Program: Cuntz's $\mathcal{E}_{\Sigma_F}$ ~~...~~ alg version

has generators $h_s \quad s \in \Gamma$
relations $h_s h_t = 0 \qquad s^{-1}t \notin F$

$$\sum_s h_s h_t = h_t = \sum_s h_t h_s$$

Γ acts on $\mathcal{E}_{\Sigma_F}$ by alg automorphisms $t(h_s) = h_{ts}$

i.e. in $\mathcal{E}_{\Sigma_F} \rtimes \Gamma$ one has $t\, h_s = h_{ts} t$. The crossproduct is like a semi-direct product, ~~so its~~ except that ~~...~~ $\mathbb{C}[\Gamma]$ is outside the crossproduct. The crossproduct should be an idempotent ring whose multiplier algebra contains $\mathbb{C}[\Gamma]$, and ~~these~~ in this way firm $\mathcal{E}_{\Sigma_F}$-modules have natural Γ actions. ~~this is the point~~

$$\mathrm{Mult}^{(P,Q,\ldots)}(A) = \left\{ (\lambda, \rho) \in \mathrm{Hom}_{A^{op}}(P,P) \times \mathrm{Hom}_A(Q,Q) \;\middle|\; \langle q, \lambda p \rangle = \langle q\rho, p \rangle \right\}$$

~~Why~~ $C = \mathcal{E}_{\Sigma_F}$ has local identities so it's flat, Yes.

So what's next?? ~~...~~ functions on $\underline{E}\Gamma$

Think this out. So now consider $B = \mathcal{E}_{\Sigma_F} \rtimes \Gamma = \bigoplus_{s \in \Gamma} \mathcal{E} s$

$s h_t s^{-1} = h_{st}$. Now look at B modules M such that $BM = M$. $B = C \rtimes \Gamma$, C is a subring of ~~an ideal in~~ B

$$B = C\Gamma = \Gamma C \qquad CB = C^2\Gamma = C\Gamma = B$$
$$BC = \Gamma C^2 = \Gamma C = B$$

C is a subalg of B which gen. B as a left or right ideal. ~~Assume~~

It should now be possible to straighten out the problems. $B = C \rtimes \Gamma$ $\quad C = \mathcal{E}_{\Sigma_F}$

In C you have the local identity $\sum_{s\in\Gamma} h_s = \text{net}\{\sum_{s\in F\text{inite}} h_s\}$.

What do you want? A Morita equiv. of B with $A = P_F$. So let M be a B-module such that $\sum_{s\in\Gamma} h_s M = M$. equiv. to $\sum_s h_s m = m$ for all m.

$$h_s = s h_1 s^{-1}$$

$$\sum_s s h_1 M = M$$

At the moment you have the operators $s \in \Gamma$ on M and h_1. But you know that $h_1 = \sum_s h_1 h_s = \sum_s h_s h_1$ where these are finite sums. This should allow me to define a substitute for $h_1^{1/2}$, namely for F big enough $u_F = \sum_{s\in F} h_s$ is a ~~local~~ identity for h_1.
$u_F h_1 = h_1 u_F = h_1$

Does this imply h_1 on M considered as v.s. is nuclear? Seems unlikely

Partitions of 1. Yes! Life is difficult.

Simplest case first, namely $C = \mathbb{C}[\hat{\Gamma}] = \bigoplus_{s\in\Gamma} \mathbb{C}e_s$. There are three Γ-actions on this algebra corresp. to the left, right, & conjugation actions of Γ on itself.

$C = \overset{\mathbb{C}[\Gamma]}{\{\text{finite support functions on } \Gamma\}} = \bigoplus_{s\in\Gamma} \mathbb{C}e_s$

a firm C-module is the same as a ~~[illegible]~~ v.s. with Γ-grading.
Γ acts on itself in 3 ways: left, right, conj, hence get 3 cross product algebras.

Your picture of the multiplier algebra is wrong, flawed.

$M(C) = \prod_{s\in\Gamma} \mathbb{C}e_s$

$$C \rtimes \Gamma \longrightarrow \tilde{C} \rtimes \Gamma \longrightarrow \Gamma$$

what is the mult. alg of $C \rtimes \Gamma$? ~~[illegible]~~

Put $B = C \rtimes \Gamma$ for either the left or right action

You guess that $M(B)$ should be a ring of operators ?

$C = \mathbb{C}(\Gamma) = \{\text{fns fin. supp on } \Gamma\} \simeq \bigoplus_{s\in\Gamma} \mathbb{C}e_s$

~~[illegible]~~ $C \rtimes \Gamma$ for ~~[illegible]~~ left or right translation action. Call this ring B. Extension of algs.

$$B \hookrightarrow \tilde{C} \rtimes \Gamma \twoheadrightarrow \mathbb{C}[\Gamma]$$

You know that B is the ring of finite matrix operators on ~~[illegible]~~ the vector space C with basis e_s.
So it has the form ~~[illegible]~~ $P \otimes Q$ with $P = C$
and $Q =$ ~~[illegible]~~ finite supp. dual

Again, $C = \mathcal{E}_{\Sigma_F}$ gen. h_s $s \in \Gamma$ $h_s h_t = 0$, $s^{-1}t \notin F$

$$\sum_s h_s h_t = h_t = \sum_s h_t h_s$$

$$B = \mathcal{E}_{\Sigma_F} \rtimes \Gamma = \bigoplus_{s\in\Gamma} Cs \qquad t h_s t^{-1} = h_{ts}$$

Now that $h_1 h_t = 0 = h_t h_1$ for $t \notin F$

so that $h_1 = \sum_s h_1 h_s = \sum_{s\in F} h_1 h_s = h_1 \sum_{s\in F} h_s$

also $h_1 = \sum_s h_s h_1 = \left(\sum_{s\in F} h_s\right) h_1$

so we have $h_1 = h_1 k = k h_1$ with $k = \sum_{s\in F} h_s$

In other words k is a local left + right unit for h_1.

Put $p = \sum_{s\in\Gamma} h_1 s k$ ~~note~~ $\cancel{h_1 s k = \sum_{t\in F} h_1 s h_t}$

note $h_1 s h_t = h_1 h_{st} s^{-1} = 0$ for $st \notin F$ i.e.

so $p = \sum_{s\in\Gamma}\sum_{t\in F} h_1 s h_t = \sum_{t\in F}\sum_{s\in Ft^{-1}} h_1 s h_t$ is a finite ~~sum~~

$$\cancel{p^2 = \sum_{s\in\Gamma}\sum_{t\in\Gamma} h_1 s \overbrace{k h_1}^{h_1} t k = \sum_{s\in\Gamma}\sum_{t\in\Gamma} h_1 s h_1 t k}$$

$$p^2 = \sum_{s\in\Gamma}\sum_{t\in\Gamma} h_1 s k h_1 t k = \sum_{s\in F}\sum_{t\in\Gamma} h_1 s h_1 t k$$

$$= \sum_{s\in F}\sum_{u\in\Gamma} h_1 s h_1 s^{-1} u k = \sum_{u\in\Gamma} h_1 \sum_{s\in F} \overbrace{s h_1 s^{-1}}^{h_s} u k = \sum_{u\in\Gamma} h_1 u k = p$$

$$p = \sum_{s\in\Gamma} h_1 s k = \sum_{s\in\Gamma}\sum_{t\in F} h_1 s t h_1 t^{-1} = \sum_{t\in F}\sum_{st\in F} h_1 s t h_1 t^{-1}$$

$$p = \sum_{s\in\Gamma} h_1 s \sum_{t\in F} t h_1 t^{-1} = \sum_{\substack{s,t \\ \text{such that } t\in F,\ st\in F}} h_1 s t h_1 t^{-1}$$

$$= \sum_{t,u \text{ such that } t\in F,\, u\in F} h_1 u h_1 t^{-1} \quad ?$$

$$p = \sum_{s\in\Gamma} h_1 s k \qquad \text{where } k = \sum_{t\in F} t h_1 t^{-1}$$

$$= \sum_{t\in F}\sum_{st\in F} h_1 s t h_1 t^{-1} = \sum_{t\in F}\sum_{u\in F} h_1 u h_1 t^{-1}$$

$$= \left(\sum_{t\in F} h_1 t\right)^2$$

Check it over again $\quad C = \mathcal{E}_{\Sigma_F}$ $\quad$ gen h_s $\quad s\in\Gamma$
rel $h_s h_t = 0$ for $s^{-1}t \notin F$

$$\sum_{s\in tF} h_s h_t = h_t = \sum_{s\in tF} h_t h_s \longrightarrow \sum_s h_s h_t = h_t = \sum_s h_t h_s$$

$$\sum_{s^{-1}t\in F} \Leftrightarrow t^{-1}s\in F \Leftrightarrow s\in tF$$

$B = C \rtimes \Gamma \qquad t h_s t^{-1} = h_{ts}.$

$$h_1 = \left(\sum_{s\in F} h_s\right) h_1 = h_1\left(\sum_{s\in F} h_s\right)$$

$$p = \sum_{s\in\Gamma} h_1 s k \qquad k = \sum_{t\in F} h_t$$

$$\therefore p = \sum_{s\in\Gamma}\sum_{t\in F} h_1 s t h_1 t^{-1} = \sum_{t\in F}\sum_{s\in\Gamma} \overbrace{h_1 s t h_1 t^{-1}}^{=0 \text{ for } st\notin F}$$

$$= \sum_{t\in F}\sum_{u\in\Gamma} h_1 u h_1 t^{-1} = \sum_{t\in F}\sum_{u\in F} h_1 u h_1 t^{-1} = \left(\sum_{s\in F} h_1 s\right)^2$$

$$ph_1 = \sum_{s\in F}\sum_{t\in F} h_1 s h_1 t h_1$$

Begin again $\quad \mathcal{E}$ $\left|\begin{array}{l} \text{gen} \quad h_s \quad s\in\Gamma \\ \text{rel} \quad h_s h_t = 0 \quad s^{-1}t \notin F \\ h_t = \sum_s h_s h_t = \sum_s h_t h_s \end{array}\right.$

Define Γ action on $\mathcal{E}$ by $\quad t h_s t^{-1} = h_{ts}$

$B = \mathcal{E} \rtimes \Gamma = \bigoplus_{s\in\Gamma} \mathcal{E} s \qquad (fs)(gt) = f(sgs^{-1})st$

$p = \sum h_1 s k$

Begin again $\quad \mathcal{E}$ $\left|\begin{array}{l} \text{gen} \quad h_s \quad s\in\Gamma \\ \text{rels} \quad h_s h_t = 0 \quad \text{if } s^{-1}t\notin F \\ \qquad h_t = \sum_s h_s h_t = \sum_s h_t h_s \end{array}\right.$

action of Γ on $\mathcal{E}$ by $\sigma_t(h_s) = h_{ts}$

form $B = \mathcal{E}\rtimes\Gamma = \bigoplus_s \mathcal{E} s$

mult. $(cs)(c't) = c(sc's^{-1})\, st.$ $\qquad \boxed{t h_s t^{-1} = h_{ts}}$

$t h_1 t^{-1} = h_t \quad \forall t.$ $\qquad \left.\begin{array}{l} h_1 h_s = 0 \quad \text{if } s\notin F \\ h_s h_1 = 0 \quad \text{if } s^{-1}\notin F \end{array}\right\}$ same

$h_1 = \underbrace{\Big(\sum_{s\in F} h_s\Big)}_{k} h_1 = h_1\Big(\sum_{s\in F} h_s\Big).$ $\quad$ Next introduce

$p = \sum_{s\in\Gamma} h_1 s k = \sum_{s\in\Gamma} h_1 s \sum_{t\in F} h_t$ $\qquad \begin{array}{l} h_1 s h_t = h_1 s t h_1 t^{-1} \\ = 0 \text{ for } st\notin F \end{array}$

$\neq 0 \implies s\in F$

$p^2 = \sum_{s,t\in\Gamma} h_1 s k h_1 t k = \sum_{s,t\in\Gamma} h_1 h_s\, stk$

$= \sum_{s\in F} h_1 \sum_t h_s stk = \sum_{s\in F} h_1 \sum_u h_s s(s^{-1}u)k$?

$$p^2 = \sum_{s,t\in\Gamma} h_1 s k \text{ [redacted] } h_1 t k = \sum_{s\in F,\, t\in\Gamma} h_1 h_s s t k$$

$$= \sum_{s\in F,\, u\in\Gamma} h_1 h_s s(s^{-1}u) k = \sum_{u\in\Gamma}\sum_{s\in F} h_1 h_s u k = \sum_{u\in\Gamma} h_1 u k = p$$

so you have $p = \sum_{s\in\Gamma} h_1 s k$ $\qquad k = \sum_{t\in F} h_t$

[redacted] There's another piece of information, namely

$$p = \sum_{s\in\Gamma}\sum_{t\in F} h_1 s t h_1 t^{-1} = \sum_{t\in F}\sum_{s\in\Gamma} h_1 s t h_1 t^{-1}$$

[redacted]

let $s = ut^{-1}$ $$= \sum_{t\in F}\sum_{u\in\Gamma} h_1 u h_1 t^{-1}$$

$$= \sum_{t\in F}\sum_{u\in F} h_1 u h_1 t^{-1} = \left(\sum_{t\in F} h_1 t\right)^2$$

where does p lies? In $B = \mathcal{E}\rtimes\Gamma = \bigoplus_{s\in\Gamma}\mathcal{E}s$

What is next? last night wondered about how canonical is the choice of $p = \sum h_1 s k$ or $\sum k s h_1$, also do you use both $h_1 = h_1 k$ and $h_1 = k h_1$?

$$p^2 = \sum_{s,t} h_1 s h_1 t k = \sum_{s,t} h_1 h_s s t k$$

$$= \sum_s \sum_t h_1 h_s s t k = \overbrace{\sum_t h_1 t k}^{p}$$

used $\sum_s h_1 h_s = h_1$ at the end and

$h_1 = k h_1 = \sum_s h_s h_1$ at the beginning.

~~[illegible]~~. What about $\sum_s h_1 s$? This is a well defined operator on $\mathcal{E}$ since $\mathcal{E} = \sum h_t \mathcal{E}$. It's a left multiplier on $\mathcal{E}$

Look at simplest case $F = \{1\}$, so

$$h_s h_t = 0 \quad \text{for } s \neq t. \qquad = h_s \quad \text{for } s = t.$$

Then what is $\sum_s h_1 s$?

Fascinating. In this case $\mathcal{E} = \mathbb{C}[\hat{\Gamma}] = \bigoplus_{s\in\Gamma} \mathbb{C}e_s$.

What probably happens is that $\sum_s s$, the norm, appears in the pairing. The pairing between $\mathbb{C}[\hat{\Gamma}]$ and $\mathbb{C}[\Gamma]$ which yields the crossproduct algebra $\mathcal{E} \rtimes \Gamma. = \mathbb{C}[\hat{\Gamma}] \otimes \mathbb{C}[\Gamma]$, basis $e_s t$

$$e_s t\, e_{s_1} = e_s e_{ts_1} t = \begin{cases} e_s t & \text{if } s = ts_1 \\ 0 & \text{otherwise} \end{cases}$$

$$(p \otimes q) p_1 = \underline{p} \langle q, p_1 \rangle$$

$$\mathcal{E} = \text{alg} \left| \begin{array}{ll} \text{gen} & h_s \quad s \in \Gamma \\ \text{rel} & h_s h_t = 0 \quad \text{if } s^{-1}t \notin F \end{array} \right.$$

$$t a t^{-1} = \sigma_t(a)$$

$$h_s = \sum_t h_s h_t = \sum_t h_t h_s$$

Γ acts on $\mathcal{E}$: $\sigma_t(h_s) = h_{ts}$, form $\mathcal{E} \rtimes_\sigma \Gamma = \bigoplus_{t\in\Gamma} \mathcal{E} t$

Simplest case. $F = \{1\}$. $h_s h_t = \begin{cases} 0 & s \neq t \\ h_s & s = t \end{cases}$

write e_s for h_s, $\mathcal{E} \rtimes \Gamma$ has basis $e_s t \quad s, t \in \Gamma$.

$$p = \sum_s h_1 s h_1 = \sum_s h_1 h_s s = h_1^2 = h_1$$

In general

$$p = \sum_s h_1 s k \qquad k = \sum_{s\in F} h_s \qquad (h_1 = h_1 k = k h_1)$$

$$= \sum_s \sum_{t\in F} h_1 s t h_1 t^{-1} = \sum_{t\in F} \sum_{u\in F} h_1 u h_1 t^{-1} = \Big(\sum_{u\in F} h_1 u\Big)^2$$

F can be arb. large.

Look at $\sum_{s\in\Gamma} h_1 s$ left acting on $\mathcal{E}$

Since $\mathcal{E} = \sum_t h_t \mathcal{E}$, and $h_1 s h_t = h_1 s t h_1 t^{-1}$
$= 0$ for $st \notin F$.

the left mult operator $\sum_s h_1 s$ on $\mathcal{E}$ is well defined, and it is a left multiplier on $\mathcal{E}$, $\sum_s h_1 s \in \mathrm{Hom}_{\mathcal{E}^{op}}(\mathcal{E}, \mathcal{E})$. What is it in the $F=\{1\}$ case? $\sum_s e_1 s$ on $\mathbb{C}[\hat{\Gamma}] = \bigoplus_t \mathbb{C} e_t$?

$$\sum_s e_1 s \sum_t e_1 t\, e_u ?$$

$$\left(\sum_s e_1 s\right) e_u = \sum_s e_1 e_{su} s = e_1 u^{-1}$$

$$\sum_s e_1 s\, e_1 u^{-1} = \sum_s e_1 e_s s^{-1} u^{-1} = e_1 u^{-1}$$

$$\sum_s e_1 s \underbrace{\sum_t e_1 t\, e_u}_{\sum_t e_1 e_{tu} t \,=\, e_1 u^{-1}} = \sum_s e_1 s\, e_1 u^{-1} = e_1 u^{-1}$$

so $\sum_s e_1 s$ maps $\mathcal{E} = \mathbb{C}[\hat{\Gamma}]$ into $\mathbb{C} e_1$

and $\left(\sum_s e_1 s\right)^2 = e_1$

You should understand, but don't, the behavior of $\sum_{s\in\Gamma} s$, the norm, something you encountered in the case of a principal bundle

Basic pairing: $\langle f, g\rangle = \mathrm{Norm}(fg)$ $\quad f, g \in C_c(Y)$

How to discuss this? Continuous fns. comp support on $\mathbb{R}$: $C_c(\mathbb{R})$ ~~commutative~~ noncomutal ring with trace, ?

Return to Γ and $\mathcal{E}$: alg gen by $h_s, s \in \Gamma$

$$\left(\begin{array}{l} \text{rel. } h_s h_t = 0 \quad s^{-1}t \notin F \\ h_t = \sum_s h_s h_t = \sum_s h_t h_s \end{array} \right.$$

Γ act on $\mathcal{E}$ $\quad \sigma^s(h_t) = h_{st}$. in $\mathcal{E} \rtimes_\sigma \Gamma$ have $t h_s t^{-1} = h_{ts}$. Note $h_1 = \left(\sum_{s \in F} h_s\right) h_1 = h_1 \left(\sum_{s\in F} h_s\right)$

$$p = \sum_s h_1 s k = \sum_s \sum_{t \in F} h_1 s h_t = \sum_s \sum_{t \in F} h_1 s t h_1 t^{-1}$$

$$= \sum_{t \in F} \sum_s h_1 s t h_1 t^{-1} = \sum_{t \in F} \left(\sum_{u \in F} h_1 u\right) h_1 t^{-1}$$

$$p^2 = \sum_{s,t} h_1 s \cancel{k} h_1 t \cancel{k} = \sum_{s,\cancel{t}} \cancel{\ \ } h_1 h_s \overset{u}{st} k = \sum_u h_1 u k$$

$s \in F$ $\quad$ What about $\sum_s k s$

$$p^2 = \sum_{\substack{s,t \\ s \in F \\ t \in F}} h_1 s h_1 t k = \sum_{\substack{s \in F \\ u}} h_1 h_s \overset{u}{\cancel{\ }} k$$

Claim that if $k = \sum_{s \in F} h_s$, then $k h_1 = h_1$

$\Rightarrow h_1 k =$

assume $\sum_s h_s h_t = h_t \quad \forall t$ 804

$$p^2 = \sum_{s,t} h_1 s\, h_1 t k$$

$$= \sum_{\substack{s\in F\\ t\in\Gamma}} h_1 h_s\, stk = \sum_{\substack{s\in F\\ t\in\Gamma}} h_1 h_s\, tk$$

Look if you assume that $\sum_s h_s h_t = h_t \quad \forall t$
then $\sum_s h_s \xi = \xi$ for any $\xi \in \mathcal{E}$.

~~Go back to notes~~ $\mathcal{E}$ gen $h_s \quad s\in\Gamma$
rel $h_s h_t = 0$ if $s^{-1}t \notin F$
$h_t = \sum_s h_s h_t$

$\mathcal{E}\rtimes\Gamma \qquad t h_s t^{-1} = h_{ts}$. Choose $K \supset F$ K finite
put $k = \sum_{s\in K} h_s$ so that $kh_1 = \sum_{s\in K} h_s h_1 = \sum_s h_s h_1 = h_1$

$$p = \sum_s h_1 s k, \quad p^2 = \sum_{s,t} h_1 s k h_1 t k = \sum_{s,t} h_1 h_s\, stk$$

$$= \sum_s \sum_u h_1 h_s\, uk$$

$$= \sum_u \Big(\sum_s h_1 h_s\, uk\Big) = \sum_u h_1 uk$$

It seems then that $h_t = \sum_s h_s h_t \implies h_t = \sum_s h_t h_s$

Question Is $\sum_s k s h_1$ also a projector

$$\sum_{s,t} k s h_1 k t h_1 = \sum_{s,t} k s h_1 t h_1 = \sum_{s,t} k h_s \cancel{s} t h_1 = \sum_t k \cancel{t} h_1$$

Yes. Is it the same as p above?

$$\sum_{s\in\Gamma} h_1 s k = \sum_{\substack{s\in\Gamma\\ t\in K}} h_1 s t h_1 t^{-1} = \sum_{t\in K}\sum_{s\in\Gamma_K} h_1 s h_1 t^{-1} \quad \Big(\sum_{s\in K} h_1 s\Big)^2$$

$$\sum_{t\in\Gamma} kth_1 = \sum_{\substack{s\in K\\ t\in\Gamma}} \overbrace{h_s th_1}^{sh_1 s^{-1}th_1} = \sum_{\substack{s\in K\\ s^{-1}t\in K}} h_s h_t t$$

So it would seem by symmetry that

$$\sum_{t\in\Gamma} kth_1 = \Big(\sum_{s\in K} sh_1\Big)^2$$

Make a program. Start with $\mathcal{E}$, ~~[illegible]~~ any alg with gen., no you need univ. alg to define Γ action

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$$\mathcal{E} = \mathbb{C}[\Gamma] = \bigoplus_{s\in\Gamma} \mathbb{C}e_s \qquad \underbrace{\sum_s e_s e_t}_{e_t^2} = e_t \qquad e_s e_t = 0,\ s\neq t$$

$$p = \sum_s e_1 s e_1 = \sum_s e_1 e_s s = e_1$$

$\sum_{s\in S} h_1 s$ as an operator on $\mathcal{E}\rtimes\Gamma$

Question: Does $\sum$ Let's unders

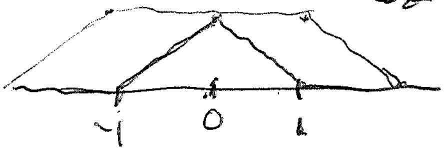
You have lots of things to understand better.

$$\sum_{s\in K} sh_1 \sum_{t\in K} th_1 = \sum_{s\in K} s\Big(\sum_{t\in K} h_1 th_1\Big)$$

$$= \sum_{s\in K} s \sum_{t\in\Gamma} h_1 th_1 = \sum_{s\in K}\sum_{t\in\Gamma} h_s st h_1 \qquad t = s^{-1}u$$

$$= \sum_{s\in K}\sum_{u\in\Gamma} h_s s(s^{-1}u) h_1 = \sum_{\substack{s\in K\\ u\in\Gamma}} h_s uh_1 = \sum_u kuh_1$$

I think you have all the tools you need to construct a Morita equivalence. Go back to the $\mathbb{Z}$-examples and work on the details. Let $\mathcal{E}$ = algebra $C_c(\mathbb{R})$ of cont. comp. support functions on $\mathbb{R}$ with $\Gamma = \mathbb{Z}$ acting by translation $(u^n f u^{-n})(x) = f(x-n)$. $\mathcal{E} \rtimes \mathbb{Z} = \bigoplus \mathcal{E} u^n$ with this mult. $h_0(x)$:



$h_m h_n = 0$ if $|m-n| \geq 2$.

$$k = \sum_{n \in \{-1,0,1\}} h_n \qquad k h_0 = h_0 k = h_0$$

Your projection p is $\sum\limits_{|n| \leq 2} k u^n h_0$ or $\sum\limits_{|n| \leq 2} h_0 u^n k$.

What if you use $h_0^{1/2}$, $p = \sum\limits_n h_0^{1/2} u^n h_0^{1/2}$

In general $p = \sum h_1^{1/2} s h_1^{1/2} = \sum h_1^{1/2} h_s^{1/2} s$

Check $\sum\limits_{s,t} h_1^{1/2} h_s s t h_1^{1/2} = \sum\limits_{s,t} h_1^{1/2} h_s t h_1^{1/2} = \sum\limits_t h_1^{1/2} t h_1^{1/2}$.

How to say this? ~~A key point~~

What seems to be true is that h_0 can be any element of $\mathcal{E} = C_c(\mathbb{R})$ such that $\sum u^n h_0 u^{-n} = 1$ and ? ~~In general~~ You need to abstract the partition of unity stuff.

In general you have an algebra $\mathcal{A}$ acted on by Γ

Return to previous difficulties.

$\mathcal{B}$ alg with Γ action

Let $\mathcal{E}$ be an algebra with a Γ-action, let $B = \mathcal{E} \rtimes \Gamma$, this is Γ-graded. Let $h_1 \in B$, put $h_s = s h_1 s^{-1}$, assume $h_t h_s = 0$ $t^{-1}s \notin F$, $h_1 = \sum_s h_1 h_s = \sum_s h_s h_1$. Assuming a good factor. $h_0^{1/2} h_1^{1/2}$ of h_1 you get $p = \sum_{s \in \Gamma} h_0^{1/2} s h_0^{1/2} \in B$, $p = p^2$.

Question: What is the significance of p? Answer. The only thing I can think of is to form pB, Bp, pBp. Is it possible that [redacted] you have a Morita equivalence between B and pBp? And that $pBp =$ [redacted] P_F. You should understand this for $\Gamma = \mathbb{Z}$ $F = \{-1, 0, 1\}$.

Start with $\mathcal{E} = C_c(\mathbb{R})$ with $\mathbb{Z}$ action $(u^n * f)(x) = f(x-n)$, let $B = \mathcal{E} \rtimes \mathbb{Z} = \bigoplus_{n \in \mathbb{Z}} \mathcal{E} u^n$. What sort of Morita equivalence do you have already. I think you have a M.eq. of B with $C(\mathbb{R}/\mathbb{Z})$, which is unital. Why? Consider $Y = \mathbb{R} \xrightarrow{\pi} X = \mathbb{R}/\mathbb{Z}$ the universal bundle, you have

$$\begin{tikzcd} Y \times_X Y \arrow[r] \arrow[d] & Y \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

cartesian, locally over X this is

$$\begin{tikzcd} \mathbb{Z} \times \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \arrow[r] & pt \end{tikzcd}$$

sheaf yields $C_c(Y \times_X Y) \xleftarrow{\sim} C_c(Y) \otimes_{C(X)} C_c(Y)$

$\wr|$

and $C_c(Y \times \mathbb{Z})$ which should turn out to be $C_c(Y) \otimes_\sigma \mathbb{C}[\mathbb{Z}] = B$. So B should arise from the dual pair with $P = C_c(\mathbb{R})$, $Q = C_c(\mathbb{R})$

with pairing $\langle f, g\rangle = \sum_{n\in\mathbb{Z}} (fg)(x-n) \in C(\mathbb{R}/\mathbb{Z})$

This stuff seems right.

Let's work on the Morita equivalence details. $Y \xrightarrow{\pi} X$ is a principal Γ-bundle with X compact. $A = C(X)$, $B = C_c(Y\times_X Y)$

$$\begin{array}{ccc} Y\times_X Y & \xrightarrow{pr_2} & Y \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\pi} & X \end{array} \qquad \begin{array}{ccc} C_c(Y\times_X Y) & \longleftarrow & C_c(Y) \\ \uparrow & & \\ C_c(Y) & & C(X) \end{array}$$

Let $b(y,y') \in C_c(Y\times_X Y)$ be a kernel.

$$pr_{1*}\, b(y,y')\underbrace{f(y')}_{pr_2^* f} = pr_{1*}\, b\, pr_2^*$$

In this way $B = C_c(Y\times_X Y)$ should operate on $E = C_c(Y)$

To work out the formulas restrict to a point $x = \pi(y) = y+\mathbb{Z}$. $Y\times_X Y = \{(y,y')\in\mathbb{R}^2 \mid y-y'\in\mathbb{Z}\}$

$x = 0+\mathbb{Z}$.

$$\begin{array}{ccc} \overset{b(y,y')}{\mathbb{Z}\times\mathbb{Z}} & \xrightarrow{pr_2} & \overset{f(y')}{\mathbb{Z}} \\ \downarrow{\scriptstyle pr_1} & & \\ \mathbb{Z} & & \end{array}$$

finite support matrices indexed by $\mathbb{Z}$.

$$(b*f)(y) = \sum_{y'} b(y,y')f(y')$$

An element b of $C_c(Y\times_X Y)$ is a function $b(y,y')$ on $\{(y,y') \mid y-y'\in\mathbb{Z}\}$, and it operates on $C_c(Y)$ by

$$(bf)(y) = \sum_{y'\in y+\mathbb{Z}} b(y,y')f(y')$$

Basically you have $\pi : Y \longrightarrow X$, $\mathbb{R} \quad \mathbb{R}/\mathbb{Z}$ have

$E = C_c(Y)$. So given $b \in C_c(Y \times_X Y)$, $b(y,y')$

$f \in E$, then $pr_{1*}\, b\, pr_2^* f = pr_{1*}\big(b(y,y')f(y')\big)$

$$= \sum_{y' \in y \in \mathbb{Z}} b(y,y')f(y')$$

Take $g, h \in C_c(Y)$. $\quad b(y,y') = g(y)h(y')$

then
$$(bf)(y) = \sum_{y' \in y+\mathbb{Z}} g(y)h(y')f(y')$$
$$= g(y) \sum_n h(y'+n)f(y'+n)$$

$$= g\langle h, f\rangle$$

~~You seem to have found something namely that a special case of the fact~~

projection p? Choose $h_0 \in Y$ so that $\pi_* h_0 = 1$

$\pi : Y \longrightarrow X$ principal Γ-bundle with X compact.

$$\begin{tikzcd}
Y \times_X Y \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & Y \arrow[d, "\pi"] \\
Y \arrow[r, "\pi"] & X
\end{tikzcd}$$

~~Comm. rings of functions~~

functions $b \in C_c(Y \times_X Y)$ become correspondences

$(b * f) = pr_{1*}\, b\, pr_2^* f$

basic example $\quad b = pr_1^*(g)\, pr_2^*(h)$

$$pr_{1*}\, pr_1^*(g)\, pr_2^*(h)\, p_2^* f = g\, \pi_*(hf)$$

Is there a trace on this alg? candidate

$tr(b) = \pi_* \Delta^* b = \sum_n' b(y+n, y+n) \in C(X)$.

If $b = pr_1^* g \, pr_2^* h$, then $\Delta^* b = gh$, so

$tr(b) = \pi_*(gh) = \langle g, h \rangle$. This seems to work nicely.

Now look at p. p is an idemp in the ring of correspondences $B = C_c(Y \times_X Y)$ whose image is should be $\mathcal{E} = C_c(Y)$. p is constructed from an elt $h_0 \in \mathcal{E}$ such that $\pi_*(h_0) = 1$.

$b = pr_1^*(h_0)$, then $pr_{1*} \, pr_2^*(h_0) \, pr_2^*(f)$

$B = C_c(Y \times_X Y)$ acts on $\mathcal{E} = C_c(Y)$

You have $pr_{1*} : B \longrightarrow \mathcal{E}$

One idea so far ~~would be is to~~ is to compare $B = \mathcal{E} \rtimes \Gamma$ with $\mathcal{E}$, there seems to be a map fs to f. of left B-modules.

$A \rtimes \Gamma = \bigoplus_{s \in \Gamma} As$ ~~seems to~~

$(A \rtimes \Gamma) \otimes_{\mathbb{C}[\Gamma]} \mathbb{C}$ still

$$(a_1 t)(a_2 s) \longmapsto (a_1 t) * a_2 = a_1 \, {}^{t}a_2$$

$$\parallel$$

$$a_1 \, {}^{t}a_2 \, ts \longmapsto a_1 \, {}^{t}a_2$$

If this correct $pr_{1*} : B \longrightarrow \mathcal{E}$ should be a left B-module map. $b_1 = g_1 \otimes h_1$, $b_2 = g_2 \otimes h_2$

$$b_1 b_2 = g_1 \otimes \langle h_1, g_2 \rangle h_2 \qquad pr_{1*}(g_2 \otimes h_2) = g_2 \pi_*(h_2)$$

$$pr_{1*}(b_1 b_2) = g_1 \underbrace{\pi_*(\langle h_1, g_2 \rangle h_2)}_{\langle h_1, g_2 \rangle \pi_*(h_2)} \qquad b_1 pr_{1*}(b_2) = g_1 \underbrace{\langle h_1, g_2 \pi_*(h_2) \rangle}_{\langle h_1, g_2 \rangle \pi_*(h_2)}$$

projection $p \in B$? $pr_{1*} : B \longrightarrow \mathcal{E}$ Can you produce a B module section of pr_{1*}? Try multiplying.

$$f \longmapsto pr_1^*(f)\, pr_2^*(h_0) \qquad f(y)\, h_0(y')$$

Be more intelligent $B = C_c(Y \times_X Y)$ $C_c(Y)$

$$\mathcal{E} = C_c(Y)$$

$$C_c(Y \times_X Y) \xleftarrow{\sim} C_c(Y) \otimes_{C(X)} C_c(Y)$$

left + right B-module isom.

So if you want a left-B module maps ~~from~~ between $B = C_c(Y \times_X Y)$ to $\mathcal{E} = C_c(Y)$, you need $C(X)$ module maps between $C_c(Y)$ and $C(X)$, and this is easy. $\pi_*(h_0) = 1$.

But in the case of $({}_A P_A, P)$ with A unital and $\langle Q, P \rangle = A$ you get a projection over $B = P \otimes_A Q$ from a choice of $1 = \sum \langle q_i, p_i \rangle$. This gives embedding of a summand of a free module

What am I going to do? ~~all you~~ Pick $p_0, q_0 \in \mathcal{E}$ such that $\pi_x(q_0 p_0) = 1$, and then $b = p_0 \otimes q_0$ ~~is a projecto~~ should be idempotent in B.

Summarize: Aim to understand Cuntz's Durham talk especially in the case $\Gamma = \mathbb{Z}$, $F = \{-1, 0, 1\}$. There seems to be an explicit Morita equivalence on the algebraic level to be detailed. Made explicit.

You've tried various ideas

First look at Cuntz discussion for Γ, F

$\mathcal{E} = \mathcal{E}_{\Sigma_F}$ univ. alg | gen by h_s $s \in \Gamma$
rels $h_s h_t = 0$ $s^{-1}t \notin F$

$$h_t = \sum_s h_s h_t = \sum_s h_t h_s$$ (found that one of these is is enough)

Γ action $s * h_t = h_{st}$

Can form $B = \mathcal{E} \rtimes \Gamma$ and construct p

(Yesterday you noticed non-canonical character of p, but today you seem to have an explanation)

What's missing in this picture is the algebra A corresponding to the functions on the base. A might be P_F or pBp. P_F is not unital like A in the geometric case, or like pBp.

Start by trying to find a version of $C_c(Y \times_X Y)$

You have $\mathcal{E} = \mathcal{E}_{\Sigma_F}$ corresp. to $C_c(Y)$. Now it should be true in the geometric situation that

$$C_c(Y \times_X Y) = C_c(Y) \rtimes \Gamma$$

$$Y \times_X Y \xleftarrow{\sim} Y \times \Gamma$$
$$(y, ys) \qquad (y, s)$$

The question is whether Cuntz's noncomm. model fits the geometric picture.

$\mathcal{E} = \mathcal{E}_{\Sigma_F}$ gen. h_s $\qquad t * h_s = h_{ts}$

Another point is that the geom. realization $|\Sigma_F|$ is the space of finite ~~probably~~ probability measures as Γ support has "width". This might be relevant to the GNS discussion.

In the n.c. theory you form $\mathcal{E} \rtimes \Gamma$ which is the analog of $B = C_c(Y \times_X Y) = C_c(Y \times \Gamma) = C_c(Y) \otimes \mathbb{C}[\Gamma]$.
You also have this projector p given by a formula (in the crossproduct alg. B)

$$p = \sum h_1^{1/2} s h_1^{1/2}$$

There is a lot to prove. For example is Bp isomorphic as B-module to $\mathcal{E}$? ~~This seems to be~~

Let's review the geom. situation where you have established the Morita equivalence, so

$$\begin{array}{ccc} B = C_c(Y \times_X Y) & & C_c(Y) \\ \downarrow \pi_* & & \downarrow \pi_* \\ C_c(Y) & & C(X) \end{array}$$

$$\left| B = P \otimes_A Q \right| \qquad \begin{array}{cc} & P \\ Q & A \end{array}$$

Discuss again. In the case of a principal bundle $Y \xrightarrow{\pi} X$ with group Γ, you establish a Morita equiv between $B = C_c(Y \times_X Y) = C_c(Y) \rtimes \Gamma$ and $C(X)$ by means of the B-module $C_c(Y)$.

You want to modify the argument so as to treat the case of $\mathcal{E} = \mathcal{E}_{\Sigma_F}$ as $B = \mathcal{E} \rtimes \Gamma$-module. What you would like is to [redacted] show Cuntz's projection p in B has the appropriate properties:

1) $Bp \simeq \mathcal{E}$ 2) $\begin{pmatrix} B & Bp \\ pB & pBp \end{pmatrix}$ is form Morita context

i.e. $Bp \otimes_{pBp} pB \xrightarrow{\sim} B$, 3) $pBp = P_F$

You had problems with this.

Look carefully at $\mathcal{E}$ as B-module. $B = \mathcal{E} \rtimes \Gamma$

It looks like $B \otimes_\Gamma \mathbb{C} = \mathcal{E}$.

$$P \otimes_A Q = C_c(Y \times_X Y) \longleftarrow \cdots\cdots C_c(Y) = Q$$

$$P = C_c(Y) \qquad\qquad C(X) = A$$

Assuming you have a Morita equivalence, then

$$\mathrm{Hom}_B(P \otimes_A M, P \otimes_A N) = \mathrm{Hom}_A(M, N)$$

$$\therefore \quad \mathrm{Hom}_B(\underbrace{P \otimes_A Q}_{B}, \underbrace{P \otimes_A A}_{P}) = \mathrm{Hom}_A(Q, A)$$

How much do you understand? Take simplest case

$F = \{1\}$. $B = \mathbb{C}[\hat{\Gamma}] \rtimes \Gamma = \bigoplus_{s,t} \mathbb{C}e_s \otimes \mathbb{C}t$

You want the corresp. picture $B = C_c(\Gamma \times \Gamma)$

$$\sum_t b(s,t) f(t) = pr_{1*}\, b\, pr_2^* f$$ sum over Γ.

$$\sum_t g(s) h(t) f(t) = g \langle h, f \rangle$$

How do I proceed? In the geometric case you have B expressed as a tensor product.
$B = C_c(\Gamma\times\Gamma)$

In this case, what is p? Geometric

$$\begin{tikzcd} B = C_c(\Gamma\times\Gamma) \arrow[r, "pr_{2*}"] \arrow[d, "pr_{1*}"] & C_c(\Gamma) = Q \arrow[d, shift right, "\pi_*"'] \\ P = C_c(\Gamma) & \mathbb{C} = A \arrow[u, shift right, "h_1"'] \end{tikzcd}$$

It seems h_1 can be arb. in $C_c(\Gamma)$ such that $\pi_*(h_1) = 1$ and the corresp. map from P to B is

$$f \longmapsto f\otimes h_1$$

Use that $B = P\otimes_A Q = C_c(\Gamma)\otimes_{\mathbb{C}} C_c(\Gamma)$ is gen by $g\otimes h$ and that $pr_{1*} = id\otimes \pi_*$ $\sum_{s\in\Gamma}$

So the maps $B = C_c(\Gamma\times\Gamma)$ and $p = id\otimes h_1\pi_*$

$$\begin{tikzcd} B = C_c(\Gamma\times\Gamma) \arrow[d, shift right, "1\otimes\pi_*"'] \\ P = \mathcal{E} = C_c(\Gamma) \arrow[u, shift right, "1\otimes h_1"'] \end{tikzcd}$$

$p(g\otimes h) = g\otimes h_1\pi_* h$

or $p(g\otimes h) = g\otimes h_1\pi_* h = (id\otimes h_1\pi_*)(g\otimes h)$

So the proj operator on B is $P\otimes (h_1\pi_*)$ on $B = P\otimes_A Q$

Now go back to ~~...~~? this I recognize as $\sum_{s\in\Gamma} h_s$

~~...~~ but something is wrong ??

Anyway the next point ~~should involve ... nothing~~ to discuss is how to get $B = \mathcal{E}\rtimes\Gamma$ in the form $P\otimes_A Q$. Your idea is to show $\mathcal{E} = Bp = (P)$
$Q = \mathcal{E}^\vee = pB$ and then $A = pBp$

But this leads to a unital $\mathbf{A}$??

Look at $$p = \sum_{s\in\Gamma} h_1^{1/2}\, s\, h_1^{1/2} = \sum_{s\in\Gamma} \underbrace{h_1^{1/2} h_s^{1/2}}_{=\,0}\, s \in \mathcal{E}\rtimes\Gamma = B$$ for $s\notin F$.

Return to geometric situation

$$\begin{array}{l} C_c(Y\times_X Y) = C_c(Y\times\Gamma) = C_c(Y)\otimes\mathbb{C}[\Gamma] \\ \big\downarrow pr_{1*} \\ C_c(Y) \end{array}$$

Let $b(y,y')\in B$

$$pr_{1*}(b)(y) = \sum_{y'\in\pi^{-1}\{y\}} b(y, y')$$

Program. You have this potential proof of Morita inv: $C_c(Y)\rtimes\Gamma \sim C(X)$ in the geometric case. Review: By glueing you have

$$B = C_c(Y\times_X Y) \xleftarrow{\sim} C_c(Y)\otimes_{C(X)} C_c(Y) = P\otimes_A Q$$

$$pr_1^*\otimes pr_2^* \longleftarrow g\otimes h$$

you have the pairing $\langle h,g\rangle = \pi_*(gh)$ this isom takes the rank 1 product on $P\otimes_A Q$ into the correspondence product

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Repeat: In the geom. case you have a potential proof of Morita equivalence between $B=C_c(Y)\rtimes\Gamma$ and $A=C(X)$ as follows. Let $P=Q=C_c(Y)$ with natural A-module structures, let $\langle\ ,\ \rangle: Q\otimes P\longrightarrow A$ be $\langle h,g\rangle = \pi_*(hg)$, where $\pi_*: C_c(Y)\longrightarrow C(X)$ sums the Γ translates of a compactly supported function to get a periodic function. Claim one has an isom $P\otimes_A Q\longrightarrow B$

$$C_c(Y)\otimes_{C(X)} C_c(Y) \xrightarrow{\sim} C_c(Y\times_X Y), \quad g\otimes h\mapsto pr_1^*(g)\,pr_2^*(h)$$

which one should be to able to establish by glueing. This iso identifies product on $P\otimes_A Q$ with the product of correspondences.

The remaining point is to identify for

$$C_c(Y\times_X Y) \simeq C_c(Y\times\Gamma) = C_c(Y)\otimes\mathbb{C}[\Gamma]$$

the correspondence product with the crossproduct.

Work this out a bit. Our description of $C_c(Y\times_X Y)$ uses kernels $b(y,y')$. First you want to understand the trivial bundle case, say $X=pt.$ $Y=\Gamma$ you need to chose left or right. $Y\times Y=\Gamma\times\Gamma$ with diagonal action. So we have $\mathbb{C}$-valued comp. supp fns. $b(s,t)$

$$(b_1 * b_2)(s,u) = \sum_t b_1(s,t)\, b_2(t,u)$$

$$s^{-1}u = s^{-1}t\, t^{-1}u$$

Start from the kernel. $f(y_1,y_2)$ $\quad sy_1 = y_2$

Given $b(s,t)$ you want to split it into functions supported on the sets $s^{-1}t=u$ for each u.

$$\Gamma\times\Gamma = \coprod_{u\in\Gamma} \Delta(1,u) \qquad s(1,u) = (s,\overset{t}{\overline{su}})$$

$$Y\times_X Y = \coprod_{u\in\Gamma} \{(y,y') \mid y'=yu\}$$

$$Y\times_X Y = \coprod_{u\in\Gamma} \{y,yu \mid y\in Y\} \qquad b_t(y,yt)\, b_u(yt,ytu)$$

$$C_c(Y\times_X Y) = \bigoplus_t C_c(Y(1,t))$$

$B = C_c(\Gamma \times \Gamma)$ = space of kernels, composition

$$(b_1 * b_2)(s,t) = \sum_{u\in\Gamma} b_1(s,u)\, b_2(u,t)$$

this composition, or product leads to a Γ-grading of the algebra B, how? look at how differences multiply

$s^{-1}t = (s^{-1}u)(u^{-1}t)$, so put

$$B_u = \{ b \in B \mid b(s,t) = 0 \text{ for } s^{-1}t \neq u \}$$

i.e. b supported on $\{(s,su) \mid s\in\Gamma\}$
$\Delta(1,u)$.

$$B_u B_v \overset{?}{\subseteq} B_{uv}$$

~~Let $b_1 \in B_u$ i.e. $b_1(s$~~

Let $b_1 \in B_u$ $\quad b_1(s,t) \neq 0 \implies s^{-1}t = u$

$b_2 \in B_v$ $\quad b_2(s_1,t_1) \neq 0 \implies s_1^{-1}t_1 = v$

$$\underbrace{b_1(s,w)}_{\neq 0}\ \underbrace{b_2(w,t)}_{\neq 0}$$

$s^{-1}w = u$
$w^{-1}t = v$
$\implies s^{-1}t = uv$

other difference $\quad st^{-1} \overset{?}{=} su^{-1}ut^{-1}$ $\quad$ Yes. $\quad$ OK

$$(b_1 * g)(s) = \sum_{t\in\Gamma} b_1(s,t)\, g(t)$$

Repeat the preceding.

$$B = C_c(\Gamma\times\Gamma) \xleftarrow{\sim} C_c(\Gamma)\otimes C_c(\Gamma)$$

$pr_1^*(g)\, pr_2^*(h) \qquad g \otimes h$

$\qquad \|$

$g(s)\, h(t)$

the action on $C_c(\Gamma)$ by $b \in B$ is the op. $pr_{1*}\, b\, pr_2^*$

$$pr_{1*}\{b\, pr_2^* f\} = pr_{1*}\{ b(s,t) f(t)\} = \sum_t b(s,t) f(t)$$

~~You are now using the standard~~

$$B = C_c(Y \times_X Y) \xleftarrow{\sim} C_c(Y) \otimes_{C(X)} C_c(Y)$$ the geometric Morita equivalence

~~First~~ Next step is to express B as the cross product alg $C_c(Y) \rtimes \Gamma$.

$$Y \times_X Y \xleftarrow{\sim} Y \times \Gamma$$
$$(y, sy \text{ or } ys) \qquad (y, s)$$

$$\begin{array}{ccc} & \Gamma \times \Gamma & \\ \swarrow pr_1 & & \searrow pr_2 \\ \Gamma & & \Gamma \end{array} \qquad pr_2^* f, \quad f$$

Go over again

$B = C_c(\Gamma \times \Gamma)$

acting on $E = C_c(\Gamma)$ by $(bf)(s) = \sum_t b(s,t) f(t)$

~~Whats so important~~

composition $(b_1 b_2)(s,u) = \sum_t b_1(s,t)\, b_2(t,u)$ think of [redacted] $f(t)$ as being $(pr_2^*(f))(s,t) = f(t)$. Independent of ~~the~~

So what is your idea? ~~Basically you have 2 ...~~

Repeat: In the geometric sit $\Gamma \to Y \xrightarrow{\pi} X$ X comp.

$$B = C_c(Y \times_X Y) \xleftarrow{\sim} C_c(Y) \otimes_{C(X)} C_c(Y) = P \otimes_A Q$$

the correspondence type, or matrix product in B is [redacted] ~~... matrix~~ given by $\langle h, g \rangle = \pi_*(hg)$

Begin again: Geom. situation $\Gamma \to Y \xrightarrow{\pi} X$ comp. One has

$$B = C_c(Y \times_X Y) \xleftarrow{\sim} C_c(Y) \otimes_{C(X)} C_c(Y) = P \otimes_A Q$$

B-action on $P = C_c(Y)$: $(b * f)(s) = \sum_t b(s,t) f(t)$

$$\begin{array}{ccc} & \Gamma \times \Gamma & \\ pr_1 \swarrow & & \searrow pr_2 \\ \Gamma & & \Gamma \end{array}$$

$$b * f = pr_{1*}(b \; pr_2^* f)$$

$$(g \otimes h) * f = pr_{1*}\{ pr_1^* g \; pr_2^*(hf)\} = g\, \overbrace{\pi_*(hf)}^{\langle h, f\rangle}$$

To prove M.eq you need $\langle Q, P \rangle = A$, find $f, g \in C_c(Y)$ so that $\langle g | f \rangle = \pi_*(gf) = 1$, then $f \otimes g \in B$ is an idemp.

Actually you know that $A \xrightarrow{f} Q \xrightarrow{\pi_*(-g)} A \Rightarrow P \to B \to P$

giving a proj ~~p~~ in B whose right ideal Bp $= C_c(Y)$. ~~...~~ ∴ geometric picture is clear ~~~~ using kernels. ~~But I you need the group Γ~~

POINT The group Γ did not enter above ~~so you have no compactness there~~ so the construction might extend to $B = C_c(Y \times_X Z) \cong C_c(Y) \otimes_A C_c(Z)$ for two covering spaces of X, to give a Morita equiv. $B \sim A$.

Next project is handle the ~~...~~ transition from $C_c(Y \times_X Y)$ to $C_c(Y) \rtimes \Gamma$, which should be isomorphic in a nice way. ~~If you get dz little.~~

O.K. you need to pass from $b(s,t)$

$$Y \times_X Y \xleftarrow{\Delta} Y$$

given, ~~so you use it~~ this yields the degree ~~0~~ component.

$$\begin{array}{ccc} Y \times_X Y & \longleftarrow & Y \times \Gamma \\ (y, ys) & \longleftarrow\!\shortmid & (y, s) \end{array} \qquad \underbrace{C_c(Y \times Y)}_{\text{acted on by } \Gamma \times \Gamma} = C_c(Y) \otimes \mathbb{C}[\Gamma] \quad \text{~~right and left.~~}$$

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ideas: [illegible] This assembly stuff which gives you a [illegible] fibre bundle over X with fibre the group ring should be related to the complex of chains on the cover space Y. You might [illegible] try to relate this to the Novikov conjecture. Maybe to understand Andrew's ~~...~~ proofs for topological invariance | Wall obstruction finiteness Pontryagin classes.

Let us return to $\mathcal{E} \rtimes \Gamma$ gen. by ~~...~~ $h_s = s h_1 s^{-1}$ $s \in \Gamma$ relations $h_s h_t = 0$ $s^{-1}t \notin F$. $h_1 = \sum_s h_s h_1 = \sum_s h_1 h_s$

Take Hilbert space viewpoint, look at a Hilbert space H with Γ action and $h_1 \geq 0$ satisfying relations above and $\sum s h_1 H$ dense in H. What does GNS say.

Consider special case $\Gamma = \mathbb{Z}$ $F = \{-1, 0, 1\}$.
H is a Hilbert space rep of $\mathbb{Z}$ i.e. u unitary
$h_0 \geq 0$ such that $h_0 u^n h_0 = 0$ for $|n| \geq 2$.

$\overline{\sum_n u^n h_0 H} = H$, $\sum_{n \in \mathbb{Z}} h_n \xi = \xi$ $\forall \xi \in H$.

To be as precise as possible about this situation h_0 is a pos. ~~herm.~~ operator, so $h_0^{1/2}$ is defined, and the subspace $\overline{h_0 H} = \overline{h_0^{1/2} H}$. What is your aim? to reconstruct H, u as simply as possible. This means understanding the operators you have on H. At the moment you have $u^n \in \Gamma$, $h_n = u^n h_0 u^{-n}$, relations

$h_0 h_n = h_0 u^n h_0 u^{-n} = 0 \iff h_0 u^n h_0 = 0$ for $n \geq 2$. You want to reconstruct H. What do you need to reconstruct the subspace $\overline{h_0 H}$? Here you need to be precise. In general if you give a subset I of a Hilbert space H such that $\bigoplus_{\xi \in I} \mathbb{C}\xi$ is dense in H, then H can be constructed from all the scalar prod. (ξ, η), and a set of numbers $\rho(\xi, \eta)$ occur exactly when (ξ_i, η_j) is ≥ 0. for any finite subset of I. You are very close I think. There's a completion process. Simplify – suppose $\Gamma = \{1\}$. and h_0 is an op. on H $\ni$ $h_0 \geq 0$ and $\overline{h_0 H} = H$. Given $h_0 \xi_i \in h_0 H$, $i = 1, \ldots, n$ you have $(h_0 \xi_i, h_0 \xi_j) = (\xi_i, h_0^2 \xi_j)$

Repeat. Given a representation of $\mathbb{Z}$ on a Hilbert space H and an operator $h_0 \geq 0$ on H such that $h_0 h_n = 0$ $|n| \geq 2$

$\sum_n h_n h_0 = h_0$, $\overline{\sum_n u^n h_0 H} = H$. These should imply $\sum h_n = 1$ on H.

Try again to focus. H, u unitary, $h_0 \geq 0$ such if $h_n = u^n h_0 u^{-n}$, then $h_0 h_n = 0$ $|n| \geq 2$, $\sum_n h_n h_0 = h_0$ and $\overline{\sum h_n H} = H$. $\sum u^n h_0 H$, so H is generated by the image $h_0 H$, also $\sum h_n = 1$ on H. This is all ~~straightforward and~~ clear. Now you want to construct H from $h_0 H$. You form ~~linear com~~ finite sums $\sum_n u^n \xi_n$ $\xi_n \in h_0 H$. Need $(\ ,\ )$ which is determined by $n \mapsto (\xi, u^n \xi)$. Because $h_0 = h_0$ one knows that $\mathrm{Ker}(h_0) = (h_0 H)^\perp$ so $h_0 u^n h_0 = 0$ means $u^n h_0 H \perp h_0 H$. So you end up with a family of closed subspaces $u^n \overline{h_0 H}$ $n \in \mathbb{Z}$.

Question. Is the Hilbert space representation H of $\mathbb{Z}$ a subspace of a "free" repentation – the orth. direct sum $\bigoplus_n u^n V$?

Consider H Hilbert sp, u unitary, $h_0 \geq 0$ such that if $h_n = u^n h_0 u^{-n}$, then $h_m h_n = 0$ $|m-n| \geq 2$, $\sum h_n = 1$, $\overline{\sum_n h_n H} = H$. Call this a $\mathbb{Z}$-equiv. partition of unity. Let $p = \sum_n h_0^{1/2} u^n h_0^{1/2}$. Then $p^2 = p = p^*$ on H, $upu^{-1} = p$. What is the meaning of p?

Where is the $\mathbb{Z}$-graded projection? Before when you looked at this you found $\sum z^{-n} p_n$ where $p_n = h_0^{1/2} u^n h_0^{1/2}$. So you ~~probably~~ have the wrong formula for p.

Goal: to understand ~~well~~ the Hilbert space version for $\Gamma = \mathbb{Z}$, $F = \{-1,0,1\}$. Consider a Hilb. space rep H of $\mathcal{E}_{\Sigma_F} \rtimes \mathbb{Z}$. This is a non unital alg. ~~so it~~ so you assume that $H = \overline{\sum h_n H}$ which should guarantee that you have a unitary repn of $\mathbb{Z}$ on H. This should be made clearer, but algebraically you know that $\sum h_n = id$ on $\bigoplus h_n H_0$, etc...

To proceed it's simplest to assume u, h_0 given on H with the desired properties: u unitary, $h_0 \geq 0$, if $h_m h_n = 0$, $|m-n|>1$, $\sum h_n = 1$. Then $h_0 H$ generates H as $\mathbb{Z}$-module, H is a completion of $\mathbb{C}[u,u^{-1}] \otimes_{\mathbb{C}} h_0 H$ with respect to scalar product determined by a function on the group $u^n \longmapsto h_0 u^n h_0$? Maybe you should introduce $h_0^{1/2}$ to get the best form. How to do this? Look at ≥ 0 forms. ~~Better~~

Picture H as built from $u^n h_0 H$. It's probably quicker to

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~~[illegible]~~ for each we have h_n a hermitian operator ≥ 0 on H, better a ~~non-neg~~ hermitian form $(\xi, h_n \xi) \geq 0$, and the sum is the ~~identity~~ scalar product $\sum_n (\xi, h_n \xi) = (\xi, \xi)$. So repeat.

H Hilbert space with u unitary and $h_0 \geq 0$ satisfying

$$\sum (u^{-n}\xi, h_0 u^{-n}\xi) = \|\xi\|^2$$

$$\sum (\xi, h_n \xi) = \|\xi\|^2 \qquad \forall \xi \in H$$

and the ~~[illegible]~~ F condition $\left(h_m^{\frac{1}{2}}\xi', h_n^{1/2}\xi\right) = 0$ for $|m-n| \geq 2$.

~~So what else is g~~ What next? $\xi \longmapsto (h_n^{1/2}\xi)$

So what to do?

Try: How much further to go? What do you want to accomplish? You have this partition of unity notion namely operators $h_n \geq 0$ with $\sum_n h_n = 1$. To relate to geometric partitions of unity you want $\forall n$ the set of $m \ni h_n h_m \neq 0$ is finite. ~~[illegible]~~ Next introduce the positive sqrts $h_n^{1/2}$. Then $\xi \longmapsto (h_n^{1/2}\xi)_{n\in\mathbb{Z}} \in \overset{(2)}{\bigoplus_{n\in\mathbb{Z}}} H$ is isometric $\sum_n \| h_n^{1/2}\xi\|^2 = \sum_n (\xi, h_n \xi) = \|\xi\|^2$. Also the image of H lies in $\overset{(2)}{\bigoplus_n} \overline{h_n^{1/2}H}$ same a $\overline{h_n H}$. Life goes on.

Put the group into the picture. ~~[illegible]~~ What remains? You need to understand the graded projection $p = \sum h_0^{1/2} u^n h_0^{1/2}$ You have to understand the graded projection. You have a picture in the $\mathbb{Z}$ case. Basically you ~~are taking~~ want to use GNS. GNS tells you that H can be reconstructed from $\overline{h_0^{1/2}H} = V$ and the function $n \longmapsto h_0^{1/2} u^n h_0^{1/2}$, $\mathbb{Z} \longrightarrow \mathcal{L}(V)$ which is a completely pos. fn. on $\mathbb{Z}$. ~~In fact what happens~~

~~is~~ Things are special because of the condition $\sum h_n = 1$.

Take the $\mathbb{Z}$-equivariant situation. You have above these emb. $H \hookrightarrow L^2(S^1, V)$ which commutes with u on both sides, ~~so you get~~ which is isometric, get $P =$ orth projection op on $L^2(S^1, V)$ with image H. and $P =$ mult by $p(z)$.

~~[illegible]~~

Move to a general Γ, so H is a unitary repn of the disc. group Γ, given $h_1 \geq 0$ such that $\sum_s \underbrace{s h_1 H}_{h_s H} = H$, also $h_s h_t = 0$ $s^{-1}t \notin F$. $\widetilde{s h_1 s^{-1}}$

Actual you want $\sum_s (\xi, \widetilde{h_s}\xi) = \|\xi\|^2$, and then you get $\sum_s \| h_1^{1/2} s^{-1}\xi\|^2 = \|\xi\|^2$, should get isometry

$$H \hookrightarrow \bigoplus \otimes h_s^{\frac{1}{2}} H \qquad ?$$

On H you have operator $s \in \Gamma$ unity, $h_s \geq 0$

such that $s h_1 s^{-1} = h_s$ $\quad h_s h_t = 0 \quad s^{-1}t \in F \quad \sum_s (\xi, h_s \xi) = \|\xi\|^2$

isom. embedding Γ-equivariant

$$H \xrightarrow{(h_s^{1/2})_s} \bigoplus_{s \in L}^{(2)} \underbrace{V_s}_{} \qquad V_s = \overline{h_s H} = \overline{h_s^{1/2} H}$$

$$\begin{tikzcd} H \arrow[rr, "h_s^{1/2}"] \arrow[dr, "h_1^{1/2}"'] & & V_s \\ & V_1 \arrow[ur, "s"'] & \end{tikzcd}$$

$$V_s = \overline{h_s H} = \overline{s h_1 \underbrace{s^{-1} H}_{}} = s\,\overline{h_1 H}$$

Then you have this ρ on $\bigoplus_{s \in L}^{(2)} V_s$

Γ equivariant

$$\begin{tikzcd} H \arrow[r, "h_s^{1/2}"] & V_s \\ H \arrow[u, "s"] \arrow[r, "h_1^{1/2}"] & V_1 \arrow[u, "s"] \end{tikzcd}$$

$$H \overset{\alpha}{\hookrightarrow} \bigoplus_{s}^{(2)} sV_1 \overset{\beta}{\twoheadrightarrow} H$$

H Hilbert space rep of Γ, $h_1 \geq 0$ such that

$\sum h_s = 1 \qquad h_s = s h_1 s^{-1} \qquad$ means $\sum_s (\xi, h_s \xi) = \|\xi\|^2$

$\sum_s \|h_s^{1/2} \xi\|^2 = \|\xi\|^2$ get isometry $\xi \longmapsto (h_s^{1/2} \xi)_s$

in $\bigoplus_{\Gamma}^{(2)} \underbrace{\overline{h_s^{1/2} H}}_{V_s = sV_1}$ also Γ-equivariant. $\qquad \overline{h_s^{1/2} H} = s \underbrace{\overline{h_1^{1/2} H}}_{V_1}$

$$\begin{tikzcd} H \arrow[r, "h_s^{1/2}"] \arrow[d, "t"] & sV_1 \arrow[d, "t"] \\ H \arrow[r, "h_{ts}^{1/2}"] & tsV_1 \end{tikzcd} \qquad t h_s^{1/2} t^{-1} = h_{ts}^{1/2}$$

$$H \xrightarrow{\alpha} \bigoplus_{\Gamma}^{(2)} V_s \xrightarrow{\beta = \alpha^*} H$$

so what do you know? $\qquad e_s$ projection on V_s

$$id_H = \alpha^* \left(\sum e_s\right) \alpha \qquad e_s = s e_1 s^{-1}$$

~~Let~~ $e_s\alpha = h_s^{1/2}$, $\alpha^* e_s = h_s^{1/2}$

$\alpha^* e_s \alpha = h_s$. The other point is the projection

$p = \alpha\alpha^*$ on $\overset{(2)}{\bigoplus}_{s\in\Gamma} sV_1$. Want

$$\begin{tikzcd} V_1 \arrow[r, hook] \arrow[rr, bend right, "h_1^{1/2}"'] & \overset{(2)}{\bigoplus} V_t \arrow[r, "\alpha^*"] & H \arrow[r, hook, "\alpha"] & \overset{(2)}{\bigoplus} V_s \arrow[l, bend left, "h_s^{1/2}"] \end{tikzcd}$$

You want to calculate the projection p

$$\bigoplus V_t \xrightarrow{\alpha^*} H \xrightarrow{\alpha} \bigoplus V_s$$

you must be careful about t action on $\bigoplus V_s$ – it combines index shift +

Γ-equivariant maps.

$$\begin{array}{ccc} \xi & \longmapsto & (h_s^{1/2}\xi)_s \mapsto t h_s^{1/2}\xi \\ \downarrow & & \\ t\xi & \longmapsto & (h_s^{1/2} t\xi)_s \end{array}$$

Repeat. H, Γ, $h_1 \geq 0$, $h_s = s h_1 s^{-1}$, $\sum h_s = 1$ in the sense of ≥ 0 herm. forms. Get $H \xrightarrow{(h_s^{1/2})} \overset{(2)}{\bigoplus}_s \underbrace{\overline{h_s^{1/2} H}}$ $V_s = sV_1$

$$\begin{array}{ccc} H & \xrightarrow{h_s^{1/2}} & V_s \\ t\downarrow & & \downarrow t \\ H & \xrightarrow{h_{ts}^{1/2}} & V_{ts} \end{array}$$

$t h_s^{1/2} = h_{ts}^{1/2} t$

$h_s^{1/2}\xi \in V_s$

$h_{ts}^{1/2} t\xi$ $\quad$ $t h_s^{1/2}\xi \in tV_s = V_{ts}$

Perhaps the way to handle this is to ~~take gilch~~ form $\overset{(2)}{\bigoplus}_s H\otimes s = H \otimes \mathbb{C}[\Gamma]$

There's a technical point to get straight relating to induced and coinduced modules

$h_1^{1/2} : V \longrightarrow H$ extends to Γ-equiv. $\bigoplus_s s\otimes V \longrightarrow H$

$$\begin{array}{ccc} s\otimes v & \longmapsto & s h_1^{1/2} v \\ t\downarrow & & t\downarrow \\ ts\otimes v & \longmapsto & ts h_1^{1/2} v \end{array}$$

and $h_1^{1/2} = \varphi : H \longrightarrow V$ coextends

$$H \longrightarrow \prod s\otimes V$$

$\varphi : H \longrightarrow V$ coextends to $\hat\varphi : H \longrightarrow \prod_{s\in\Gamma} s\otimes V$

$\hat\varphi : \xi \longmapsto \left(s \longmapsto s\otimes\varphi(s^{-1}\xi)\right)$

or $\hat\varphi(\xi)_s = s\otimes\varphi(s^{-1}\xi)$ check equivariant

$\hat\varphi(t\xi) = \left(s \longmapsto s\otimes\varphi(s^{-1}t\xi)\right)$

$\cancel{(t\hat\varphi(\xi))_s = (s \mapsto s\otimes t\varphi}$

$$\begin{array}{c} (s\otimes v_s)_{s\in S} \\ \downarrow t \\ (ts\otimes v_s)_{s\in S} \end{array}$$

H Γ-module, $\varphi : H \longrightarrow V$ linear ~~module m~~

$\mathrm{Hom}(H,V) = \mathrm{Hom}_\Gamma(H, \mathrm{Hom}(\mathbb{C}[\Gamma], V))$

$\varphi \longmapsto \left(\xi \longmapsto (s \longmapsto \varphi(s\xi))\right)$

In my situation $V = \overline{h_0^{1/2}H}$ and $\varphi = h_0^{1/2}$

then $\xi\in H$ goes to $s \longmapsto h_0^{1/2}s\xi$

$$\begin{tikzcd} H \arrow[r, "\hat\varphi"] \arrow[d, "t"'] & \prod_s V \arrow[d, "t=?"] \\ H & \prod_s V \end{tikzcd}$$

$\xi \longmapsto \hat\varphi(\xi) = \{s \mapsto \varphi(s\xi)\}$

$t\xi \longmapsto \hat\varphi(t\xi) = \{s \mapsto \varphi(st\xi)\}$.

$t(s \mapsto v_s) = (s \mapsto v_{st})$

Let's get the pieces together

$H \xrightarrow{h_s^{1/2}} V_s = sV_1$

$\overline{h_s^{1/2}H} = s\overline{h_1^{1/2}H} = sV_1$

$$\begin{tikzcd} H \arrow[r, "h_s^{1/2}"] \arrow[d, "t"'] & sV_1 \arrow[d, "t"] \\ H \arrow[r, "h_{ts}^{1/2}"] & tsV_1 \end{tikzcd}$$

$th_s^{1/2} = h_{ts}^{1/2}t$

You want to use $\bigoplus_s sV_1 \subset \bigoplus_s^{(2)} sV_1 \subset \prod_s sV_1$

basically you have $V_1 \xrightarrow{h_1^{1/2}} H$ extending to $\overline{h_s^{1/2}H}$

Γ-eq map $\bigoplus_s^{(2)} sV_1 \twoheadrightarrow H$ s-th comp. is $sV_1 = \overline{sh_1^{1/2}s^{-1}H}$

$\bigoplus_s^{(2)} h_s^{1/2}H$

Not clear yet.

Go back to $V_1 = \overline{h_1^{1/2}H}$ $V_s = sV_1 = \overline{h_s^{1/2}H}$

$$H \xrightarrow[\alpha]{\text{isometric}} \bigoplus_s^{(2)} \overline{h_s^{1/2}H} \xrightarrow{\beta} H$$

$\xi \longmapsto (s \mapsto h_s^{1/2}\xi)$

$\xi \longmapsto \alpha_s(\xi) = h_s^{\frac{1}{2}}\xi$

Let $\sum v_s \in \bigoplus h_s^{1/2}H$ $\quad \left(i(\xi), \sum v_s\right) = \sum_s \left(h_s^{1/2}\xi, v_s\right)$

$= \left(\xi, \sum_s h_s^{1/2} v_s\right)$

$\beta \sum v_s$

Better notation. Put $V_1 = \overline{h_1^{1/2}H}$ $\quad V_s = sV_1 = \overline{h_s^{1/2}H}$

$$H \xrightarrow{\alpha} \bigoplus_{s\in\Gamma}^{(2)} V_s \xrightarrow{\beta} H$$

$\xi \longmapsto (s \mapsto h_s^{1/2}\xi)$ $\qquad \beta\alpha = id_H$

$(s \mapsto v_s) \longmapsto \sum_s h_s^{1/2} v_s$

~~so what is your projection of.~~

α has components $\alpha_s = h_s^{1/2} : H \longrightarrow V_s = sV$

β ——————— $\beta_t = h_t^{1/2} : V_t \longrightarrow H$

$p = \alpha\beta$ has components

$\alpha_s\beta_t : V_t \longrightarrow H \longrightarrow V_s$ $\qquad$ by Γ invariance

$$\begin{tikzcd} V_t \arrow[r, "\beta_t"] \arrow[d, "u"'] & H \arrow[r, "\alpha_s"] \arrow[d, "u"] & V_s \arrow[d, "u"] \\ V_{ut} \arrow[r, "\beta_{ut}"] & H \arrow[r, "\alpha_{us}"] & V_{us} \end{tikzcd}$$

$u h_s^{1/2} = h_{us}^{1/2} u$

OK. $u \alpha_s \beta_t u^{-1} = \alpha_{us}\beta_{ut}$

still ~~so~~ confused. Go over again

H, Γ acts, $h_s \geq 0$ $\quad t h_s t^{-1} = h_{ts}$ $\quad \sum h_s = 1$.

$h_s = s h_1 s^{-1}$. $\quad V_s = \overline{h_s^{1/2} H} = \overline{s h_1^{1/2} s^{-1} H} = s V_1$

$$H \xrightarrow{\alpha} \bigoplus_s^{(2)} V_s \xrightarrow{\alpha^*} H \qquad \alpha(\xi) = (s \mapsto h_s^{1/2}\xi)$$

$$\sum_s \|h_s^{1/2}\xi\|^2 = \sum_s (\xi, h_s \xi) = \|\xi\|^2$$

α isometry.

$$\langle (s \mapsto \eta_s \in V_s), (s \mapsto h_s^{1/2}\xi) \rangle = \sum_s (\eta_s, h_s^{1/2}\xi) = \sum_s (h_s^{1/2}\eta_s, \xi)$$

$$\alpha^*(s \mapsto \eta_s) = \sum_s h_s^{1/2}\eta_s \qquad \alpha^*\alpha = \sum h_s = 1.$$

~~we want the invariance~~

equivariance $\quad t\alpha\xi = t(s \mapsto h_s^{1/2}\xi)$?

how does t act on $\bigoplus_s^{(2)} V_s$? $\quad$ system of imprimitivity

$s \mapsto \eta_s \in V_s \qquad t\eta_s \in V_{ts} \qquad (t\eta)_s \in V_s$

$\therefore (t\eta)_{ts} = t\eta_s \qquad \boxed{(t\eta)_s = t\eta_{t^{-1}s} \qquad t\eta = t\eta t^{-1}}$

check $\quad (t\alpha\xi)_s = t(\alpha\xi)_{t^{-1}s} = t h_{t^{-1}s}^{1/2}\xi = h_s^{1/2} t\xi$

$(\alpha t \xi)_s = h_s^{1/2} t\xi$ $\qquad$ You now want to convert all V_s to V_1 which should give ~~be~~ simpler formulas

$$\begin{tikzcd} H \arrow[r, "\alpha"] \arrow[dr, "\alpha'"'] & \bigoplus^{(2)}_s \underbrace{V_s}_{= sV_1} \arrow[d, "\theta"] \\ & \bigoplus^{(2)}_s V_1 \end{tikzcd} \qquad \begin{tikzcd} \xi \arrow[r, mapsto, "\alpha"] \arrow[dr, mapsto, "\alpha'"'] & (s \longmapsto h_s^{1/2}\xi) \arrow[d, mapsto] \\ & (s \longmapsto \underbrace{s^{-1}h_s^{1/2}\xi}_{h_1^{1/2}s^{-1}\xi}) \end{tikzcd}$$

$$\alpha'(\xi)_s = h_1^{1/2}s^{-1}\xi$$

check $\sum_s \| h_1^{1/2}s^{-1}\xi\|^2 = \sum_s (\xi, s^{+1}h_1^{1/2}h_1^{1/2}s^{-1}\xi) = \sum_s (\xi, h_s\xi)$

$$\begin{array}{ccc} (s \longmapsto \eta_s) & \in \bigoplus_s V_s = \bigoplus_s sV_1 \\ \Big\downarrow \theta & \\ (s \longmapsto s^{-1}\eta_s) & \in \bigoplus_s V_1 \end{array}$$

$\eta_s = (\alpha\xi)_s = h_s^{1/2}\xi$

$(\theta\alpha\xi)_s = s^{-1}h_s^{1/2}\xi$
$= h_1^{1/2}s^{-1}\xi$

t acting on $(s \longmapsto \eta_s) \in \bigoplus V_s$

is $s \longmapsto t\eta_{t^{-1}s}$

$(s \longmapsto \eta_s) \in \bigoplus_s sV_1$ $\qquad t(s \mapsto \eta_s) = (s \mapsto t\eta_{t^{-1}s})$

$\Big\downarrow \theta$

$(s \longmapsto \underbrace{s^{-1}\eta_s}_{\bar\eta_s}) \in \bigoplus_s V_1 \ni (s \longmapsto \underbrace{s^{-1}t\eta_{t^{-1}s}}_{\bar\eta_{t^{-1}s}})$

So apparently t acting on $\bigoplus_s V_1$ is $(t\bar\eta)_s = \bar\eta_{t^{-1}s}$

$$(t(\alpha'\xi))_s = (\alpha'\xi)_{t^{-1}s} = h_1^{1/2}s^{-1}t\xi = (\alpha'(t\xi))_s$$