,u) \mid tu = s\}} \Sigma^b_t \otimes \Sigma^a_u = \begin{cases} \mathbb{C} & \text{if } t=b \text{ and } u=a \\ 0 & \text{if not} \end{cases}$$

$$= \begin{cases} \mathbb{C} & \text{if } ba = s \\ 0 & \text{if not} \end{cases} = \Sigma^{ba}$$

Alternative: Given $V_{a^{-1}t} \longrightarrow W_t \quad \forall t$

and $W_{b^{-1}s} \longrightarrow X_s \quad \forall s.$

$$V_{a^{-1}b^{-1}s} \longrightarrow W_{b^{-1}s} \longrightarrow X_s \quad \forall s$$
$$(ba)^{-1}s$$

Alternative: Here use $\underline{\mathrm{Hom}}'(V, W)$ defined by
$$\mathrm{Hom}_{\hat{\Gamma}}(U, \underline{\mathrm{Hom}}'(V, W)) = \mathrm{Hom}_{\hat{\Gamma}}(V \otimes U, W)$$

$\underline{\mathrm{Hom}}'(V,W)_a = \mathrm{Hom}_{\Gamma}(V\otimes\Sigma^a, W)$ where

$$(V\otimes\Sigma^a)_s = \bigoplus_{s=\cancel{tu}} V_{\cancel{t}}\otimes\Sigma^a_{\cancel{u}} \quad \begin{matrix}\mathbb{C} & \text{if } s=a \\ 0 & \text{otherw.}\end{matrix} \quad = V_{sa^{-1}}$$

$$V\otimes\Sigma^a \longrightarrow W \qquad W\otimes\Sigma^b \longrightarrow X$$

$$V\otimes\underbrace{\Sigma^a\otimes\Sigma^b}_{\Sigma^{ab}} \longrightarrow W\otimes\Sigma^b \longrightarrow X$$

~~$V_{s(ab)}$ ... X_s~~

$$V_{sa^{-1}} \longrightarrow W_s \qquad W_{tb^{-1}} \longrightarrow X_{\cancel{t}}$$

$$V_{\underbrace{tb^{-1}a^{-1}}_{t(ab)^{-1}}} \longrightarrow W_{tb^{-1}} \longrightarrow X_t$$

I'm confused. If V is Γ-graded, then ~~the~~ there are apparently two Γ-graded algebras $\underline{\mathrm{Hom}}(V,V)$ and $\underline{\mathrm{Hom}}'(V,V)$. degree ~~a~~ elements of the former are maps $V_{a^{-1}s}\longrightarrow V_s$ $\forall s$, and of the latter are map $V_{sa^{-1}}\longrightarrow V_s$ $\forall s$.

Maybe what's useful to remember is that there are two basic ways to shift indexing - left + right translation.

~~...~~ This gives two types of homogeneous operators namely

$V_{a^{-1}s} \longrightarrow V_s$ $\forall s$ composition: $V_{\underbrace{a^{-1}b^{-1}}_{(ba)^{-1}}s} \xrightarrow{\deg(a)} W_{b^{-1}s} \xrightarrow{\deg(b)} X_s$, $\deg ba$

or $V_{sa^{-1}} \longrightarrow V_s$ $\forall s$. comp. $V_{s\underbrace{b^{-1}a^{-1}}_{(ab)^{-1}}} \xrightarrow{\deg a} W_{sb^{-1}} \xrightarrow{\deg b} X_s$, $\deg ab$

You should now go back to a free Γ-module $M = \bigoplus_s sV$. You need to describe Γ-invariant projections on M.

Start with a Γ-module M, whether left or right is irrelevant, via $sm = ms^{-1}$. Assume given a subspace V of M such that $\bigoplus_{s\in\Gamma} sV \longrightarrow M$ is an isomorphism. This means that M is the free Γ module gen. by the v.s. V.

Yesterday what did you learn? You looked at the tensor category of $\hat{\Gamma}$-modules (= Γ-graded modules)

$(V\otimes W)_s = \bigoplus_{s=tu} V_t \otimes W_u$. Get left + right ~~shift~~ translation

$$(\mathbb{C}a \otimes V)_s = V_{a^{-1}s} \qquad (V\otimes \mathbb{C}a)_s = V_{sa^{-1}}$$

~~allowing~~ yielding two notions of ~~map~~ homogeneous map of degree a: $V_{a^{-1}s} \longrightarrow W_s$ $\quad V_{sa^{-1}} \longrightarrow W_s$

Next go back to a free Γ-mod $M = \bigoplus sV$. Point. left and right Γ-modules are the same via $sm = ms^{-1}$, so there are two obvious ways to grade a free Γ module. First define the free Γ-module gen by vs V to be $\mathbb{C}[\Gamma]\otimes V$.

Let M be a free Γ-module, more precisely $M \simeq \mathbb{C}[\Gamma]\otimes V$, so $\exists$ subspace V of M such that $M = \bigoplus_{s\in\Gamma} sV$. Here you use left action, ~~but you~~ and $M = \bigoplus_{s\in\Gamma} Vs$ for the right. What's the point? The point is that these are two Γ-gradings. You didn't say this right.

~~$M = \bigoplus_{s\in\Gamma} sV$ equivalently then $M = \bigoplus_{s\in\Gamma} V s^{-1}$~~

Let M be a Γ module. You can view Γ as operating on the left or the right via: $sm = ms^{-1}$. Let M be free, i.e. $\exists$ subspace V s.t. $\bigoplus_{s\in\Gamma} sV \xrightarrow{\sim} M$ equivalently $\bigoplus_{s\in\Gamma} Vs = M$. Thus you have two Γ-gradings on M which are related by inverse since $sV = Vs^{-1}$.

You are ~~looking at~~ interested in idempotent operators on ~~the~~ a free Γ-module which commute with the Γ-action.

~~[illegible]~~ Recap. Equivalence between left + right Γ-modules via $sm = ms^{-1}$. ~~[illegible]~~ ~~[illegible]~~ Notion of free Γ-module generated by v.s. V: $M = \mathbb{C}[\Gamma]\otimes V$. Up to canon ~~isom~~ same as a Γ-module M with subspace V s.t. $\bigoplus_{s\in\Gamma} sV \xrightarrow{\sim} M$. This gives ~~a~~ Γ-grading with $M_s = sV$, making M a Γ-graded v.s. have $tM_s = M_{ts}$ so M is a left ~~[illegible]~~ Γ graded Γ-module.

~~[illegible]~~. Summary.

Γ-modules = Γ-graded vector spaces

$$V = \bigoplus_{s\in\Gamma} V_s$$

form a cat with ~~maps~~ morphs = linear maps preserving grading

tensor product
$$(V\otimes W)_s = \bigoplus_{s=tu} V_t \otimes W_u$$

~~noncom~~ comm in general when Γ noncomm.

~~[illegible]~~ tensoring or left or right with $\mathbb{C}a$ $(a\in\Gamma)$ leads to shift or susp. ops.

then to ~~maps~~ two kinds of ~~my~~ maps $V \to W$ of degree a:

$$V_{a^{-1}s} \longrightarrow W_s,\ \forall s \qquad \text{or} \qquad V_{sa^{-1}} \longrightarrow W_s,\ \forall s.$$

~~Answer~~ Question: What is the degree of a composition of homog. maps?

left shift.

$$\begin{array}{ccccc} V_{b^{-1}a^{-1}s} & \xrightarrow{\beta} & W_{a^{-1}s} & \xrightarrow{\alpha} & X_s \\ \| & & & \nearrow_{\alpha\beta} & \\ V_{(ab)^{-1}s} & & & & \end{array}$$

so

$$\mathrm{Hom}(W,X)_a \times \mathrm{Hom}(V,W)_b \longrightarrow \mathrm{Hom}(V,X)_{ab}$$

$$\alpha,\ \beta \qquad\qquad \alpha\beta$$

right shift

$$V_{sa^{-1}b^{-1}} \xrightarrow{\beta} W_{sa^{-1}} \xrightarrow{\alpha} X \qquad a^{-1}b^{-1} = (ba)^{-1}$$

$$\mathrm{Hom}(W,X)_a \times \mathrm{Hom}(V,W)_b \longrightarrow \mathrm{Hom}(V,X)_{ba}$$

$$\alpha,\ \beta \qquad\qquad \alpha\beta$$

rewrite ~~in~~ using (right) Kasparov composition

$$\mathrm{Hom}(V,W)_b \times \mathrm{Hom}(W,X)_a \longrightarrow \mathrm{Hom}(V,X)_{ba}$$

You are interested ultimately in ~~a projection~~ idempotent operators on a free Γ-module. left or right ~~does~~ for the Γ action does not make any difference, nor does the order of ~~maps~~ composition matter

free Γ-module: $M = \mathbb{C}[\Gamma] \otimes V$

$$\mathrm{Hom}_\Gamma(\mathbb{C}[\Gamma]\otimes V, \mathbb{C}[\Gamma]\otimes V) = \mathrm{Hom}(V, \mathbb{C}[\Gamma]\otimes V)$$

$$\mathbb{C}[\Gamma]\otimes V\otimes V^*$$

Look at the group ring $\mathbb{C}[\Gamma]$

First consider a free Γ-module ?

Assume you understand Γ-graded vector spaces, i.e. $\hat{\Gamma}$-modules. Now consider a vector space M equipped with Γ-action. ?

The idea is to replace the ?

Generalization. The shifting $V_{a^{-1}s}$ generalizes to $V = \bigoplus_{s \in \Delta} V_s$ where Δ is a Γ torsor.

So far you have looked at vector spaces with Γ-grading. Next look at Γ-modules and compatibility.

Start with splitting $V = \bigoplus_{k \in K} V_k$ grading of V wrt K, say Γ operates on V permuting the V_k, so Γ acts on the set K. K is a Γ-set so can be split into orbits, each orbit is described by a representation of stabilizer. Mackey's imprimitivity theory. Interesting case for you is where K is a Γ-torsor.

Free module: where $K = \Gamma$ i.e. K is a Γ torsor with basept chosen.

Consider a free Γ-module $M \simeq \bigoplus_{s \in \Gamma} sV$. Then M has a Γ-grading with $M_s = sV$ such that $tM_s \subset M_{ts}$, whence M is a graded left $\mathbb{C}[\Gamma]$-module

You are interested in operators on the Γ-module $M = \mathbb{C}[\Gamma]\otimes V$ which commute with Γ action:

$$\mathrm{Hom}_\Gamma\left(\mathbb{C}[\Gamma]\otimes V, \mathbb{C}[\Gamma]\otimes V\right) = \mathrm{Hom}\left(V, \mathbb{C}[\Gamma]\otimes V\right)$$

Given $\quad \theta : V \longrightarrow \mathbb{C}[\Gamma]\otimes V \quad$ one has

$$\theta(v) = \sum s\, \theta_s v \qquad \forall v$$

for unique $\theta_s \in \mathrm{End}(V)$. Assume $\{s \mid \theta_s \neq 0\}$ finite

$$\theta = \sum_s s\theta_s \in \mathbb{C}[\Gamma]\otimes \mathrm{End}(V).$$

Important is how θ extends uniquely to a Γ-module endo $\tilde{\theta}$ of $\mathbb{C}[\Gamma]\otimes V$, namely

$$\tilde{\theta}(tv) = t\,\theta(v) = \sum_s ts\theta_s v$$

so $\sum s\theta_s \in \mathbb{C}[\Gamma]\otimes \mathrm{End}(V)$ becomes the op

$$tv \longmapsto ts\theta_s v$$

You still haven't focussed properly. You persist using left Γ action. Since left and right Γ actions are equivalent this should be OKAY. But the Γ grading should be changed

So let M be the free Γ-module gen by V and ~~right~~ write it using standard cross product notation

$$M = \bigoplus M_s \qquad \text{where} \qquad M_s = Vs$$

Start at the beginning with the Hilbert space situation.

~~[illegible]~~ Begin with a Hilbert space H, with group Γ acting by unitary operators, with a closed subspace $V \subset H$ such that $\sum_{s \in \Gamma} sV$ dense in H.

Assume h_1 is positive operator on H such that $\overline{h_1 H} = V$, let $h_s = s h_1 s^{-1}$, and assume

$$\sum_{s \in \Gamma} h_s = 1 \qquad \text{(sum of pos. ops makes sense).}$$

Let

$$H \xrightarrow{\alpha} \bigoplus_{s\in\Gamma}^{(2)} sV \xrightarrow{\alpha^*} H$$

$$\alpha(\xi) = \bigoplus_{s\in\Gamma} h_s^{1/2}\xi$$

makes sense because $h_s^{1/2}\xi \subset s h_1^{1/2} s^{-1} H \subset sV$

and $\|\alpha(\xi)\|^2 = \sum_s \|h_s^{1/2}\xi\|^2 = \sum_s (\xi, h_s\xi) = \|\xi\|^2$

so α is an isometry $\Rightarrow$ α^* orth proj onto H.

If $\sum_{s\in\Gamma} s\eta_s \in \bigoplus_{s\in\Gamma}^{(2)} sV$, then $\left(\alpha^* \bigoplus_s s\eta_s, \xi\right) =$

$$\left(\bigoplus_s s\eta_s, \bigoplus_s h_s^{1/2}\xi\right) = \sum_s \left(s\eta_s, s h_1^{1/2} s^{-1}\xi\right) = \sum_s \left(s h_1^{1/2}\eta_s, \xi\right)$$

$$\therefore \boxed{\alpha^*\left(\bigoplus_s s\eta_s\right) = \sum_s s h_1^{1/2}\eta_s}$$

Again: H Hilbert space, Γ group acting on H by unitaries, $h_1 \geq 0$ on H, $h_s = s h_1 s^{-1}$ for $s \in \Gamma$, Assume $\sum_{s\in\Gamma} h_s = 1$ on H (well-defined since $h_s \geq 0$).

Put $V = \overline{h_1^{1/2} H}$, closed subspace of H, $sV = \overline{s h_1^{1/2} s^{-1} H} = \overline{h_s^{1/2} H}$

Define

$$H \xrightarrow{\alpha} \bigoplus_s^{(2)} sV \qquad \alpha(\xi) = \bigoplus_s h_s^{1/2}\xi$$

$$\|\alpha(\xi)\|^2 = \sum_s \|h_s^{1/2}\xi\|^2 = \sum_s (\xi, h_s \xi) = \|\xi\|^2.$$

$\therefore$ α isometry, $\alpha^* =$ orthog projection onto H

Calc. $\boxed{\alpha^*\left(\bigoplus_s s\eta_s\right) = \sum_{s\in\Gamma} h_s^{1/2} s\eta_s = \sum_s s h_1^{1/2} \eta_s}$

So you have

$$\begin{tikzcd} H \arrow[r, "\alpha"] \arrow[rr, bend left, "id"] & \bigoplus\limits_{s\in\Gamma}^{(2)} sV \arrow[r, "\alpha^*"] & H \end{tikzcd}$$

~~Action of Γ on $\bigoplus\limits_{s\in\Gamma}^{(2)} sV$ is $t \bigoplus_s s\eta_s = \bigoplus_s ts\eta_s = \bigoplus_s s\eta_{t^{-1}s}$~~

~~$\alpha(t\xi) = \sum_s h_s^{1/2} t\xi$ $\quad t\alpha(\xi) = t \bigoplus_s h_s^{1/2}\xi = \bigoplus_s h_{ts}^{1/2} t\xi$~~

The notation $\bigoplus_s s\eta_s$ is not so good. You mean the function $(s\eta_s)_{s\in\Gamma}$ ~~that $t(s\eta_s)$ function $s \mapsto ts\eta_{s^{-1}}$~~

such that

Start again, but find good notation. H, Γ, h_1 as above

$V = \overline{h_1^{1/2} H}$ $\quad$ $sV = \overline{h_s^{1/2} H} \subset H$ $\quad$ embedding α takes ξ to the fn. $s \longmapsto h_s^{1/2}\xi \in sV$. ~~$\bigoplus\limits_{s\in\Gamma}^{(2)} sV \cong \{\eta : \Gamma \to H \mid$~~

~~η_s η_s ξ s~~

What is $\bigoplus^{(2)}_{s\in\Gamma} sV$?

its the set functions φ from Γ to H such that $\forall s\in\Gamma$, $\varphi(s)\in sV$, with ℓ^2 norm. $\sum_{s\in\Gamma}\|\varphi(s)\|^2$

If you put $s^{-1}\varphi(s)=\eta_s$, then you get $\{\eta:\Gamma\to V\}$ with ℓ^2 norm.

$$H \xhookrightarrow{\alpha} L^2(\Gamma;V) \xrightarrow{\alpha^*} H$$

$$\xi \longmapsto \alpha(\xi)_s = s^{-1}h_s^{1/2}\xi = h_1^{1/2}s^{-1}\xi \qquad (\varphi) \longmapsto \sum s h_1^{1/2}\varphi(s)$$

$$\big(\alpha(\xi),\varphi\big) = \sum_s \big(h_1^{1/2}s^{-1}\xi, \varphi(s)\big) = \sum_s\big(\xi, s h_1^{1/2}\varphi(s)\big)$$

α is Γ-equiv.

How does Γ act on $L^2(\Gamma;V)$. it has to be

$$(t\varphi)(s) = \varphi(t^{-1}s)$$

$$\big(t(\alpha(\xi))\big)(s) = \alpha(\xi)(t^{-1}s) = h_1^{1/2}s^{-1}t\xi = \alpha(t\xi)(s)$$

$$\alpha^*(t\varphi) = \sum_s s h_1^{1/2}(t\varphi)(s) = \sum_s s h_1^{1/2}\varphi(t^{-1}s)$$

$$= \sum_{s} ts h_1^{1/2}\varphi(s) = t\alpha^*\varphi$$

now do $\alpha\alpha^*$ which is an operator on $L^2(\Gamma;V)$ commuting with Γ action. $(t\varphi)(s)=\varphi(t^{-1}s)$. Example: Let $\theta:\Gamma\longrightarrow\mathcal{L}(V)$ Let $\theta_s\in\mathcal{L}(V)$ Can you make θ act on $\varphi:\Gamma\longrightarrow V$ ~~commuting~~ so as to commute with Γ action on φ.

Go over this: $\varphi : \Gamma \to V$ $\quad (\cancel{t}u\varphi)(s) = \varphi(\cancel{t}u^{-1}s)$

$\theta : \Gamma \to \mathcal{L}(V)$. Consider $\theta(s)\varphi(t)$. Out of these products you <u>want to</u> construct a $(\theta * \varphi) : \Gamma \to V$

$\underbrace{L_u\varphi}$ You have $\varphi \in L^2(\Gamma, V)$ and Γ-action $\overbrace{(u\varphi)}(s) = \varphi(u^{-1}s)$. Let T be an operator on $L^2(\Gamma, V)$ commuting with this Γ-action. Example: $(R_v\varphi)(s) = \varphi(sv)$. Then

$$(L_uR_v\varphi)(s) = (R_v\varphi)(u^{-1}s) = \varphi(u^{-1}sv)$$
$$(R_vL_u\varphi)(s) = (L_u\varphi)(sv) = \varphi(u^{-1}sv).$$

Another example ~~is~~ is a $\theta \in \mathcal{L}(V)$ where

$(\theta\varphi)(s) = \theta\varphi(s)$. $\quad (L_u\theta\varphi)(s) = (\theta\varphi)(u^{-1}s) = \theta\,\varphi(u^{-1}s) = \theta L_u\varphi$

But θ also commutes with R_v.

$$(\theta R_v \varphi)(s) = \theta(R_v\varphi)(s) = \theta\varphi(sv)$$
$$(R_v\theta\,\varphi)(s) = (\theta\varphi)(sv) = \theta\varphi(sv).$$

Put these together.

$$\sum_v \theta_v R_v \qquad \text{finite sums.} \qquad \text{i.e. } \theta_v \neq 0 \quad v \in \text{finite set.}$$

So apparently the

Repeat. $H, \Gamma, h_1 \geq 0, h_s = sh_1s^{-1}, \sum h_s = 1$ on H.

$V = \overline{h_1^{1/2}H} \qquad sV = \overline{h_s^{1/2}H}$

$$\begin{tikzcd} H \arrow[r, "(h_s^{1/2})"] \arrow[dr, "(h_1^{1/2}s^{-1})"'] & \bigoplus^{(2)}_{s\in\Gamma} sV \arrow[d, "s"] \\ & \bigoplus^{(2)}_s V = L^2(\Gamma, V) \end{tikzcd}$$

$(\alpha\xi)(s) = h_1^{1/2}s^{-1}\xi$

$\|\alpha\xi\|^2 = \sum_s \underbrace{\|h_1^{1/2}s^{-1}\xi\|^2}_{(\xi,\, sh_1s^{-1}\xi) = (\xi, h_s\xi)} = \|\xi\|^2$

$\alpha^*\varphi = \sum_s s h_1^{1/2}\varphi(s).$ $\qquad$ Γ action $\quad \varphi \in L^2(\Gamma, V)$

$$\boxed{(L_t\varphi)(s) = \varphi(t^{-1}s)}$$

$$(L_t(\alpha\xi))(s) = (\alpha\xi)(t^{-1}s) = h_1^{1/2}s^{-1}t\xi = \alpha(t\xi)$$

$$(R_u\varphi)(s) = \varphi(su) \qquad (L_tL_u\varphi)(s) = (L_u\varphi)(t^{-1}s) = \varphi(\overset{(tu)^{-1}}{u^{-1}t^{-1}}s) = (L_{tu}\varphi)(s)$$

$$(R_t(R_u\varphi))(s) = (R_u\varphi)(st) = \varphi(stu) = (R_{tu}\varphi)(s)$$

$[L_t, R_u] = 0 \qquad \theta \in \mathcal{L}(V) \qquad (\theta\varphi)(s) = \theta\varphi(s).$

$[L_t, \theta] = [R_u, \theta] = 0.$ So you get operators θR_u on $L^2(\Gamma, V)$ commuting with left action:

$$\mathcal{L}(V) \otimes \mathbb{C}[\Gamma] \longrightarrow \mathcal{L}(L^2(\Gamma, V))$$

$$\theta \otimes s \longmapsto \theta R_s$$

This notation is clearly what Joachim uses. Now where are we?? You have this projection $\alpha\alpha^*$ on $L^2(\Gamma, V)$, and you want to get it in the image of the map above. This should involve the overlap condition: ~~that~~ $h_s^{1/2}h_t^{1/2} = 0$ for $s^{-1}t \notin F$

$$H \xrightarrow{\alpha} L^2(\Gamma, V) \xrightarrow{\alpha^*} H$$

$$\xi \longmapsto (\alpha\xi)(s) = h_1^{1/2}s^{-1}\xi \qquad \alpha^*f = \sum_s s h^{1/2} f(s)$$

$$(\alpha\alpha^*f)(s) = h_1^{1/2}s^{-1}\sum_t t h^{1/2} f(t) = \sum_t \underbrace{(h_1^{1/2}s^{-1}t h_1^{1/2})}_{=0 \text{ for } s^{-1}t \notin F} f(t)$$

$$(\alpha\alpha^*f)(u^{-1}s) = \sum_t h_1^{1/2}s^{-1}ut\, h_1^{1/2} f(ut)$$

$$(\alpha\alpha^*f)(s) = \sum$$

So far we have reached the formula

$$(\alpha\alpha^* f)(s) = h_1^{1/2} s^{-1} \sum_t t\, h_1^{1/2} f(t)$$

$$= \sum_t \left(h_1^{1/2} s^{-1} t\, h_1^{1/2}\right) f(t) = \sum_u \left(h_1^{1/2} u\, h_1^{1/2}\right) f(su)$$

So something is not clear.

Alternative. Maybe use a different embedding.

$$H \xhookrightarrow{\alpha} L^2(\Gamma, V) \qquad \cancel{R_t(R_u\alpha) = R_t(\alpha u) = \alpha ut}$$

$$\xi \longmapsto (\alpha\xi)(s) = h_1^{1/2} s\xi \qquad R_t R_u \alpha = R_t \alpha u = \alpha tu$$

$$(R_t(\alpha\xi))(s) = (\alpha\xi)(st) = h_1^{1/2} st\xi$$

$$\| \quad (\alpha(t\xi))(s) = h_1^{1/2} st\xi \qquad \therefore R_t\alpha = \alpha t$$

$$(R_t(R_u\varphi))(s) = (R_u\varphi)(st) = \varphi(stu) = (R_{tu}\varphi)(s).$$

$$(R_t(\alpha\xi))(s) = (\alpha\xi)(st) = h_1^{1/2} st\xi = \alpha(t\xi)(s)$$

$$(R_t(R_u(\alpha\xi)))(s) = R_u(\alpha\xi)(st) = (\alpha\xi)(stu) = R_{tu}\alpha$$

should write $R_t(\alpha) = \alpha\circ t$. Then $R_t(R_u\alpha) = R_t(\alpha\circ u)$?

$$(\alpha(\xi))(s) = h_1^{1/2} s\xi$$

$$(R_t(\alpha(\xi)))(s) = (\alpha(\xi))(st) = h_1^{1/2} st\xi = \alpha(t\xi)(s)$$

$$R_u(R_t(\alpha(\xi))) = R_u(\alpha(t\xi)) = \alpha(ut\xi) = R_{ut}(\alpha(\xi))$$

$$(R_u(R_t\varphi))(s) = (R_t\varphi)(su) = \varphi(sut) = (R_{ut}\varphi)(s)$$

Analyze the situation. You have a notation for the situation. On the H side you have Γ acting and $h_1 \geq 0$ such that $\sum_s h_s = 1$ on H. What about the V side. Try for something 'intrinsic', i.e. like the subspaces sV, the translates of V under Γ. Take the free case where the ~~~~ translates are orthogonal. You think indexing by s^{-1} might help.

$\varepsilon = \pm 1$

$$L^2(\Gamma, V) \xrightarrow{\beta} H \qquad (f : \Gamma \to V) \longmapsto \sum_s h_1^{1/2} s^{\varepsilon} f(s)$$

$$\overset{\|}{\bigoplus_{\Gamma}^{(2)} V}$$

$$(\beta^* \xi, f) = \left(\xi, \sum_s \underbrace{h_1^{1/2} s^{\varepsilon} f(s)}\right)$$

$$= \sum_s \left(s^{-\varepsilon} h_1^{1/2} \xi, f(s)\right)$$

$$= \left(\beta^*\xi, f\right)$$

$$(\underset{\alpha}{\beta^*}\xi)(s) = h_1^{1/2} s^{-\varepsilon} \xi$$

so $$(\beta\beta^*)(\xi) = \beta(s \mapsto h_1^{1/2} s^{-\varepsilon}\xi) = \sum_s s^{\varepsilon} h_1^{1/2} h_1^{1/2} s^{-\varepsilon} \xi$$

$$= \xi$$

$$(\beta^*\beta) f = \beta^* \sum_t t^{\varepsilon} h_1^{1/2} f(t)$$

$$= \left(s \longmapsto h_1^{1/2} s^{-\varepsilon} \sum_t t^{\varepsilon} h_1^{1/2} f(t)\right)$$

$$= \left(s \longmapsto \sum_t h_1^{1/2} s^{-\varepsilon} t^{\varepsilon} h_1^{1/2} f(t)\right)$$

so if you take $\varepsilon = -1$, then you'll have something in convolution form namely

$$(pf)(s) = \sum_t \left(h_1^{1/2}\, s t^{-1} h_1^{1/2}\right) f(t)$$

Some progress made. Given H, Γ, h_1 etc.

$$\begin{tikzcd}
H \arrow[r, "\alpha"] & \bigoplus_s^{(2)} sV \arrow[r, "\beta"] & H \\
H \arrow[r, "\alpha"] & \bigoplus_s^{(2)} V \arrow[u, "s"] \arrow[r, "\alpha^*"] & H
\end{tikzcd}$$

$$\xi \longmapsto \left(s \mapsto h_s^{1/2}\xi\right) \qquad sf(s) \longmapsto$$

$$\xi \longmapsto \left(s \mapsto h_1^{1/2} s^{\varepsilon}\xi\right)$$

$$f \longmapsto \sum_s s^{-\varepsilon} h_1^{1/2} f(s)$$

Composite $\quad \xi \longmapsto \sum_s s^{-\varepsilon} h_1 s^{\varepsilon}\xi = \xi.$

$$\alpha(t\xi) = \left(s \longmapsto h_1^{1/2} s^{\varepsilon} t\xi\right)$$

two actions of Γ on $L^2(\Gamma, V)$

$$(tf)(s) = f(t^{-1}s)$$
$$(tf)(s) = f(st)$$

Let $f(s) = h_1^{1/2} s^{\varepsilon}\xi$

left $(tf)(s) = h_1^{1/2}(t^{-1}s)^{\varepsilon}\xi$

right $(tf)(s) = h_1^{1/2}(st)^{\varepsilon}\xi$

use ~~left~~ right action $\quad \varepsilon = -1 \quad (\alpha(t\xi))(s) = h_1^{1/2} s^{-1} t\xi$

$(tf)(s) = f(t^{-1}s)$

Then $\alpha(\xi)(s) = h_1^{1/2} s^{-1}\xi$

$$(t(\alpha\xi))(s) = (\alpha\xi)(t^{-1}s) = h_1^{1/2}(t^{-1}s)^{-1}\xi$$

$$\alpha(\xi) = \left(s \longmapsto h_1^{1/2} s^{-1}\xi\right)$$

$$tf = \left(s \longmapsto f(t^{-1}s)\right)$$

$$\beta f = \sum_s s h_1^{1/2} f(s)$$

$$t\beta f = \sum_s t s h_1^{1/2} f(s) = \sum s h_1^{1/2} f(t^{-1}s) = \beta(tf)$$

$$(\alpha\beta f)(s) = \alpha\left(\sum_t t h_1^{1/2} f(t)\right) = h_1^{1/2} s^{-1} \sum_t t h_1^{1/2} f(t)$$

Confused again.

$$\mathbb{H} \longrightarrow L^2(\Gamma, V) \longrightarrow H$$

$$f \xmapsto{\ \beta\ } \sum_s s^\varepsilon h_1^{1/2} f(s)$$

$$\xi \xmapsto{\ \alpha\ } \left(s \mapsto h_1^{1/2} s^{-\varepsilon}\xi\right)$$

$$\beta\alpha\xi = \sum_s s^\varepsilon h_1^{1/2} h_1^{1/2} s^{-\varepsilon}\xi = \sum_s s^\varepsilon h_1 s^{-\varepsilon}\xi = \xi.$$

$$\alpha\beta f = \left(s \mapsto h_1^{1/2} s^{-\varepsilon} \sum_t t^\varepsilon h_1^{1/2} f(t)\right)$$

$$= \left(s \mapsto \underbrace{\sum_t \left(h_1^{1/2} s^{-\varepsilon} t^\varepsilon h_1^{1/2}\right) f(t)}_{\sum_t \left(h_1^{1/2} s t^{-1} h_1^{1/2}\right) f(t)}\right) \qquad \varepsilon = -1.$$

$$\alpha(t\xi) = \left(s \mapsto h_1^{1/2} s t \xi\right) = R_t \alpha(\xi) \quad \left(= s \mapsto h_1^{1/2}(st)\xi\right)$$

$$\beta(tf) = \sum_s s^{-1} h_1^{1/2} \underbrace{(tf)(s)}_{f(st)} = \sum_s s^{-1} h_1^{1/2} f(st) = \sum t s^{-1} h_1^{1/2} f(s) = t\beta(f)$$

Conclusion: Let Γ act on $L^2(\Gamma, V)$ via $(tf)(s) = f(st)$ i.e. R_t

---

Assembly map. $H, \Gamma, h_1 \geq 0, \quad V = \overline{h_1^{1/2} H}$

$$L^2(\Gamma, V) \xrightarrow{\ \beta\ } H$$

$$f \longmapsto \sum_{s\in\Gamma} s^\varepsilon h_1^{1/2} f(s) \qquad \varepsilon = \pm 1$$

$$(\beta f, \xi) = \sum_s \left(f(s), h_1^{1/2} s^{-\varepsilon}\xi\right) \qquad \therefore (\beta^*\xi)(s) = h_1^{1/2} s^{-\varepsilon}\xi$$

$$\beta\beta^*\xi = \sum_{s\in\Gamma} s^\varepsilon h_1^{1/2} h_1^{1/2} s^{-\varepsilon}\xi = \xi$$

equivariance $\quad t\beta(f) = \sum_s ts^{\varepsilon} h_1^{1/2} f(s) = \sum_s ts^{\varepsilon} h_1^{1/2} f(s)$

$$= \sum_u t(t^{-1}u) h_1^{1/2} f(t^{-1}u) = \beta(\underbrace{u \mapsto f(t^{-1}u)}_{L_t f})$$

$$\boxed{\varepsilon = +1 \quad t\beta(f) = \beta(L_t f)}$$

$$\boxed{\varepsilon = -1 \quad t\beta(f) = \beta(R_t f)}$$

$$t\beta(f) = \sum_s ts^{-1} h_1^{1/2} f(s)$$

$$= \sum_u \underbrace{t(ut)^{-1}}_{u^{-1}} h_1^{1/2} f(ut) = \beta(R_t f)$$

$$(\beta^*(\beta f))(s) = h_1^{1/2} s^{-\varepsilon} \sum_t t^{\varepsilon} h_1^{1/2} f(t)$$

$$= \sum_t h_1^{1/2} s^{-\varepsilon} t^{\varepsilon} h_1^{1/2} f(t)$$

$$= \sum_t (h_1^{1/2} st^{-1} h_1^{1/2}) f(t)$$

You want to take $\varepsilon = -1$ to get convolution form.

---

begin again $\quad H, \Gamma, h_1 \geq 0, \quad h_s = sh_1s^{-1}, \quad \sum_s h_s = 1$ on H

$V = \overline{h_1^{1/2} H}, \quad sV = \overline{h_s^{1/2} H}$.

$$\alpha : H \longrightarrow \ell^2(\Gamma, V) = \bigoplus_{s\in\Gamma}^{(2)} V \qquad \boxed{(\alpha\xi)(s) = h_1^{1/2} s^{\varepsilon} \xi} \qquad \varepsilon = \pm 1$$

$$\boxed{\alpha^* f = \sum_s s^{-\varepsilon} h_1^{1/2} f_s} \qquad (\alpha^* f, \xi) = \sum_s (s^{-\varepsilon} h_1^{1/2} f_s, \xi)$$

$$\alpha^*\alpha\xi = \sum_s s^{-\varepsilon} h_1 s^{\varepsilon}\xi = \sum_s h_{s^{\varepsilon}}\xi = \xi.$$

$$(\alpha\alpha^* f)(s) = \left(\alpha\left(\sum_t t^{-\varepsilon} h_1^{1/2} f_t\right)\right)(s) = \sum_t h_1^{1/2} s^{\varepsilon} t^{-\varepsilon} h_1^{1/2} f_t$$

$\therefore$ if $\varepsilon = 1$.

$(\alpha\xi)(s) = h_1^{1/2} s\xi$	$\alpha^*\alpha\xi = \xi$
$\alpha^* f = \sum_s s^{-1} h_1^{1/2} f(s)$	$(\alpha\alpha^* f)(s) = \sum_t (h_1^{1/2} st^{-1} h_1^{1/2}) f(t)$

Put

$$(tf)(s) = (R_t f)(s) = f(st). \quad \text{Then} \quad t\alpha^* f = \sum_s ts^{-1} h_1^{1/2} f(s)$$

$$= \sum_s (st^{-1})^{-1} h_1^{1/2} f(st^{-1}t) = \sum_u u^{-1} h_1^{1/2} f(ut) = \alpha^* R_t f$$

$$\alpha(t\xi)(s) = h_1^{1/2} st\xi = (\alpha\xi)(st) = (R_t\alpha\xi)(s)$$

$$\boxed{\alpha t = R_t \alpha} \qquad \boxed{t\alpha^* = \alpha^* R_t}$$

$\alpha t u \xi = R_t \alpha u \xi = R_t R_u \alpha \xi = R_{tu}\alpha\xi$. seems O.K.

So it all seems to work. One way to check this is to look at operators on $\mathbb{C}[\Gamma]\otimes V$ which commute with R_t operators. This contains linear comb. of operators

$$(L_t \theta f)(s) = \theta f(st)$$

and really is the tensor product alg $\mathbb{C}[\Gamma]\otimes \mathrm{End}(V)$. What's left?

Go back over things. $H, \Gamma, h_1 \geq 0, \sum h_s = 1$ on H

$V = h_1^{1/2} H$, $\quad \alpha : H \longrightarrow \ell^2(\Gamma, V) \qquad (\alpha\xi)(s) = h_1^{1/2} s^{\varepsilon}\xi$

$\alpha^* : \longleftarrow \qquad \alpha^*\eta = \sum_s s^{-\varepsilon} h_1^{1/2} f(s)$

$\alpha^*\alpha = id \qquad (\alpha\alpha^* f)(s) = \sum_t \underbrace{\left(h_1^{1/2} s^{\varepsilon} t^{-\varepsilon} h_1^{1/2}\right)}_{\text{two possibilities}} f(t)$

$\varepsilon = 1 \quad h_1^{1/2} s t^{-1} h_1^{1/2}$ right invariant

$\varepsilon = -1 \quad h_1^{1/2} s^{-1} t h_1^{1/2}$ left ——

Check. $\varepsilon = 1 \quad (\alpha(t\xi))(s) = h_1^{1/2} st\xi = (\alpha\xi)(st) = (R_t(\alpha\xi))(s)$

$\varepsilon = -1 \quad (\alpha(t\xi))(s) = h_1^{1/2} s^{-1} t\xi = (\alpha\xi)(t^{-1}s) = (L_t(\alpha\xi))(s)$

$\varepsilon = 1 \quad (\alpha\alpha^* f)(s) = \sum_t \left(h_1^{1/2} \underset{\text{right inv.}}{st^{-1}} h_1^{1/2}\right) f(t) \qquad (st^{-1})^{-1} = ts^{-1}$

~~Work out the~~

Supports. $\quad h_s h_t = s h_1 s^{-1} t h_1 t^{-1} = 0$

$\longrightarrow h_1 s^{-1} t h_1 = 0 \implies h_1^{1/2} s^{-1} t h_1^{1/2} = 0$

So you learn a little, namely that the projection $\alpha\alpha^*$ is supported in a left (resp right) invariant "tube" depending on your choice of notation. ~~So back~~ Let's try to reach Cuntz's notation.

$B = \mathcal{E}_{\Sigma_F} \rtimes \Gamma$ $\quad C^*$-alg (nonunital) whose ~~[illegible]~~ reps on a Hilb. space H (satisfying $BH = H$) should be equivalent to a Γ action + ~~[illegible]~~ $h_1 \geq 0 \ni \sum h_s = 1$ and also $h_1 s h_1 = 0$ for $s \notin F$. ~~[illegible]~~ You get ~~[illegible]~~ a projection in B, namely $p = \sum_{s \in F} h_1^{1/2} s\, h_1^{1/2} = \sum_s h_1^{\frac12} h_s^{\frac12} s$

What is happening? Hilb. reprs of B should be the same as data H, Γ, h_1 s.t. $\sum h_s = 1$, $h_1^{1/2} s\, h_1^{1/2} = 0$, $s \notin F$.

Question: What is pH?

Look at a projection in a Γ-graded algebra $B = \bigoplus_{s \in \Gamma} B_s$

$$p = \sum_s p_s \qquad p_s \in B_s$$

$$p^2 = \sum_{s,t} p_s p_t = \sum_u \sum_{st=u} p_s p_t \qquad p_u = \sum_{st=u} p_s p_t = \sum_s p_s p_{s^{-1}u}$$

Amazing ~~to~~ what you don't understand (good or firm)

Look at $B = \mathcal{E}_{\Sigma_F} \rtimes \Gamma$. You believe that a Hilb. repn. of B is given by the data $H, \Gamma, h_1^{1/2}$ such that $h_1^{1/2} s\, h_1^{1/2} = 0$ for $s \notin F$ and $\sum_s s h_1 s^{-1} = 1$. ~~[illegible]~~ ~~[illegible]~~ You know that $p = \sum_s h_1^{1/2} s\, h_1^{1/2}$ is idempotent. What is p on H?

First step. Let's ~~try~~ to eliminate V. This should be easy because V can be any subspace between $h_1^{1/2} H$ and H.

Let H be a Hilbert space with unitary action of Γ and operator $h_1^{1/2} \geq 0$ such that $\sum_s s h_1 s^{-1} = 1$ and $h_1^{1/2} s h_1^{1/2} = 0$ for $s \notin F$. Let $p = \sum_{s \in \Gamma} h_1^{1/2} s h_1^{1/2}$.

Then $p^2 = \sum_{s,t} h_1^{1/2} s h_1 t h_1^{1/2} = \sum_{s,t} h_1^{\frac{1}{2}}(s h_1 s^{-1}) st h_1^{1/2}$

$$= \sum_s h_1^{\frac{1}{2}}(shs^{-1}) \sum_t st h_1^{1/2} = \sum_s h_1^{1/2}(shs^{-1}) \sum_u u h_1^{1/2}$$

$$= h_1^{1/2} \sum_u u h_1^{1/2} = p.$$

p is an element of $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$, hence an operator on H. p should be the orthogonal projection onto $V = \overline{h_1^{1/2} H}$.

$$pH = h_1^{1/2} \sum_u u h_1^{1/2} H \subset h_1^{1/2} H \subset \overline{h_1^{1/2} H}$$

$$p h_1^{1/2} \xi = \sum h_1^{1/2} s h_1 \xi$$

Given $H, \Gamma, h_1^{1/2} \geq 0 \ni h_1^{1/2} s h_1^{1/2} = 0 \quad s \notin F$

$$\sum_{s \in \Gamma} s h_1 s^{-1} = 1$$

Put $p_s = h_1^{1/2} s h_1^{1/2}$. Then

$$\sum_{st=u} p_s p_t = \sum_{st=u} h_1^{\frac{1}{2}} s h_1 t h_1^{1/2} = \sum_s h_1^{\frac{1}{2}} s h_1 s^{-1} u h_1^{1/2}$$

$$= h_1^{\frac{1}{2}} u h_1^{1/2} = p_u$$

You get a Γ-graded projection with support F, with values in $h_1^{\frac{1}{2}} \mathcal{L}(H) h_1^{\frac{1}{2}}$

Continue with trying to set up a Morita equivalence

$\mathcal{E}$ generators h_s relations $\left|\begin{array}{l} h_s h_t = 0 \quad s^{-1}t \notin F \\ h_s = \sum_t h_t h_s = \sum_t h_s h_t \end{array}\right.$

Γ action $s h_t s^{-1} = h_{st}$

Set up Morita equivalence for Hilb. representations.

$\mathcal{E}_{\Sigma_F} \rtimes \Gamma \qquad H, \Gamma, h_1^{1/2} \geq 0, \ \sum s h_1 s^{-1} = 1$

Keep to simple situations. Go back to $\mathbb{Z}$ case, you want to understand ~~how to~~ what happens when you ~~replace~~ change from $C_c(\mathbb{R})$ to Cuntz's $\mathcal{E}_{\Sigma_F}$. Recall $\mathcal{E}_{\Sigma_F}^{ab} = C_c(\mathbb{R})$

Now you know that $C_c(\mathbb{R}) \rtimes \mathbb{Z}$ is Morita equiv to $C(\mathbb{R}/\mathbb{Z})$. Recall $C_c(\mathbb{R}) \rtimes \mathbb{Z} = C_c(\mathbb{R} \times \mathbb{Z}) = C_c(\mathbb{R} \times_{\mathbb{Z}} \mathbb{R})$ $\overset{\sim}{\Leftarrow} C_c(\mathbb{R}) \otimes_{C(\mathbb{R}/\mathbb{Z})} C_c(\mathbb{R})$. You want a noncommutative generalization. Other points: $C(\mathbb{R}/\mathbb{Z})$ is unital, so that $C_c(\mathbb{R})$ is a finite proj. right (resp left) module over the cross product $C_c(\mathbb{R}) \rtimes \mathbb{Z}$. It should be the image of a projection. You get a projection by choosing $k \in C_c(\mathbb{R})$ s.t. $\pi_*(k) = 1 \in C(\mathbb{R}/\mathbb{Z})$ then you have $P \otimes_A Q$

$$\begin{array}{ccc} & & P \\ & & \uparrow\downarrow \\ A \text{ is a summand of } P & Q & A \end{array}$$

so Q ——— $P \otimes_A Q$

You see a problem looming in ? the non. comm. setting — the analog of A is not unital

Alg. $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$ has left and right local identities

~~It should be clear that e.~~

Something you've forgotten is when $\mathbb{Z}$ is a flat $\tilde{A}$-module. Let A be a left ideal in R unital. When is R/A a flat R-module. Special case: When is R/A a proj. R-module.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & R \arrow[l, bend right, "\cdot e"'] \arrow[r] & R/A \arrow[l, bend right, "\exists \cdot x"'] \arrow[r] & 0 \end{tikzcd}$$

$\exists x \in R$ such that $Ax = 0$, and $x - 1 \in A$

so $(x-1)x = 0$ ie $x^2 = x$. Put $e = 1 - x$.

Better [redacted] $\exists e \in A$ such that $ae = a$ $\forall a \in A$. $A \supset Re \supset Ae = A$

Summary so far. A left ideal of R unital, then R/A is R-projective $\iff$ $A = Re$, e such that $e \in A$ and $e^2 = e$.

Alt. $\exists e \in A \ni A(1-e) = 0$.

When is R/A R-flat?

when $\forall a_1, \dots, a_n \ \exists a \ni \forall i \ a_i(1-a) = 0$

$\forall a_1 \ \exists a \ni a_1(1-a) = 0$ Yes.

Ind. Hyp. $a_i(1-a') = 0$ $i = 1, \dots, n-1$

Choose $a'' \ni a_n(1-a')(1-a'') = 0$

YES!

then $a_i(1-a) = 0$ $a = a' + a'' - a'a''$.

So what's next.

matrix criterion for flatness.

$$\begin{tikzcd} R \arrow[r] \arrow[rr, bend left] \arrow[dr, "0"'] & R^n \arrow[r, dashed] \arrow[d] & R^m \arrow[dl, dashed] \\ & M & \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\cdot a"] & R \arrow[r, "(x_j)"] \arrow[d, "1"'] & R^m \arrow[dl, "(\xi_j)"] \\ & R/A & \end{tikzcd}$$

$$a \in A \implies \exists (x_j) \quad a(x_j) = 0$$

$$\ni \ 1 \equiv \sum x_j \xi_j \quad \text{mod } A$$

get.

$$\forall a \in A \quad \exists \ x_j \in R, \ a x_j = 0$$

$$\exists \ y_j \in R, \quad 1 - \sum x_j y_j \in A$$

$$\exists a' \quad 1 - a' = \sum x_j y_j$$

$$\implies \quad a(1-a') = \sum a x_j y_j = 0$$

So this condition: $\forall a \ \exists a' \quad a(1-a') = 0$
local right identities is equivalent to R/A left flat whenever A embedded in R as left ideal

$$0 \to A \to R \to R/A \to 0$$

$$0 \to \underbrace{M \otimes_A A \to M}_{} \to M/MA \to 0$$

M firm $\iff$ $M = MA$.

to construct a Morita equivalence. $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$

$C = \mathcal{E}_{\Sigma_F}$ gen. $h_s^{1/2}$, $s \in \Gamma$ $\ni$ $h_s^{1/2} h_t^{1/2} = 0$ for $s^{-1}t \notin F$ and $h_s = \sum_t h_t h_s = \sum_t h_s h_t$, which implies $\mathcal{E}_{\Sigma_F}$ has local left + right identities. This ~~should be~~ implies that a module M over $C = \mathcal{E}_{\Sigma_F}$ is firm $\iff$ $\mathcal{E}_{\Sigma_F} M = M$. i.e. $\sum h_s^{1/2} M = M$. From $M = C \otimes_C M$ you should get an action of Γ on M. ? Why?

Time to do this carefully. Suppose A nonunital with Γ action, $B = A \rtimes \Gamma = A \otimes \mathbb{C}[\Gamma]$. Extension:

$$\underbrace{A \rtimes \Gamma}_{B} \hookrightarrow \underbrace{\tilde{A} \rtimes \Gamma}_{R} \twoheadrightarrow \mathbb{C}[\Gamma]$$

semi direct product type extension. firm B-modules $=$ B-firm R-modules. Claim B has local identities, say right $\iff$ R/B is R flat. OK a B module M is firm $\iff$ $BM = M$.

Wait: You know $C = \mathcal{E}_{\Sigma_F}$ has local idents, what about $B = C \rtimes \Gamma$ given $b = \sum_s c_s s$ finite so $\exists\, e \ni (1-e) c_s = 0$. Clear

So a $B = C \rtimes \Gamma$ module M is firm iff $\sum h_s = 1$ on M

Go thru calculation. Start with a firm B-module H, so you have $H, \Gamma, h_1^{1/2}$, $\begin{cases} h_1^{1/2} s h_1^{1/2} = 0 & s \notin F \\ \sum s h_1 s^{-1} = 1 & \text{on } M. \end{cases}$

$$H \xrightarrow{\alpha} \mathbb{C}[\Gamma] \otimes H \xrightarrow{\beta} H \qquad (\alpha \xi)(s) = h_1^{1/2} s^{-1} \xi$$

$\mathbb{C}[\Gamma] \otimes H = \{f : \Gamma \to H\}$ fin. support $\qquad \beta(f) = \sum_s s\, h_1^{\frac{1}{2}} f(s) \qquad \beta\alpha = 1$ on H

$$t\beta(f) = \sum_s ts\, h_1^{1/2} f(s) = \sum_{t^{-1}s} s\, h_1^{1/2} f(t^{-1}s) = \sum s h_1^{1/2} (L_t f)(s)$$
$$= (\beta L_t f)(s)$$

$$(\alpha\beta f)(s) = h_1^{1/2} s^{-1} \sum_t t\, h_1^{1/2} f(t) = \sum_t \underbrace{\left(h_1^{1/2} s^{-1} t\, h_1^{1/2}\right)}_{0 \text{ for } s^{-1}t \notin F} f(t)$$

What can you say? $p = \alpha\beta$ is a projector on $\mathbb{C}[\Gamma] \otimes H$ in fact on $\mathbb{C}[\Gamma] \otimes h^{1/2} H$.

Alternative notation $\quad \beta(f) = \sum_s s^{-1} h_1^{1/2} f(s) \;\Big|\; (\alpha\xi)(s) = h_1^{1/2} s\, \xi$

$$t\beta(f) = \sum_s t s^{-1} h_1^{1/2} f(s) = \sum_{st} t t^{-1} s\, h_1^{1/2} f(st) = (\beta R_t f)(s)$$

$$(\alpha\beta f)(s) = h_1^{1/2} s \sum_t t^{-1} h_1^{1/2} f(t) = \sum_t \underbrace{h_1^{1/2}(st^{-1}) h_1^{1/2}}_{0 \text{ for } st^{-1} \notin F} f(t)$$ right inv.

~~Question~~ $\quad f(s) = g(s^{-1})$

$$\cancel{(\alpha\beta f)(s^{-1}) = \sum_t h_1^{1/2}(s^{-1}t) h_1^{1/2} f(t) = \sum_t h_1^{1/2}(st^{-1}) h_1^{1/2} f(t^{-1})}$$

$$\cancel{(\alpha\beta f)(s^{-1}) = \sum \left(h_1^{1/2} s t\, h_1^{1/2}\right) f(t)}$$

matrices $\quad h_1^{1/2} s^{-1} t\, h_1^{1/2} \qquad h_1^{1/2}(s t^{-1}) h_1^{1/2}$

$\qquad\qquad h_1^{1/2} s t^{-1} h_1^{1/2} \qquad h_1^{1/2}(s^{-1} t) h_1^{1/2}$

Still confused. Try to focus upon the problem. The idea is that a firm $B = C \rtimes \Gamma$-module H seems to amount to a ~~kind of projection operator~~ type of module, namely, a vector space V equipped with operators

$p_s = h_1^{1/2} s h_1^{1/2}$ $\quad \sum_s p_s p_{s^{-1}t} = \sum_s h_1^{1/2} s h_1 s^{-1} t h_1^{1/2} = h_1^{1/2} t h_1^{1/2} = p_t$.

So there's a certain ~~non~~ ring P_F gen'd by element p_s, $s \in \Gamma$ subject to $p_s = 0$ $s \notin F$ and $\sum_s p_s p_{s^{-1}t} = p_t$ $\forall t$. This ring P_F is clearly idempotent as each generator is quadratic expression of the others. So what next??

$C = \mathcal{E}_{\Sigma_F}$ generators $h_s^{1/2}$ $s \in \Gamma$ etc.

C has local left + right idents, so a C module H is firm iff $CH = H$. Let $B = C \rtimes \Gamma$. extn.

$$C \rtimes \Gamma \hookrightarrow \tilde{C} \rtimes \Gamma \twoheadrightarrow \mathbb{C}[\Gamma]$$

but $C \rtimes \Gamma$ should have local units also. So a firm B-module H should amount to a Γ module H with op $h_1^{1/2}$ $\ni$ $h_1^{1/2} s h_1^{1/2} = 0$ $s \notin F$, $\sum s h_1 s^{-1} = 1$.

Write this carefully sometime

Anyway you can form

$$\alpha(t\xi)(s) = h_1^{1/2} s^{-1} t \xi = h_1^{1/2}(t^{-1}s)\xi = (L_t \alpha\xi)(s)$$

$$H \xrightarrow{\alpha} \mathbb{C}[\Gamma] \otimes H \xrightarrow{\beta} H$$

$$\mathbb{C}[\Gamma] \otimes H = \{f : \Gamma \to H \text{ finite support}\}$$

$(\alpha\xi)(s) = h_1^{1/2} s^{-1} \xi$ $\qquad \beta f = \sum_s s h_1^{1/2} f(s)$

$\beta\alpha = \mathrm{id}_H$ $\qquad (\alpha\beta f)(s) = \sum_t \left(h_1^{1/2} s^{-1} t h_1^{1/2}\right) f(t)$

$$\boxed{\begin{aligned} t(\beta f) &= \sum_s t s h_1^{1/2} f(s) \\ &= \sum s h_1^{1/2} f(t^{-1}s) \\ &= (\beta L_t f)(s) \end{aligned}}$$

So you end with the function $p_s = h_1^{1/2} s h_1^{1/2}$ satisfying $\sum_s p_s p_{s^{-1}t} = \sum_s h_1^{1/2} s h_1 s^{-1} t h_1^{1/2} = p_t$ and $p_s = 0$ for $s \notin F$. Let $A = P_F$ be the alg with these gens + relations. So you have a homom. $A \longrightarrow B$, any B-mod H restricts to an A-module M which we replace by $AM = h_1^{1/2} M$?

You want to ~~[scribbled out]~~ go backwards. Let V be an A module i.e. equipped with operators p_s, $s \in \Gamma$ satis support + idempotence conditions. Then on $\mathbb{C}[\Gamma] \otimes V$ you should have an idempotent operator commuting with ~~the~~ Γ-action.

$$\mathbb{C}[\Gamma] \otimes V = \{f : \Gamma \to V \text{ fin. supp}\}$$

Suppose you have $k : \Gamma \longrightarrow \mathcal{L}(V)$ fin. supp.

$$(kf)(s) = \sum_t k(s^{-1}t) f(t)$$

$$(kL_u f)(s) = \sum_t k(s^{-1}t) f(u^{-1}t) = \sum_t k(s^{-1}ut) f(t)$$

$$(L_u kf)(s) = \sum_t k(\underbrace{(u^{-1}s)^{-1}t}_{s^{-1}ut}) f(t)$$

$$\mathbb{C}[\Gamma] \otimes \mathcal{L}(V) \longrightarrow \mathcal{L}(\mathbb{C}[\Gamma] \otimes V)$$

Composition $\quad (k_1 k_2 f)(s) = \sum_t k_1(s^{-1}t)(k_2 f)(t)$

$$= \sum_t k_1(s^{-1}t) \sum_u k_2(t^{-1}u) f(u)$$

$$= \sum_u \Big\{ \underbrace{\sum_t k_1(s^{-1}t) k_2(t^{-1}u)}_{\sum_{xy = s^{-1}u} k_1(x) k_2(y)} \Big\} f(u)$$

$$\sum_{xy = s^{-1}u} k_1(x) k_2(y) = (k_1 * k_2)(s^{-1}u)$$

So ~~[scribbled out]~~ $f \mapsto k*f$ is idemp iff $k*k = k$.

~~[struck out] these together~~

So now given V with ops $p_s \in \mathcal{L}(V)$, $s \in \Gamma$ satis support + idemp. conds, then $(p*f)(s) = \sum_t p(s^{-1}t) f(t)$

H

suppose H is a $B = C \rtimes \Gamma_1$ module, $C = \mathcal{E}_{\Sigma_F}$
which is firm: $BH = H$, or $CH = H$, $\sum h_s = 1$ on H.

then get $$H \xrightarrow{\alpha} \mathbb{C}[\Gamma]\otimes H \xrightarrow{\beta} H \qquad \beta f = \sum_s s\, h_1^{1/2} f(s)$$

$\mathbb{C}[\Gamma]\otimes H = \{f : \Gamma \to H \text{ fin. supp}\}$ $\qquad (\alpha\xi)(s) = h_1^{1/2} s^{-1}\xi$

$$(\alpha\beta f)(s) = \sum_t \left(h_1^{1/2} s^{-1} t\, h_1^{1/2}\right) f(s) \qquad \alpha t = L_t \alpha,\ \beta L_t = t\beta.$$

So you go from H, with operators s, $h_1^{1/2}$, i.e. H as B module to H with operator $p_s = h_1^{1/2} s\, h_1^{1/2}$ satisfying
$\sum_s p_s p_{s^{-1}t} = \sum_s h_1^{1/2}(s h_1 s^{-1}) t\, h_1^{1/2} = p_t$, $p_s = 0$ if $s \notin F$. If $A = P_F$ is the universal ring, then get H as A-module.
i.e. have homom. $$A \longrightarrow B$$ $$p_s \longmapsto h_1^{1/2} s\, h_1^{1/2}$$

Next you go in opposite direction, start with an A-module V, i.e. v.s. with p_s as above. Use the p_s to define a projection on $\mathbb{C}[\Gamma]\otimes V$.

$$(pf)(s) = \sum_{t\in\Gamma} p_{s^{-1}t} f(t)$$ find $p^2 = p$ and $L_t p = p L_t$.

Recover the algebraic viewpoint, where you avoid $h_1^{1/2}$. $\mathcal{E}_{\Sigma_F}$ gen h_s, $s \in \Gamma$ | $h_s h_t = 0$ for $s^{-1}t \notin F$
$$h_s = \sum_t h_s h_t$$

Write in terms of h_1 | $0 = h_s h_t = s h_1 s^{-1} t h_1 t^{-1} \iff h_1 s^{-1}t h_1 = 0$

$h_1 s h_1 = 0 \quad s \notin F$ $\qquad h_1 = \sum_{t\in F} h_1 h_t$

$k = \sum_{t\in F'} h_t \qquad F' \supset F$

$$H \xrightarrow{\alpha} \mathbb{C}[\Gamma]\otimes H \xrightarrow{\beta} H$$

$(\alpha\xi)(s) = k s^{-1}\xi$ $\qquad f \longmapsto \sum s\, h_1 f(s)$

$$\xi \longmapsto (s \mapsto k s^{-1}\xi)$$

$$\beta\alpha\xi = \sum_s s\overbrace{h_1 k}^{h_1} s^{-1}\xi = \sum_s h_s\xi = \xi$$

$$(\alpha\beta f)(s) = \sum_t k s^{-1} t h_1 f(t) \qquad p_s = k s h_1 \text{ is such that } p_s k = p_s \text{ for all } s.$$

~~Note that hkh_1~~ So for all gen^{p_s} of A you have $p_s k = p_s$ $\qquad p_1 = kh_1$

$$p_1 = \sum_s p_s p_{s^{-1}} = \sum_s ksh_1ks^{-1}h_1 = \sum_s k\, sh_1s^{-1}h_1 = kh_1$$

See if you can reconstruct the proof that $h_1 = h_1 k$

i.e. $h_1 = h_1 \sum_{s\in F'} h_s \implies h_1 = kh_1$. Involves $\sum s$

Let's change ~~notation on~~ order so that $kh_1 = h_1$

$$\beta f = \sum_s sk f(s) \qquad (\alpha\xi)(s) = h_1 s^{-1}\xi$$

$$\beta\alpha\xi = \sum_s \underbrace{skh_1s^{-1}}_{sh_1s^{-1}}\xi = \xi$$

$$(\alpha\beta f)(s) = \sum_t h_1 s^{-1} t k f(t) \qquad p_s = h_1 s k$$

$$\sum_s p_s p_{s^{-1}t} = \sum_s h_1 s k h_1 s^{-1} t k$$

~~Check things carefully: Recall~~

~~$p_s p_{s^{-1}t} \neq 0 \implies s\in F,\ s^{-1}t \in F$~~ ~~NO~~ ~~$t^{-1}s\in F$~~

~~$\implies s \in F \cap tF$~~

Suppose $p_s = h_1 s k \qquad k = \sum_{t\in F} th_1t^{-1}$

$$p_s = h_1 s \sum_{t\in F} th_1t^{-1} = \sum_{t\in F} h_1 s t h_1 t^{-1}$$

$$k = \sum_{s\in F} h_s \qquad kth_1 = \sum_{s\in F} h_s h_t t \qquad h_s t h_1 \neq 0 \implies s^{-1}t \in F$$

$$\therefore\ kth_1 = \sum_{s\in F\cap tF} sh_1s^{-1}th_1$$

Let $K \supset F$ consider $\left(\sum_{s\in K} sh_1\right)^2$

$$= \sum_{\substack{s\in K\\ t\in K}} sh_1 th_1 = \sum_{\substack{s\in K\\ t\in \Gamma}} sh_1 s^{-1} s t h_1 = \sum_{s\in K}\sum_{t\in\Gamma} sh_1 s^{-1} t h_1$$

$$= \sum_{t\in\Gamma}\left(\sum_{s\in K} h_s\right) th_1 = \sum_{t\in\Gamma} k t h_1$$

Start again. $h_s h_t = 0$ $\quad s^{-1}t \notin K$ $\quad K = K^{-1}$ cont. 0.

$$\sum_{s\in K} sh_1 \sum_{t\in K} th_1 =$$

$$= \sum_{s\in K} sh_1 \sum_{t\in K} th_1 = \sum_{s\in K}\sum_{t\in K} sh_1 th_1 = \sum_{s\in K}\sum_{t\in\Gamma} sh_1 th_1$$

$$= \sum_{s\in K}\sum_{t\in\Gamma} sh_1 s^{-1} s t h_1 = \sum_{s\in K}\sum_{u\in\Gamma} \underbrace{sh_1 s^{-1} u h_1}_{\neq 0 \Rightarrow s^{-1}u\in K}$$

$$= \sum_{\substack{s\in K\\ u\in sK}} sh_1 s^{-1} u h_1 \qquad (s,u)\in K\times KK \qquad = \sum_{u\in KK}\left(\sum_{s\in K} h_s\right) u h_1$$

$$= \sum_{u\in KK} k u h_1$$

$h_s h_t = 0$ $\quad s^{-1}t \in F$ $\qquad \neq 0 \Rightarrow t\in F$

$$\sum_{s\in F} sh_1 \sum_{t\in F} th_1 = \sum_{(s,t)\in F\times F} sh_1 s^{-1} s t h_1$$

$$\sum_{s\in F} sh_1 \sum_{t\in F} th_1 = \sum_{s\in F}\sum_{t\in F} \underbrace{sh_1th_1}_{\neq 0 \Rightarrow t\in F} = \sum_{\substack{s\in F\\ t\in\Gamma}} sh_1th_1$$

$$= \sum_{s\in F}\sum_{u\in\Gamma} h_s\, uh_1 = \sum_{u\in\Gamma}\underbrace{\sum_{s\in F} h_s}_{k}\, uh_1 = \sum_{u\in\Gamma} kuh_1$$

B: Γ, $h_s = sh_1s^{-1}$, $h_sh_t = 0$ $\quad s^{-1}t\notin F$

$$h_1 = \sum_{s\in F} h_sh_1 = \sum_{s\in\Gamma} h_sh_1$$

$$k = \sum_{s\in F} h_s \qquad kh_1 = h_1$$

$$H \xrightarrow{\alpha} \mathbb{C}[\Gamma]\otimes H \xrightarrow{\beta} H$$

$$\{f : F\to H\} \longmapsto \sum_s skf(s)$$

$$p_s = h_1sk \qquad (\alpha\xi)(s) = h_1s^{-1}\xi$$

$$\beta\alpha\xi = \sum_s s\overbrace{kh_1}^{h_1}s^{-1}\xi = \sum h_s\xi = \xi.$$

$$(\alpha\beta f)(s) = \sum_t h_1s^{-1}tk\, f(t) \qquad p_s = h_1sk$$

$$\sum_s \underbrace{p_sp_{s^{-1}t}}_{\neq 0 \Rightarrow s\in F,\ s^{-1}t\in F} = \sum_s h_1skh_1s^{-1}tk = h_1tk = p_t$$

You are confused. Go over things again $C = \mathcal{E}_{\Sigma_F}$

$=$ alg gen by h_s $s\in\Gamma$ | relations $h_sh_t = 0$ for $s^{-1}t\notin F$.

Γ action on C: $th_st^{-1} = h_{ts}$ | $h_s = \sum_t h_th_s$ (maybe also $h_s = \sum_t h_sh_t$)

$B = C\rtimes\Gamma$, C has local left identities $\left(1-\sum_{t\in sF} h_t\right)h_s = 0$

same should be true for B

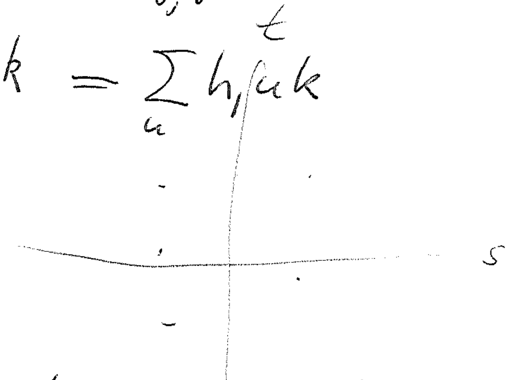
$$C = \sum h_sC \qquad B = C\otimes\mathbb{C}[\Gamma] = \sum h_sB$$

Question: What's the meaning of $\sum_s h_1sk = \sum_s p_s$?

This is a projection $\sum_s h_1 s k \sum_t h_1 t k = \sum_{s,t} h_1 s h_1 t k$

$$= \sum_{s,t} h_1 h_s s t k = \sum_{s,u} h_1 h_s u k = \sum_u h_1 u k$$

$u = st$



$$p = \sum_s h_1 s k \qquad kp = p$$

$$p = \sum_s h_1 s \sum_{t \in F} h_t = \sum_{\substack{s \in \Gamma \\ t \in F}} h_1 s t h_1 t^{-1} = \sum_{t \in F} \sum_{s \in \Gamma} h_1 s t h_1 t^{-1}$$

$$= \sum_{t \in F} \sum_{u \in \Gamma} h_1 u h_1 t^{-1} = \sum_{t \in F} \sum_{u \in F} = \left(\sum_{s \in F} h_1 s \right)^2$$

$$p \sum_{t \in F} h_1 t = \sum_{\substack{s \in \Gamma \\ t \in F}} h_1 s k h_1 t = \sum_{\substack{s \in \Gamma \\ t \in F}} h_1 s h_1 t = \sum_{s \in F} h_1 s \sum_{t \in F} h_1 t$$

Go over the above. Recall $h_s h_t = 0 \quad s^{-1}t \notin F$

$h_s = \sum_{t \in sF} h_t h_s \qquad h_s h_t \neq 0 \Rightarrow s^{-1} t \in F$

$h_1 = k h_1 \qquad k = \sum_{t \in F} h_t \qquad (\alpha t \xi)(s) = h_1 s^{-1} t \xi = (\alpha \xi)(t^{-1} s)$

$\therefore \boxed{\alpha t = L_t \alpha} \quad \boxed{\beta L_t = t \beta}$

$$H \xrightarrow{\alpha} \mathbb{C}[\Gamma] \otimes H \xrightarrow{\beta} H \qquad \beta(L_t f) = \sum_s t s k f(t^{-1} s) = t \beta f$$

$(f : \Gamma \to H)$ fin. supp $\longmapsto \beta f = \sum_s s k f(s)$

$\xi \longmapsto (\alpha \xi)(s) = h_1 s^{-1} \xi \qquad \beta \alpha \xi = \sum_s \underbrace{s k h_1 s^{-1}}_{h_s} \xi = \xi$

$$(\alpha \beta f)(s) = \sum_t (h_1 s^{-1} t k) f(t) \qquad p_s = h_1 s k$$

$$\sum_s p_s p_{s^{-1}t} = \sum_s h_1 s h_1 s^{-1} t k = h_1 t k = p_t$$

$p_s p_{s^{-1}t} \neq 0 \Longrightarrow s \in F$ and $s^{-1} t \in F$

You would like to show $h_1 k = h_1$?

$$p = \sum_s h_1 s k \qquad p h_1$$

$$h_1 k = \sum_{t\in F} h_1 h_t = \sum_{s\in\Gamma} h_1 h_s$$

$$k h_1 = h_1$$

$$\sum_{s\in F} h_s h_1 = h_1$$

$$p = \sum_{s\in\Gamma} h_1 s k$$

$$= \sum_{s\in\Gamma} h_1 s \sum_{t\in F} h_t = \sum_{\substack{s\in\Gamma\\ t\in F}} h_1 s t h_1 t^{-1} = \sum_{t\in F}\sum_{s\in\Gamma} h_1 s t h_1 t^{-1}$$

$u = st$
$s = ut^{-1}$

$$= \sum_{t\in F}\sum_{u\in\Gamma} h_1 u h_1 t^{-1}$$

$$= \sum_{t\in F}\sum_{u\in F} h_1 u h_1 t^{-1} = \left(\sum_{u\in F} h_1 u\right)^2.$$

$$p\sum_{t\in F} h_1 t = \sum_{s\in\Gamma} h_1 s k \sum_{t\in F} h_1 t = \sum_{\substack{s\in \cancel{\Gamma} F\\ t\in F}} h_1 s h_1 t = p$$

$$kp = \sum_s k h_1 s k = \sum_s h_1 s k = p$$

$$k h_1 = h_1$$

$$p_s h_1 = h_1 s k h_1 = h_1 s h_1$$

~~Prove that $k h_1 = h_1$ implies $h_1 k = h_1$. Better~~

$$p = \sum_{\substack{s\in\Gamma\\ t\in F}} h_1 s h_t$$

Assumptions: $\quad h_s h_t = 0 \quad$ for $\quad s^{-1}t \notin F$

alt: $\quad h_1 s h_1 = 0 \quad$ for $\quad s \notin F$.

put $k = \sum_{t\in F} h_t$, then $\quad k h_1 = h_1 \quad$ i.e. $\sum_{t\in F \text{ or } \Gamma} h_t h_1 = h_1$

$$p = \sum_{s\in\Gamma} h_1 s k, \quad p^2 = \sum_{\substack{s\in\Gamma \text{ or } F\\ t\in\Gamma}} h_1 s h_1 t k = \sum_{\substack{s\in F\\ t\in\Gamma}} h_1 h_s \cancel{s} t k =

$p^2 = \sum_{s\in\Gamma} h_1 h_s t \sum_{u\in F} h_u$ $\qquad h_1 h_s t h_u \neq 0$

$t \in$ some F' ind. of s, u $\qquad \Rightarrow s\in F,\ u\in F,\ s^{-1}tu \in F$

so use $\sum_{s\in\Gamma} h_s h_{tu} t'$

$p^2 = \sum_{t\in\Gamma} h_1 t k = p$ $\qquad p = \sum_t h_1 t \sum_{u\in F} h_u$

$p = \sum_s h_1 s k = \sum_s h_1 s \sum_{t\in F} t h_1 t^{-1} = \sum_{\substack{s\in\Gamma \\ t\in F}} h_1 s t h_1 t^{-1}$

$p = \sum_{s\in\Gamma} h_1 s k = \sum_{s\in\Gamma}\sum_{t\in F} h_1 s h_t = \sum_{s\in\Gamma}\sum_{t\in F} h_1 s t h_1 t^{-1}$

$= \sum_{t\in F}\sum_{s\in\Gamma} h_1 s t h_1 t^{-1} = \sum_{t\in F}\sum_{s\in F} h_1 s h_1 t^{-1} = \left(\sum_{s\in F} h_1 s\right)^2$

$p\sum_{t\in F} h_1 t = \sum_{s\in\Gamma}\sum_{t\in F} h_1 s k h_1 t = \left(\sum_{s\in F} h_1 s\right)^2 = p.$

So you have an op $q = \sum_{s\in F} h_1 s$ such $p = q^2$, $pq = q^2$

$\therefore q^3 = q^2$

Roots of $\lambda^3 - \lambda = 0$ are $\lambda = 0, 1, -1$

If you use the projection p to split the space into $p = 0$, $p = 1$. Then you expect there to be $q = \pm 1$ eigenspace; indicates choice of F may be relevant.

Can you prove $h_1 k = h_1$? $\qquad q^2 = p$, $pq = p$

Again $p = \sum_{s\in\Gamma} h_1 s k = \sum_{s\in\Gamma}\sum_{t\in F} h_1 s t h_1 t^{-1}$

$$= \sum_{t\in F}\sum_{s\in\Gamma} h_1 s t h_1 t^{-1} = \sum_{t\in F}\sum_{s\in F} h_1 s h_1 t^{-1} = \left(\overset{q}{\sum_{s\in F} h_1 s}\right)^2$$

$\boxed{p = q^2}$.

$$pq = \sum_{s\in\Gamma} h_1 s k \sum_{t\in F} h_1 t = \sum_{s\in\Gamma}\sum_{t\in F} h_1 s \cancel{k} h_1 t$$

$$= \sum_{s\in F}\sum_{t\in F} h_1 s h_1 t = \left(\sum_{s\in F} h_1 s\right)^2 = q^2.$$

$pq = q^2$ so $q^3 = pq = q^2$. Characteristic poly is $\lambda^3 - \lambda^2 = \lambda^2(\lambda - 1)$ which means a splitting of the module into $q=1$ eigenspace and a $q^2 = 0$ eigenspace.

$\cancel{h_1 k = h_1 \Rightarrow h_1 k h_1 = h_1^2}$

~~Repeat the Start with the~~ Variant of preceding where $p = \sum_s k s h_1 = \sum_s \sum_{t\in F} h_t s h_1 = \sum_s\sum_{t\in F} t h_1 t^{-1} s h_1$

$$= \sum_{t\in F}\sum_s t h_1 t^{-1} s h_1 = \left(\underbrace{\sum_{t\in F} t h_1}_{q}\right)^2$$

$$pq = \sum_s k s h_1 \sum_{t\in F} t h_1 \cancel{= \sum_s \sum_{t\in F} k s h_1 s t h_1}$$

$$\cancel{= \sum_s\sum_{t\in F} k h_s s t h_1 = k\sum_{t\in F}\left(\sum_s h_s\right) t h_1 = \sum_t k t h_1}$$

$$\cancel{pq = \sum_s\sum_{t\in F} k h_s s t h_1} = \sum_s \sum_{t\in F} k s t h_t^{-1} h_1 \quad ?$$

$$\cancel{= \sum_{t\in F}\sum_s k s t h_t^{-1} h_1}$$

Variant. $p = \sum_{s\in\Gamma} k s h_1 = \sum_{s\in\Gamma}\sum_{t\in F} h_t s h_1 \overset{\shortparallel}{=} t h_1 t^{-1} s h_1$

$= \sum_{t\in F}\sum_{s\in\Gamma} t h_1 t^{-1} s h_1 = \sum_{t\in F}\sum_{\substack{\cancel{s\in tF}\\ u\in F}} t h_1 \overset{u}{\cancel{t^{-1}s}} h_1$

$= \left(\sum_{t\in F} t h_1\right)^2 = q^2$

$pq = \sum_{s\in\Gamma} k s h_1 \sum_{t\in F} t h_1 = \sum_{s\in\Gamma}\sum_{t\in\Gamma} k s h_1 t h_1$

$\cancel{= \sum_{t\in\Gamma}\sum_{s\in\Gamma} k s h_1 t h_1} = \sum_{s\in\Gamma}\sum_{\substack{t\in\Gamma\\ s^{-1}u}} k h_s \overset{u}{st} h_1$

$= \sum_{u\in\Gamma}\sum_{s\in\Gamma} k h_s u h_1 = \sum_{u\in\Gamma} k u h_1 = p$

so again $q^3 = q^2 = p$

Let's go on to ~~Hess.~~

$H \xrightarrow{\alpha} \mathbb{C}[\Gamma] \underset{\Psi}{\otimes} H \xrightarrow{\beta} H$

$(s \mapsto f(s))$

$(\alpha\xi)(s) = h_1 s^{-1}\xi$

$\alpha t \xi = L_t \alpha \xi$

$\beta f = \sum s k f(s)$

$t\beta f = \sum_s t s k f(t^{-1}s) = \beta L_t f$

$\beta\alpha\xi = \sum_s s \underbrace{k h_1}_{h_1} s^{-1}\xi = \xi$

$(\alpha\beta f)(s) = h_1 s^{-1}\sum_t t k f(s) = \sum_t (h_1 s^{-1} t k) f(s).$

$p_s = h_1 s k$

$\sum_s p_s p_{s^{-1}t} = \sum_s h_1 s \underbrace{k h_1}_{h_1} s^{-1} t k =$ [redacted] $\sum_s h_1 h_s t k$

$= h_1 t k = p_t$

$p_s = h_1 s \sum_{t\in F} h_t = \sum_{t\in F} h_1 s t h_1 t^{-1}$

Let's see if something can be done about Morita equivalence. ~~Begin to~~

$$h_s h_t = h_1 s^{-1} t h_1 t$$

H, Γ, $h_1^{1/2}$, $\left|\begin{array}{ll} \sum h_s = 1 & \text{on } H. \\ h_s h_t = 0 & s^{-1}t \notin F. \end{array}\right.$

$$H \xrightarrow{\alpha} \mathbb{C}[\Gamma]\otimes H \xrightarrow{\beta} H$$

$$(\alpha \xi)(s) = h_1^{1/2} s^{-1} \xi$$

$$\beta f = \sum_s s h_1^{1/2} f(s)$$

$$(\alpha\beta f)(s) = \sum_t \left(h_1^{1/2} s^{-1} t h_1^{1/2}\right) f(t)$$

From this data you get $p_s = h_1^{1/2} s h_1^{1/2}$ $\left|\begin{array}{l} \text{support } p_s = 0 \quad s \notin F. \\ \text{idemp.} \\ \sum_s p_s p_{s^{-1}t} = p_t \end{array}\right.$

$$\sum_s p_s p_{s^{-1}t} = h_1^{1/2} s h_1 s^{-1} t h_1^{1/2} = p_t$$

So you find that H ~~the B~~ your module over $B = \mathcal{E}_{\Sigma_F} \rtimes \Gamma$. can be reconstructed from itself and the family p_s.

This seems too abstract.

~~This seems too~~

So go back to $\mathbb{Z}$ where you have a geometric picture and look ~~at the for something~~ at the noncommutative version of the ~~the~~ $\mathbb{Z}$-tree

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Let's start again with $\Gamma = \mathbb{Z}$ and $F = \{-1, 0, 1\}$. The aim is to construct a "noncommutative" Morita equiv. Let us ~~first~~ begin with Hilbert space representations. You have a Hilbert space H with a unitary operator u and a positive operator $h_0^{1/2} \geq 0$ ~~such that~~ satisfying a "orthog. condition": $h_0^{1/2} u^n h_0^{1/2} = 0$ for $|n| > 1$, and a generator condition: $\sum_{n\in\mathbb{Z}} u^n h_0^{1/2} H$ is dense in H. No you want

$\forall \xi \in H$ that $\sum h_n \xi = \xi$ $\qquad h_n = u^n h_0 u^{-n}$

Partitions of unity condition.

Recap. H Hilb., u unitary op on H, $h_n = u^n h_0 u^{-n} \geq 0$ equivariant partition of 1: $\sum h_n = 1$ in the sense of positive herm. ~~[illegible]~~ operators, quasi-orthogonality:
$h_0^{1/2} u^n h_0^{1/2} = 0 \qquad |n| \geq 2.$

First idea is GNS. You should be able to reconstruct this data from a positive definite function on $\mathbb{Z}$ with values in operators on the image $h_0^{1/2} H$. ~~So let's see what~~ You should know the formulas well. Basic map.

$$H \xrightarrow{\alpha} L^2(\mathbb{Z}, V) \xrightarrow{\beta} H$$
$$\{f : \mathbb{Z} \to V\} \longmapsto \sum_{n \in \mathbb{Z}} u^n h_0^{1/2} f(n)$$

$(\alpha\xi)(n) = h_0^{1/2} u^{-n} \xi$. Then $\beta = \alpha^*$ and $\beta\alpha = id_H$

equivariance of α: $\big(\alpha(u^k\xi)\big)(n) = h_0^{1/2} u^{-n} u^k \xi = \underbrace{(\alpha\xi)(n-k)}_{(T_k(\alpha\xi))(n)}$

$$\beta(T_k f) = \sum_n u^n h_0^{1/2} f(n-k) = \sum_n u^{k+n} h_0^{1/2} f(n)$$
$$= u^k(\beta f).$$

Then $\alpha\beta$ is a projector on $L^2(\mathbb{Z}, V)$ commuting with translation. Think of $L^2(\mathbb{Z}, V)$ as L^2 functions on the circle $S^1 = \hat{\mathbb{Z}}$. ~~the commutant of~~

What is $\alpha\beta$? $\big((\alpha\beta)f\big)(n) = \sum_\ell \overbrace{h_0^{1/2} u^{-n+\ell} h_0^{1/2}}^{g(n-\ell)} f(\ell)$

~~and the quasi orthog cond. says only $|\ell| \leq 1$.~~ So the operator $\alpha\beta$ on $L^2(\mathbb{Z}, V)$ is the convolution operator ~~[illegible] with kernel~~ corresp. to multiplication by the function

$\sum_\ell (h_0^{1/2} u^{-n} h_0^{1/2} u^n z$? ~~Identify f with the Laurent~~

~~[illegible] the transform $\sum z^n f(n)$~~

$$(pf)(n) = \sum_{\ell \in \mathbb{Z}} \rho(n-\ell) f(\ell) \qquad \rho(n) = h_0^{1/2} u^{-n} h_0^{1/2}$$

$$\sum_n (pf)(n) z^n = \sum_{n,\ell \in \mathbb{Z}} \rho(n-\ell) z^{n-\ell} f(\ell) z^\ell = \hat{\rho}(z) \hat{f}(z)$$

You want to understand exactly what arises. Keep close to condition $F = \{-1, 0, 1\}$. This should help the algebraic version. Yes - the alg version uses $\mathbb{C}[\Gamma] \otimes V$, or $\mathbb{C}[\Gamma] \otimes H$, instead of L^2. Somehow you have to pin down V. Since

$$H \xrightarrow{\alpha} \mathbb{C}[\Gamma] \otimes V \xrightarrow{\beta} H.$$

$$\xi \longmapsto (\alpha \xi)(n) = h_0^{1/2} u^{-n} \xi$$

$$f \longmapsto \beta f = \sum_n u^n h_0^{1/2} f(n)$$

For α to be defined you need $h_0^{1/2} u^{-n} \xi$ to be zero for almost all n. For β to be onto you need $H = \sum_{n \in \mathbb{Z}} u^n h_0^{1/2} H$. Suppose β onto H, look at α on β $\underbrace{u^k h_0^{1/2} H}_{V}$, i.e. you take f to ~~have support~~ be $\neq 0$ only at k, $f(n) = \begin{cases} 0 & n \neq k \\ 1 & n = k. \end{cases}$

$$(\alpha \beta f)(n) = \sum_\ell \left(h_0^{1/2} u^{-n+\ell} h_0^{1/2} \right) \delta_{\ell-k} = h_0^{1/2} u^{-n+k} h_0^{1/2}$$

which is $\neq 0$ only for $|n-k| \leq 1$.

So now what happens? The kind of

$$\hat{\rho}(z) = \sum \left(h_0^{1/2} u^{-n} h_0^{1/2} \right) z^n$$

you are getting are Laurent polynomial projectors.