scratch work, go back to the old problem, $S: S^1 \to S^1$ smooth of degree zero, write

$\log S = \sum c_n e^{in\theta}$ $\quad \overline{c_n} = -c_{-n}$, so have

$\log S = f - \bar f$ $\quad$ where $f = \frac{c_0}{2} + \sum_{n \geq 1} c_n e^{in\theta} +$ arb. real constant which we can ~~normalize~~ choose so that $\int |e^f|^2 \frac{d\theta}{2\pi} = 1$. Then $\log S = -f + \bar f$ $\quad S = \frac{e^{\bar f}}{e^f}$

~~Basic picture~~. $d\mu = \frac{1}{|g|^2} \frac{d\theta}{2\pi}$ $\quad S = \frac{\bar g}{g}$ where g analytic invertible.

Change departure. Focus on a simple case

$$S = \frac{1 - hz^{-1}}{1 - \bar h z} \qquad \rho = \frac{1}{|1 - \bar h z|^2} \qquad g = \frac{1}{1 - \bar h z}$$

Focus on the essentials. Where to begin? You should start with $g = e^f$ $\;f$ smooth on $\bar D$ analytic in D, eg. $f(z) = \sum_{n \geq 0} a_n z^n$ $\quad n^k |a_n| \to \infty$ $\forall k$
In fact if it helps you can take g analytic on $(1+\varepsilon)D$. So $g = e^f$ is the basic data and it is normalized so that $\int \frac{1}{|g|^2} \frac{d\theta}{2\pi} = 1$. Thus you get $\rho = \frac{1}{|g|^2}$, $S = \frac{\bar g}{g}$.

Now form $H = L^2(S^1, d\mu)$, sequences of polys p_n, q_n. Also can use $H = L^2(S^1) g$ or $H = L^2(S^1) \bar g$.
So you get picture - polys in numerator g in denominator.

Discuss more carefully. You begin with g and get a prob. measure $d\mu = \frac{1}{|g|^2} \frac{d\theta}{2\pi}$, whence an inner product on Laurent polys, whence an $H = L^2(S^1, d\mu)$ with bifilt. $F_{mn} = \langle z^{-m}, \ldots, z^n \rangle$, also your sequences p_n, q_n

Repeat. You begin with $q = e^f$ f smooth on S^1 extending analytically inside, normalized so that $\int \frac{1}{|q|^2}\frac{d\theta}{2\pi} = 1$. Then $d\mu = \frac{1}{|q|^2}\frac{d\theta}{2\pi}$ is a prob. meas whence you get $H = L^2(S^1, d\mu)$ with bifiltration $F_{mn} = \langle z^{-m}, \dots, z^n \rangle$, and orth. poly. system $\begin{pmatrix} p_n \\ q_n \end{pmatrix}$. You want to prove that $q_n \to q$ among other things.

Let's use the scattering picture. The scattering arises from $L^2(S^1) \xrightarrow{\sim} L^2(S^1, d\mu)$ and also from $f \mapsto f\bar{q}$.

$$f \longmapsto fq$$

It is ~~obv.~~ obviously isometric and since $\frac{1}{q} \in L^2(S^1)$ it is onto. Important point about q is that mult by q, $\bar{q}$ are invertible on H_+, H_- resp. Let's do some calculations.

Question. Suppose you change the boundary condition $p_0 = q_0$, i.e. you act by S^1 on h_0. How does S change? Where to start?

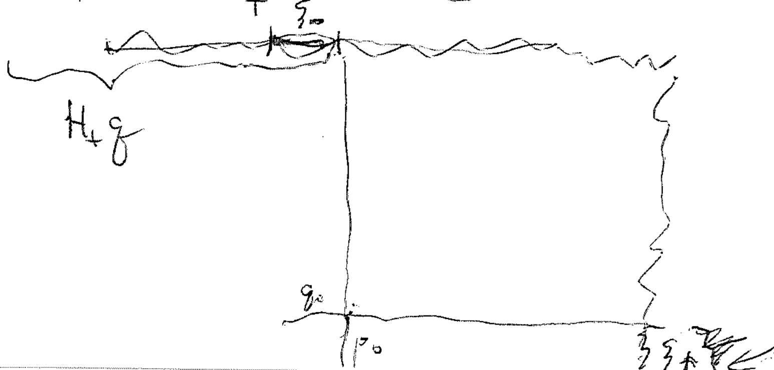
Partial unitary $aX = (\mathbb{C}\xi)^\perp$ $bX = (\mathbb{C}u\xi)^\perp$

$ua = b$ $\therefore$ $u = ba^*$ on aX. Have family of contractions

$$c_h = ba^* + u\xi h \xi^*.$$

There are lot's of things you can't do. Suppose S given, can you find q, using the fact that $H = H_+ \oplus SH_-$. Note this decomp fits with

$$\begin{array}{c} H_+ \oplus SH_- \\ \downarrow S \\ qH_+ \oplus \bar{q}H_- \end{array}$$



$\bar{q}H_-$

You need to get ~~beyond~~ around the obstruction

Given $g = e^f$ f analytic on $\bar{D}$ to simplify, you want to prove that the corresponding h sequence decays exponentially (?).

You normalize g so that $d\mu = \frac{1}{|g|^2}\frac{d\theta}{2\pi}$ is a prob. measure.

~~You have the following situation.~~

Construct following: Hilbert space $H = L^2(S^1, d\mu)$
$\xi_- = g$, $\xi_+ = \bar{g}$, forgot $u = z$. Also have $H^2_\pm(S^1, d\mu)$. Study this carefully.

What should be true?

$$\langle g \mid z^n \rangle = \int \bar{g}\, z^n \frac{1}{|g|^2}\frac{d\theta}{2\pi} = \int \frac{z^n}{g}\frac{d\theta}{2\pi}$$

$$= \begin{cases} 0 & n > 0 \\ \frac{1}{g(0)} & n = 0 \end{cases} \qquad \frac{z^n}{g}(0)$$

~~You can normalize~~ If you start with S and ~~then~~ write $-\log S = f - \bar{f}$ f analytic on $\bar{D}$ then f is unique up to an additive real constant. Also if you write $-\log g = f + \bar{f}$, then this f is unique up to an imaginary constant. So either S or g fix the non constant Taylor series coeff of f, and the constant term can be fixed by requiring $\frac{1}{g(0)} > 0$ $\int \frac{1}{|g|^2}\frac{d\theta}{2\pi} = 1$. This should not be significant to worry about at this stage.

What might be important is a difference between the measure case and the partial unitary situation.

Fix $H = L^2(S^1, d\mu)$, $u = z\cdot$, $\xi_- = g$, $\xi_+ = \bar{g}$.

But before you give $\xi_\pm$ which depend upon g you should mention $H^2_\pm(S^1, d\mu)$ in the general case, and the respective orthonormal bases $\{p_0, p_1, \dots\}$, $\{g_0, z^{-1}g_1, z^{-2}g_2, \dots\}$

$$\underset{\bar{p}_0}{\overset{\shortparallel}{g_0}},\quad \underset{\bar{p}_1}{\overset{\shortparallel}{z^{-1}g_1}},\quad \underset{\bar{p}_2}{\overset{\shortparallel}{z^{-2}g_2}}$$

(assume for this $d\mu$ inf support).

Now you want to check in the g case that

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[d] & H^2_+(d\mu) \arrow[d] \\ H^2_+(d\mu) \arrow[r] & L^2(d\mu) = H. \end{tikzcd}$$

is bicartesian (algebraically)

H^2_+ outgoing | in the strong sense of yielding unit. representations

H^2_- incoming |

You need details. $\langle g, z^n g\rangle = \int \bar{g}\, z^n g \frac{1}{|g|^2}\frac{d\theta}{2\pi} = \delta_n$

We know that

$\{1 = p_0, p_1, \dots\}$ is an orth basis for $H^2_+(S^1, d\mu)$

How to see that $\{z^{-1}g, z^{-2}g, \dots\}$, which is an orth set, extends this bases to an orth basis of $L^2(S^1, d\mu)$. Can you see that $L^2(S^1)g = H$?

$L^2(S^1) \longrightarrow H = L^2(d\mu)$ isometric and $\frac{1}{g}$ is bdd etc.

$\phi \longmapsto \phi g$

Now you use the correct representation

$$L^2(S^1) \xrightarrow{\sim} L^2(d\mu),$$ thus

$$\phi \longmapsto \phi g$$ where

$$\frac{1}{g}\psi \longleftarrow \psi$$

$$\frac{z^n}{g} \longleftrightarrow z^n$$

$\frac{z^n}{g} \in H_+$ for $n \geq 0$. So the square

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[d] & H^2_+(d\mu) \arrow[d] \\ \overline{H^2_+(d\mu)} \arrow[r] & L^2(d\mu) \end{tikzcd}$$

becomes

$$\begin{tikzcd} \mathbb{C}\frac{1}{g} \arrow[r] \arrow[d] & H_+ \arrow[d] \\ \frac{z}{g}H_- \arrow[r] & L^2(S^1) \end{tikzcd}$$

in the nom. repn.

but if now use g, which is bdd + invertible on $L^2(S^1)$ you get

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[d] & gH_+ = H_+ \arrow[d] \\ zH_- \arrow[r] & L^2(S^1) \end{tikzcd}$$

Idea. Consider g on $L^2(S^1)$, this is given by a triangular matrix. i.e.

$$\langle z^m, g z^n \rangle = \int z^{n-m} g \frac{d\theta}{2\pi}$$

$$= \begin{cases} 0 & n > m \\ g(0) & n = m \end{cases}$$

The picture emerging is that algebraically $H = L^2(S^1)$ (maybe $\mathbb{C}[z, z^{-1}]$), $u = z\cdot$, $F_{mn} = \langle z^{-m}, \ldots, z^n \rangle$ so the main subspaces of interest are in standard form. The inner product is generated by g.

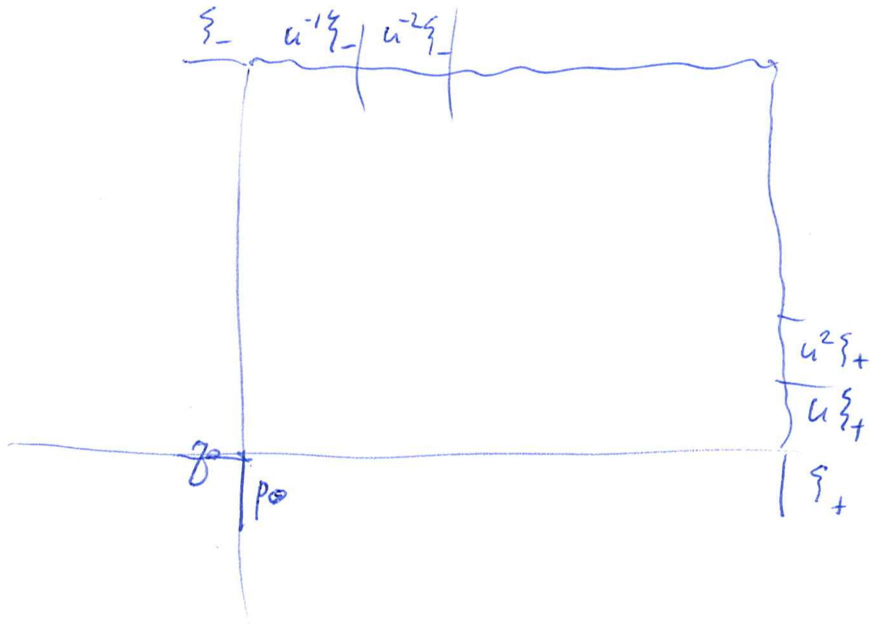
So what question

Picture: $H = L^2(S^1, d\mu) \xrightarrow{\sim} L^2(S^1)$

$$z^k \longmapsto \frac{z^k}{g}$$

Idea: $L^2(S^1) = H_+ \oplus \bar{S} H_-$

$\boxed{f \in L^2(S^1)}$ 6



$$H_-\xi_- \oplus H_+\xi_+ = H \quad ?$$

$$\begin{array}{ccc} H_- q \oplus H_+ \bar{q} & = & L^2(S^1, d\mu) \\ \downarrow & & \wr\big\downarrow \frac{1}{q} \\ H_- \oplus H_+ \underbrace{\tfrac{\bar{q}}{q}}_{S} & & L^2(S^1) \end{array}$$

Review. You have $d\mu = \frac{1}{|q|^2}\frac{d\theta}{2\pi}$ a prob. measure $q = e^{\log q}$ $\log q$ analytic on $\bar{D}$. Then have $H = L^2(S^1, d\mu)$, $u = u.$, $\xi_- = q$, $\xi_+ = \bar{q}$. ~~bdd~~ ~~operators.~~

$$L^2(S^1) \longrightarrow L^2(d\mu) \quad \text{isom.}$$
$$f \longmapsto fq$$

Basically a st. pos. bdd self adj oper. gives a new inner product

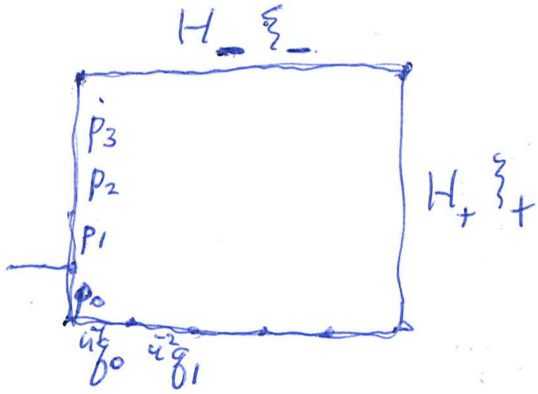
So let's try to exploit

$$H_-\xi_- \oplus H_+\xi_+ = H.$$

i.e. $$H_- q \oplus H_+ \bar{q} = L^2(S^1)$$

or $$\underbrace{\frac{1}{\bar q} H_-}_{H_-} \oplus \underbrace{\frac{1}{q} H_+}_{H_+} = L^2(S^1)$$

Still quite confused. Basically any $\eta \in H$ can be expressed uniquely in terms of the basis $u^{<0}\xi_-$ and $u^{\geq 0}\xi_+$. You need a specific formula. It looks like you have a bicart square



$$\langle p_0, \dots, p_n, \dots \rangle = \mathbb{C}[z]^{\text{closure}}$$

$$\langle u^{-1}q_0, u^{-2}q_1, \dots \rangle = z^{-1}\mathbb{C}[z^{-1}]^{\text{closure}}$$

$$\therefore \quad \langle p_0, \dots, p_i, \dots \rangle = \overline{\mathbb{C}[z]} = H_+$$

$$\langle u^{-1}q_0, \dots \rangle = \overline{z^{-1}\mathbb{C}[z^{-1}]} = z^{-1}H_-$$

$$H_+ \oplus H_- = z^{-1}H_- \oplus H_+$$

Inside $H^2(d\mu)$ you have $\xi = 1$

$\xi_- = g$, $\xi_+ = \bar{g}$. Then

$$\langle p_0, p_1, \dots \rangle = H_+ \xi_0$$

$$\langle u^{-1}\xi_0, u^{-2}\xi_-, \dots \rangle = H_- \xi_0$$

Is $H_+ \xi_0 \oplus H_- \xi_- = H = H_- \xi_0 \oplus H_+ \xi_+$

orthog. direct sum   o.d.s.

Apply incom. ref. $\xi_- \mapsto 1$, $\xi_+ \mapsto \frac{\bar{g}}{g}$, $\xi \mapsto \frac{1}{g}$

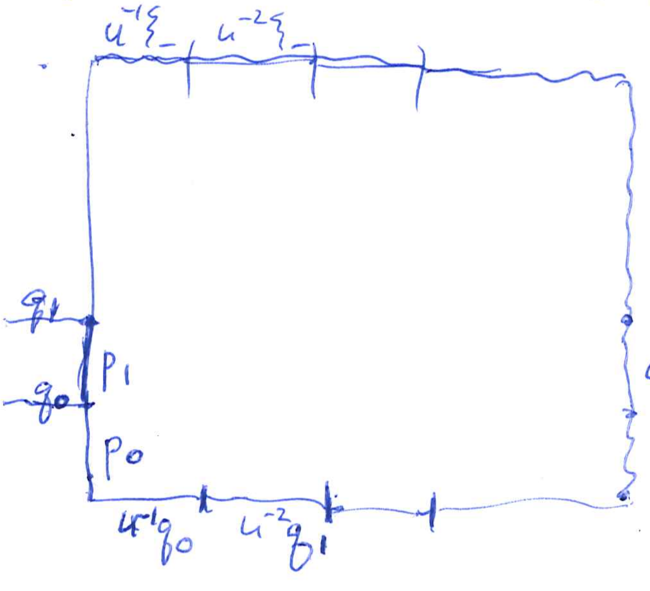
$$H_+ \xi_0 + H_- \xi_- \xrightarrow{\sim} H_+ \frac{1}{g} + H_-$$

$$g\xi_0 = \xi_- \qquad \bar{g}\xi = \xi_+ \qquad \frac{\bar{g}}{g}\xi_- = \bar{g}\xi = \xi_+$$

$$H_- \xi_0 + H_+ \xi_+ \xrightarrow{\sim} H_- \frac{1}{g} + H_+ \frac{\bar{g}}{g}$$

$$\int \bar{h}_- \frac{1}{\cancel{\bar{g}}} h_+ \frac{\cancel{\bar{g}}}{g} \frac{d\theta}{2\pi} = 0.$$

Repeat this calculation $\exists$ two orth bases for H



$$\{p_0, p_1, \dots\} \cup \{u^{-1}\xi_-, u^{-2}\xi_-, \dots\}$$

$$\{u^{-1}q_0, u^{-2}q_1, \dots\} \cup \{\xi_+, u\xi_+, \dots\}$$

This leads to a unitary isom.

$$H_+\xi_0 \oplus H_-\xi_- \longrightarrow H_-\xi_0 \oplus H_+\xi_+$$
$$\searrow H^{<s}$$

Check this using the ~~...~~ incoming representation

$$H = L^2(S^1, d\mu) \longrightarrow L^2(S^1)$$
$$f \longmapsto \frac{f}{g}$$

Thus
$$\xi_- = g \longmapsto 1$$
$$\xi_0 = 1 \longmapsto \frac{1}{g}$$
$$\xi_+ = \bar{g} \longmapsto \frac{\bar{g}}{g}$$

So
$$H_+\frac{1}{g} \overset{\perp}{\oplus} H_- = L^2(S^1) \quad ? \qquad \text{YES}$$

$$H_-\frac{1}{g} \overset{\perp}{\oplus} H_+\frac{\bar{g}}{g} = L^2(S^1) \quad ?$$
$$\Updownarrow$$
$$H_-\frac{1}{g} \overset{\perp}{\oplus} H_+ = L^2(S^1) \qquad \text{YES.}$$

Let's see if I can get ~~..~~ rid of the condition $p_0 = q_0 = \xi_0$. You want ~~..~~ to ~~...~~ use the partial unitary, dilated to a unitary.

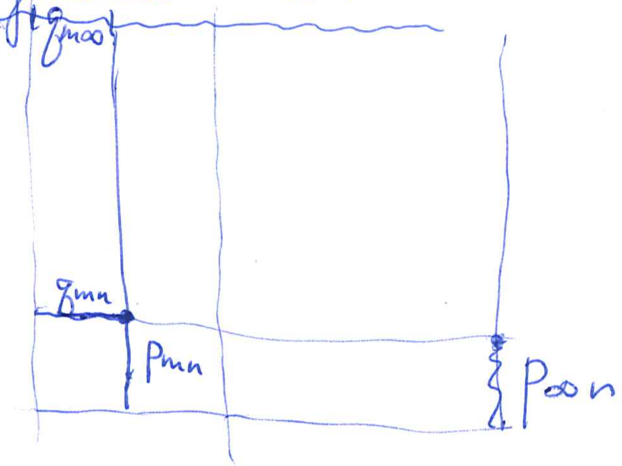
What does this mean? You continue the

Apparently things generalize to

$$\{p_n, p_{n+1}, \cdots\} \cup \{u^{-1}\xi_-, u^{-2}\xi_-, \cdots\}$$

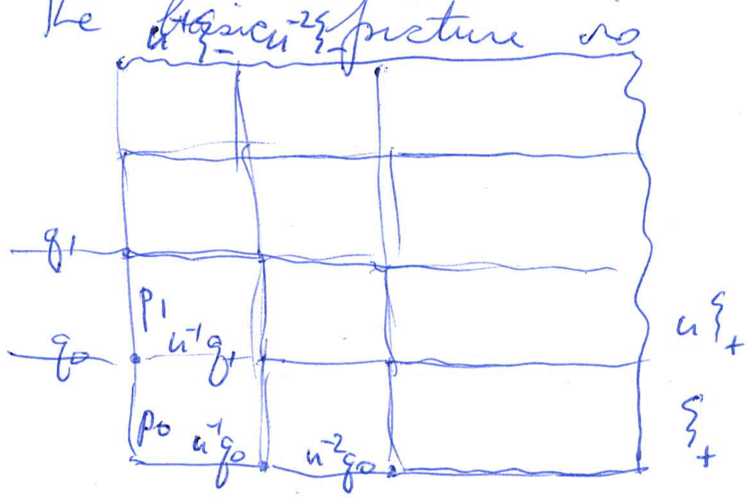
$$\simeq \{u^{-1}q_n, u^{-2}q_{n+1}, \cdots\} \cup \{u^n\xi_+, u^{n+1}\xi_+, \cdots\}$$

The general case: 4:22 -15 Dec.



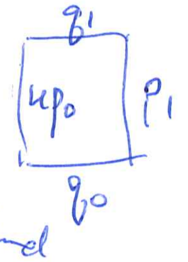
The thing you avoid is to work out the sequences $\{p_0, p_1, \dots\}$ $\{u^{-1}q_0, u^{-2}q_1, \dots$ from the other side. $\{u^{-1}\xi_-, u^{-2}\xi_-, \dots\}$, $\{\xi_+, u\xi_+, \dots\}$

The ~~basic~~ picture is

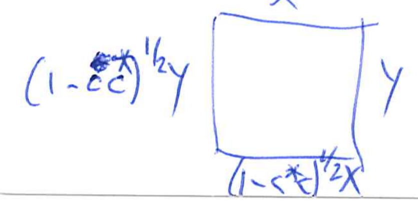


Analyze this if you can. You have orth bases indexed by $\mathbb{Z}_{\geq 0}$.

The last idea I had yesterday was to treat $\{u^{-1}\xi_-, u^{-2}\xi_-, \dots\}$ and $\{\xi_+, u\xi_+, \dots\}$ similar to p, q in

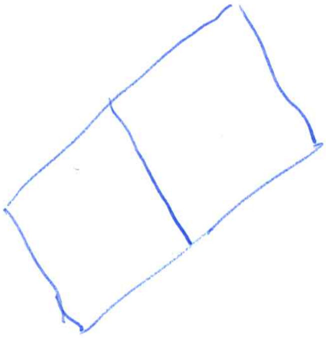


basically you have your old friend



except you will ~~use~~ probably use another operator b such that $b^*b = 1 - c^*c$

Let's work on bicartesian squares, rather bifiltrations. You want to start with



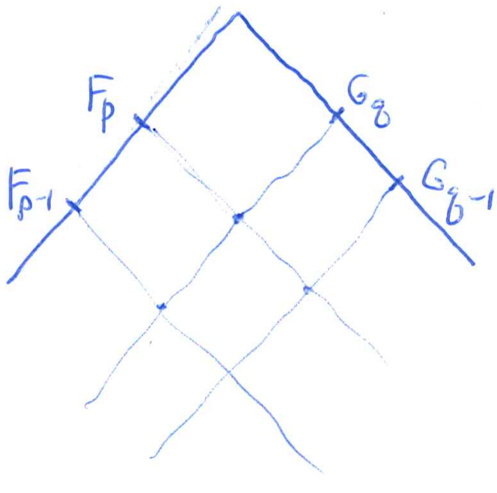
VITAMINS

What is the general pattern? You know algebraically ~~[illegible]~~ simple transitivity properties

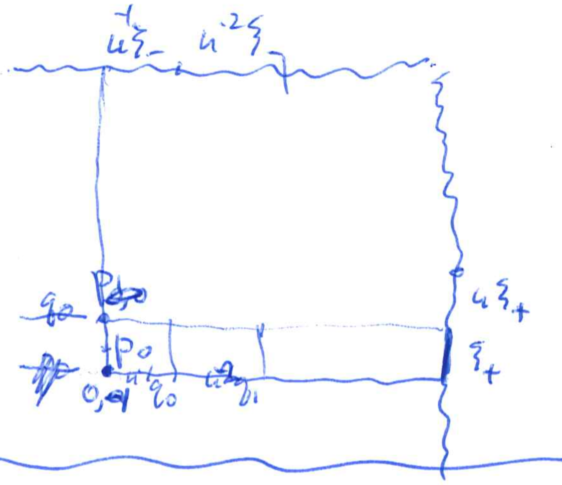
Let's look at a f.d. situation. Namely two increasing filtrations F_p G_q

$$gr^G_q(F_p/F_{p-1}) = \mathrm{Im}\{F_p \cap G_q \to F_p/F_{p-1}\} \Big/ \mathrm{Im}\{F_p \cap G_{q-1} \to F_p/F_{p-1}\}$$

$$= \frac{F_p \cap G_q + F_{p-1}}{F_p \cap G_{q-1} + F_{p-1}} \longleftarrow \frac{F_p \cap G_q}{\underbrace{F_p \cap G_q \cap (F_p \cap G_{q-1} + F_{p-1})}_{F_p \cap G_{q-1} + F_{p-1} \cap G_q}}$$



Assume each square is bicartesian



You should probably think of decreasing filtrations.

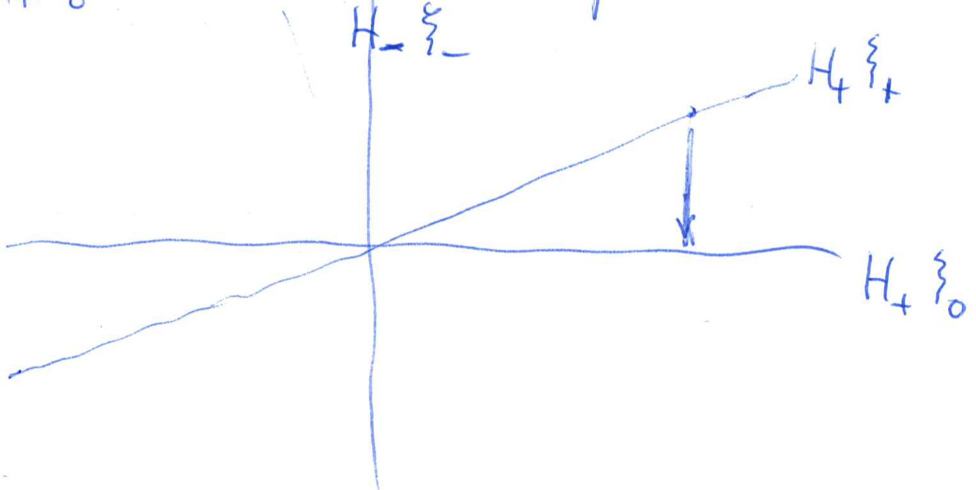
Let's work out the details. You start with g analytic invertible on $\bar{D}$. Then you have

H, u, ξ_-, ξ_+ $\qquad H = L^2(S^1, d\mu)$, $\quad d\mu = \frac{1}{|g|^2}\frac{d\theta}{2\pi}$

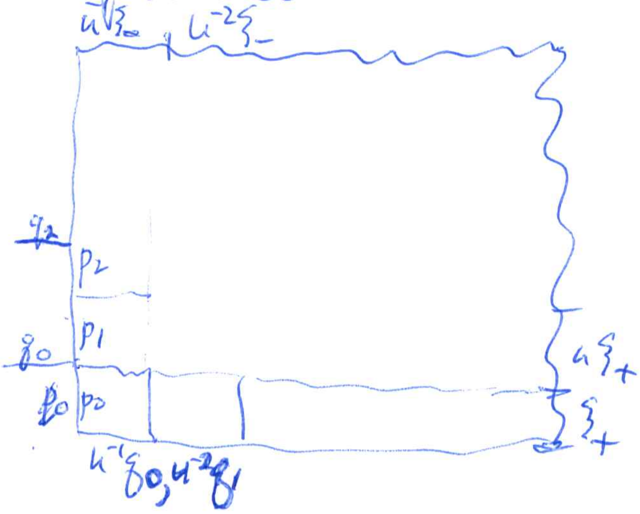
$$\xi_- = g \quad , \quad \xi_0 = 1 \quad , \quad \xi_+ = \bar{g}$$

$$H_+\xi_0 \oplus H_-\xi_- = H = H_-\xi_0 \oplus H_+\xi_+$$

You want to understand projection of $H_+\xi_+$ onto $H_+\xi_0$ and the other way round. The point is that $H_+\xi_+$ is a complement to $H_-\xi_-$ and $H_+\xi_0$ is the orth. comp

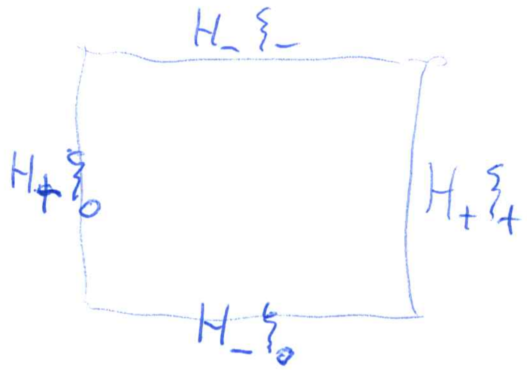


Basic picture



$$\langle p_0, p_1, \ldots \rangle \overset{\perp}{\oplus} \langle u^{-1}\xi_-, u^{-2}\xi_- \rangle = H$$

$$= \langle u^{-1}q_0, u^{-2}q_1, \ldots \rangle \overset{\perp}{\oplus} \langle \xi_+, u\xi_+, \ldots \rangle$$



Now you know that

$$H_+\xi_0 \oplus H_-\xi_- = H_-\xi_- \oplus H_+\xi_+$$

so there is a unitary matrix relating the two orthonormal bases, which is something like a scattering matrix. Let's try to get information about $\binom{p_n}{q_n}$ from the scattering. Keep in mind that staircase is another orthonormal basis. ~~For examples~~

Can we show that the ~~staircase corresp~~ orthonormal set $p_0, u^{-1}q_1, u^{-2}q_2, \ldots ; u\xi_+, u^2\xi_+, \ldots$ is a basis. ~~The problem is that you need~~

~~You know that the~~ You should ~~know that~~ be able to identify the orthogonal complements of $\quad u^{-m}H_- \xi_- + u^n H_+ \xi_+$.

$$\left(H_-\xi_- + u^n H_+ \xi_+\right)^\perp = H_+ \cap z^n \frac{\bar{g}}{g} H_-$$

$$= \frac{1}{g}\left(g H_+ \cap z^n \bar{g} H_-\right) = \frac{1}{g}\left(H_+ \cap z^n H_-\right)$$

$$= \frac{1}{g}\langle 1, z, \ldots, z^{n-1}\rangle$$

I should be able to explicitly carry out this isomorphism. ~~If you take $u = 1/z$ then~~ Thus you get exactly that the orthogonal complement of the closed sub-space spanned by $u^{\geq n}\xi_+$ and $u^{<0}\xi_-$ is spanned by $p_0, \ldots, p_{n-1}$

Now we have to go over this ~~carefully~~ repeatedly.

$$H = L^2(S^1, d\mu) \qquad d\mu = \frac{1}{|g|^2}\frac{d\theta}{2\pi}, \quad \int d\mu = 1.$$

$$\xi_- = g, \quad \xi_0 = 1, \quad \xi_+ = \bar{g}.$$

$$L^2(S^1) \xrightarrow[\sim]{\cdot g} H \xleftarrow[\sim]{\cdot \bar{g}} L^2(S^1)$$

incoming representation is $\quad H \xrightarrow{\sim} L^2(S^1)$

$$\xi_- = g \longmapsto 1$$

$$\therefore \xi_0 = \frac{1}{g} g \longmapsto \frac{1}{g}$$

this is not a good approach. Use $\xi_- = g$ and $\xi_+ = \bar{g}$ are cyclic vectors for the u action. you are interested in "half spaces" generated by ξ_- and ξ_+ namely

$$\cancel{H_-\xi_- + u^n H_+ \xi_+}$$

$$\cancel{= H_- g + z^n H_+ \bar{g} = |g|^2\left(H_-\frac{1}{\bar g} + z^n H_+ \frac{1}{g}\right)}$$

$$\cancel{= |g|^2}$$

$$H_-\xi_- + u^n H_+ \xi_+ = H_- g + u^n H_+ \bar{g}$$

$$= |g|^2\left(H_- \frac{1}{\bar g} + z^n H_+ \frac{1}{g}\right)$$

$$= |g|^2\left(H_- + z^n H_+\right)$$

and this has orthog complement in H
those f?

$$0 = \int \bar f\, \cancel{|g|^2}(H_- + z^n H_+)\frac{1}{\cancel{|g|^2}}\frac{d\theta}{2\pi}$$

i.e. $f \in \langle 1, z, .., z^{n-1}\rangle$

Thus $\langle p_0, .., p_{n-1}\rangle$ in H = orth complement of $H_-\xi_- + u^n H_+ \xi_+$.

$$H = \langle 1, .., z^{n-1}\rangle \oplus \left(H_- g + z^n H_+ \bar g\right)$$

What kind of questions to ask?

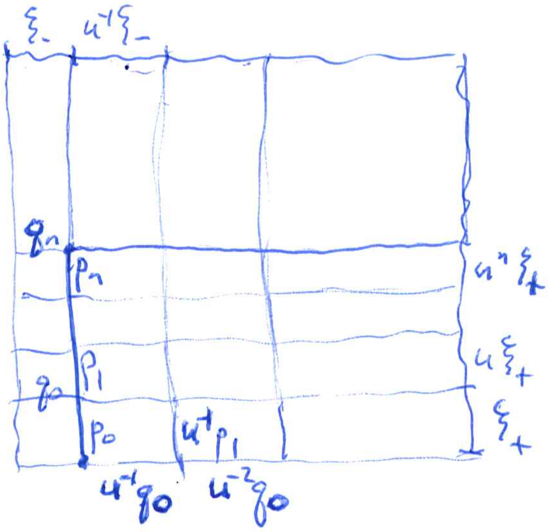
Can you write $p_n \in H_- g + z^n H_+ \bar g$

$$\frac{p_n}{q} \in H_- + z^n H_+ \bar{q}$$

You want to find $p_n \in H_- q + z^n H_+ \bar{q}$

i.e. $p_n \perp 1, z, \dots, z^{n-1}$, and also you want $p_n \perp \left(H_- q + z^{n+1} H_+ \bar{q}\right)$

$$\frac{p_n}{|q|^2} \in H_- + z^n H_+$$



$$H_+ \xi_- = (H_- \xi_-)^\perp$$

$$u^{n+1} H_- \xi_+ = \left(u^{n+1} H_+ \xi_+\right)^\perp = \left(u^{>n} \xi_+\right)^\perp$$

You want to find.

$$H_+ \xi_- \cap u^{n+1} H_- \xi_+ \xrightarrow{\sim} H_+ \cap z^{n+1} H_- \frac{\bar{q}}{q} \subset L^2(S^1)$$

$$\| $$

$$\frac{1}{q}\left(H_+ \cap z^{n+1} H_-\right) = \frac{P_n}{q}.$$

You are on the right track. What to do next? How about the partial unitary case? You have q given, but you will do something different with it.

$$\langle p \geq n\rangle \xi_0 \oplus H_- \xi_- = (F_n \xi_0)^\perp = \langle u^{-1} g_n, u^{-2} g_{n+1}, \dots\rangle \xi_0 \oplus u^{n+1} H_+ \xi_+$$

Anyway Review: Start with g analytic invertible on $\bar{D}$ normalized so that $d\mu = \frac{1}{|g|^2}\frac{d\theta}{2\pi}$, $\int d\mu = 1$. Then you have a Hilbert space $H = L^2(d\mu)$, unitary op $u = z\cdot$, cyclic vector $\xi_0 = 1$, incoming subspace $H_- g$, outgoing subspace $H_+ \bar{g}$. Here $H_\pm \subset L^2(S^1)$ have ~~are spanned~~ the orthonormal bases $\{z^n \mid n \geq 0\}$, $z^{<0}$, resp.

Results $\xi_- = g$, $\xi_+ = \bar{g}$ are cyclic vectors, yielding isos.

$$L^2(S^1) \xrightarrow[\cdot g]{\sim} L^2(d\mu) \xleftarrow[\cdot \bar{g}]{\sim} L^2(S^1)$$

or better incoming + outgoing rep.

$$\begin{array}{ccccc} L^2(S^1) & \xleftarrow[\sim]{\cdot \frac{1}{g}} & L^2(d\mu) & \xrightarrow[\sim]{\cdot \frac{1}{\bar{g}}} & L^2(S^1) \\ \cup & & & & \cup \\ H_- & & & & H_+ \end{array}$$

In a scattering situation you get filtration by looking at the support. You can project a state onto the outgoing (in) subspace and look at the support.

So in the outgoing (resp. incoming) case ~~of~~ you look at ~~the~~

$\xi \in L^2(d\mu)$ such that $\frac{\xi}{\bar{q}} \in (z^n H_+)^\perp = z^n H_-$

resp such that $\frac{\xi}{q} \in (z^{-m} H_-)^\perp = z^{-m} H_+$.

~~The intersection is~~ ~~$z^{-m}H_+ \cap z^n H_-$~~

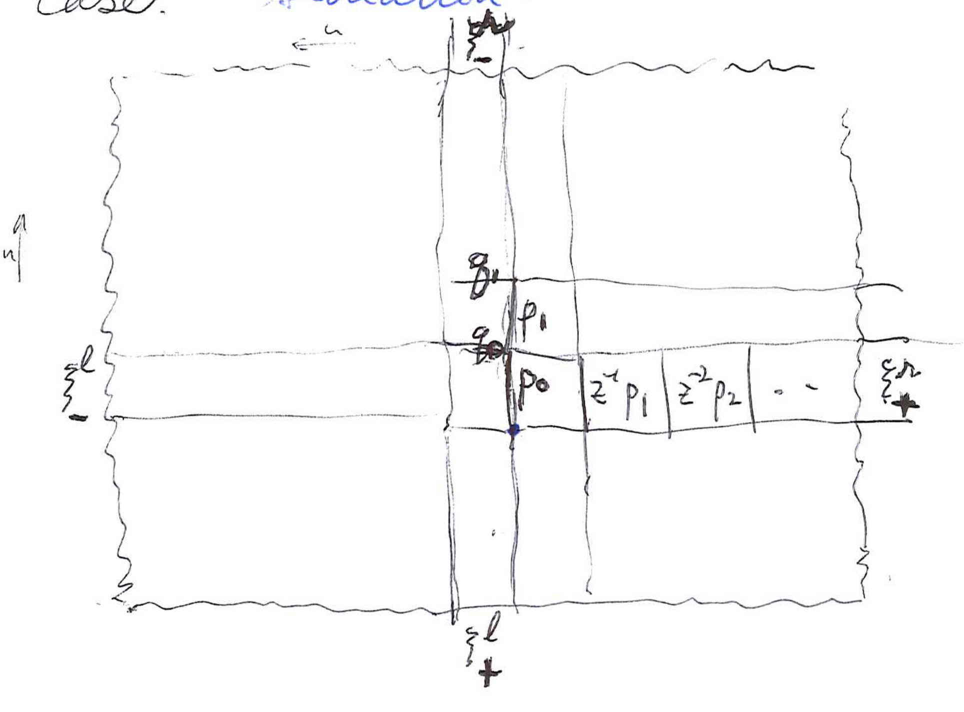
$\therefore \; \xi \in z^n \bar{q} H_- \simeq z^n H_-$

$\xi \in z^{-m} q H_+ \simeq z^{-m} H_+$

i.e. $\xi \in \langle z^{-m}, \dots, z^{n-1} \rangle$

Anyway what next?

So let's try to understand the two-sided case. situation:



$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$\begin{pmatrix} z^{-n} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} \cancel{z^{n}}\,1 & h_n z^{-n} \\ \cancel{\bar{h}_n}\;\bar{h}_n z^n & 1 \end{pmatrix} \cancel{\begin{pmatrix} z^n & 0 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} z^{-n+1} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

You need the analogue of g.
What do we have? Basically you have

$$L^2(S^1)\,\xi^r_- \oplus L^2(S^1)\,\xi^\ell_- \;\simeq\; L^2(S^1)\,\xi^r_+ \oplus L^2(S^1)\,\xi^\ell_+$$

given by ~~a~~ a 2×2 S-matrix.
This situation should be similar to

$$\begin{tikzcd} \cdot \arrow[r, dash, "q_n"] \arrow[d, dash, "u p_{n-1}"'] & \cdot \arrow[d, dash, "p_n"] \\ \cdot \arrow[r, dash, "q_{n-1}"'] & \cdot \end{tikzcd}$$

$$h_n = (q_n, p_n)$$

$$p_n - q_n h_n = k_n u p_{n-1}$$

$$p_n = k_n u p_{n-1} + q_n h_n$$

$\perp$

$$\therefore\quad |k_n|^2 + |h_n|^2 = 1$$

If $k_n > 0$, then you have a unitary 2×2 matrix

$$\begin{pmatrix} p_n \\ q_{n-1} \end{pmatrix} = \begin{pmatrix} k_n & h_n \\ h_n' & k_n' \end{pmatrix} \begin{pmatrix} u p_{n-1} \\ q_n \end{pmatrix}$$

If also $k_n' > 0$
then $k_n' = k_n$
and $h_n' = -\bar{h}_n$

$$\begin{pmatrix} z^{-n} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix} \begin{pmatrix} z^{-n+1} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

Go over what you know

$$\frac{1}{k_n}\begin{pmatrix} z & h_n \\ \bar h_n z & 1\end{pmatrix}\cdots\cdots\frac{1}{k_m}\begin{pmatrix} z & h_m \\ \bar h_m z & 1\end{pmatrix} = P_{n-m+1}\,d$$

$$\in \begin{pmatrix} \langle 1,\dots,z^{d-1}\rangle z & \langle 1,\dots,z^{d-1}\rangle \\ \langle 1,\dots,z^{d-1}\rangle z & \langle 1,\dots,z^{d-1}\rangle \end{pmatrix}$$

$$T_{nm} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar h_n z^{n} & 1\end{pmatrix}\cdots\cdots\frac{1}{k_m}\begin{pmatrix} 1 & h_m z^{-m} \\ \bar h_m z^{m} & 1\end{pmatrix} = \begin{pmatrix} \bar d & \bar c \\ c & d\end{pmatrix}$$

$$T_{nm} \in \begin{pmatrix} z^{-n} & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} \langle 1,\dots,z^{d-1}\rangle z & \langle 1,\dots,z^{d-1}\rangle \\ \langle 1,\dots,z^{d-1}\rangle z & \langle 1,\dots,z^{d-1}\rangle \end{pmatrix}\begin{pmatrix} z^{m-1} & 0 \\ 0 & 1\end{pmatrix}$$

$$\in \begin{pmatrix} z^{-n}\langle 1,\dots,z^{\overset{n-m+1}{d-1}}\rangle z^{m} & z^{-n}\langle 1,\dots,z^{\overset{n-m+1}{d-1}}\rangle \\ \langle 1,\dots,z^{d-1}\rangle z^{m} & \langle 1,\dots,z^{d-1}\rangle \end{pmatrix}$$

$$\in \boxed{\begin{pmatrix} \langle z^{-n+m},\dots,z^{0}\rangle & \langle z^{-n},\dots,z^{-m}\rangle \\ \langle z^{m},\dots,z^{n}\rangle & \langle 1,\dots,z^{n-m}\rangle \end{pmatrix}}$$

What are you trying for?

$$\begin{pmatrix} z^{-n}p_n \\ q_n\end{pmatrix} = T_{nm}\begin{pmatrix} z^{-m+1}p_{m-1} \\ q_{m-1}\end{pmatrix}$$

You will get a form for the transfer matrix $T_{\infty,-\infty}$ which expresses?

$$\begin{pmatrix} \xi^r_+ \\ \xi^r_- \end{pmatrix} = \lim_{n\to+\infty} \begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix} = \lim_{\substack{n\to+\infty \\ m\to-\infty}} T_{n,m+1} \begin{pmatrix} u^{-m} p_m \\ q_m \end{pmatrix}$$

$$= T_{\infty,-\infty} \begin{pmatrix} \xi^l_- \\ \xi^l_+ \end{pmatrix} = \begin{pmatrix} \bar{d} & \bar{c} \\ c & d \end{pmatrix} \begin{pmatrix} \xi^l_- \\ \xi^l_+ \end{pmatrix}$$

find S

$$\xi^r_+ = \bar{d}\,\xi^l_- + \bar{c}\,\xi^l_+$$
$$\xi^r_- = c\,\xi^l_- + d\,\xi^l_+$$

$$\xi^l_+ = -\frac{c}{d}\xi^l_- + \frac{1}{d}\xi^r_-$$

$$\xi^r_+ = \bar{d}\,\xi^l_- + \bar{c}\left(-\frac{c}{d}\xi^l_- + \frac{1}{d}\xi^r_-\right)$$

$$= \left(\frac{\overbrace{d\bar{d} - \bar{c}c}^{1}}{d}\right)\xi^l_- + \frac{\bar{c}}{d}\xi^r_-$$

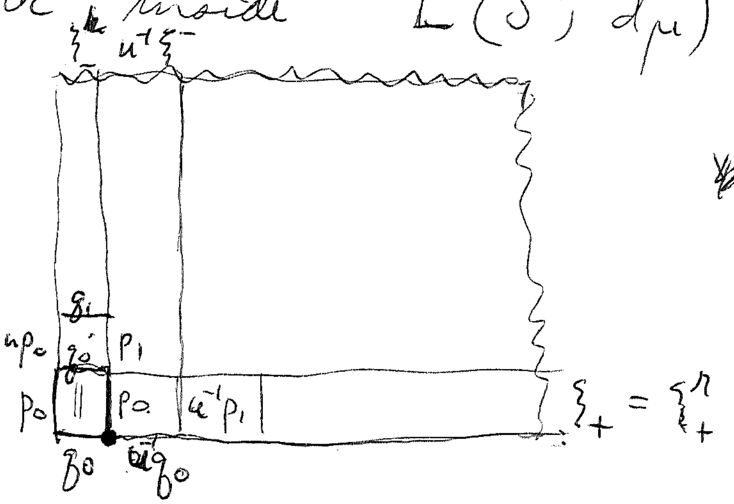
$$\begin{pmatrix} \xi^r_+ \\ \xi^l_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{\bar{c}}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi^l_- \\ \xi^r_- \end{pmatrix}$$

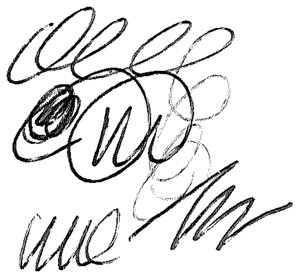
Suppose that we start with $L^2(S^1, \frac{1}{|g|^2}\frac{d\theta}{2\pi})$ form the partial unitary by removing the b condition $p_0 = q_0$

$$c_h = ba^* + u(\xi_0)h\,\xi_0^*$$

Basically you have this $X \overset{a}{\underset{b}{\rightrightarrows}} Y$

Work inside $L^2(S^1, d\mu)$





$$\begin{array}{r} 18. \\ 3.48 \\ \hline 21.48 \end{array}$$

You've found that $\xi_-^l = p_0$

Start with q construct $L^2(S^1, d\mu)$, u then the partial unitary obtained by removing the boundary condition $p_0 = q_0$, and then dilate.

Model $Y = L^2(S^1)$

$$\xi_- = 1, \quad \xi_0 = \frac{1}{q}, \quad \xi_+ = \frac{\bar{q}}{q}.$$

$$L^2(S^1) \xleftarrow[\sim]{\frac{1}{q}} L^2(d\mu) \xrightarrow[\sim]{\frac{1}{\bar{q}}} L^2(S^1)$$

$\xi_- = 1$ $\quad\quad$ $\xi_- = q$

$\xi_0 = \frac{1}{q}$ $\quad\quad$ $\xi_0 = 1$

$\xi_+ = \frac{\bar{q}}{q}$ $\quad\quad$ $\xi_+ = \bar{q}$

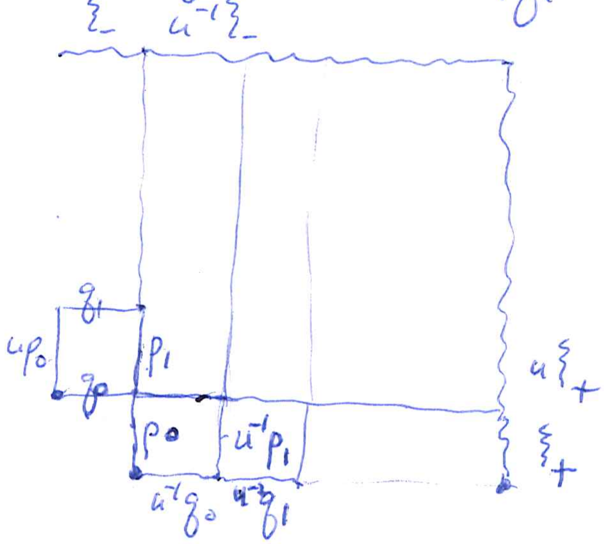
~~See if you can get~~ Picture

$$Y = \langle p_0, p_1, \cdots \rangle \overset{\perp}{\oplus} \langle u^{-1}\xi_-, u^{-2}\xi_-, \dots \rangle$$

$$= \langle u^{-1}q_0, u^{-2}q_1, \cdots \rangle \overset{\perp}{\oplus} \langle \xi_+, u\xi_+, \cdots \rangle$$

X

Given q $\quad d\mu = \frac{1}{|q|^2}\frac{d\theta}{2\pi}$ etc.



$$L^2(S^1, d\mu) = \langle p_0, p_1, \dots \rangle \overset{\perp}{\oplus} \langle u^{-1}\xi_-, u^{-2}\xi_-, \dots \rangle$$

$$= \langle u^{-1}q_0, u^{-2}q_1 \dots \rangle \overset{\perp}{\oplus} \langle \xi_+, u\xi_+, \dots \rangle$$

~~partial unitary ... corresponds~~

Go back over the perturbation

H, u, ξ

$$H = aX \oplus \mathbb{C}\xi \qquad X = (\mathbb{C}\xi)^\perp$$

$$= bX \oplus \mathbb{C}u\xi \qquad ba^{-1} = u|aX$$

$$u = ba^* + u(\xi)\xi^*$$

$$(\lambda - u)^{-1} = \frac{1}{\lambda - ba^* - u(\xi)\xi^*}$$

$$\xi^*\frac{1}{\lambda - u} = \xi^*\frac{1}{\lambda - ba^*} + \xi^*\frac{1}{\lambda - ba^*}u(\xi)\xi^*\frac{1}{\lambda - ba^*} + \cdots$$

$$= \left(1 + \xi^*\frac{1}{\lambda - ba^*}u(\xi) + \cdots\right)\xi^*\frac{1}{\lambda - ba^*}$$

$$\xi^*\frac{1}{\lambda - u}u(\xi) = \frac{1}{1 - \xi^*\frac{1}{\lambda - ba^*}u(\xi)}\xi^*\frac{1}{\lambda - ba^*}u\xi$$

Perhaps it is possible to ~~calc~~ find ~~[illegible]~~ $\langle u^k \xi_-, u^l \xi_+ \rangle$.

~~Some things are clear. Some things are clear.~~

You work in $L^2(S^1, d\mu)$ with $\xi_- = z$, $\xi_0 = 1$, $\xi_+ = \bar{z}$. And you have the two orth bases $\{p_0, p_1, \dots\} \cup \{u^{-1}\xi_-, u^{-2}\xi_-, \dots\}$ $\{u^{-1}q_0, u^{-2}q_1, \dots\} \cup \{\xi_+, u\xi_+, \dots\}$. The latter can be shifted by u to be the conjugate of the former.

The next step is to find the partial unitary. Look at the ~~dis~~ disc. Dirac eqn.

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = k_n^{-1} \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z p_{n-1} \\ q_{n-1} \end{pmatrix}$$

Possible viewpoint - all these vectors you work with are functions of z, many are Laurent polys, all should at least be continuous functions on S^1. ~~So when you are~~

Let's get back to the ideas.

First situation, smooth probability measure $d\mu = \rho \frac{d\theta}{2\pi} = \frac{1}{|g|^2} \frac{d\theta}{2\pi}$ where ~~[illegible]~~ analytic invertible on $\bar{D}$, i.e. on $(1+\varepsilon)D$, some $\varepsilon > 0$.

$H = L^2(S^1, d\mu)$ $\quad \xi_- = g, \ \xi_0 = 1, \ \xi_+ = \bar{g}$

digress. Suppose given $d\mu$ a prob. measure, inf supp you form the orth poly system

~~[illegible]~~ Consider $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ $\quad Y = aX \oplus \mathbb{C}\xi_+$
$\quad = bX \oplus \mathbb{C}\xi_-$

Dilate this to

$$\langle \cdots, u^{-2}\xi_-, u^{-1}\xi_- \rangle \oplus aX \oplus \mathbb{C}\xi_+ \oplus \langle u\xi_+, \cdots \rangle$$

$$\text{"} \quad \oplus \mathbb{C}\xi_- \oplus bX \oplus \quad \text{"}$$

to get an (H, u) with incoming and outgoing representations. Incoming and Outgoing reps.

Eigenvector equation

$$u^{-2}v_{-,-2} + u^{-1}v_{-,-1} + ax + v_{+,0} + uv_{+,1} + u^2 v_{+,2} + \cdots = \eta$$

$$+ u^{-1}v_{-,-2} + v_{-,-1} + bx + uv_{+,0} \qquad = u\eta$$

$$\eta = \lambda^2 u^{-2}v_- + \lambda u^{-1}v_- + ax + v_+ + \frac{1}{\lambda}uv_+ +$$

$$\lambda^2 u^{-1}v_- + \lambda u^{-1}v_- + (v_- + \frac{1}{\lambda}bx) + \frac{1}{\lambda}uv_+ + \frac{1}{\lambda^2}u^2 v_+$$

Start again. Consider $Y = aX \oplus V_+ = V_- \oplus bX$ a partial unitary. $V_\pm = \mathbb{C}\xi_\pm$, $\|\xi_\pm\| = 1$.

$$H = \langle \dots, u^{-2}\xi_-, u^{-1}\xi_- \rangle \oplus aX \oplus \langle \xi_+, u\xi_+, \dots \rangle$$

$$= \langle \dots, u^{-2}\xi_-, u^{-1}\xi_-, \xi_- \rangle \oplus bX \oplus \langle u\xi_+, u^2\xi_+, \dots \rangle$$

$$H = H_-\xi_- \oplus aX \oplus H_+\xi_+ = zH_-\xi_- \oplus bX \oplus zH_+\xi_+$$

Describe ~~scattering~~ outgoing representation

$$\langle u^{-n}\xi_+ \,|\, ax \rangle = \xi_+^* u^n ax$$

The basic idea is to look at $u^n(ax)$ as $n \to +\infty$ and see what comes out.

$$u^{-2}V_+ \oplus u^{-1}V_+ \oplus \,|\, V_+ \oplus uV_+ \oplus$$

$$\cap \qquad\qquad \parallel$$

$$\dots \oplus u^{-2}V_- \oplus u^{-1}V_- \oplus aX \,|\oplus|\, V_+ \oplus uV_+ \oplus \dots$$

$$\dots \oplus u^{-1}V_- \oplus V_- \oplus bX \oplus uV_+ \oplus \dots$$

$$\cup$$

$$\oplus\, u^{-1}V_- \oplus V_- \oplus uV_- \oplus u^2V_- \oplus \dots$$

~~Start with~~ $y \in Y$. $\quad \xi_+\xi_+^* y \;\; \pi_+$

$$y = aa^*y + (1 - aa^*)y$$

$$uy = ba^*y + u\pi_+y$$

$$= aa^*ba^*y + \pi_+ba^*y + u\pi_+y$$

$$u^2y = (ba^*)^2y + u\pi_+ba^*y + u^2\pi_+y$$

$$u^3y = ba^*(ba^*)^2y + u\pi_+(ba^*)^2y + u^2\pi_+(ba^*)y + u^3\pi_+y$$

$$y = u^{-3}(ba^*)^3y + u^{-2}\pi_+(ba^*)^2y + u\pi_+(ba^*)y + \pi_+y$$

so the outgoing repn. is

$$y \longmapsto \sum_{n\geq 0} u^{-n} \pi_+ (ba^*)^n y$$

$$\longmapsto \sum_{n\geq 0} z^{-n} \pi_+ (ba^*)^n y = \pi_+ \frac{1}{1-z^{-1}ba^*} y \in L^2(S^1) V_+$$

Better
$$y \longmapsto \xi_+^* \frac{1}{1-z^{-1}ba^*} y$$ is the outgoing representation which we know is isometric iff $(ba^*)^n y \to 0$ $\forall y$.

Now look at $c_h = ba^* + \xi_- h \xi_+^*$ for $|h| \leq 1$, a path of contractions joining ba^* to the unitary extension of the partial unitary such that $u(\xi_+) = \xi_-$

Question. Consider

$$\begin{pmatrix} z^{-n} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^{+n} & 1 \end{pmatrix} \begin{pmatrix} z^{-n+1} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

with a variable bdry condition $\frac{p_0}{q_0} = e^{i\phi}$.
This should be the same as conjugating with $\begin{pmatrix} e^{i\phi/2} & \\ & e^{-i\phi/2} \end{pmatrix}$

$$\begin{pmatrix} e^{i\phi/2} z^{-n} p_n \\ e^{-i\phi/2} q_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & e^{i\phi} h_n z^{-n} \\ e^{-i\phi} \bar{h}_n z^n & 1 \end{pmatrix} \begin{pmatrix} e^{i\phi/2} z^{-n+1} p_{n-1} \\ e^{-i\phi/2} q_{n-1} \end{pmatrix}$$

Check.

$$\begin{pmatrix} e^{i\phi} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & e^{i\phi} h_n \\ e^{-i\phi} \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\phi} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

Example $\quad Y = L^2(S^1) \qquad \xi_+ = 1, \quad \xi_- = z.$

$$aX = \xi_+^\perp = H_- \oplus zH_+$$
$$bX = \xi_-^\perp = zH_- \oplus z^2 H_+$$

Instead take $\quad Y = L^2(S^1) \qquad \xi_+ = z^{-1}, \quad \xi_- = 1$

Then
$$aX = \xi_+^\perp = z^{-1}H_- \oplus H_+$$
$$bX = \xi_-^\perp = H_- \oplus zH_+$$

So
$$H = H_-\xi_- \oplus aX \oplus H_+\xi_+$$
$$\simeq H_- \oplus (z^{-1}H_- \oplus H_+) \oplus H_+$$

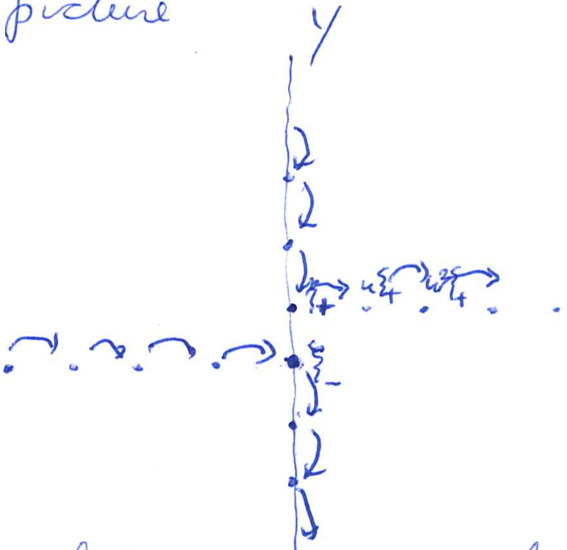
$$Y = aX \oplus V_+ = \langle \dots, z^{-2}, z^{-1}, z, z^2, \dots \rangle \oplus \langle 1 \rangle$$
$$= \langle z \rangle \oplus \langle \dots, z^{-1}, 1, z^2, z^3, \dots \rangle$$

$$H = \langle \dots, z^{-2}, z^{-1}, z, z^2, \dots \rangle \oplus \langle 1, z, z^2, \dots \rangle \xi_+$$
$$= \langle \dots, z^{-1}, 1, z^2, z^3, \dots \rangle \oplus \langle \dots, z^{-2}, z^{-1} \rangle \xi_-$$

$$H = H_-\xi_- \oplus L^2(S^1) \oplus zH_+\xi_+$$

$$\dots, u^{-2}\xi_-, u^{-1}\xi_-, z, z^2, \dots; \dots, z^{-1}, 1, z\xi_+, z^2\xi_+, \dots$$

another picture



$aX \oplus \mathbb{C}\xi_+$

[redacted]

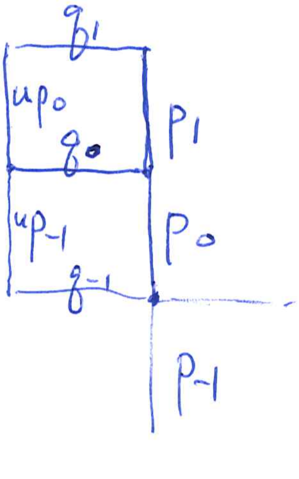
$\mathbb{C}\xi_- \oplus bX$

Recall what you are doing. You have a partial unitary $Y = aX \oplus \mathbb{C}\xi_+ = \mathbb{C}\xi_- \oplus bX$, which you dilate to get H.

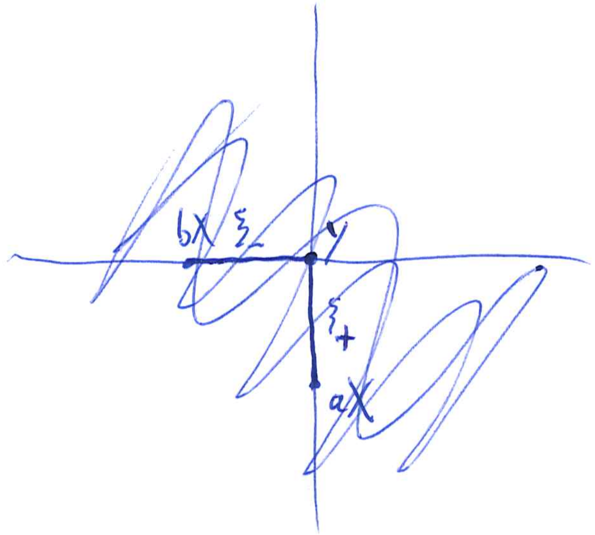
~~It's better to start~~ This ~~starting~~ point lacks the property that H is generated under u by $\xi_\pm$. ~~Not instead~~ Something to work out: Given such a partial unitary, you construct a Szegö system $\begin{pmatrix} p_n \\ q_n \end{pmatrix}$ with $\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$. This should yield the non bound state part.

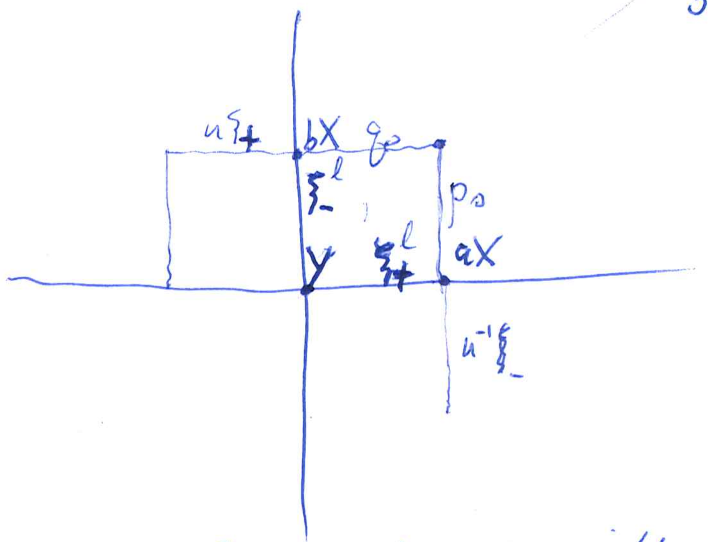
Let's check this. Given $Y = aX \oplus \mathbb{C}\xi_+ = \mathbb{C}\xi_- \oplus bX$.
Picture? $\xi_- = q_0$, $\xi_+ = p_0$? Picture of H

$$\langle \dots, u^{-2}\xi_-, u^{-1}\xi_- \rangle \oplus \underbrace{Y}_{\substack{\| \\ aX \oplus \mathbb{C}\xi_+ = \mathbb{C}\xi_- \oplus bX}} \oplus \langle u\xi_+, u^2\xi_+, \dots \rangle$$



$u^{-1}\xi_- \perp \xi_+$ $\qquad \xi_- \perp u\xi_+$





Try setting it up directly. You begin with
$Y = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$ partial unitary is
$ba^* = ba^{-1} : aX \to bX$. Look at $\langle \xi_+, \xi_- \rangle = \xi_+^* \xi_-$.
Recall that ~~one~~ one scattering function is

$$\xi_+^* \frac{z}{z - ba^*} \xi_-$$

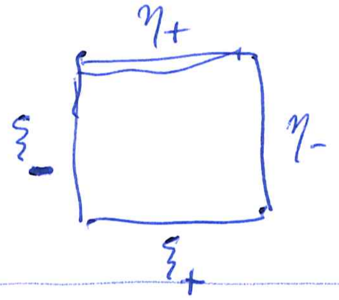
~~then~~ Put $h = \xi_+^* \xi_-$ and

$$\tilde{\eta}_- = \xi_- - \xi_+ \xi_+^* \xi_- = \xi_- - \xi_+ h \qquad \tilde{\eta}_- \perp \xi_+$$

$$\tilde{\eta}_+ = \xi_+ - \xi_- \xi_-^* \xi_+ = \xi_+ - \xi_- \bar{h} \qquad \tilde{\eta}_+ \perp \xi_-$$

Normalize $\quad \xi_- = \tilde{\eta}_- + \xi_+ h \qquad 1 = \|\tilde{\eta}_-\|^2 + |h|^2$

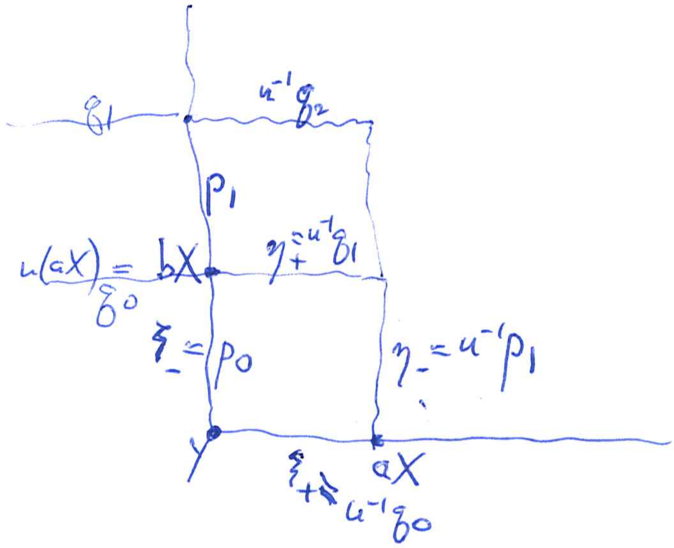
$$\therefore \|\tilde{\eta}_-\| = k = \sqrt{1 - |h|^2}$$



$$\begin{pmatrix} \eta_- \\ \eta_+ \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & -h \\ -\bar{h} & 1 \end{pmatrix} \begin{pmatrix} \xi_- \\ \xi_+ \end{pmatrix}$$

$$k\eta_+ = \xi_+$$

Review: Begin with $Y = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$



$$\begin{pmatrix}\eta_+\\ \eta_-\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar h & 1\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

$$h = \langle \eta_- | \eta_+ \rangle = -\langle \xi_- | \xi_+ \rangle$$

$$k\eta_- = \bar h \xi_+ + \xi_-$$

$$k\eta_+ = \xi_+ + h\xi_- \Rightarrow$$

$$0 = \langle k\eta_- | \xi_+ \rangle = \langle \bar h \xi_+ + \xi_- | \xi_+ \rangle = h + \langle \xi_- | \xi_+ \rangle$$

$$0 = \xi_-^*(k\eta_+) = \xi_-^*\xi_+ + \underbrace{\xi_-^* h \xi_-}_{h}$$

$$\therefore \xi_-^*\xi_+ = -h$$

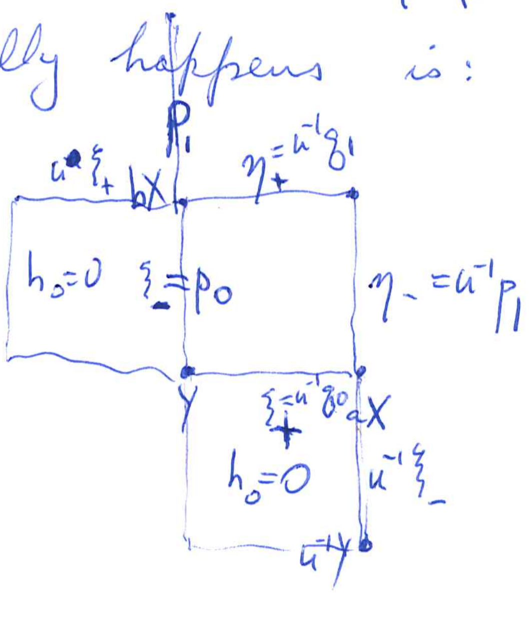
$$\begin{pmatrix}p_1\\ q_1\end{pmatrix} = \frac{1}{k_1}\begin{pmatrix}1 & \bar h_1\\ h_1 & 1\end{pmatrix}\begin{pmatrix}up_0\\ q_0\end{pmatrix}$$

$$\bar h_1 = \langle u^{-1}p_1 | u^{-1}q_1 \rangle = \langle p_1 | q_1 \rangle$$

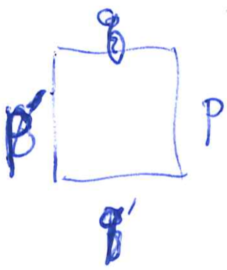
Repeat. Given $Y = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$, dilate to

$$H = H_-\xi_- \oplus Y \oplus zH_+\xi_+ \qquad aX = (H_-\xi_- \oplus H_+\xi_+)^\perp$$

What really happens is:



Now you ask $\bigcup_{k,l} u^k H_- \xi_- + u^l H_+ \xi_+$ dense in H.
Then $\overline{L^2(S^1)\xi_+ + L^2(S^1)\xi_-} = H$, whence
H known from scattering fn. $S(z) = \xi_+^* \frac{1}{1-z^{-1}ba^*}\xi_-$



transfer matrix

$$\begin{pmatrix}p\\q\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar h & 1\end{pmatrix}\begin{pmatrix}p'\\q'\end{pmatrix}$$

scatt matrix

$$\begin{pmatrix}p\\q'\end{pmatrix} = \begin{pmatrix}k & h\\ -\bar h & k\end{pmatrix}\begin{pmatrix}p'\\q\end{pmatrix}$$

$$p = kq' + hq$$

$$(q|p) = h$$

$$kq = \bar h p' + q'$$
$$kp = p' + h(kq - \bar h p')$$
$$= k(kp' + hq)$$

$$q' = kq - \bar h p'$$
$$p = kp' + hq$$

dldDE $$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} z p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = T_{nm}\begin{pmatrix} p_m \\ q_m \end{pmatrix} \qquad n \geq m$$

$$T_{nm} = \underbrace{\frac{1}{k_n}\begin{pmatrix} z & h_n \\ \bar{h}_n z & 1 \end{pmatrix}}_{\sigma_n\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}} \cdots\cdots \frac{1}{k_{m+1}}\begin{pmatrix} z & h_{m+1} \\ \bar{h}_{m+1} z & 1 \end{pmatrix}$$

$$\sigma_{n-1}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \sigma_{m+2}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \underbrace{\sigma_{m+1}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}}_{\in M_2\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}}$$

$$\sigma\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} F_n \\ F_n \end{pmatrix} = \sigma\begin{pmatrix} z F_n \\ F_n \end{pmatrix} \subset \begin{pmatrix} F_{n+1} \\ F_{n+1} \end{pmatrix}$$

$\therefore$

$$\underbrace{\sigma_{m+3}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \underbrace{\sigma_{m+2}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\sigma_{m+1}}_{\in M_2(F_1)}}_{M_2(F_2}$$

$$T_{nm} \in M_2(F_{n-m-1})\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \in \begin{pmatrix} zF_{n-m-1} & F_{n-m-1} \\ zF_{n-m-1} & F_{n-m-1} \end{pmatrix}$$

$$\begin{pmatrix} z^{-n} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & z^{-n} h_n \\ \bar{h}_n z^n & 1 \end{pmatrix}\begin{pmatrix} z^{-n+1} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$\tilde{T}_{nm} = \begin{pmatrix} z^{-n} & 0 \\ 0 & 1 \end{pmatrix} T_{nm} \begin{pmatrix} z^m & 0 \\ 0 & 1 \end{pmatrix}$$

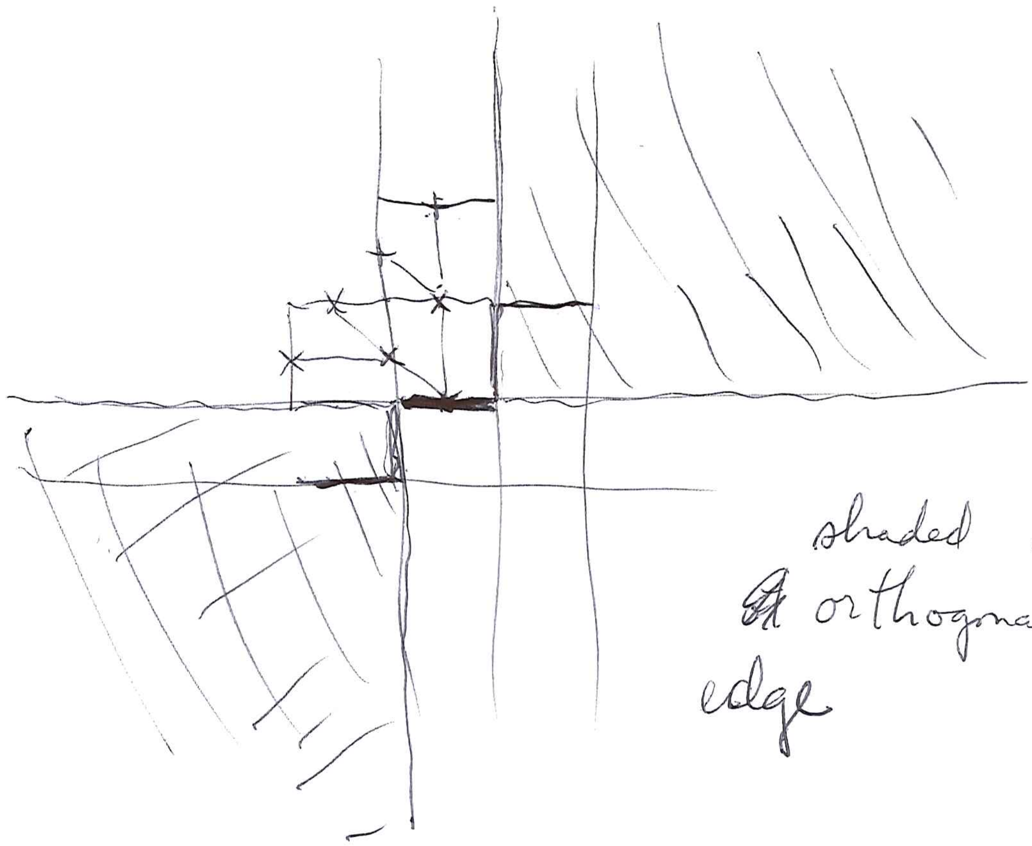
$kg - \bar{h}p' = g'$

$p = \frac{1}{k}(p' + h(kg - \bar{h}p')) = k p' + h g$

$$\begin{pmatrix} z^{-n} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z F_{n-m-1} & F_{n-m-1} \\ z F_{n-m-1} & F_{n-m-1} \end{pmatrix}\begin{pmatrix} z^{m} & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} z^{-n+m+1} F_{n-m-1} & z^{-n} F_{n-m-1} \\ z^{m+1} F_{n-m-1} & F_{n-m-1} \end{pmatrix}$$

$$= \begin{pmatrix} \langle 1, z^{-1}, \dots, z^{-n+m+1} \rangle & \langle z^{-n}, z^{-n+1}, \dots, z^{-m-1} \rangle \\ \langle z^{m+1}, \dots, z^{n} \rangle & \langle 1, z, \dots, z^{n-m-1} \rangle \end{pmatrix}$$



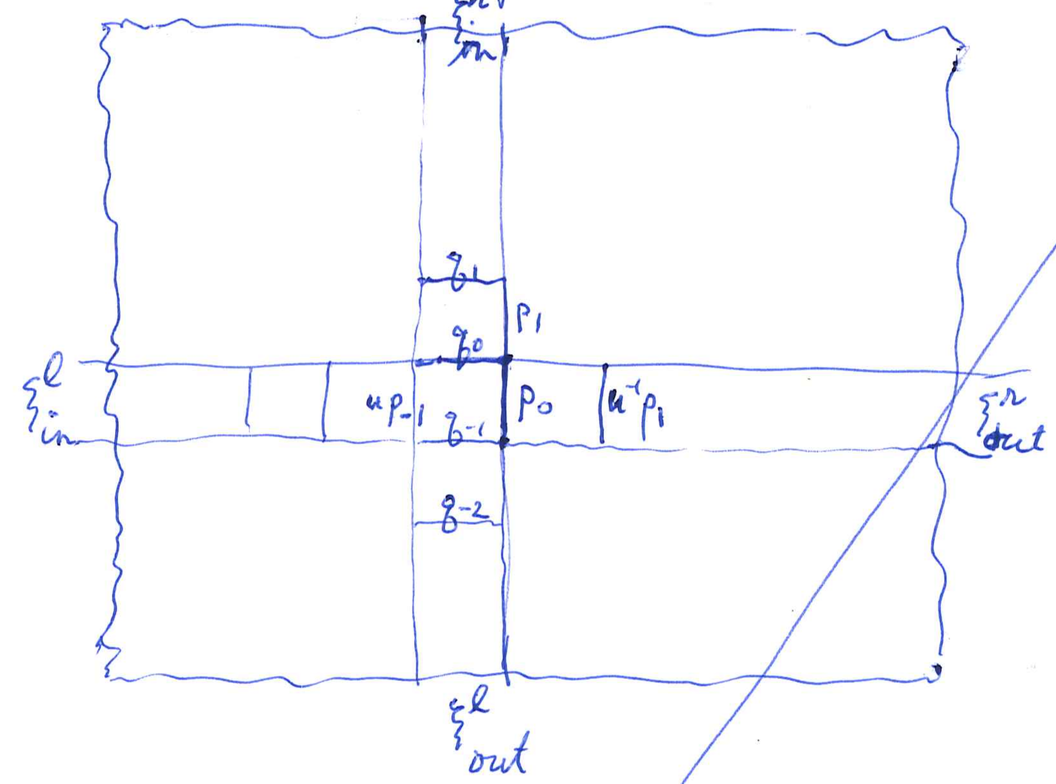
shaded region orthogonal to the edge

Get two sided scattering straight

Take finite support (h_n). For $|n|$ large

$\begin{pmatrix} z^{-n}p_n \\ q_n \end{pmatrix}$ constant. i.e.

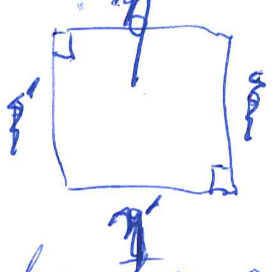
In our Hilbert space



$\xi_+^*\xi_-$

Point $\begin{pmatrix} \xi^r_{out} \\ \xi^r_{in} \end{pmatrix} = T_{\infty,-\infty}\begin{pmatrix} \xi^l_{in} \\ \xi^l_{out} \end{pmatrix} = \begin{pmatrix} d^* & c^* \\ c & d \end{pmatrix}\begin{pmatrix} \xi^l_{in} \\ \xi^l_{out} \end{pmatrix}$

There's a new "square" here.

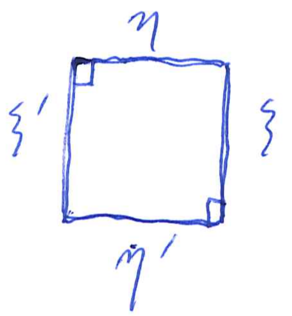


$\eta^*\xi$

where $h =$ is a function on S^1. So you are doing C^*modules. Instead of the inner product being a scalar it lies in $C(S^1)$.

$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} \xi' \\ \eta' \end{pmatrix}$ NO different d^*, d

You have to set this up carefully to avoid taking positive square roots.



$$\begin{pmatrix}\xi\\ \eta'\end{pmatrix} = \underbrace{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}}_{\text{unitary}}\begin{pmatrix}\xi'\\ \eta\end{pmatrix} \qquad \eta^*\xi = \beta$$

$$\eta' = \gamma\xi' + \delta\eta$$

$$\eta' - \gamma\xi' = \delta\eta$$

$$\eta = \frac{1}{\delta}(\eta' - \gamma\xi')$$

$$\xi = \alpha\xi' + \beta\frac{1}{\delta}(\eta' - \gamma\xi')$$

$$= \left(\alpha - \beta\frac{1}{\delta}\gamma\right)\xi' + \frac{\beta}{\delta}\eta'$$

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \underbrace{\begin{pmatrix}\alpha - \frac{\beta\gamma}{\delta} & \frac{\beta}{\delta}\\ -\frac{\gamma}{\delta} & \frac{1}{\delta}\end{pmatrix}}_{\text{transfer}}\begin{pmatrix}\xi'\\ \eta'\end{pmatrix}$$

$$\begin{vmatrix}\alpha - \frac{\beta\gamma}{\delta} & \frac{\beta}{\delta}\\ -\frac{\gamma}{\delta} & \frac{1}{\delta}\end{vmatrix} = \frac{\alpha\delta - \beta\gamma}{\delta^2} + \frac{\beta\gamma}{\delta^2} = \frac{\alpha}{\delta}$$

$$\begin{pmatrix}\alpha - \frac{\beta\gamma}{\delta} & \frac{\beta}{\delta}\\ -\frac{\gamma}{\delta} & \frac{1}{\delta}\end{pmatrix} = \begin{pmatrix}d^* & c^*\\ c & d\end{pmatrix} \qquad b$$

$$\delta = \frac{1}{d} \qquad \beta = \delta c^* = \frac{c^*}{d} \qquad \gamma = -\delta c = -\frac{c}{d}$$

$$d^* = \alpha - \frac{\beta\gamma}{\delta} = \alpha - \frac{c^*}{d}\left(-\frac{c}{d}\right)d = \alpha + \frac{c^*c}{d} \qquad \underset{= 1/d}{\alpha} = d^* - \frac{c^*c}{d} = \frac{d^*d - c^*c}{d}$$

So if $\begin{pmatrix} d^* & c^* \\ c & d \end{pmatrix}$ is the transfer matrix

then $\begin{pmatrix} \frac{1}{d} & \frac{c^*}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}$ is the unitary scattering matrix.

$$\xi = a\xi' + b\eta' \qquad \eta' = \frac{1}{d}(\eta - c\xi')$$

$$\eta = c\xi' + d\eta' \qquad = -\frac{c}{d}\xi' + \frac{1}{d}\eta$$

$$\xi = a\xi' + b\left(-\frac{c}{d}\xi' + \frac{1}{d}\eta\right) \qquad \xi = \left(a - \frac{bc}{d}\right)\xi' + \frac{b}{d}\eta$$

$$\frac{ad - bc}{dd} + \frac{b}{d}d\left(+\frac{c}{d}\right) = a$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{b^*}{d} & \frac{1}{d} \end{pmatrix} \qquad \frac{1 + |b|^2}{|d|^2} = 1.$$

~~Summary~~ What comes next? Basically you start with a $\beta(z)$, smooth function on S^1, s.t. $1 - |\beta|^2 > 0$, form the corresp. δ

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{\bar{b}}{d} & \frac{1}{d} \end{pmatrix} \qquad \left|\frac{1}{d}\right|^2 + \left|\frac{b}{d}\right|^2 = \frac{1 + |b|^2}{|d|^2} = 1.$$

Anyway what should I do? You want to reconstruct (h_n) from the scattering, the reflection coeff. β.

$$\beta = \begin{pmatrix} \xi^n \\ \xi_{out} \end{pmatrix}^* \begin{pmatrix} \xi^n \\ \xi_{in} \end{pmatrix}$$

Recall $\tilde{T}_{nm} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^{+n} & 1 \end{pmatrix} \cdots \frac{1}{k_{m+1}}\begin{pmatrix} 1 & h_m z^{-m-1} \\ \bar{h}_m z^{m+1} & 1 \end{pmatrix}$

$$= \begin{pmatrix} d^* & c^* \\ c & d \end{pmatrix} \qquad n > m$$

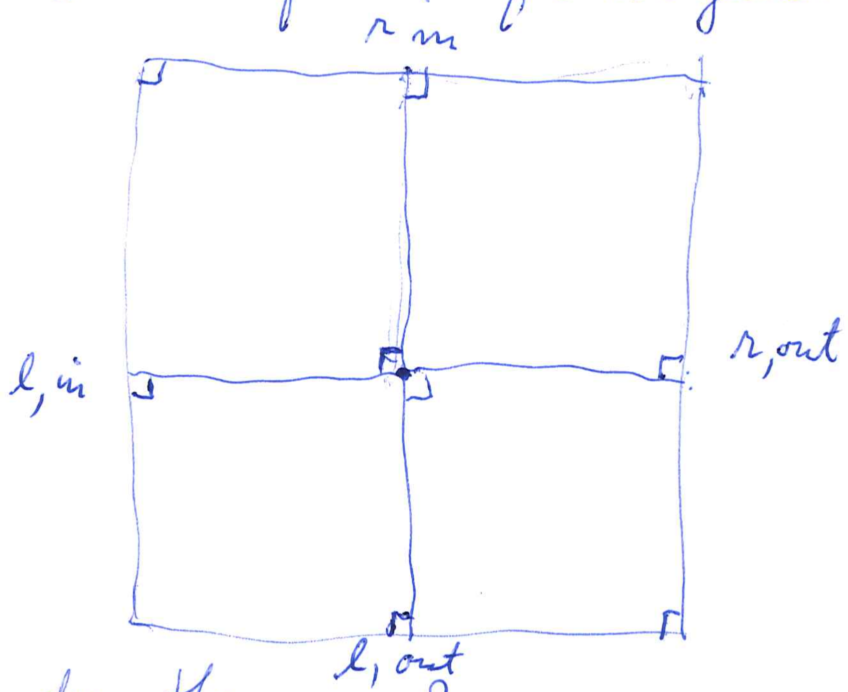
where $c \in [z^{m+1}, .., z^n]$ $\quad d \in [1, \cdots, z^{n-m-1}]$

We want to factor

$$\tilde{T}_{nm} = \tilde{T}_{n0}\,\tilde{T}_{0m}$$

assuming $n > 0 > m$

Your idea of the factorization looks like



What does this mean?

$$\begin{pmatrix} \xi^r_{out} \\ \xi^r_{in} \end{pmatrix} = \lim_{n \to +\infty} \begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix} = \tilde{T}_{n0}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$\tilde{T}_{n0} = \begin{pmatrix} d^* & c^* \\ c & d \end{pmatrix}$$ with z replaced by u.

$c \in [z^1, .., z^n]$ $\quad d \in [1, .., z^{n-1}]$

Consider a finite support (h_n). Idea: Compare the ~~cyclic~~ three generating sets $\begin{pmatrix}\xi^l_{in}\\ \xi^l_{out}\end{pmatrix}$ $\begin{pmatrix}p_0\\ q_0\end{pmatrix}$ $\begin{pmatrix}\xi^r_{out}\\ \xi^r_{in}\end{pmatrix}$.

Analogy with half line case:

$H, u.$ ξ_- ξ_0 ξ_+

$H = L^2(d\mu)$, $\lim q_n = \xi_-$, $\lim u^{-n} p_n = \xi_+$.

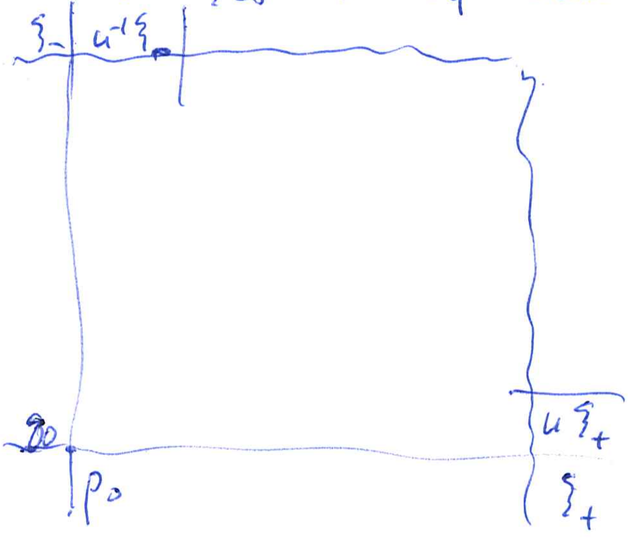
Again I won't use the h_n but rather the smooth scattering data, i.e. the analog of the function g.

Go over the rank 1 case. Scattering function $S(z)$ which is a smooth loop in S^1 of degree 0 which you factor to get g.

$$S(z) = \lim \frac{z^{-n} p_n}{q_n}$$

If you start with S, then what? What path goes from S to the Hilbert space with array of unit vectors?

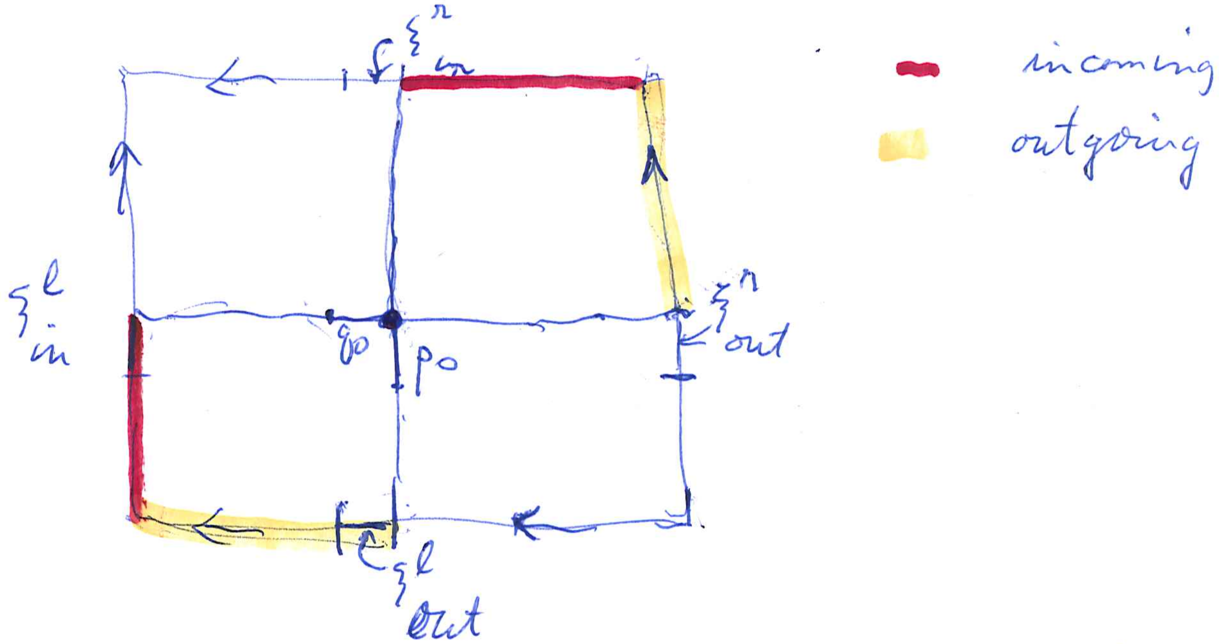
Answer is the filtration



$$F_{00} = (H_-\xi_-)^\perp \cap (zH_+\xi_+)^\perp$$
$$= (H_-\xi_- + zH_+\xi_+)^\perp$$

$$H_-\xi_- + zH_+\xi_+ \xrightarrow{\sim} H_- + zH_+ S$$

I am still puzzled by all of this. What to do. Think perturbation, e.g. (h_n) to first order. What happens if $h_n = 0$ $\forall n$. Then $u^{-n}p_n$, q_n are constant fn. of n.



For each (m,n) you have an orthogonal splitting. Work out the algebra

Transfer matrix. $\begin{pmatrix} d^* & c^* \\ c & d \end{pmatrix}$ scat. $\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}$

$$S = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}$$

a smooth loop in $U(2)$.

$$\det S = \frac{1+cc^*}{d^2} = \frac{d\bar{d}}{d^2} = \frac{\bar{d}}{d}$$

This has index 0, as d is analytic invertible on $\bar{D}$

$$\begin{pmatrix} \xi^r_{out} \\ \xi^r_{in} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi^l_{in} \\ \xi^l_{out} \end{pmatrix} \qquad \begin{pmatrix} \xi^r_{out} \\ \xi^l_{out} \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi^l_{in} \\ \xi^r_{in} \end{pmatrix}$$

Can you see that $\begin{pmatrix} uH_- \xi^l_{in} \\ H_- \xi^r_{in} \end{pmatrix} \dotplus \begin{pmatrix} H_+ u\xi^r_{out} \\ H_+ \xi^l_{out} \end{pmatrix} = H$

Thus you want ! soln for

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} f_- \\ g_- \end{pmatrix} + \begin{pmatrix} f_+ \\ g_+ \end{pmatrix} = \text{arb. elt of } L^2(S^1)^{\oplus 2}$$

Can you relate this to factorization of the transfer matrix? Actually you should check the line bundle with clutching fn. S is trivial.

need $S v_- = v_+$ to have only trivial solns.

$$-\frac{c}{d} f_- + \frac{1}{d} g_- \in H_+$$

d invertible on H_+

so $-cf_- + g_- \in H_+$

also

$f_- + bg_- \in H_+$

$$\begin{vmatrix} 1 & b \\ -c & 1 \end{vmatrix} = 1 + cc^* = d\bar{d}$$

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} f_- \\ g_- \end{pmatrix} = \begin{pmatrix} f_+ \\ g_+ \end{pmatrix}$$

$$\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix}\begin{pmatrix} f_- \\ g_- \end{pmatrix} = \begin{pmatrix} df_+ \\ dg_+ \end{pmatrix}$$

$f_- + bg_- \in H_+$

$-bcf_- + bg_- \in bH_+$

$$\begin{pmatrix} f_- \\ g_- \end{pmatrix} = \frac{1}{d\bar{d}}\begin{pmatrix} 1 & -b \\ c & 1 \end{pmatrix}\begin{pmatrix} df_+ \\ dg_+ \end{pmatrix}$$

$$\bar{d}\begin{pmatrix} f_- \\ g_- \end{pmatrix} = \begin{pmatrix} 1 & -b \\ c & 1 \end{pmatrix}\begin{pmatrix} f_+ \\ g_+ \end{pmatrix}$$

Basic problem. Suppose you have a dIdDE with (h_n) fin. support, whence $q_n = \xi^r_{in}$ $n>>0$

$q_n = \xi^l_{out}$ $n<<0$, $u^{-n}p_n = \xi^r_{out}$ $n>>0$

and $u^{-n}p_n = \xi^l_{in}$ $n<<0$. You have the transfer matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\ni \begin{pmatrix} \xi^r_{out} \\ \xi^r_{in} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi^l_{in} \\ \xi^l_{out} \end{pmatrix}$$

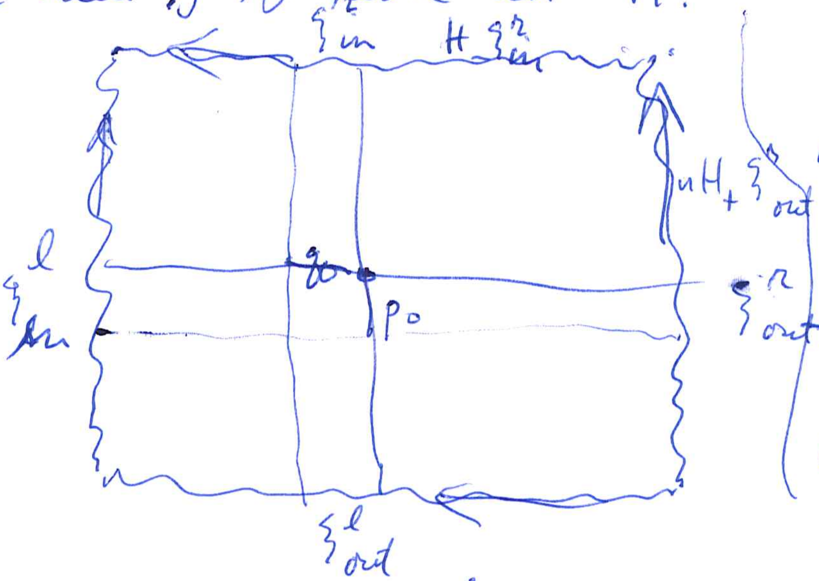
$a = d^*$, $b = c^*$, $ad - bc = 1$

which can be put into scattering form

$$\begin{pmatrix} \xi^r_{out} \\ \xi^l_{out} \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ \frac{-c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi^l_{in} \\ \xi^r_{in} \end{pmatrix}$$

The problem is to express $\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$ in terms of $\begin{pmatrix} \xi^l_{in} \\ \xi^r_{in} \end{pmatrix}$

The idea is to look in H:



There's a ~~splitting~~ splitting of H into two subspaces

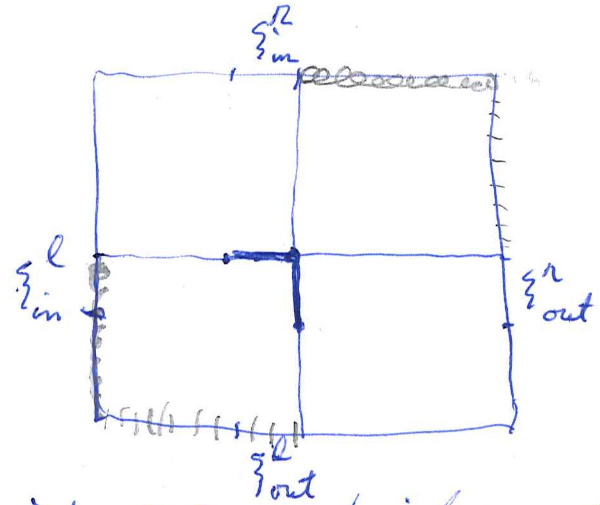
$$\left(uH_+\xi^r_{out} + H_-\xi^r_{in}\right) \text{ and } \left(H_+\xi^l_{out} + uH_-\xi^l_{in}\right)$$

rearrange into the subspaces

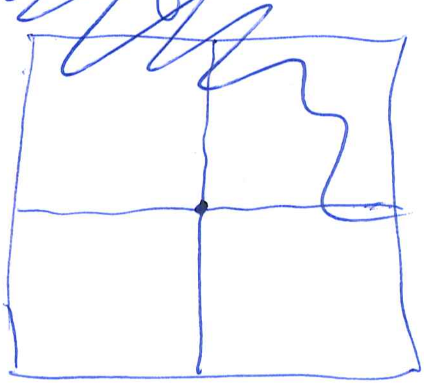
$$H_+ u\xi^r_{out} + H_+\xi^l_{out} \quad \text{and} \quad H_- u\xi^l_{in} + H_-\xi^r_{in}$$

Review: First suppose $h_n = 0 \;\; \forall n$, so that $q_n = \xi^r_{in} = \xi^l_{out}$ $\quad u^{-n} p_n = \xi^r_{out} = \xi^l_{in}$ for all n. So $H = L^2(S^1)^{\oplus 2}$. ~~The basic idea~~

Picture



~~think of~~



~~Better~~ the point 0,0 specifies incoming and outgoing subspaces

incoming $\quad H_- \xi^r_{in} + H_- u \xi^l_{in}$

outgoing $\quad H_+ \xi^l_{out} + H_+ u \xi^r_{out}$

When all $h_n = 0$ these are complementary. (You can get rid of the u by moving $(0,0)$ to $(0,-1)$?)

$$\begin{pmatrix} \xi^r_{out} \\ \xi^l_{out} \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi^l_{in} \\ \xi^r_{in} \end{pmatrix}$$

See what the subspaces are

$$\begin{pmatrix} zf_+ & g_+ \end{pmatrix} \begin{pmatrix} \xi^r_{out} \\ \xi^l_{out} \end{pmatrix} \doteq \begin{pmatrix} zf_+ & g_+ \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$= \begin{pmatrix} zf_+ \frac{1}{d} + g_+\left(-\frac{c}{d}\right) \\ \cancel{zf_+ \frac{b}{d} + g_+ \frac{1}{d}} \end{pmatrix} = \begin{pmatrix} zf_+ - cg_+ \\ \cancel{bzf_+ + g_+} \end{pmatrix} \frac{1}{d}$$

row vector

$$H_+ \xi^l_{out} + H_+ u \xi^r_{out} = \frac{1}{d}(zf_+ - cg_+)\xi^l_{in} + \frac{1}{d}(bzf_+ + g_+)\xi^r_{in}$$

Check this zero intersects with $H_- u \xi^l_{in} + H_- \xi^r_{in}$

$$\frac{1}{d}(zf_+ - cg_+) \in zH_-$$

$$\frac{1}{d}(bzf_+ + g_+) \in H_-$$

simplifies to

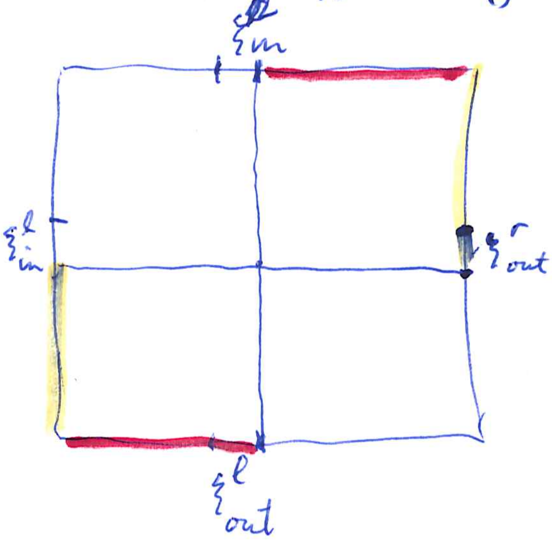
$$z\tilde{f}_+ - c\tilde{g}_+ \in zH_- \qquad \tilde{f}_+ = \frac{f_+}{d}$$

$$bz\tilde{f}_+ + \tilde{g}_+ \in H_-$$

$$\tilde{f}_+ - z^{-1}c\tilde{g}_+ \in H_-$$

$$bz\tilde{f}_+ + \tilde{g}_+ \in H_-$$

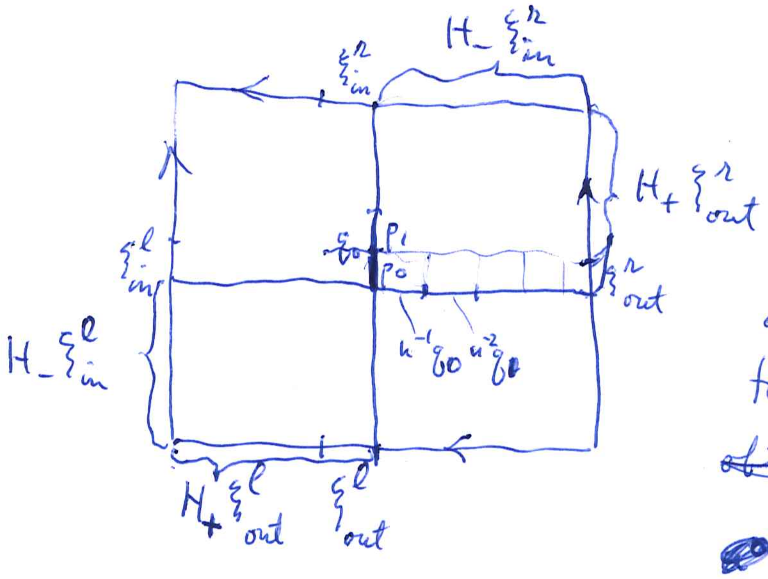
Set this up differently



$$H_+ \xi^r_{out} + H_+ \xi^l_{out}$$

$$H_- \xi^l_{in} + H_- \xi^r_{in}$$

There are four subspaces here which ~~complementary~~ should split H.



Aim to understand this

You seek a kind of orthogonalization process. $p_0, p_1, \ldots$ is an orthonormal sequence orthogonal to $H_-\xi^r_{in}$. ~~In fact you obtain~~ You get p_0 by making ξ^r_{out} orthogonal to $H_-\xi^r_{in}$ NO

$$\xi^r_{out} - \sum_{k=1}^{\infty} u^{-k}\xi^r_{in}\left(u^{-k}\xi^r_{in} \,\middle|\, \xi^r_{out}\right)$$

$$\xi^r_{out} \perp uH_+\xi^r_{out}$$

$$p_0 \perp H_-\xi^r_{in} + uH_+\xi^r_{out}$$

$$\tilde{p}_0 \in \left(H_-\xi^r_{in} + uH_+\xi^r_{out}\right)^{\perp} \cap \left(\xi^r_{out} + H_-\xi^r_{in} + uH_+\xi^r_{out}\right)$$

$$\tilde{p}_0 = \quad + \sum_{k\geq 1} a_{-k} u^{-k}\xi^r_{in} + \sum_{l\geq 0} b_l u^l \xi^r_{out}$$

$b_0 = 1$

$$\tilde{p}_0 = \xi^r_{out} + \sum_{l\geq 1} b_l u^l \xi^r_{out} + \sum_{k\geq 1} a_{+k} u^{-k}\xi^r_{in}$$

b_l, a_{+k} chosen to minimize $\|\tilde{p}_0\|^2$.

$$\tilde{p}_0 = f_+(u)\xi^r_{out} + g_-(u)\xi^r_{in}$$

$f_+ \in 1 + zH_+$

$g_- \in H_-$

You have a concrete problem now to solve

$$0 = \left(u^{\ell} \xi^{r}_{out} \middle| \tilde{p}_0 \right) = b_\ell + \sum_{k \geq 1} \left(u^{\ell} \xi^{r}_{out} \middle| u^{-k} \xi^{r}_{in} \right) a_{+k} \qquad \text{for } \ell \geq 1$$

$$0 = \left(u^{-k} \xi^{r}_{in} \middle| \tilde{p}_0 \right) = \sum_{\ell \geq 0} \left(u^{-k} \xi^{r}_{in} \middle| u^{+\ell} \xi^{r}_{out} \right) b_\ell + a_{+k} \qquad \text{for } k \geq 1.$$

So you have a matrix. $S_{k+\ell} = \left(u^{-k} \xi^{r}_{in} \middle| u^{\ell} \xi^{r}_{out} \right)$

and you want to solve

$$b_\ell + \sum_{k \geq 1} \bar{S}_{k+\ell}\, a_k = 0 \qquad \ell \geq 1$$

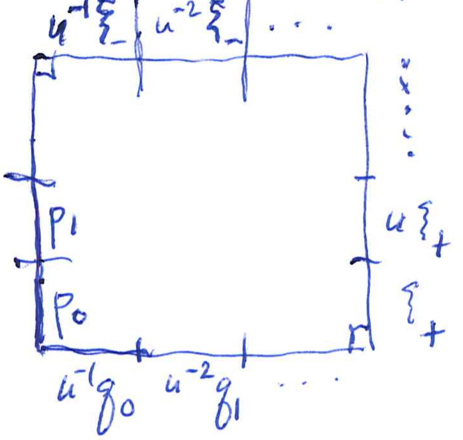
$$a_k + S_k b_0 + \sum_{\ell \geq 1} S_{k+\ell}\, b_\ell = 0 \qquad k \geq 1$$

$$\tilde{p}_0 = \sum_{k \geq 1} a_k u^{-k} \xi^{r}_{in} + \sum_{\ell \geq 0} b_\ell u^{\ell} \xi^{r}_{out}$$

$$0 = \left(u^{-k} \xi^{r}_{in} \middle| \tilde{p}_0 \right) = a_k + \sum_{\ell \geq 0} b_\ell S_{k+\ell} \qquad k \geq 1$$

$$0 = \left(u^{\ell} \xi^{r}_{out} \middle| \tilde{p}_0 \right) = \sum_{k \geq 1} a_k \bar{S}_{k+\ell} + b_\ell \qquad \ell \geq 1$$

At this point I see pretty much how to get around the obstacle. The point you missed: ~~Given~~ Given $H_-\xi^n_{in} + H_+\xi^n_{out}$ you want to construct opposite orthonormal bases



You want to solve

$$\tilde{p}_0 = f_+(u)\xi_+ + f_-(u)\xi_-$$

with $\quad f_+(0) = 1 \qquad (uH_+\xi_+, \tilde{p}_0) = (H_-\xi_-, \tilde{p}_0) = 0$

$$\tilde{p}_0 = \sum_{l \geq 0} b_l u^l \xi_+ + \sum_{k>0} a_k u^{-k}\xi_-$$

$l \geq 0$

$$0 = (u^l \xi_+, \tilde{p}_0) = b_l + \sum_{k>0} a_k \overbrace{(u^l\xi_+, u^{-k}\xi_-)}^{\bar{S}_{lk}}$$

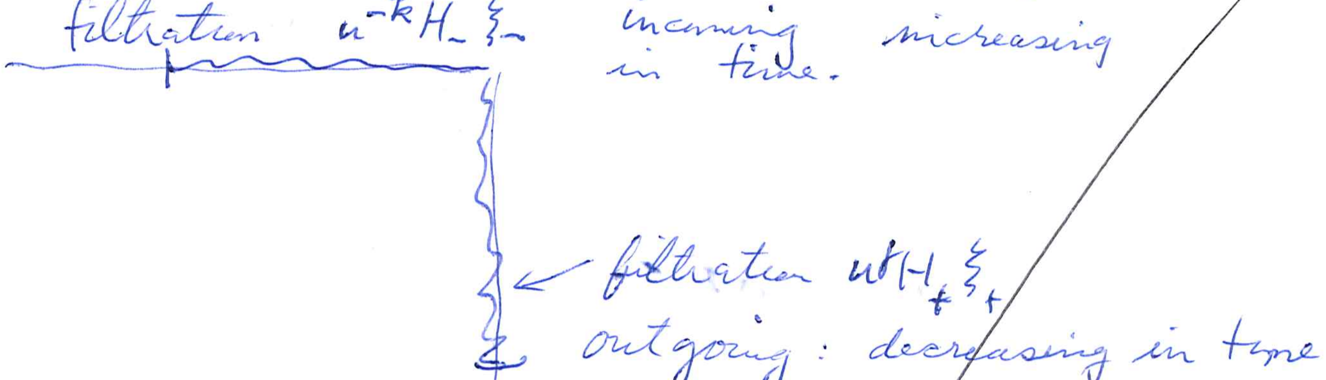
$$0 = (u^{-k}\xi_-, \tilde{p}_0) = \sum_{l \geq 0} b_l \underbrace{(u^{-k}\xi_-, u^l\xi_+)}_{S_{kl}} + a_k$$

Let $\underline{b} = (b_l)_{l \geq 1}$. Then $\quad b + S^*a = 0$

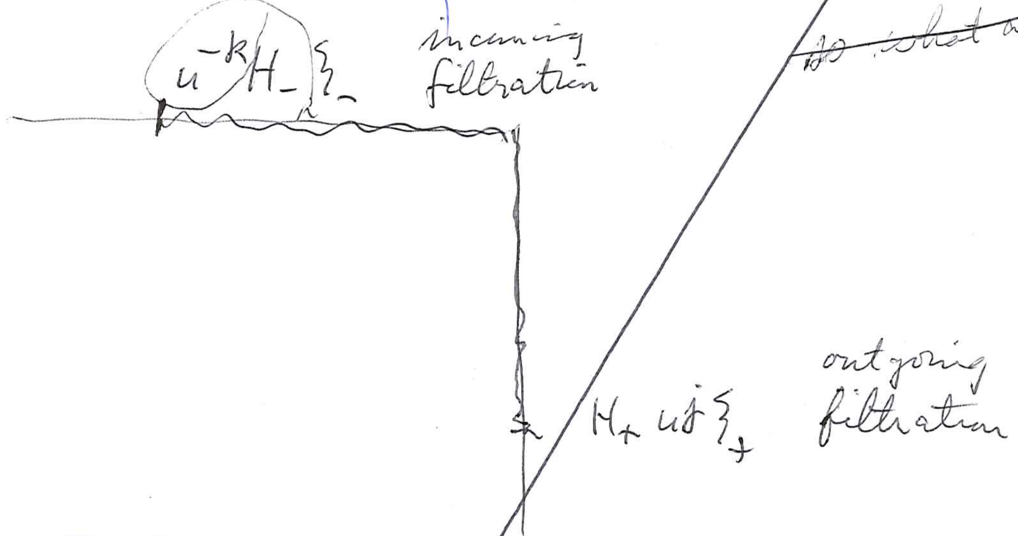
~~$b + S^*a$~~ and $\quad a + Sb + S_{*,0}\, b_0 = 0$

$$\boxed{a - SS^*a + S_{*,0}\, b_0 = 0}$$

Go over what you learned. Given



Given



~~so what am~~

This is inside a Hilbert space $H = L^2(S^1)\xi_+ + L^2(S^1)\xi_-$ whose inner product results from a contraction, reflection coefficient $S: L^2(S^1)\xi_+ \longrightarrow L^2(S^1)\xi_-$ $S_{kj} = (u^{-k}\xi_-, u^{j}\xi_+)$, (which commutes with u i.e. S_{kj} depends only on $k+j$, ignore for the moment) $\|f_+\xi_+ + f_-\xi_-\|^2 =$

$$\begin{pmatrix} f_+ \\ f_- \end{pmatrix}^* \begin{pmatrix} 1 & S \\ S^* & 1 \end{pmatrix} \begin{pmatrix} f_+ \\ f_- \end{pmatrix}.$$

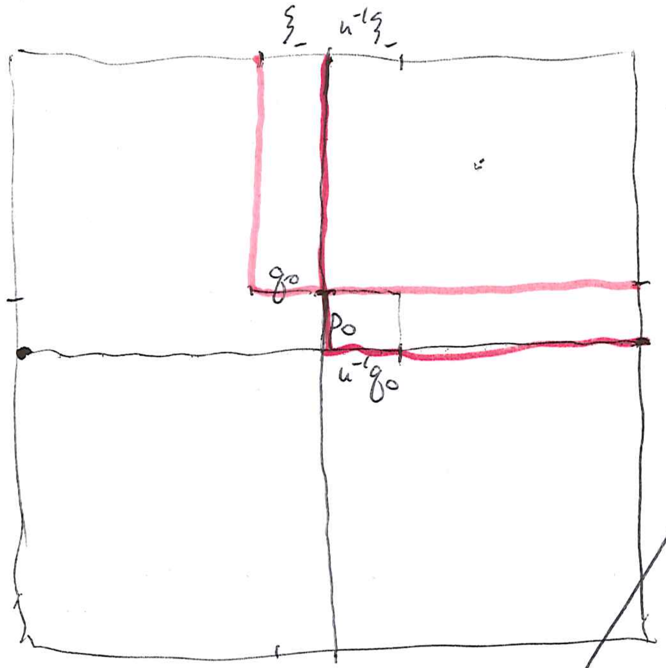
Now you put in your filtration.

The only concern here is probably that H should be the alg. sum of $L^2(S^1)\xi_\pm$ which means basically that $(1-S^*S)^{1/2}, (1-SS^*)^{1/2}$ have closed image, hence are bounded away from 0.

~~Talk~~

Consider unitary ref. $S = \frac{\bar{g}}{g}$

$$S_{kj} = (u^{-k}\xi_-, u^{j}\xi_+)$$



$$p_0 \in (H_+\xi_+ + H_-\xi_-) \ominus (H_+u\xi_+ + H_-\xi_-)$$

$$p_0 = b_0\xi_+ + \sum_{j\geq 1} b_j u^j \xi_+ + \sum$$

$$p_0 = b_0\xi_+ + (\text{something in } H_+u\xi_+ + H_-\xi_-)$$

$$q_0 = a_0\xi_- + (\text{——— " })$$

$$(q_0|p_0) = (q_0|b_0\xi_+) = (a_0\xi_- + \text{something in } H_-\xi_-|b_0\xi_+)$$

Today perhaps you can write up the inverse scattering transform.

First. $\{u^{-k}\xi_-\}$ orthonormal

$H_- u^{-k}\xi_-$ increasing (incoming)

$\{u^j\xi_+\}$ orthonormal

$H_+ u^j \xi_+$ decreasing filt. (~~outgoing~~)

The fact that you have filtrations amounts to an ordering on these orthonormal sets.

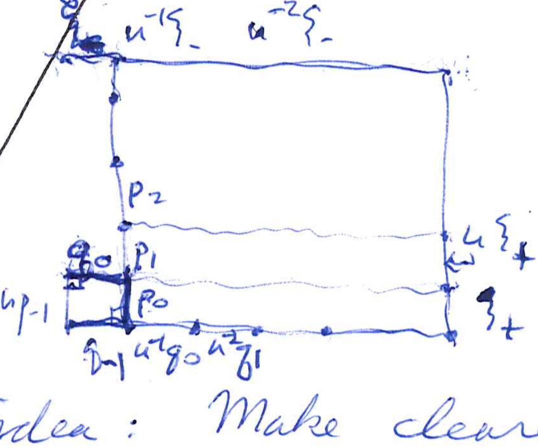
$$S_{k+j} = (u^{-k}\xi_- \mid u^j \xi_+) \qquad \text{contraction}$$

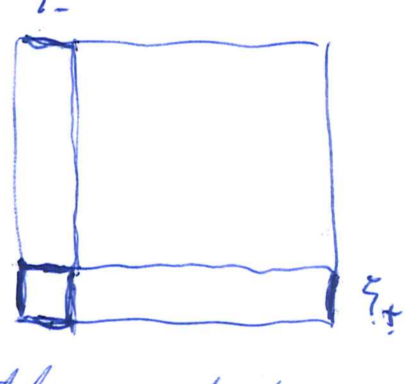
Alg you have $V_\pm = L^2(S^1)\xi_\pm$ glued together via $S : V_+ \longrightarrow V_-$. Assuming $1 - S^*S$, $1 - SS^* > 0$ and invertible you have a transversal situation. Description

$$\| f_+(u)\xi_+ + f_-(u)\xi_- \|^2 = \begin{pmatrix} f_+ \\ f_- \end{pmatrix}^* \begin{pmatrix} 1 & S^* \\ S & 1 \end{pmatrix} \begin{pmatrix} f_+ \\ f_- \end{pmatrix}$$

$\xi_-^* f_-^* f_+ \xi_+$

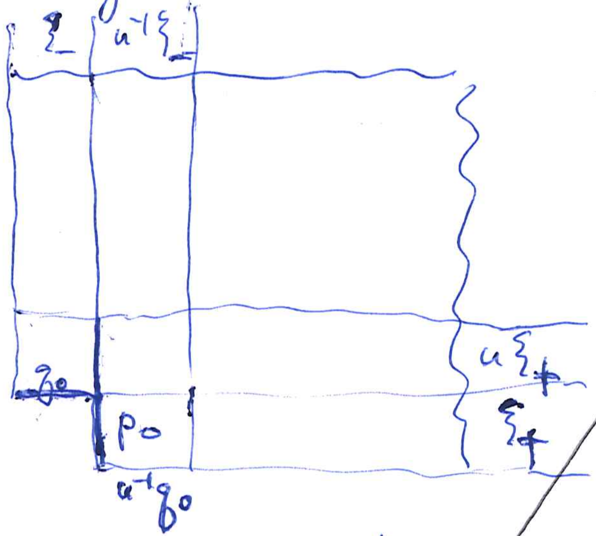
Suppose you restrict to ~~[illegible]~~ the subspace $H_+\xi_+ + H_-\xi_-$ so $f_\pm \in H_\pm$. What can you say about the opposite sides.

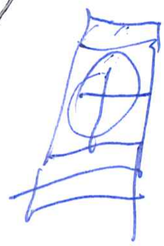




Another idea: Make clearer the relation between S restricted between $H_+\xi_+$ and $H_-\xi_-$ and the Schur expansion.

Problem: Suppose $S = \frac{\bar{g}}{g}$ with g invertible + analytic over $\bar{D}$. Then you get usual





except that p_0, g_0 are dependent

incoming space $u^{-k}H_-\xi_-$

outgoing space $u^{j}H_+\xi_+$

Then $\left(u^{-k}H_-\xi_- + u^{j}H_+\xi_+\right)^{\perp}$ $\qquad \xi_- = S(u)\xi_+$

$$= u^{-k}H_+\xi_- \cap u^{j}H_-\xi_+$$

$$S_{k+j} = \left(u^{-k}\xi_-, u^{j}\xi_+\right)$$

$L^2\left(S^1, \frac{1}{|g|^2}\frac{d\theta}{2\pi}\right)$ $\qquad \left(u^{-k}\frac{\xi_-}{g}, u^{j}\frac{\xi_+}{\bar{g}}\right) = \int z^{k+j}\,\bar{g}^{2}\frac{1}{|g|^2}\frac{d\theta}{2\pi}$

$$= \int z^{k+j}\,\frac{\bar{g}}{g}\,\frac{d\theta}{2\pi} = S_{k+j}$$

$$u^{-k}H_+\xi_- \cap u^{j}H_-\xi_+ \xrightarrow{\sim} z^{-k}H_+ g \cap z^{j}H_-\bar{g}$$

$$= z^{-k}H_+ \cap z^{j}H_-$$

$$\delta_{kj} = \left(u^{-k}\xi_-,\ u^j\xi_+\right)$$

$$= \int z^{+k}\,\bar{q}\ z^{j}\,\bar{q}\ \frac{1}{|q|^2}\,\frac{d\theta}{2\pi}$$

$$= \int z^{j+k}\,\frac{\bar{q}}{q}\,\frac{d\theta}{2\pi}$$

$$\delta_{k+j} = \xi_-^{*}\ z^{k+j}\ \xi_+$$

$$\delta_n = \xi_-^{*}\ u^{n}\ \xi_+$$

$$f_+(u)\,\xi_+ = \sum c_n u^n \xi_+ = \sum_k u^{-k}\xi_-\left(u^{-k}\xi_-\right)^{*}\sum c_n u^n \xi_+$$

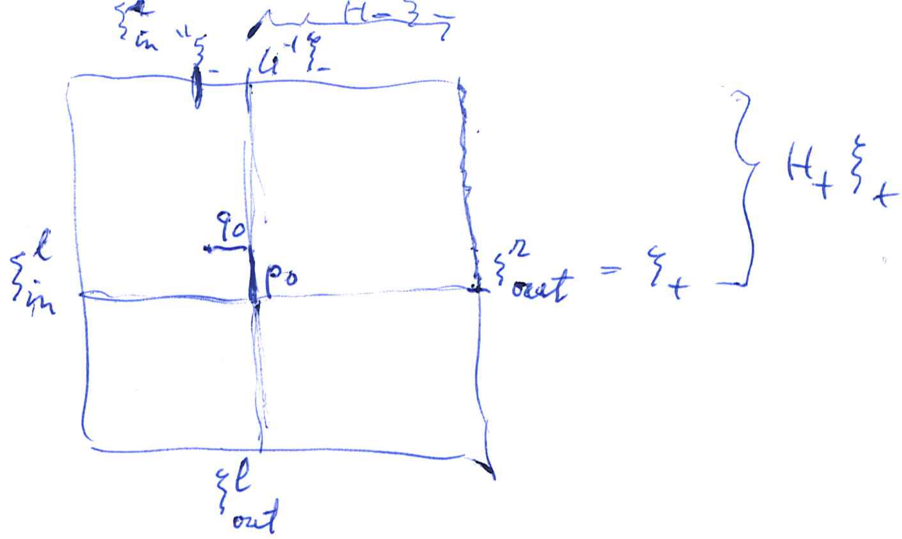
$$= \sum u^{-k}\xi_-\ \xi_-^{*}\sum c_n\, u^{k+n}\xi_+$$

$$= \sum u^{-k}\xi_-\ \sum c_n \delta_{k+n}$$

$$= \sum_k u^{-k}\xi_-\ \sum_n S_{k+n} c_n$$

$$= \sum_{k,n} S_{k+n} c_n u^{-k}\xi_-$$

$$= \sum_{k\geq 1}\Big(\sum_n$$



<u>OKAY</u>

$$\begin{pmatrix}\xi^r_{out}\\ \xi^l_{out}\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\xi^l_{in}\\ \xi^r_{in}\end{pmatrix}$$

note that $\xi^r_{out} = \xi_+$ is $\perp$ $u^{<0}\xi^l_{in}$ so that

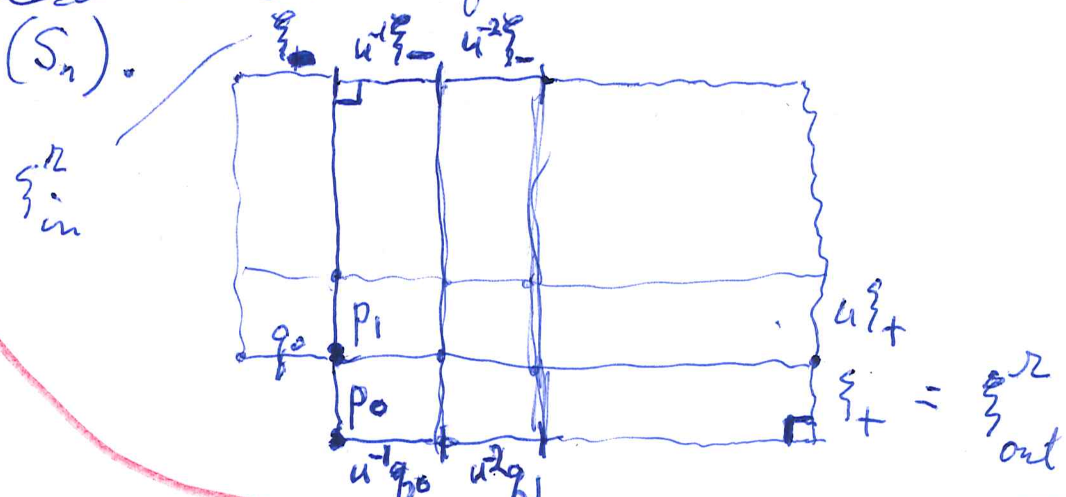
ξ^r_{out} expressed in terms of the orthon. basis $u^{\mathbb{Z}}\xi^l_{in}$, $u^{\mathbb{Z}}\xi^r_{in}$ has no terms $u^{<0}\xi^l_{in}$

$$\therefore \quad \xi^r_{out} = \alpha\,\xi^l_{in} + \beta\,\xi^r_{in} \qquad \alpha = \sum_{n\geq 0}\alpha_n u^n$$

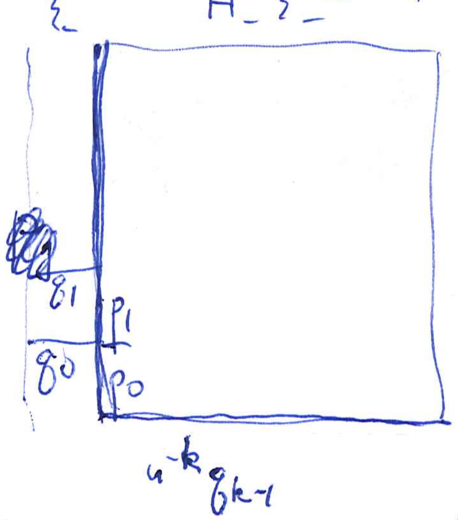
ξ^l_{out} exp. in terms of $u^{\mathbb{Z}}\xi^l_{in}$ $u^{\mathbb{Z}}\xi^r_{in}$

$$\xi^l_{out} = \gamma\,\xi^l_{in} + \delta\,\xi^r_{in}$$

My aims: equivalence between the (h_n) and (S_n).



$H_-\xi_- = \langle u^{-k}\xi_-\rangle_{k\geq 1}$



$H_+\xi_+ = \langle u^j\xi_+\rangle_{j\geq 0}$

There is some big scattering type matrix relating the two orthonormal bases for $H_+\xi_+ + H_-\xi_-$. First basis is

$$p_0, p_1, \dots ; u^{-1}\xi_-, u^{-2}\xi_-, \dots$$

2nd basis $u^{-1}q_0, u^{-2}q_1, \dots ; \xi_+, u\xi_+, \dots$

One example is $L^2\left(S^1, \frac{1}{|g|^2}\frac{d\theta}{2\pi}\right)$ with $p_0 = q_0 = 1$

$L^2(S^1, d\mu)$ $\quad p_0 = q_0 = 1.$

$$\langle p_0, p_1, \dots, p_n\rangle = [z^0, \dots, z^n]$$

$\therefore \langle p_0, p_1, \dots\rangle = \overline{\mathbb{C}[z]}$ in $L^2(S^1, d\mu)$

Now $\overline{\mathbb{C}[z]}$ is outgoing, so get

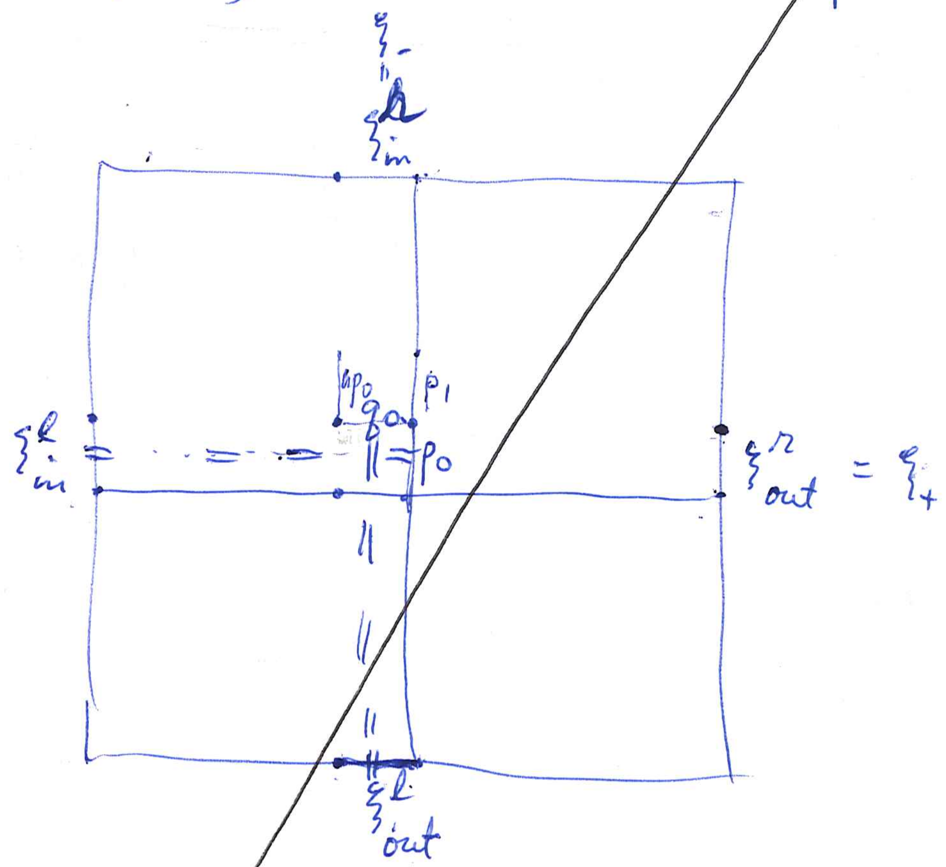
$q = \lim q_n \perp up_j, j \geq 0$

$\;\;\overset{\shortparallel}{\xi_-}$ $\quad$ Thus $\left(u^{-1}\xi_- \,\middle|\, \overline{\mathbb{C}[z]}\right) = 0$

Take an S and restrict attention to $S_{k+j} = (u^{-k}\xi_- \mid u^{j}\xi_+)$ $k+j > 0$.

These coefficients should determine $h_1, h_2, \dots$ Set $h_0, h_{-1}, \dots = 0$. Thus $p_0 = \xi^l_{in}$

$q_0 = \xi^l_{out}$



This is a scattering situation so det. by

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \qquad \begin{pmatrix} \xi_+ \\ q_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} p_0 \\ \xi_- \end{pmatrix}$$

$$\xi_+ = \alpha \overset{p_0}{\xi^l_{in}} + \beta \xi_- \qquad \alpha, \delta \in H_+$$

$$\xi^l_{out} = q_0 = \gamma p_0 + \delta \xi_-$$

ξ^r

Go back to notation

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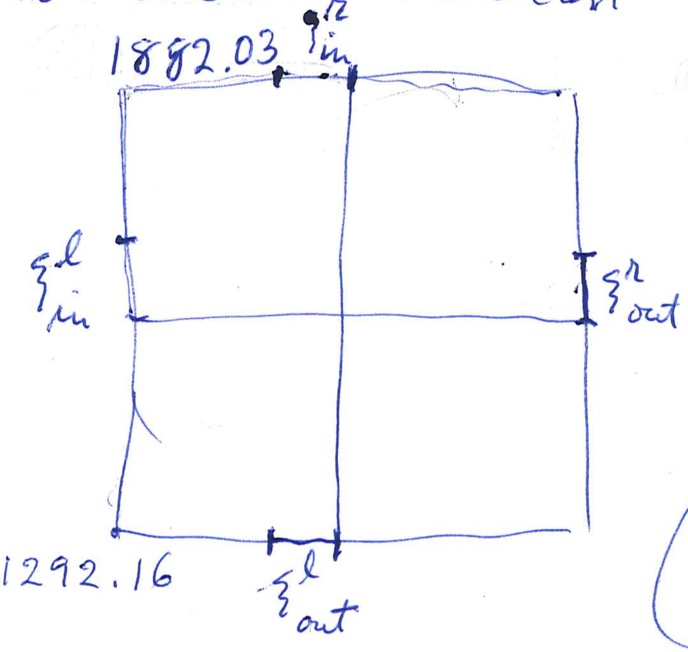


Then

$$\begin{pmatrix} \xi^r_{out} \\ \xi^l_{out} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \xi^l_{in} \\ \xi^r_{in} \end{pmatrix}$$

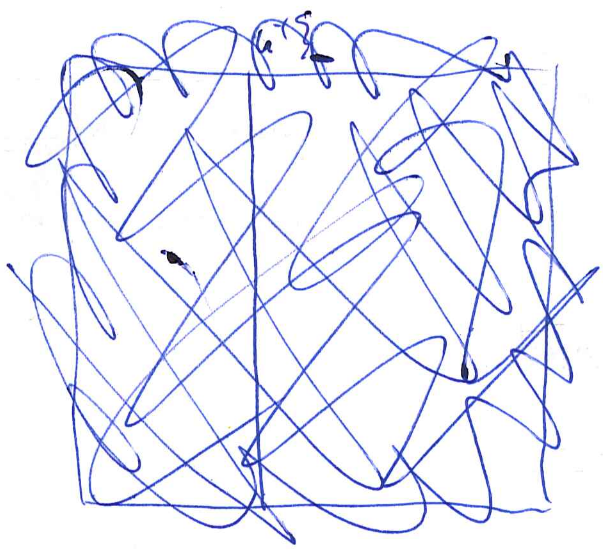
$$\alpha, \delta \in H_+$$

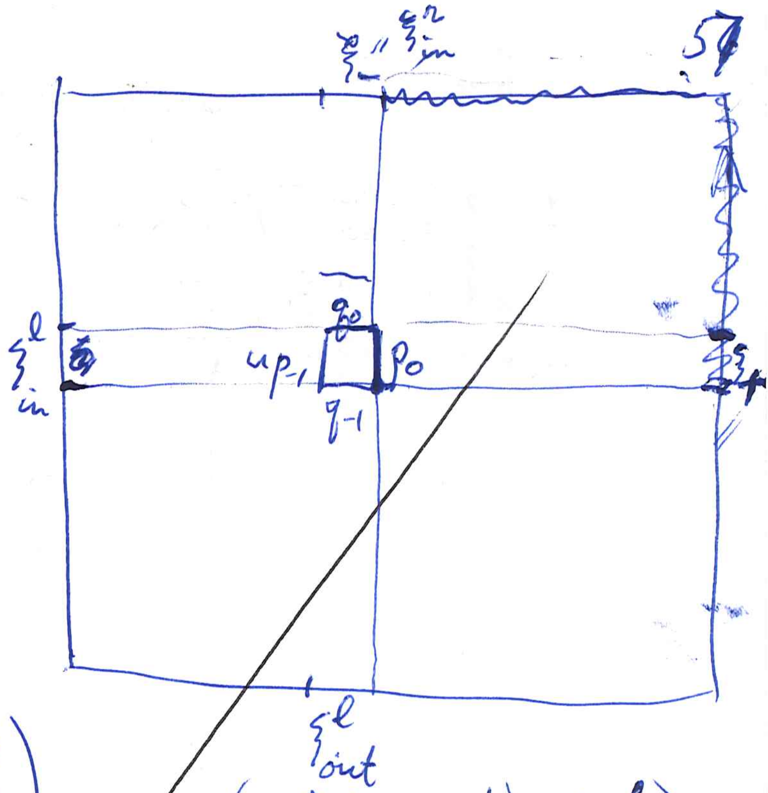
$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U(2, C(S^1))$$

You want to understand why $\alpha = \delta$.
~~the~~ If you write

$$\begin{pmatrix} \xi^r_{out} \\ \xi^l_{out} \end{pmatrix} = \begin{pmatrix} \gamma & \delta \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} \xi^r_{in} \\ \xi^l_{in} \end{pmatrix}$$

$\alpha = \delta$ means symmetry of the unitary matrix $\begin{pmatrix} \gamma & \delta \\ \alpha & \beta \end{pmatrix}$. Does this mean there is a symplectic form around? Should be so because we know that $\alpha = \delta$ is equivalent to the transfer matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ having $\det = 1$. So it should be true that there is a natural volume form on our rank 2 $\mathbb{C}(S^1)$ modules.





$$\begin{pmatrix}\xi^r_{out}\\ \xi^l_{out}\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\xi^l_{in}\\ \xi^r_{in}\end{pmatrix} = \begin{pmatrix}\frac{ad-bc}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi^l_{in}\\ \xi^r_{in}\end{pmatrix}$$

$$\begin{pmatrix}\xi^r_{out}\\ \xi^r_{in}\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi^l_{in}\\ \xi_{out}\end{pmatrix}$$

$$\begin{pmatrix}\xi_+\\ q_0\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}p_0\\ \xi_-\end{pmatrix} \qquad \begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}p_0\\ q_0\end{pmatrix}$$

$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \begin{pmatrix}d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

$$p_0 = d\xi_+ - b\xi_-$$

$$\boxed{p_0 = \sum_{j\geq 0} a_j u^j \xi_+ + \sum_{k\geq 1} b_k u^{-k}\xi_- \in H_+\xi_+ + H_-\xi_-}$$

$$\underset{\overset{\shortparallel}{0}\ \text{for } k\geq 1}{(u^{-k}\xi_- \mid p_0)} = \sum_{j\geq 0} a_j \underbrace{(u^{-k}\xi_- \mid u^j\xi_+)}_{\delta_{j+k}} + b_k$$

Certain things are clear

$$p_0 \in H_+\xi_+ + H_-\xi_-$$

$$a^{-1} q_0 \in H_+\xi_+ + H_-\xi_-$$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} z p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} z p_{-1} \\ q_{-1} \end{pmatrix}$$

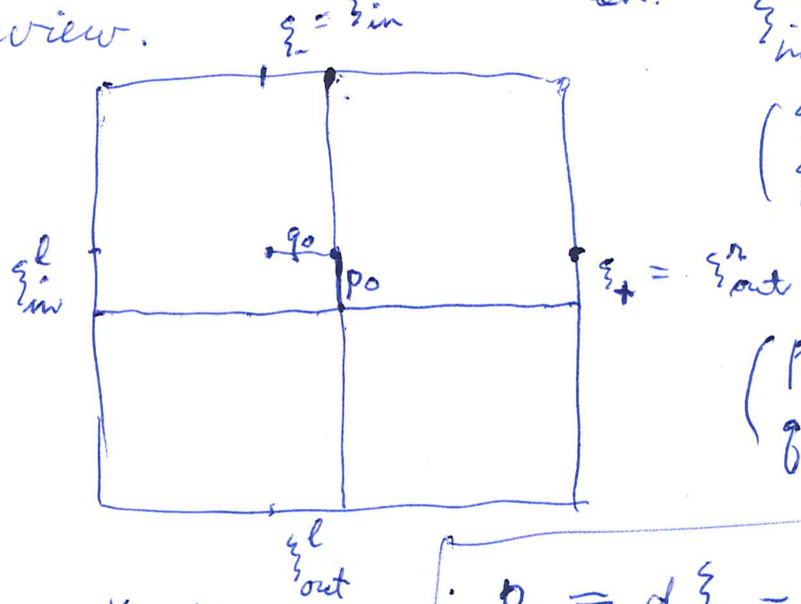
$$p_0 \in H_+\xi_+ + H_-\xi_-$$

$$q_0 \in u H_+\xi_+ + d H_-\xi_-$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

a, d

Review.



$\xi^l_{in} = p_0 \qquad \xi^l_{out} = q_0$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} d^* & b^* \\ c & d \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d & -b^* \\ -c & d^* \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$d \in H_+ \qquad -c^* \in H_-$

$-z^{-1}c \in H_+ \qquad z^{-1}d^* \in H_-$

$$\boxed{\begin{aligned} p_0 &= d\xi_+ - c^*\xi_- \\ q_0 &= -c\xi_+ + d^*\xi_- \end{aligned}}$$

$d \in H_+ \qquad -c^* \in H_-$

$-c \in zH_+ \qquad d^* \in zH_-$

Suppose now you begin with $\bar{g}$ g invertible analytic on $\overline{D}$, let

$S = \frac{\bar{g}}{g}$, focus on $S_n = (\xi_- | u^n \xi_+)$ $n \geq 1$.

$$S_n = \int \bar{g} z^n \bar{g} \frac{1}{|g|^2} \frac{d\theta}{2\pi} = \int z^n \frac{\bar{g}}{g} \frac{d\theta}{2\pi}$$

$$S(z) = \sum S_n z^{-n}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} d^* & c^* \\ c & d \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

You should get

$$\begin{pmatrix} \bar{g} \\ g \end{pmatrix} = \begin{pmatrix} d^* & c^* \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\therefore g = c + d$

Still not clear how to get c, d from g.

$$\begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & h_n u^{-n} \\ \bar{h}_n u^n & 1 \end{pmatrix} \begin{pmatrix} u^{n-1} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$\bar{g} = \frac{\overset{dd^*}{\cancel{1 + c^* c}} + c^* d}{d} = d^* + c^*.$$

$$g = c + d$$

$$dd^* - cc^* = 1$$

both g, d are invertible analytic $c \in zH_+$

Look at scattering picture

$$\begin{pmatrix} \xi_+ \\ q_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & +\frac{c^*}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} p_0 \\ \xi_- \end{pmatrix} \qquad \begin{pmatrix} \bar{g} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{c^*}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} 1 \\ g \end{pmatrix}$$

$$\bar{g} = \frac{1 + c^* g}{d} \qquad 1 = \frac{-c}{d} + \frac{1}{d} g$$

~~So~~ you need to somehow use the fact that only half of the Fourier coeffs of S are relevant. So ask some intelligent questions. Given $S = \frac{\bar{q}}{q}$ you glue $L^2(S^1)\xi_+$ and $L^2(S^1)\xi_-$ via glue $H_+ \xi_+$ and $H_- \xi_-$ so that

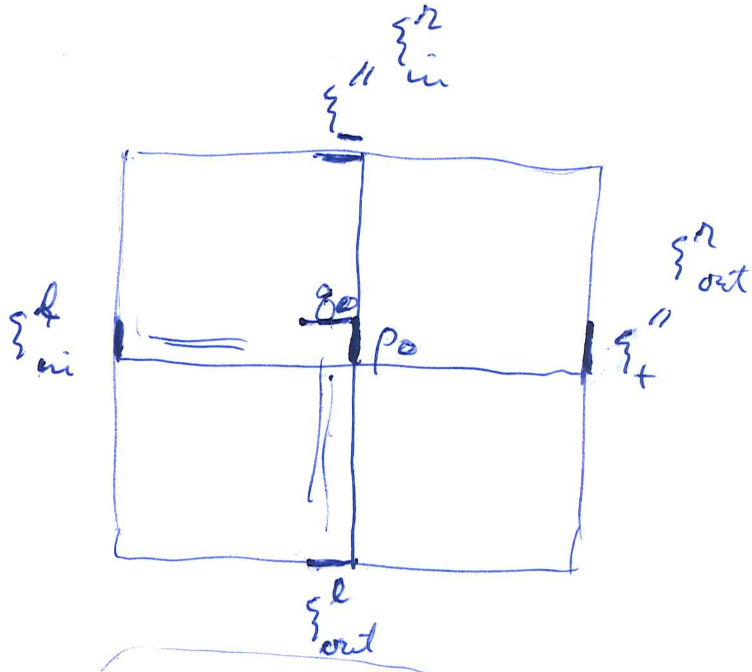
$$(u^{-k}\xi_- \mid u^j \xi_+) = (\xi_- \mid u^{j+k}\xi_+) = \int z^{j+k}\frac{\bar{q}}{q}\frac{d\theta}{2\pi}$$

for $k \geq 1$, $j \geq 0$. How can we alter $\frac{\bar{q}}{q}$ without changing these numbers.

$$S_n = \int z^n S(z)\,\frac{d\theta}{2\pi} \qquad n \geq 1$$

Answer δS can be any F.S. $\sum_{n \leq 0} c_n z^n$

$$\frac{\bar{q}}{q} = \sum_{n \in \mathbb{Z}} c_n z^n$$



$$\xi_+ = \frac{1}{d}p_0 + \frac{c^*}{d}\xi_-$$

$$q_0 = \frac{-c}{d}p_0 + \frac{1}{d}\xi_-$$

$$(\xi_- \mid u^n \xi_+) = \left(\overset{\xi^r_{in}}{\xi''_-} \mid \cancel{u^n \frac{1}{d}\xi^\ell_{in}}\right) + \underbrace{\left(\xi_- \mid u^n \frac{c^*}{d}\xi_-\right)}_{\| \; \int z^n \frac{c^*}{d}\frac{d\theta}{2\pi}}$$

So $\frac{c^*}{d} \equiv \frac{\bar{q}}{q}$ modulo H_+ ?

You want $\int \cancel{z}^n \frac{c^*}{d} \frac{d\theta}{2\pi} = \int z^n \frac{\bar{g}}{g} \frac{d\theta}{2\pi}$ $\quad n \geq 1$

$$\frac{c^*}{d} - \frac{\bar{g}}{g} = \sum_{n \geq 0} g_n z^n$$ we know $c^* \in H_-$

obvious choice is $d = g$. c^* = non const part of $\bar{g}$.

<u>Repeat</u> ~~exp~~ E = Hilb. space = $L^2(S^1, \frac{1}{|g|^2} \frac{d\theta}{2\pi})$

$$\xi_- = g \ , \quad \xi_+ = \bar{g} \qquad S_n = (\xi_- | u^n \xi_+)$$

$$= \int \bar{g}\, z^n \bar{g} \frac{1}{|g|^2} \frac{d\theta}{2\pi} = \int z^n \frac{\bar{g}}{g} \frac{d\theta}{2\pi}$$ $\boxed{\text{YES}}$

This is the appropriate reflection coeff for this Hilbert spaces, ~~I want to~~ which has the property that $H_+ \xi_+ \oplus_{alg.} H_- \xi_- = E$.

However I want to replace it by a ref

$$\begin{pmatrix} \xi^r_{out} \\ \xi^l_{out} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \xi^l_{in} \\ \xi^r_{in} \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ g_0 \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{d} & \frac{c^*}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} p_0 \\ \xi_- \end{pmatrix}$$