except for notation the important point is that mult by g $f \longmapsto gf$, $L^2(S^1)$ $L^2(S^1)(1-|\beta|^2)\frac{d\theta}{2\pi}$ is a Hilb. space isom. (comm. with $u = z\cdot$), Moreover preserves filtration so $gH_+ = H_+$ $\therefore \exists\, g' \in H_+$ s.t. $gg' = 1$, so $g^{-1} \exists$ in H_+. Next $|g|^2(1-|\beta|^2) = 1$

$\alpha = \frac{1}{g}$ $\begin{pmatrix} \alpha & \beta \\ -\frac{\alpha}{\bar\alpha}\bar\beta & \alpha \end{pmatrix}$. $(\xi_+ - \xi_-\beta)g^{-1} = \xi'_-$

~~so what next?~~ involution

$$\|\xi_+ f + \xi_- g\|^2 = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \beta^* \\ \beta & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

$$\| \quad \int \begin{pmatrix} f \\ g \end{pmatrix}^t \begin{pmatrix} 1 & \beta \\ \beta^* & 1 \end{pmatrix} \quad ?$$

$$\|\xi_+ g^* + \xi_- f^*\|^2 = \int \begin{pmatrix} g^* \\ f^* \end{pmatrix}^* \begin{pmatrix} 1 & \beta^* \\ \beta & 1 \end{pmatrix} \begin{pmatrix} g^* \\ f^* \end{pmatrix}$$

$$= \int \begin{pmatrix} g \\ f \end{pmatrix}^t \begin{pmatrix} 1 & \beta^* \\ \beta & 1 \end{pmatrix} \begin{pmatrix} g^* \\ f^* \end{pmatrix} \quad ?$$

$$= \int \begin{pmatrix} g \\ f \end{pmatrix}^* \begin{pmatrix} 1 & \beta \\ \beta^* & 1 \end{pmatrix} \begin{pmatrix} g \\ f \end{pmatrix}$$

Still have to understand properties of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\frac{1}{k_n}\begin{pmatrix}1 & \bar{h}_n\\ h_n & 1\end{pmatrix}\begin{pmatrix}u^{-n} q_{n-1}\\ u^{-n+1}p_{n-1}\end{pmatrix} = u^{-n}\frac{1}{k_n}\begin{pmatrix}1 & \bar{h}_n\\ h_n & 1\end{pmatrix}\begin{pmatrix}q_{n-1}\\ u p_{n-1}\end{pmatrix}$$

$$\sigma\begin{pmatrix}p_n\\ q_n\end{pmatrix} = \begin{pmatrix}u^{-n} q_n\\ u^{-n} p_n\end{pmatrix} \quad ? \qquad \text{YES.}$$

back to the $SL(2,\mathbb{Z})$-tree, but first you want to look at $\mathbb{Z}$-trees. Review trans. line.

$$\partial_x E + l\,\partial_t I = 0 \qquad\qquad g\,\partial_x E + g^{-1}\partial_t I = 0$$

$$\partial_x I + l^{-1}\partial_t E = 0 \qquad\qquad g^{-1}\partial_x I + g\,\partial_t E = 0$$

$$g = l^{-1/2}$$

$$(\partial_x + \partial_t)(gE + g^{-1}I) = 0$$

$$(\partial_x - \partial_t)(gE - g^{-1}I) = 0$$

$$\begin{pmatrix}g & g^{-1}\\ -g & +g^{-1}\end{pmatrix}\begin{pmatrix}E\\ I\end{pmatrix} = \begin{pmatrix}Ae^{-sx}\\ Be^{sx}\end{pmatrix}e^{st}$$

You want ~~to compare~~ ~~section~~ segments of transmission line connected together.

$$\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}g_0 & 0\\ 0 & g_0^{-1}\end{pmatrix}\begin{pmatrix}E_x\\ I_x\end{pmatrix} = \begin{pmatrix}A_0 e^{-sx}\\ B_0 e^{sx}\end{pmatrix}e^{st} \qquad 0 < x < 1$$

$$= \begin{pmatrix}e^{-sx} & 0\\ 0 & e^{sx}\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}g_0 & \\ & g_0^{-1}\end{pmatrix}\begin{pmatrix}E_0\\ I_0\end{pmatrix}$$

$$\begin{pmatrix}E_x\\ I_x\end{pmatrix} = \begin{pmatrix}g_0^{-1} & 0\\ 0 & g_0\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}^{-1}\begin{pmatrix}e^{-sx} & 0\\ 0 & e^{sx}\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}g_0 & 0\\ 0 & g_0^{-1}\end{pmatrix}\begin{pmatrix}E_0\\ I_0\end{pmatrix}$$

You would like to understand what lies behind this coupling. There are two things to compare. Impedance | transmission lines with reflection + transmission at each junction, versus trans. line segments of different impedance. What sort of things should you aim for? The first idea is that you should end up with grid spaces somehow. So what do I do next? Review your idea from a few days ago, connected with

$$\frac{1}{2}\begin{pmatrix}1 & +1\\ -1 & 1\end{pmatrix}\frac{1}{k}\begin{pmatrix}1 & h\\ h & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ +1 & 1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\frac{1}{k}\begin{pmatrix}1+h & -1+h\\ h+1 & -h+1\end{pmatrix}$$

$$=\frac{1}{k}\begin{pmatrix}1+h & 0\\ 0 & 1-h\end{pmatrix}=\begin{pmatrix}\frac{1+h}{k} & 0\\ 0 & \frac{1-h}{k}\end{pmatrix}.$$

How did this occur before?

$$\frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\frac{1}{k}\begin{pmatrix}1 & h\\ h & 1\end{pmatrix}\begin{pmatrix}z^{1/2} & 0\\ 0 & z^{-1/2}\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}z^{1/2} & -z^{1/2}\\ z^{-1/2} & z^{-1/2}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1+h}{k} & 0\\ 0 & \frac{1-h}{k}\end{pmatrix}\underbrace{\begin{pmatrix}\frac{z^{1/2}+z^{-1/2}}{2} & \frac{-z^{1/2}+z^{-1/2}}{2}\\ \frac{-z^{1/2}+z^{-1/2}}{2} & \frac{z^{1/2}+z^{-1/2}}{2}\end{pmatrix}}_{(1-s^2)^{-1/2}\begin{pmatrix}1 & s\\ s & 1\end{pmatrix}}$$

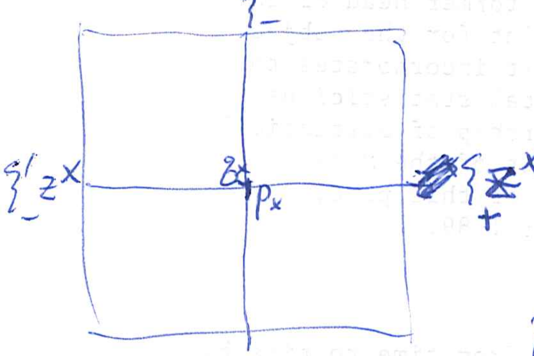
$$s=\frac{-z^{1/2}+z^{-1/2}}{z^{1/2}+z^{-1/2}}=\frac{-z+1}{z+1}$$

You want to string these together and link with a grid space. So what to do? Begin with general coupling $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}z^{1/2} & 0\\ 0 & z^{-1/2}\end{pmatrix}$. First understand this, the Hilbert space picture behind this.

Now you have a real problem – to make sense of all this, and then to ~~work out the~~ do the applications. The basic idea is that

$$p_x = \xi'_- z^x \overset{H_+}{(1-f)} + \xi_- \overset{H_+}{(-g)} \perp \left(\xi'_- z^x H_+ + \xi_- H_+\right)$$

$$q_x = \xi'_- z^x(-\phi) + \xi_-(1-\psi) \perp \left(\xi'_- z^x H_+ + \xi_- H_+\right)$$



get

$$0 = \int \begin{pmatrix} z^x H_+ \\ H_+ \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} z^x(1-f) & z^x(-\phi) \\ -g & 1-\psi \end{pmatrix}$$

$$0 = \int \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}^* \begin{pmatrix} 1 & z^{-x}\bar{b} \\ bz^x & -1 \end{pmatrix} \begin{pmatrix} 1-f & -\phi \\ -g & 1-\psi \end{pmatrix}$$

$$\begin{pmatrix} 1 & T_x^* \\ T_x & -1 \end{pmatrix} \begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_+^*(z^{-x}\bar{b}) \\ \varepsilon_+^*(bz^x) & 0 \end{pmatrix}$$

$$T_x = \varepsilon_+^* \, b z^x \, \varepsilon_+ .$$

---

idea: Recall: F_pA alg with increasing filtration, you form $\bigoplus_p h^p F_p A \subset A[h]$, get a deformation between $\mathrm{gr}\,A$ at $h=0$ and A at $h=1$. Is there some analogous thing. ~~Let's~~ Problem of degree ≥ 0. ~~Not for a modu~~ No problem. F_pV $p \in \mathbb{Z}$

$\subset F_pV \subset F_{p+1}V \subset \cdots$ $\qquad \bigoplus_{p\in\mathbb{Z}} h^p F_p V \subset \bigoplus_{p\in\mathbb{Z}} h^p V$

Continuous version replaces direct sum over $\mathbb{Z}$ with functions $f : \mathbb{R} \to V$ such that $f(x) \in F_x V$

Question. ~~Suppose~~ Consider ξL^2 with

$$\|\xi f\|^2 = \int f^* \rho f \qquad \text{closed subspace } zH_+$$

find $\xi f = \xi(1-\phi) \perp zH_+$

$$\int (zH_+)^* \rho (1-\phi) = 0$$

$$\varepsilon_+^* \rho (1-\phi) = 0 \qquad \boxed{(\varepsilon_+^* \rho\, \varepsilon_+)\phi = \varepsilon_+^*(\rho).}$$

you can solve this because $\varepsilon_+^* \rho\, \varepsilon_+$ is pos. def. self adjoint

$$\rho f \in \mathbb{C} \oplus H_-$$

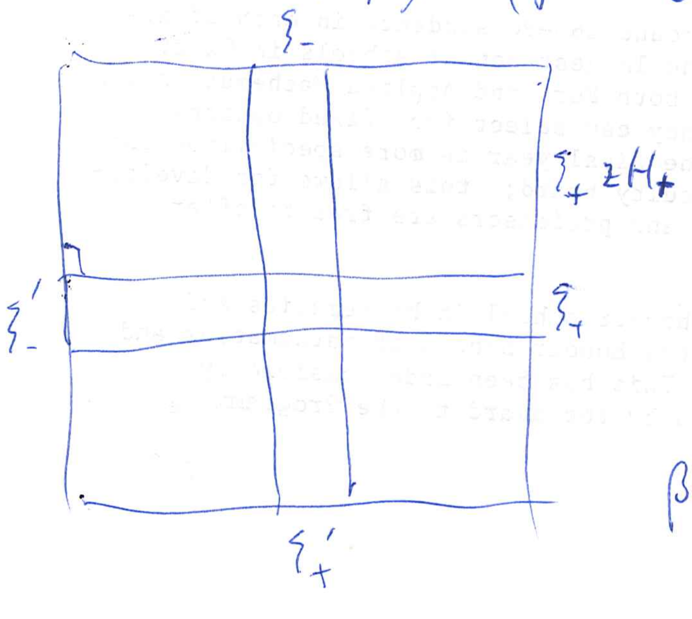
<u>MORAL</u>: What's important in all this orthogonal projection stuff is invertibility for pos. def. herm. ops. i.e. $\begin{pmatrix} 1 & T^* \\ T & 1 \end{pmatrix}$ with $\|T\| < 1$

or $1 + TT^*$, $1 + T^*T$. Contraction is not so important.

Today lecture To reconstruct scattering matrix.

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} \qquad \text{form } \beta$$

$\alpha = \delta$ invertible on H_+



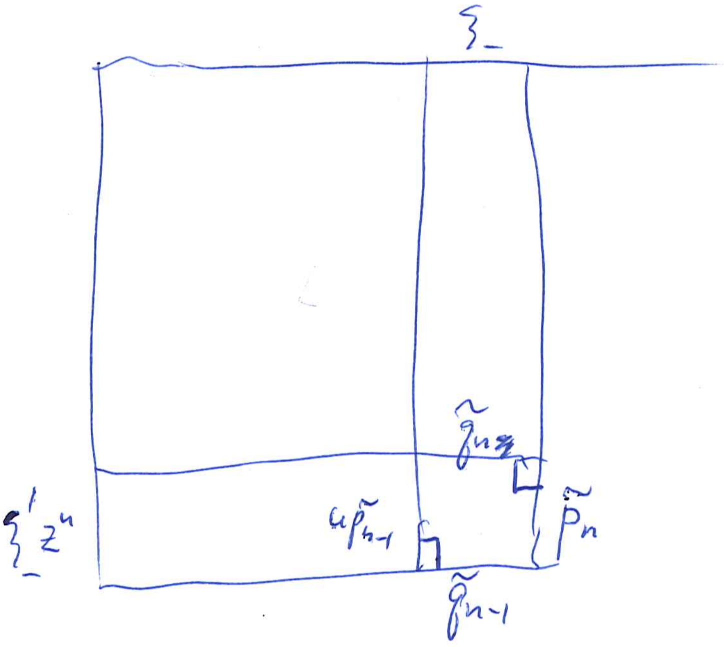
$$W = \xi_+ zH_+ + \xi_- L^2$$

$$\xi'_- = \xi_+ f + \xi_- g$$

$$\int \begin{pmatrix} zH_+ \\ L^2 \end{pmatrix}^* \begin{pmatrix} 1 & \bar\beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = 0$$

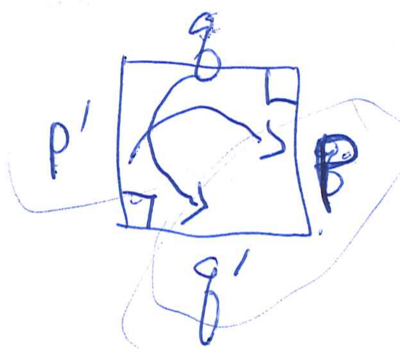
$$\beta f + g = 0 \qquad \varepsilon_+^*(f + \bar\beta g) = 0$$

$$\varepsilon_+^*((1 - |\beta|^2) f) = 0$$



$\begin{pmatrix} \bar{d} & b \\ \bar{b} & d \end{pmatrix} : S^1 \longrightarrow su(1,1)$

$|d|^2 = 1 + |b|^2 \qquad \log(1+|b|^2)$



$$p' = (>0)p + (\ )q$$

$$q = (\ )p' + (>0)q'$$

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p' \\ q' \end{pmatrix}$$

$$p = ap' + bq' \qquad p' = \frac{1}{a}(p - bq')$$

You want to express $\begin{pmatrix} p \\ q' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} p' \\ q \end{pmatrix}$

and this must be unitary matrix

New projects: Wronskian

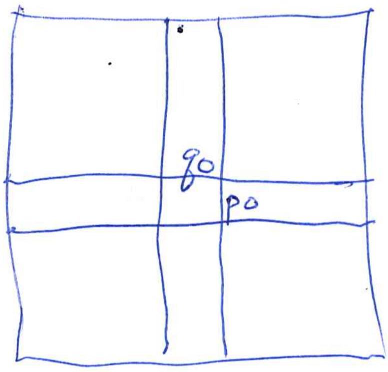
factoring matrices | transfer / scattering

$\oplus\, t^n F_n V \subset \oplus\, t^n V$ device

plumbing along the $SL(2,\mathbb{Z})$ tree (connection with TQFT? Frobenius algs. YBE's

metaplectic reprs.

WRONSKIAN - consider grid space ~~...~~



corresp to sequence (h_n). Pick a <u>center</u>. Point is that once you shift from

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} u p_{n-1} \\ q_{n-1} \end{pmatrix}$$

to

$$\begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n u^{-n} \\ \bar{h}_n u^{n} & 1 \end{pmatrix}\begin{pmatrix} u^{-n+1} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

the transfer matrices between sites lie in $SU(1,1)$, so there is a volume form preserved. ~~I recall that ... IH~~

Recall ~~...~~ $SU(1,1)$ structure

IH
W/n
σ

What is an $SU(1,1)$ structure on V, $\dim(V) = 2$? ~~...~~ simply a volume $\omega \in \Lambda^2 V$ and a conjugation σ on V ~~such that~~, then I is defined by

$$I(v_1, v_2) = \frac{\sigma(v_1) \wedge v_2}{\omega}$$ obviously sesqui-linear

herm. symm? $$\overline{I(v_1, v_2)} = \frac{v_1 \wedge \sigma(v_2)}{\sigma(\omega)}, \quad I(v_2, v_1) = \frac{\sigma(v_2) \wedge v_1}{\omega}$$

you need $\sigma(\omega) = -\omega$. Conversely given $I(v_1, v_2)$ herm. symm. and σ conj. $\ni$ $\overline{I(v_1, v_2)} = I(\sigma v_1, \sigma v_2)$

$$\overline{I(v_1,v_2)} = I(v_2, v_1)$$

Then $I(\sigma v_1, v_2) = I(\sigma v_2, v_1)$?

$$I(v, v') = \overline{I(v', v)}$$ doing something wrong.

σ conjugation on V, $I(v_1, v_2)$ ~~herm~~ sesq.

$\Rightarrow$ $I(\sigma v_1, v_2)$ bilinear, I herm. $\Rightarrow \overline{I(\sigma v_1, v_2)} = I(v_2, \sigma v_1)$

$\Rightarrow \overline{I(\sigma v, v)} = I(v, \sigma v)$

σ conjugation on $V \simeq \mathbb{C}^2$

$I(v_1, v_2)$ herm. form on V.

I sesqui. $\Longrightarrow$ $\underline{I(\sigma v_1, v_2)}$ bilinear

I herm $\Longrightarrow$ $\overline{I(\sigma v_1, v_2)} = I(v_2, \sigma v_1)$

~~now take~~ $\Longrightarrow$ $\overline{I(\sigma v, v)} = I(v, \sigma v)$

$\Longrightarrow$ ($\sigma v = v \Longrightarrow \overline{I(v,v)} = I(v,v)$ as real)

You are doing something wrong.. So

assume $I(\sigma v_1, v_2)$ skew-symm.

i.e. $I(\sigma v_1, v_2) = -I(\sigma v_2, v_1)$

~~But~~ $I(\sigma v_1, \sigma v_2) = -I(v_2, v_1) = -\overline{I(v_1, v_2)}$

$I(\sigma v, \sigma v) = -I(v,v)$ real

Try this Assume ~~$I(\sigma v_1, v_2)$~~

$$I(\sigma v_1, \sigma v_2) = -\overline{I(v_1, v_2)} = -I(v_2, v_1)$$

$$I(\sigma v_1, v_2) = -I(\sigma v_2, v_1)$$ $\therefore$ skew symm.

$$I(\sigma v_1, v_2) = \frac{v_1 \wedge v_2}{\omega}$$ forces $\sigma(\omega) = -\omega$

$$I(v_1, v_2) = \frac{\sigma v_1 \wedge v_2}{\omega}$$ ~~$I(v_1, v_2)$~~

$$\overline{I(v,v)} = \frac{\overline{\sigma v \wedge v}}{\omega} = \frac{v \wedge \sigma v}{\sigma(\omega)} = \frac{\sigma v \wedge v}{-\sigma(\omega)} = I(v,v)$$

Review: $\dim_{\mathbb{C}} V = 2$. I hermitian form on V, σ conjugation, then $I(\sigma v_1, v_2)$ is bilinear and we can ask that it be skew-symm.

$$I(\sigma v_1, v_2) + I(\sigma v_2, v_1) = 0$$

equiv. $I(\sigma v_1, \sigma v_2) = -\overline{I(v_2, v_1)} = -\overline{I(v_1, v_2)}$

Thus $\frac{1}{i} I(\sigma v_1, \sigma v_2) = \overline{\frac{1}{i} I(v_1, v_2)}$ ~~is a~~ should be the complexification of a skew form on $V^{\sigma}_{\mathbb{R}}$.

Go over enough 'til it's clean.

I hermitian form, σ conjugation

condition $I(\sigma v, v) = 0$ all v.

Get a skew form $\int v_1 \wedge v_2 = I(\sigma v_1, v_2)$

so $I(v_1, v_2) = \int \sigma(v_1) \wedge v_2$

$$I(v_2, v_1) = \int \sigma(v_2) \wedge v_1 = \int \sigma(v_2 \wedge \sigma v_1) = -\int \sigma(\sigma v_1 \wedge v_2)$$

So you want $\int \sigma = -\int$

Start other way, given $\phi : \Lambda^2 V \longrightarrow \mathbb{C}$ $\quad \phi\sigma = -\phi$

define $I(v_1, v_2) = \phi(\sigma v_1 \wedge v_2)$ $\quad$ I is sesquilin.

$$I(v_2, v_1) = \phi(\sigma v_2 \wedge v_1) = \phi(\sigma(v_2 \wedge \sigma v_1))$$

$$\cancel{= \phi\sigma(-\sigma v_1 \wedge v_2) = -I(v_1, v_2)} = (-\phi\sigma)(\sigma v_1 \wedge v_2)$$

$$\overline{I(v_1, v_2)} = \bar{\phi}(\sigma v_1 \wedge v_2) \qquad \bar{\phi} = -\phi\sigma$$

σ conjugation on V, $\omega' : \Lambda^2 V \longrightarrow \mathbb{C}$

linear fnl $\ni$ ~~[illegible]~~ $\omega'\sigma(v_1 \wedge v_2) = -\bar{\omega}(v_1 \wedge v_2)$

then $I(v_1, v_2) = \omega'(\sigma v_1 \wedge v_2)$ is a hermitian form

sesquilinear ✓

$$I(v_2, v_1) = \omega'(\sigma v_2 \wedge v_1) = \omega'\sigma(\overset{-\sigma v_1 \wedge v_2}{v_2 \wedge \sigma v_1})$$

$$= -\omega'\sigma(\sigma v_1 \wedge v_2) = \overline{\omega(\sigma v_1 \wedge v_2)}$$

$$= \overline{I(v_1, v_2)}.$$

Converse direction

$$\omega'(v_1 \wedge v_2) = I(\sigma v_1, v_2)$$

σ is time reversal.

So consider now a disc. D.E.

$$\begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n} \underbrace{\begin{pmatrix} 1 & h_n u^{-n} \\ \bar{h}_n u^n & 1 \end{pmatrix}}_{S^1 \longrightarrow SU(1,1)} \begin{pmatrix} u^{-n+1} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

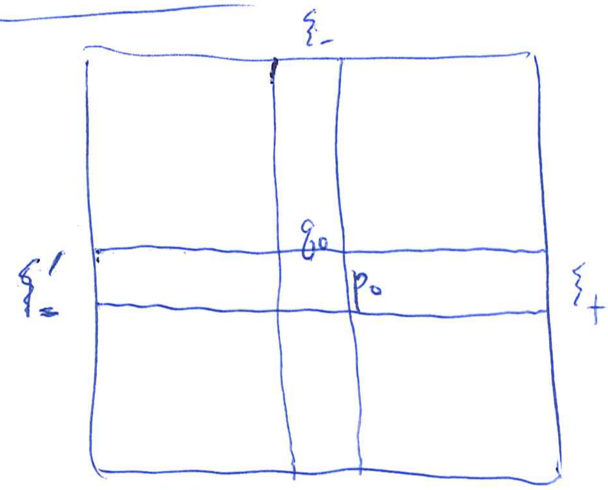
Define $\sigma(p_0) = q_0 \qquad \sigma(q_0) = p_0$

~~should follow that~~ Since

$$\begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} \bar{d}_n & b_n \\ \bar{b}_n & d_n \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

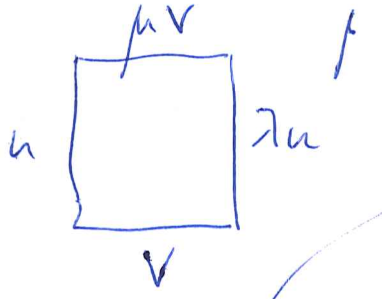
$$\sigma(q_n) = \sigma(\bar{b}_n p_0 + d_n q_0) = b_n q_0 + \bar{d}_n p_0 = u^{-n} p_n$$

$\sigma(\xi_+) = \xi_- \qquad \sigma(\xi'_-) = \xi'_+$



what about constant grid space $\mu = \frac{1}{k}\left(\frac{k\lambda - k^2}{k\lambda - 1}\right) =$

$k\mu - 1 = \frac{1-k^2}{k\lambda - 1}$



$\sigma(v) = u$

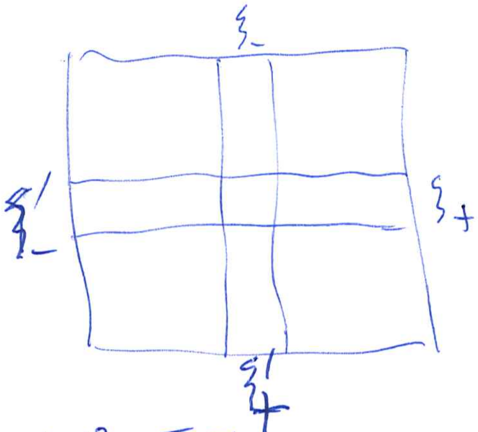
$(k\lambda - 1)\,u = h v$
$(k\mu - 1)\,v = \bar{h} u$

are these relations preserved?

$(k\mu - 1)\,v = \bar{h} u$ you seem to want $\sigma \lambda \sigma^{-1} = \mu$

If $U = \mu \lambda^{-1}$ then $\sigma U \sigma^{-1} = \lambda \mu^{-1} = U^{-1}$. This σ should be compatible with $\| \ \|^2$, since it transforms the increasing staircase below the main diagonal to the inc. staircase above.

Let's check at the boundary in scattering situation



$\sigma(\xi_+ f + \xi_- g) = \xi_+ \bar{g} + \xi_- \bar{f}$

$$\|\xi_+ \bar{g} + \xi_- \bar{f}\|^2 = \int \begin{pmatrix} \bar{g} \\ \bar{f} \end{pmatrix}^* \begin{pmatrix} 1 & \bar{\beta} \\ \beta & 1 \end{pmatrix} \begin{pmatrix} \bar{g} \\ \bar{f} \end{pmatrix}$$

$$= \int (g \ f) \begin{pmatrix} 1 & \bar{\beta} \\ \beta & 1 \end{pmatrix} \begin{pmatrix} \bar{g} \\ \bar{f} \end{pmatrix} = \int |g|^2 \rho$$

$$\begin{matrix} |f|^2 & \bar{f}\bar{\beta} g \\ \bar{g}\beta f & |g|^2 \end{matrix} = \begin{pmatrix} \bar{f} \\ \bar{g} \end{pmatrix} \begin{pmatrix} 1 & \bar{\beta} \\ \beta & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

$$(g \ f) \begin{pmatrix} \bar{g} + \bar{\beta}\bar{f} \\ \beta \bar{g} + \bar{f} \end{pmatrix} = \int \begin{matrix} g\bar{g} + g\bar{\beta}\bar{f} + \\ f\beta\bar{g} + f\bar{f} \end{matrix}$$

$$IH(\xi_+ f + \xi_- g) = \|f\|^2 - \|g\|^2$$
$$\| $$
$$IH(\xi_+ \bar{g} + \xi_- \bar{f}) = \|\bar{f}\|^2 - \|\bar{g}\|^2$$

$$\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\sigma(\xi'_-) = \sigma(d\xi_+ - b\xi_-) = \underbrace{-\bar{b}}_{-c}\xi_+ + \underbrace{\bar{d}}_{a}\xi_- = \xi'_+$$

Use model for constant grid space: $\bar{E} = L^2(S^1)$

[redacted] $\bar{E} \simeq L^2(S^1)$

$$\lambda \longmapsto z. \qquad u \longmapsto \frac{h}{kz-1}$$

$$\mu \longmapsto \frac{z-k}{kz-1}. \qquad v \longmapsto 1$$

Continuous case. $L^2 = H_- \oplus H_+$ Instead of

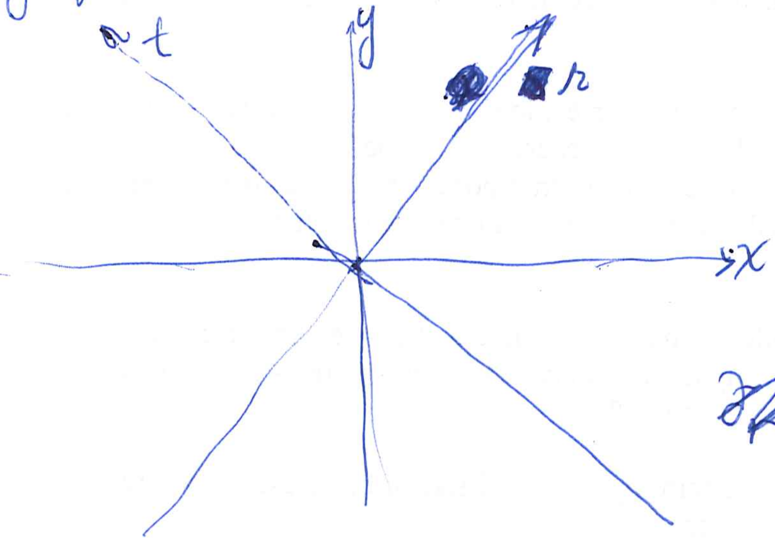
$f(z) = \sum a_n z^n$ $z \in S^1$ you have $f(\omega) = \int a_t z^t \, dt$

where $z^t = e^{it\omega}$ instead of $z^n = e^{in\theta}$

Dirac equation is $\partial_t \begin{pmatrix} z^{-t} p_t \\ q_t \end{pmatrix} = \begin{pmatrix} 0 & h_t z^{-t} \\ \bar{h}_t z^t & 0 \end{pmatrix} \begin{pmatrix} z^{-t} p_t \\ q_t \end{pmatrix}$

$$\begin{pmatrix} z^{-t} p_t \\ q_t \end{pmatrix} = \frac{1}{k_t} \begin{pmatrix} 1 & h_t z^{-t} \varepsilon \\ \bar{h}_t z^t \varepsilon & 1 \end{pmatrix} \begin{pmatrix} z^{-t+\varepsilon} p_{t-\varepsilon} \\ q_{t-\varepsilon} \end{pmatrix} \qquad \text{OK}$$

wrong parameter t or x.



$$\partial_y = \partial_t + \partial_r$$

$$\partial_x = -\partial_t + \partial_r$$

$$f(t,r) = f(-x+y, x+y)$$

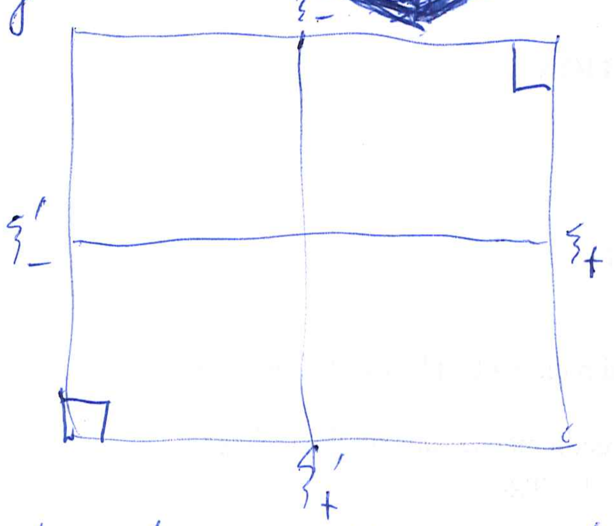
$$\partial_x f = -\partial_t f + \partial_r f$$

$$\partial_y f = \partial_t f + \partial_r f$$

Basic equation (differential) is $\quad z^n = e^{i\theta_n}$ 856

$$\partial_t \begin{pmatrix} z^{-t} p_t \\ q_t \end{pmatrix} = \begin{pmatrix} 0 & h_t z^{-t} \\ \bar{h}_t z^t & 0 \end{pmatrix} \begin{pmatrix} z^{-t} p_t \\ q_t \end{pmatrix} \qquad z^t = e^{i\omega t}$$

get a [redacted] transfer matrix.



$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

to first order in (h_t)

$$b = \int h_t z^{-t} dt \qquad h_t = \int \frac{d\omega}{2\pi} b e^{it\omega}$$

Let's set up the Szego business. First you have to get stuff straight. Use $\mathrm{IH}(\xi'_- f + \xi_- g) = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$

$$\| \xi'_- f + \xi_- g \|^2 = \|f\|^2 + \|g\|^2$$

$$\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} \qquad \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{bmatrix} \frac{1}{d} & \frac{+b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{bmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\mathrm{IH}(\xi'_- f + \xi_- g) = \mathrm{IH}(\xi_+ df - \xi_- bf + \xi_- g) = \|df\|^2 - \|-bf + g\|^2$$

~~what comes next ?? nothing. What to do next ???~~

$$\log(1+|b|^2) = \int_{-\infty}^{\infty} dt \; a_t e^{i\omega t} = \alpha + \bar{\alpha}$$

where $\alpha(\omega) = \int_0^\infty dt \; a_t e^{i\omega t}$ $\qquad d(\omega) = e^{\alpha(\omega)}$

$$|d|^2 = 1 + |b|^2. \qquad \xi_+ = (\xi'_- + \xi_- b) d^{-1}$$

$$\mathrm{IH}(\xi'_- H_+ + \xi_- L^2, \xi'_- f + \xi_- g) = \int \begin{pmatrix} H_+ \\ L^2 \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}^{\mathbb{R}_{>0} + H_+} = 0$$

$$bf = g \qquad \varepsilon_+^*(f + \bar{b} g) = \varepsilon_+^*(f(1+|b|^2)) = 0.$$

You want to understand the smoothness properties, growth properties ? of h.

Philosophy: Assume the non-linear aspects of the transform $(h_t) \longmapsto b(\omega)$, which to first order is $b(\omega) = \int dt\, h_t e^{i\omega t}$, The F.T., doesn't matter. Thus if $b \in$ Schwarz space $\mathcal{S}$ then (h_t) also is. Then $\log(1+|b|^2) \in \mathcal{S}$, but when you split: $\log(1+|b|^2) = \alpha + \bar{\alpha}$,

$\alpha(\omega) = \int_0^\infty dt\, a_t e^{i\omega t}$ $(a_t) \in \mathcal{S}$, and

α is smooth, analytic in UHP, but the imaginary part of α should only be $O(\frac{1}{\omega})$ corresp. to jump from 0 to a_0.

$$\int_0^\infty dt\, e^{i\omega t} = \left[\frac{e^{i\omega t}}{i\omega}\right]_0^\infty = \frac{i}{\omega}$$

Supposedly under the iso-spectral flows d doesn't change. Does $\exists$ nice examples?

$$\begin{pmatrix} \cosh & \sinh \\ \sinh & \cosh \end{pmatrix} \quad ?$$

---

continue with orthogonal projection. What can you do?

$$\int \begin{pmatrix} H_+ \\ L^2 \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = 0 \qquad \begin{array}{l} f \in \mathbb{R}_{>0} + H_+ \\ g \in L^2 \end{array}$$

$$bf = g$$

$$\varepsilon_+^*(f(1+|b|^2)) = 0$$

What does one learn from this? You should learn something about continuous filtrations. You start with $\xi'_- L^2 + \xi_- L^2$ and IH, but the first step is to pass to $(\xi'_- + \xi_- b)L^2$ which is the orthogonal of $\xi_- L^2$ for IH. IH is pos. def. on this subspace $IH((\xi'_- + \xi_- b)f) = \int \begin{pmatrix} f \\ bf \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ bf \end{pmatrix} = \int f^*(1+|b|^2)f$

Abstract this situation, take $\rho > 0$ form $L^2(\mathbb{R}, \rho \frac{d\omega}{2\pi})$

Consider filt. $e^{i\omega t} H_+$. Assume $\frac{1}{\varepsilon} \geq \rho \geq \varepsilon$ so that $L^2(\mathbb{R}, \frac{d\omega}{2\pi}) = L^2(\mathbb{R}, \rho\frac{d\omega}{2\pi})$ same TVS but different $\| \|^2$.

$$f(\omega) = \int dt\, e^{i\omega t} \hat{f}(t) = \int_{-\infty}^{0} + \int_{0}^{\infty} \in H_- \oplus H_+$$

You have the filtration $e^{i\omega t} H_+$, decreases as t increases of $L^2(\mathbb{R})$. Somehow you ~~are splitting~~ can split this filtration for the inner product $\|f\|^2 = \int |f|^2 \rho \frac{d\omega}{2\pi}$.

Meaning? Construct the orthogonal projection

Strange method. You produce $f \in 1 + H_+$ which is $\perp$ to H_+ in some sense

$$\int (H_+)^* \rho \overset{H_+}{(1-\phi)} = 0 \qquad \varepsilon_+^*(\rho(1-\phi)) = 0$$

why is this solvable? $H_+ \xrightarrow{\varepsilon_+} L^2$ $\qquad \varepsilon_+^*(\rho\phi) = \varepsilon_+^*(\rho)$

$(\varepsilon_+^* \rho\, \varepsilon_+)\phi = \varepsilon_+^*(\rho)$, this can be solved because

$$\varepsilon \leq \varepsilon_+^* \rho\, \varepsilon_+ \leq \frac{1}{\varepsilon} \quad \text{on } H_+$$

~~At some point you should try to~~ define $(1-\phi) \cdot L^2(\frac{d\omega}{2\pi}) \longrightarrow L^2(\rho \frac{d\omega}{2\pi})$ and show this is unitary, so that $1-\phi \in L^\infty$

Return to transmission lines, putting structures on the $SL(2,\mathbb{Z})$-tree. ~~Transmission~~

$$\begin{pmatrix} z^{-n}p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} z^{-n} & z^{-n}h_n \\ \bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} z^n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z^{-n+1}p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$= \frac{1}{k_n}\begin{pmatrix} 1 & z^{-n}h_n \\ \bar{h}_n z^n & 1 \end{pmatrix}\begin{pmatrix} z^{-n+1}p_{n-1} \\ q_{n-1} \end{pmatrix}$$

The idea was to introduce $\psi_n \in \mathbb{C}^2$ for n even and a line depending on choice of $z^{1/2}$ for n odd.

$$\begin{pmatrix} \tilde{p}_n \\ \tilde{q}_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}\begin{pmatrix} \tilde{p}_{n-1} \\ \tilde{q}_{n-1} \end{pmatrix}$$

$$\begin{pmatrix} z^{-\frac{n}{2}}p_n \\ z^{-\frac{n}{2}}q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}\begin{pmatrix} z^{\frac{-n+1}{2}}p_{n-1} \\ z^{\frac{-n+1}{2}}q_{n-1} \end{pmatrix}$$

Back to transmission line | to discuss quantization

$$-\partial_x E = \lambda\, \partial_t I \qquad \lambda\gamma = 1$$
$$-\partial_x I = \gamma\, \partial_t E \qquad \text{take } \gamma = \lambda = 1.$$

Energy $\int\left(\frac{1}{2}E^2 + \frac{1}{2}I^2\right)dx$

There should also be a skew-symmetric form. Power EI

$$\partial_x E + \partial_t I = 0$$
$$\partial_x I + \partial_t E = 0$$

$$\partial_t \int_a^b \frac{1}{2}(E^2+I^2)\,dx = \int_a^b \underbrace{\big(E(-\partial_x I) + I(-\partial_x E)\big)}_{-\partial_x(EI)}dx$$

$$= EI\big|_a - EI\big|_b$$

~~osc~~ treat as a harmonic oscillator somehow, modes

$\cancel{\partial_x E + i\omega I = 0}$
$\cancel{\partial_x I + i\omega E = 0}$

$$(\partial_x + \partial_t)(E+I) = 0$$
$$(\partial_x - \partial_t)(E-I) = 0$$

$$\begin{pmatrix} E+I \\ E-I \end{pmatrix} = \begin{pmatrix} Ae^{-sx} \\ Be^{sx} \end{pmatrix} e^{st}$$

So if you fix $s = -i\omega$ frequency then there is a 2-diml solution space with that frequency. Here you are looking at global solutions

Repeat Equations of motion are $\wedge$; for each ω there is a complex 2-diml space of solutions – left and right moving, which are independent. ~~Focus on~~ pick right moving now you have simple $\begin{pmatrix} Ae^{s(t-x)} \\ 0 \end{pmatrix}$. You have this

complex [redacted] line with time evolution for each $\omega \in \mathbb{R}$. How do you quantize?

Suppose you have simple harm. osc. $H = \frac{p^2}{2m} + \frac{k}{2} q^2$

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q} = -kq \qquad \dot{q} = \frac{p}{m} \qquad \ddot{q} = \frac{1}{m}(-kq)$$

$$\omega^2 = \frac{k}{m} \qquad k = m\omega^2$$

the other ingredient you need for quant is the ~~those~~ commutation relation $[p,q] = \frac{\hbar}{i}$. ~~Also~~ Maybe you should examine quantizing the EM field. The equations of motion are something like ~~d~~ $dA = d^*A = 0$

A is a 1-form with components E, I.

Look at notes from summer 1999, quant. of harm. osc. ~~Other idea~~ Consider phase space. Equation of motion is ~~a~~ normally 2nd order linear DE on config. space Q, becomes first order linear flow on phase space. Phase space Ω is $\mathbb{R}$ v.s., with energy ~~form~~ H making it a real Hilb. space, time evolution = skew-adj operator X. ~~[illegible]~~ Have polar decomp.

$$X = |X| J \qquad |X| = (-X^2)^{1/2} \qquad J^2 = -1.$$

V phase space - real vector space with time evolution e^{tX}, X is diagonalizable ~~[illegible]~~ in $V \otimes_{\mathbb{R}} \mathbb{C}$, so X ~~[illegible]~~ splits into 2-planes (assuming X nondeg, all ~~eigenvalues~~ $\neq 0$) where e^{tX} is a rotation. ~~Thus~~ ~~[illegible]~~ Then ~~[illegible]~~ X has polar decomp $X = |X| J$, $|X| = (-X^2)^{1/2}$, $J^2 = -1$. ~~[illegible]~~ ~~polar decomp.~~ Now $|X|$ is the Hamiltonian operator since the time evolution in ~~the~~ ~~[illegible]~~ a complex line with $X = \omega$ is $e^{it\omega}$. What does this mean, what you are saying? There is a basic background you should say first.

Normal exposition for a harmonic oscillator. The phase space is a real vector space $V_{\mathbb{R}}$ equipped with a symplectic form $A : V_{\mathbb{R}} \longrightarrow V_{\mathbb{R}}^*$ and a positive quadratic form ~~[illegible]~~ H, the energy. Time evolution operator X is defined by $\frac{1}{2} A X = H$.

e.g. Let $V =$ ~~[illegible]~~ $\mathbb{R}^2$ coords q, p Let $H = \frac{1}{2m} p^2 + \frac{k}{2} q^2$, $\begin{pmatrix} q' \\ p' \end{pmatrix}^t A \begin{pmatrix} q \\ p \end{pmatrix} = -q'p + p'q$, then

$$\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} X = \begin{pmatrix} \frac{k}{2m} & 0 \\ 0 & \frac{k}{2m} \end{pmatrix}$$

$$\therefore \quad X \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

$$\dot{q} = \frac{p}{m} \qquad \dot{p} = -kq$$

Now $\frac{1}{2}AX = H \Rightarrow AX$ symm. $\Rightarrow AX = -X^tA$ 862
or $X^tA + AX = 0 \Rightarrow A$ preserved (as bilinear form) by the flow e^{tX}. Same for H. $X^tAX + AXX = 0$
$X^tH + HX = 0$ means X skew-symm wrt the inner product given by H so that you get splitting into 2 planes orth. wrt H.

What you would like to understand is how much of the picture comes from what you can see. You dont see the symplectic form A or the energy H, but you do see the motion, so the operators $X, |X|, J$, the frequencies and the space of modes of a given frequency are visible.

Continue with standard picture. You quantize how? Symmetric algebra construction from a 1-particle space, which is the complex vector space V_r equipped with J and the hermitian inner product whose imaginary part is A. Trace $e^{t|X|}$ idea

Consider a harmonic oscillator: real vector space V_r, operator X such that $-X^2$ is diagonalizable with >0 eigenvalues, so that you have polar decomp $X = |X|J$, $|X| = (-X^2)^{1/2}$ $J^2 = -1$. If eigenvalues of $-X^2$ are ω_j^2 $\omega_j > 0$, then $|X|$ has the eigenvalues ω_j. It appears that $|X|$ is the Hamiltonian operator on the 1-particle space which is V_r with $i = J$. ~~[illegible]~~ So far V_r has no inner product, but it is possible to form thermal averages $\dfrac{\mathrm{Trace}(e^{-\beta|X|}T)}{\mathrm{Trace}(e^{-\beta|X|})}$

Suppose you fix X and vary the symplectic form A, or what amounts to the same thing, varying $H = \frac{1}{2}AX$. A must be preserved by X meaning $X^tA + AX = 0$, I guess this means A is a symplectic form in each eigenspace for $|X|$.

The symplectic form is where Planck's constant enters. For a simple harmonic oscillator Phase space is of complex dim 1 and time evol. is $e^{i\omega t}$. You are missing what is needed to convert frequency ω to energy.

Bosonic + fermionic theory. In either case you have V_r, A, H $\quad A: V_r \to V_r^*$ skew symm

$H: V_r \to V_r^*$ symm (+ pos.?), $\boxed{\frac{1}{2}AX = H}$,

e.g. $\quad Q \underset{k}{\overset{m}{\rightrightarrows}} Q^*$ $\quad$ Q conf. space (real v.s.)
m, k pos quad. forms.

$$V_r = Q \oplus Q^*$$

Repeat: A harmonic oscillator has a real phase space V and a time flow operator X such that $-X^2$ is diagonalizable with positive eigenvalues. Put $|X| = (-X^2)^{1/2}$

$J = \dfrac{X}{|X|}$, then $J|X| = |X|J$, $J^2 = -1$. So V becomes a complex vector space such that X has eigenvalues $i\omega$, $\omega > 0$.

From the time evolution on phase space you can recover the 1-particle quantum state space as complex vector with Hamiltonian $|X|$, but so far you don't have an inner product, which you need for probabilities.

PUZZLE: the full quantum state space is essentially the symmetric tensor space $S(V)$, you have the Hamiltonian operator on $S(V)$ from $|X|$, so can construct thermal averages using the trace.

$\mathrm{Tr}\, e^{-\beta|X|}$. ? ?

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confusion reigns, so let's defer all of this stuff for a while

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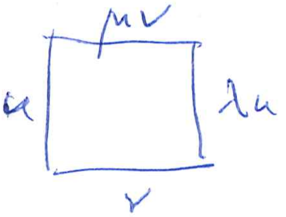
Return to constant h grid space



$$\begin{pmatrix}\lambda u\\ \mu v\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar h & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} \qquad \begin{aligned}(k\lambda-1)u &= hv\\ (k\mu-1)v &= \bar h u\end{aligned}$$

Grid space is isom. to $\mathbb{C}[z, z^{-1}, (z-k)^{-1}, (kz-1)^{-1}]$.

What you should work on now is the $SU(1,1)$ structure on the grid space E associated to a disc D.E., which you know is a free module of rank 2 over $\mathbb{C}[z,z^{-1}]$

The rec. relation
$$\begin{pmatrix}z^{-n}p_n\\ q_n\end{pmatrix} = \underbrace{\frac{1}{k_n}\begin{pmatrix}1 & h_n z^{-n}\\ \bar h_n z^n & 1\end{pmatrix}}_{S^1 \longrightarrow SU(1,1)}\begin{pmatrix}z^{-n+1}p_{n-1}\\ q_{n-1}\end{pmatrix}$$

tell us that if $A = \mathbb{C}[z,z^{-1}]$, then $z^{-n}p_n \wedge q_n \in \Lambda^2_A E$ is indep. of n.

How to get insight into this business. First analyze $SU(1,1)$-structure on $V \simeq \mathbb{C}^2$. $SU(1,1)$ and $SL(2,\mathbb{R})$ are conjugate subgroups of $SL(2,\mathbb{C})$. A starting point: Let $H(\ ,\ )$ be a ~~[illegible]~~ hermitian form. Better is to ~~[illegible]~~ begin with a conjugation σ on V and a volume $\omega : \Lambda^2 V \xrightarrow{\sim} \mathbb{C}$ satisfying $\overline{\omega(v_1 \wedge v_2)} = -\omega(\sigma v_1 \wedge \sigma v_2)$. Then you get ~~a~~ sesq. form $H(v_1,v_2) \overset{\sim}{=} \omega(\sigma v_1 \wedge v_2)$ and

~~[illegible]~~ $\overline{H(v_1,v_2)} = \overline{\omega(\sigma v_1 \wedge v_2)}$

$= -\omega(v_1 \wedge \sigma v_2) = \omega(\sigma v_2, v_1) = H(v_2, v_1)$. Conversely if $H(v_1,v_2)$ given, put $\omega(v_1 \wedge v_2) = H(\sigma v_1, v_2)$ then ω is bilinear. $\overline{\omega(v_1,v_2)} = \overline{H(\sigma v_1, v_2)}$

$= H(v_2, \sigma v_1) = H(\sigma(\sigma v_2), \sigma v_1) = \omega(\sigma v_2, \sigma v_1)$?

~~If define~~ $\cancel{\omega(v_1, v_2) = H(\sigma v_1, v_2)}$

Begin again. Suppose σ given on V, and $\omega : \Lambda^2 V \xrightarrow{\sim} \mathbb{C}$. Define $H(v_1,v_2) = \omega(\sigma v_1 \wedge v_2)$.

Prop. of H: sesquilinear, $H(v_2, v_1) = \omega(\sigma v_2 \wedge v_1)$

$= -\omega(v_1 \wedge \sigma v_2) = -\omega(\sigma(\sigma v_1) \wedge \sigma v_2)$. Assume

$\boxed{\bar{\omega}(v_1 \wedge v_2) = -\omega(\sigma v_1 \wedge \sigma v_2).}$ If so, then $\gamma = \overline{\omega(\sigma v_1, v_2)}$

$= \overline{H(v_1,v_2)}$, so $H(v_2, v_1) = \overline{H(v_1,v_2)}$ $\therefore$ H herm. symm.

Conversely given H herm. form put
$\omega(v_1, v_2) = H(\sigma v_1, v_2)$. ω is bilinear

$V \simeq \mathbb{C}^2$, σ conjugation on V, $\omega : \Lambda^2 V \to \mathbb{C}$
nondeg skew form satisfying $\omega(\sigma v_1 \wedge \sigma v_2) = -\overline{\omega(v_1 \wedge v_2)}$
Then $H(v_1, v_2) = \omega(\sigma v_1 \wedge v_2)$ is ~~nondeg~~ Hermitian.
H is sesqui-linear, $\overline{H(v_2, v_1)} = \overline{\omega(\sigma v_2 \wedge \sigma^2 v_1)}$
$= -\omega(v_2 \wedge \sigma v_1) = \omega(\sigma v_1 \wedge v_2) = H(v_1, v_2)$. This is what's important, but suppose you have a hermitian form $H(v_1, v_2)$. ~~Is the bilinear form~~
~~$H(\sigma v_1, v_2)$ alternating?~~

Repeat what you want to remember, namely
If $\omega : \Lambda^2 V \xrightarrow{\sim} \mathbb{C}$ satisfies $\omega \circ \sigma = -\omega$ i.e.
$\omega(\sigma v_1, \sigma v_2) = -\overline{\omega(v_1, v_2)}$, then
$H(v_1, v_2) = \omega(\sigma v_1, v_2)$ is hermitian symm.

$$\overline{H(v_2, v_1)} = \overline{\omega(\sigma v_2, \sigma^2 v_1)} = -\omega(v_2, \sigma v_1) = \omega(\sigma v_1, v_2) = H(v_1, v_2)$$

~~Def~~ Consider now the grid space E belonging to
$$\begin{pmatrix} z^{-n} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix} \begin{pmatrix} z^{-n+1} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

Define $W : \Lambda^2_A E \longrightarrow A$ $\qquad A = \mathbb{C}[z, z^{-1}]$

$V \simeq \mathbb{C}^2$ with σ e.g. $\sigma\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}\bar b\\ \bar a\end{pmatrix}$

$\omega : \Lambda^2 V \longrightarrow \mathbb{C}$ eg $\begin{pmatrix}a\\b\end{pmatrix} \wedge \begin{pmatrix}a'\\b'\end{pmatrix} \longmapsto \begin{vmatrix}a & a'\\ b & b'\end{vmatrix}$

$$\omega(\sigma v, \sigma v') = -\overline{\omega(v, v')}$$

$$\omega\left(\begin{pmatrix}\bar b\\ \bar a\end{pmatrix} \wedge \begin{pmatrix}\bar b'\\ \bar a'\end{pmatrix}\right) = \begin{vmatrix}\bar b & \bar b'\\ \bar a & \bar a'\end{vmatrix} = \overline{\begin{vmatrix}b & b'\\ a & a'\end{vmatrix}} = -\overline{\begin{vmatrix}a & a'\\ b & b'\end{vmatrix}}$$

then $H(v, v') = \omega(\sigma v, v')$ is hermitian

$$H\left(\begin{pmatrix}a\\b\end{pmatrix}, \begin{pmatrix}a'\\b'\end{pmatrix}\right) = \begin{vmatrix}\bar b & a'\\ \bar a & b'\end{vmatrix} = \bar b b' - \bar a a'$$

---

ex. $V = \mathbb{C}$ $\sigma\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}\bar a\\ \bar b\end{pmatrix}$ $\omega(v, v') = v^t \begin{pmatrix}0 & +i\\ -i & 0\end{pmatrix} v$

$$\omega\left(\begin{pmatrix}a\\b\end{pmatrix}, \begin{pmatrix}a'\\b'\end{pmatrix}\right) = \begin{pmatrix}a & b\end{pmatrix}\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\begin{pmatrix}a'\\b'\end{pmatrix} = i(ab' - ba')$$

$$H\left(\begin{pmatrix}a\\b\end{pmatrix}, \begin{pmatrix}a'\\b'\end{pmatrix}\right) = i(\bar a b' - \bar b a') = \begin{pmatrix}a\\b\end{pmatrix}^* \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\begin{pmatrix}a'\\b'\end{pmatrix}$$

$$H\left(\begin{pmatrix}a\\b\end{pmatrix}, \begin{pmatrix}a\\b\end{pmatrix}\right) = i(\bar a b - \bar b a) = -2\,\mathrm{Im}(\bar a b)$$

Your mistake ~~before~~ yesterday: possible H have 3 dims, ~~but the possible~~ maybe 2 if you impose reality condition, possible ω have dim 1.

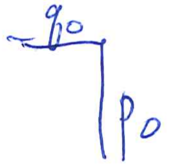
$$H\left(\begin{pmatrix}a\\b\end{pmatrix}, \begin{pmatrix}a'\\b'\end{pmatrix}\right) = \begin{pmatrix}a\\b\end{pmatrix}^* \begin{pmatrix}r & s\\ \bar s & r'\end{pmatrix}\begin{pmatrix}a'\\b'\end{pmatrix} \qquad r, r' = 0.$$

Now you want to move on to the Wronskian

E = grid space $= Ap_0 + Aq_0$ $\qquad A = \mathbb{C}[u, u^{-1}]$.

~~want to explain~~ $\{u^n p_0, u^n q_0\}$ orth. basis for IH, so it should be simple. The point apparently is that IH is not a general hermitian form, ~~but~~ rather it arises from a pair σ, ω on E. Interesting to note that IH is independent of the choice of center, but σ, ω do.

$$\sigma \{u\} \sigma^{-1} = u^{-1} \qquad \text{better} \qquad \sigma f \sigma^{-1} = \bar{f}$$



$$\sigma(p_0) = q_0$$

$$\begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$\sigma \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \sigma^{-1} \overset{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}{=} \begin{pmatrix} \bar{a}_n & \bar{b}_n \\ \bar{c}_n & \bar{d}_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \bar{d}_n & \bar{c}_n \\ \bar{b}_n & \bar{a}_n \end{pmatrix}$$

arg. You have

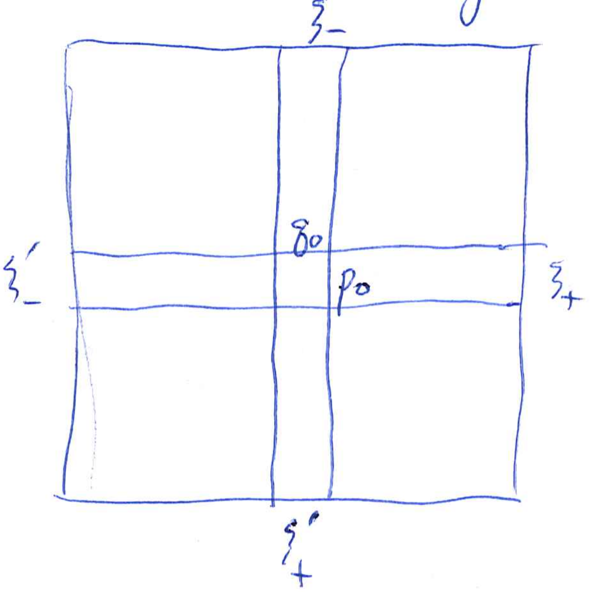
$$\begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$\sigma(u^{-n} p_n) = \bar{a}_n q_0 + \bar{b}_n p_0 = d_n q_0 + c_n p_0 = q_n$$

Also $\quad u^{-n} p_n \wedge q = (a_n p_0 + b_n q_0) \wedge (c_n p_0 + d_n q_0)$

$$= \underbrace{\begin{vmatrix} a_n & c_n \\ b_n & d_n \end{vmatrix}}_{=1.} p_0 \wedge q_0$$

Take a scattering situation



$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \lim_{n\to+\infty}\begin{pmatrix}u^{-n}p_n\\ q_n\end{pmatrix}$$

$$\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} \qquad n\to-\infty$$

$$\boxed{\sigma(\xi_+) = \xi_- \qquad \sigma(\xi'_-) = \xi'_+}$$

$f, g \in \mathbb{C}[u,u^{-1}]$

$$IH(fp_0 + gq_0) = \int\left(|f|^2 - |g|^2\right) = \int\begin{pmatrix}f\\ g\end{pmatrix}^*\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}f\\ g\end{pmatrix}$$

$$Wr\big(\underbrace{\sigma(fp_0+gq_0)}_{\bar f q_0 + \bar g p_0},\ fp_0+gq_0\big) = \left(|g|^2 - |f|^2\right)^{\omega}(p_0\wedge q_0)$$

Take $\omega(p_0\wedge q_0) = -1$.

$$\boxed{\int Wr(\sigma\xi,\xi) = IH(\xi,\xi).}$$

Look at constant h grid space. You first need to establish σ on any grid space with unitary symmetry u. Time reflection symmetry. Recall the presentation of E, invertible operators λ, μ, $\ni$ $(k\lambda-1)(k\mu-1) = 1-k^2$. Obvious conjugation with $\sigma\lambda\sigma^{-1} = \mu$, $\sigma\mu\sigma^{-1} = \lambda$. Note $u = \mu\lambda^{-1}$, $\sigma u\sigma^{-1} = \lambda\mu^{-1} = u^{-1}$. OK

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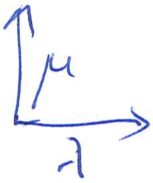
$(k\lambda - 1)w = \bar{h} v$

$(k\mu - 1)v = \bar{h} w$

$$\begin{array}{ccc} & \mu v & \\ w & \boxed{} & \lambda w \\ & v & \end{array}$$



~~[illegible]~~ You have the conjugation oper σ on E. Next you need the Wronskian

One one hand you have $B =$ * alg. gen. by unitaries λ, μ $\ni (k\lambda-1)(k\mu-1) = 1-k^2$

contains * subalg $A = \mathbb{C}[u, u^{-1}]$ where $u = \mu\lambda^{-1}$.

B is free of rank 2 over A. You would like to calculate $\Lambda^2_A B$, i.e. a skew-pairing ~~[illegible]~~. Describe what's happening. You have ~~[illegible]~~ σ on

$$B = \mathbb{C}[\lambda, \mu, \lambda^{-1}, \mu^{-1}] / ((k\lambda-1)(k\mu-1) = 1-k^2)$$

$$A = \mathbb{C}[u, u^{-1}] \qquad u = \mu\lambda^{-1}$$

$u = \infty$ means $\lambda = k^{-1}, \ \mu = \infty$

or $\lambda = 0, \ \mu = k$

~~Simple enough to do the ...~~

~~What does ...~~

~~[illegible]~~ Review. Given $V \simeq \mathbb{C}^2$ with conjugation σ e.g. $\sigma\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \bar{z}_2 \\ \bar{z}_1 \end{pmatrix}$

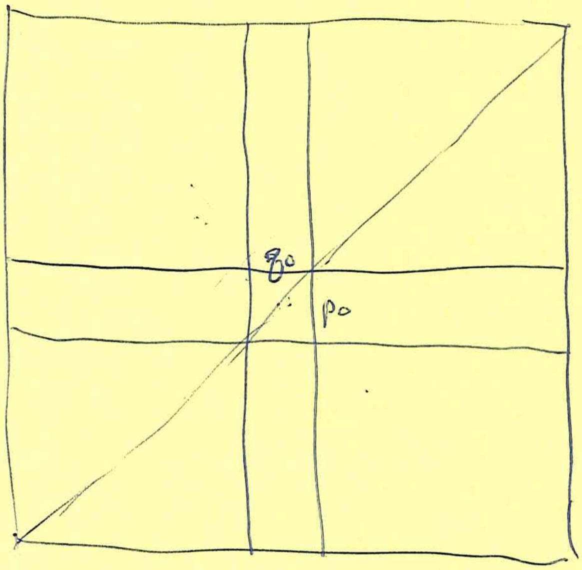
and $\omega : \Lambda^2 V \to \mathbb{C}$ $\quad \omega\left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} z_1' \\ z_2' \end{pmatrix}\right) = \begin{vmatrix} z_1 & z_1' \\ z_2 & z_2' \end{vmatrix}$

satisfy $\overline{\omega(v_1, v_2)} = -\omega(\sigma v_1, \sigma v_2)$ ✓

get herm. form

$$H(v_1, v_2) = \omega(\sigma v_1, v_2)$$

$$\overline{\omega(\sigma v, v)} = -\omega(\sigma^2 v, \sigma v) = \omega(\sigma v, v)$$



Review:

$V \simeq \mathbb{C}^2$ $\quad \sigma$ conj.

e.g. $\sigma\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \bar z_2 \\ \bar z_1 \end{pmatrix}$

$\omega : \Lambda^2 V \xrightarrow{\sim} \mathbb{C}$

eg $\omega\left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} z_1' \\ z_2' \end{pmatrix}\right) = \begin{vmatrix} z_1 & z_1' \\ z_2 & z_2' \end{vmatrix}$

$$\boxed{\overline{\omega(v, v')} = -\,\omega(\sigma v, \sigma v')}$$

~~Ht~~ herm. form $\quad H(v, v') = \omega(\sigma v, v')$

$$H(v, v) = H\left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) = \begin{vmatrix} \bar z_2 & z_1 \\ \bar z_1 & z_2 \end{vmatrix} = |z_2|^2 - |z_1|^2$$

Claim IH on the grid space arising from disc D.E.

$$\begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix} = \underbrace{\frac{1}{k_n}\begin{pmatrix} 1 & h_n u^{-n} \\ \bar h_n u^n & 1 \end{pmatrix}}_{S^1 \longrightarrow SU(1,1)} \begin{pmatrix} u^{-n+1} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

should you have z instead of u?

$$E = A p_0 \oplus A q_0 \qquad A = \mathbb{C}[u, u^{-1}] \subset C(S^1)$$

σ defined on the grid space when $h_{mn} = h_{m+n}$

What do you really want to do? You believe that the indefinite hermitian form is connected with energy flow, power, somehow it is geometric, simpler in any case than the energy. This is vague, but concretely you ~~can ask for~~ want to calculate IH for the continuous constant h grid.

Discuss cont. D.E.

$$\partial_t \psi = \begin{pmatrix} \partial_x & -h \\ +\bar h & -\partial_x \end{pmatrix} \psi \qquad \begin{pmatrix} \partial_t - \partial_x & 0 \\ 0 & \partial_t + \partial_x \end{pmatrix} \psi = \begin{pmatrix} 0 & -h \\ +\bar h & 0 \end{pmatrix} \psi$$

What do you seek? Wronskian.

$$\begin{pmatrix} \partial_x & -h \\ +\bar{h} & -\partial_x \end{pmatrix} \psi = s\psi = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \psi$$

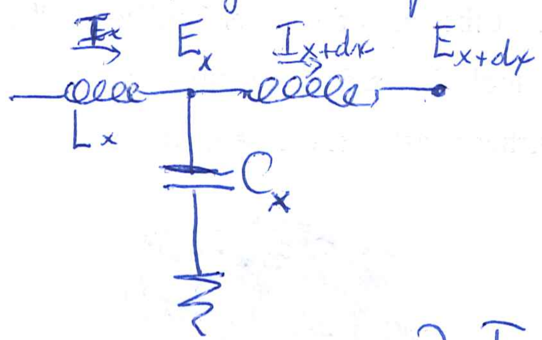
$$\begin{pmatrix} \partial_x & -h \\ -\bar{h} & \partial_x \end{pmatrix} \psi = \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix} \psi$$

$$\partial_x \psi = \begin{pmatrix} s & +h \\ +\bar{h} & -s \end{pmatrix} \psi$$

tr matrix $= 0$
so ~~f~~ volume preserved

If ψ^1, ψ^2 are 2 solutions, then $\det \left| \psi^1 \ \psi^2 \right|$ is constant.

Study transmission line with varying impedance but unit signal speed.



$$I_x - I_{x+dx} = C_x \, dx \, \dot{E}_x$$

$$E_x - E_{x+dx} = dx \, \ell_{x+dx} \, \dot{I}_{x+dx}$$

$$-\partial_x I = c_x \partial_t E$$

$$-\partial_x E = \ell_x \partial_t I$$

end up with

$$\boxed{\begin{aligned} \partial_t E + g \partial_x I &= 0 \\ \partial_t I + g^{-1} \partial_x E &= 0 \end{aligned}}$$

$$\partial_x E + \partial_t I = 0$$

$$\partial_x I + \partial_t E = 0$$

$$(\partial_x + \partial_t)(E + I) = 0$$

$$(\partial_x - \partial_t)(E - I) = 0$$

$$\begin{pmatrix} E + I \\ E - I \end{pmatrix} = \begin{pmatrix} A e^{-sx} \\ B e^{sx} \end{pmatrix} e^{st}$$

varying impedance trans. line, unit signal speed

$$-\partial_x I = l_x^{-1}\dot{E} \qquad -\partial_x E = l_x \dot{I}$$

$$l\,\partial_x I = -sE \qquad l^{-1}\partial_x E = -sI$$, can eliminate

$$l^{-1}\partial_x\, l\,\partial_x I = -s\left(l^{-1}\partial_x E\right) = s^2 I$$

Put $\boxed{g = l^{1/2}}$. $\quad g^2\partial_x I = -sE$ becomes

$$g\,\partial_x g^{-1}(gI) = -s(g^{-1}E) \quad \text{and} \quad g^{-2}\partial_x E = -sI$$

becomes $\underbrace{g^{-1}\partial_x g}_{\partial_x + \frac{g'}{g}}(g^{-1}E) = -s(gI)$,

$$\boxed{\phi = g^{-1}\partial_x g} \qquad \boxed{\begin{aligned}(\partial_x + \phi)(g^{-1}E) &= -s(gI)\\ (\partial_x - \phi)(gI) &= -s(g^{-1}E)\end{aligned}}$$

$$\partial_x\left(g^{-1}E + gI\right) + \phi\left(g^{-1}E - gI\right) = -s\left(g^{-1}E + gI\right)$$
$$\partial_x\left(g^{-1}E - gI\right) + \phi\left(g^{-1}E + gI\right) = s\left(g^{-1}E - gI\right)$$

~~[illegible]~~ You want to ~~to~~ understand IH in the cont. setting. ~~What are~~

You want to understand IH.

$$\partial_x \begin{pmatrix} z^{-x}p_x \\ q_x \end{pmatrix} = \begin{pmatrix} 0 & hz^{-x} \\ \bar{h}z^{x} & 0 \end{pmatrix} \begin{pmatrix} z^{-x}p_x \\ q_x \end{pmatrix}$$

left + right moving waves with reflection

$$(\partial_x+\partial_t)\alpha = -\phi\beta$$
$$(\partial_x-\partial_t)\beta = -\phi\alpha$$

ϕ ind of t.

$$\partial_t\alpha = -\partial_x\alpha - \phi\beta$$
$$+\partial_t\beta = +\partial_x\beta + \phi\alpha$$

$$\partial_t\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}-\partial_x & -\phi\\ \phi & \partial_x\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

How to get started? Review Wronskian. This takes place over the circle. You have a grid space E with time shift, specializing z. Your grid space is a rank 2 free module over fns on S^1. There's something interesting here. Let's play with the ideas. You have a grid space E which is a free module of rank 2 over $\mathbb{C}[z,z^{-1}]$. Just think of E as a vector bundle over the circle, where the fibre at z is the space of solutions of the Dirac eqn. at z. Go over the structure

Today's program: Write up Wronskian, σ, IH for disc D.E. grid space. Your aim should be to understand the situation geometrically as something over the unit circle.

Let's use the scattering picture, assume $h_n = 0$ for $|n|$ large. This way the four limits of $z^{-n}p_n, q_n$ are already in the grid space E, which is a free module of rank 2 over $\mathbb{C}[z,z^{-1}]$, with bases ξ_+, ξ_- and ξ'_-, ξ_+. (There are problems with scattering bases ξ'_-, ξ_- and ξ_+, ξ'_+ because d is not a unit in $\mathbb{C}[z,z^{-1}]$.) What do you want? For each z you have a 2 diml space of solutions of the dDE.

Keep to the scattering situation where description is easy. Idea: V_z = the solution space at $z \in S^1$ is 2 dimensional. structure? ~~Have~~

Choosing $\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$ as center, ??

Review. $V = \mathbb{C}^2$ σ conjugation $\sigma\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \bar z_2 \\ \bar z_1 \end{pmatrix}$

$\omega : \Lambda^2 V \xrightarrow{\sim} \mathbb{C}$, $\omega\left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix}\right) = \begin{vmatrix} z_1 & z'_1 \\ z_2 & z'_2 \end{vmatrix}$, note

$\overline{\omega(v,v')} = -\omega(\sigma v, \sigma v')$, so $H(v,v') = \omega(\sigma v, v')$ is hermitian

$$\omega\left(\sigma\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix}\right) = \begin{vmatrix} \bar z_2 & z'_1 \\ \bar z_1 & z'_2 \end{vmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix}$$

Wronskian

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basic idea
$$\begin{pmatrix} z^{-n}p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar h_n z^n & 1 \end{pmatrix}\begin{pmatrix} z^{-n+1}p_{n-1} \\ q_{n-1} \end{pmatrix}$$

Thus if $|z| = 1$, then V_z is 2-diml with σ, ω, H

Discuss scattering situation. You want a continuous ~~situation~~ example. Start with transmission line unit signal space

$$\partial_x E + \rho \dot{I} = 0 \qquad \partial_x I + \rho^{-1}\dot{E} = 0 \qquad \rho(x) > 0$$

$g = \rho^{1/2}$

$$g^{-1}\partial_x g\,(g^{-1}E) + (gI)^{\cdot} = 0$$

$$g\,\partial_x g^{-1}(gI) + (g^{-1}E)^{\cdot} = 0$$

$$g^{-1}\partial_x(g) = \partial_x \log g = \frac{g'}{g} = f$$

$$\begin{pmatrix} g^{-1}\partial_x g & s \\ s & g\,\partial_x g^{-1} \end{pmatrix}\begin{pmatrix} g^{-1}E \\ gI \end{pmatrix} = 0$$

$$\begin{pmatrix} \partial_x & s \\ s & \partial_x \end{pmatrix}\begin{pmatrix} g^{-1}E \\ gI \end{pmatrix} = \begin{pmatrix} -f\,g^{-1}E \\ +f\,gI \end{pmatrix}$$

$$(\partial_x + s)(g^{-1}E + gI) = -f\,(g^{-1}E - gI)$$

$$(\partial_x - s)(g^{-1}E - gI) = f\,(g^{-1}E + gI)$$

$$\psi = \begin{pmatrix} g^{-1}E + gI \\ g^{-1}E - gI \end{pmatrix} \qquad \text{Then}$$

$$\partial_x \psi = \begin{pmatrix} s & -f \\ f & -s \end{pmatrix}\psi$$

$$\boxed{\begin{pmatrix} \partial_x & f \\ -f & -\partial_x \end{pmatrix}\psi = s\,\psi}$$

Energy etc. $EN = \frac{1}{2}\int (\rho^{-1}E^2 + \rho I^2)\,dx$

$$= \frac{1}{2}\int \{(g^{-1}E)^2 + (gI)^2\}\,dx$$

$$\partial_t(EN) = \int (\rho^{-1}E\dot{E} + \rho I\dot{I})\,dx$$

$$-\partial_x I E + -I\partial_x E = -\partial_x(IE)$$

Focus on the problem, which seems to be changing from ∂_x, ∂_t to $\partial_x + \partial_t$ and $\partial_t - \partial_x$ things are puzzling because

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continuous case $L^2 = H_- \oplus H_+$

IH(ξ_- problem

IDEA: Szego fmla $\delta(0) = \cdots$ might lead, point the way, develop, to a treatment of ξ

Explore briefly implications of the fact that IH can be expressed using linear algebra over $\mathbb{C}[u,u^{-1}]$, i.e. σ, Wronskian, better to say, using that the grid space E is a rank 2 free module over $\mathbb{C}[u,u^{-1}]$. with $SU(1,1)$ structure,

Grid space should be viewed as a rank vector bundle over the circle

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Consider $\begin{pmatrix} \partial_x & ih \\ i\bar{h} & -\partial_x \end{pmatrix}\psi = ik\psi$ h compact support on x line

$$\begin{pmatrix} ?\, e^{ikx} \\ ?\, e^{-ikx} \end{pmatrix} \xleftarrow{x \to -\infty} \psi \xrightarrow{x \to +\infty} \begin{pmatrix} ?\, e^{ikx} \\ ?\, e^{-ikx} \end{pmatrix}$$

Two-dimensional space of solutions, eigenfunctions, denote it V_k, an element ψ of V_k is a type of linear functional on the appropriate grid space, so ψ can be evaluated on elements of grid space (in principle), whence you get numbers

$$\begin{pmatrix} \psi^1_{x=0} \\ \psi^2_{x=0} \end{pmatrix} = \psi \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \qquad \psi(\xi_+) = \lim_{x \to +\infty} e^{-ikx} \psi^1_x$$

$$\psi(\xi_-) = \lim_{x \to +\infty} e^{ikx} \psi^2_x$$

What are you ideas? For each $k \in \mathbb{R}$ you consider V_k, solutions of the homogeneous D.E, but you also have in mind solving the inhomog. eqn., equivalently Green's function. But G fn. needs bdry conditions at $\pm\infty$. There should be some recipe, eg replace k by $k \pm i0_+$ and take L^2 ∂ conditions. This should amount to deciding between incoming and outgoing ∂ conditions.

Consider previous continuous example

$$\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix} \psi \qquad \text{constant coeff}$$

$$\omega \psi = \begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix} \psi \qquad \begin{vmatrix} k-\omega & 1 \\ 1 & -k-\omega \end{vmatrix} = \omega^2 - k^2 - 1 = 0$$

$$\omega = \pm\sqrt{1+k^2}$$

$$\begin{pmatrix} \omega-k & 0 \\ 0 & \omega+k \end{pmatrix}\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$$

$(\omega-k)\psi^1 = \psi^2$

$(\omega+k)\psi^2 = \psi^1$

to setup up the general solution, suppose you use F.T. in x.

$$\frac{1}{i}\partial_t \hat{\psi} = \begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}\hat{\psi}$$

$$\hat{\psi} = \exp\left(i\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}t\right)\hat{\psi}_{t=0}$$

two functions of k

$$\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}\begin{pmatrix} 1 & -1 \\ \omega-k & \omega+k \end{pmatrix} = \begin{pmatrix} \omega & -1(-\omega) \\ (\omega-k)\omega & (\omega+k)(-\omega) \end{pmatrix}$$

$\frac{1+k^2-k\omega}{\omega^2}$ $\quad -1-k\omega-k^2$ $\quad -\omega^2-k\omega$

$$= \begin{pmatrix} 1 & -1 \\ \omega-k & \omega+k \end{pmatrix}\begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$$

$$\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}\begin{pmatrix} 1 & -1 \\ \omega-k & \omega+k \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ \omega-k & \omega+k \end{pmatrix}\begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$$

$\omega+k+\omega-k = 2\omega$

$$\frac{1}{2\omega}\begin{pmatrix} \omega+k & 1 \\ -\omega+k & 1 \end{pmatrix}\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}\begin{pmatrix} 1 & -1 \\ \omega-k & \omega+k \end{pmatrix}$$

$$\cancel{\begin{pmatrix} 1 & -\omega+k \\ \omega-k & 1 \end{pmatrix}}$$

$\cancel{\det = 1+(\omega-k)^2}$

$$\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ \omega-k & \omega+k \end{pmatrix}\begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}\begin{pmatrix} \omega+k & 1 \\ -\omega+k & 1 \end{pmatrix}\frac{1}{2\omega}$$

$$\boxed{e^{i\begin{pmatrix} k & 1 \\ 1 & -k\end{pmatrix}t} = \begin{pmatrix} 1 & -1 \\ \omega-k & \omega+k\end{pmatrix}\begin{pmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t}\end{pmatrix}\begin{pmatrix} \omega+k & 1 \\ -\omega+k & 1\end{pmatrix}\frac{1}{2\omega}}$$

pretty messy.

Let's pause and examine the forced simple harmonic oscillator

First case $\quad \ddot{x} + \omega_0^2 x = \mathrm{Re}\left(Ae^{i\omega t}\right)$

$$\ddot{x} + \omega_0^2 x = Ae^{i\omega t} \qquad x = Be^{i\omega t}$$

$$\left(-\omega^2 + \omega_0^2\right)B = A$$

$$\therefore \quad B = \frac{A}{-\omega^2+\omega_0^2} \qquad x = \mathrm{Re}\left(\frac{A}{-\omega^2+\omega_0^2}e^{i\omega t}\right)$$

Hamiltonian approach $\qquad m=1 \quad k=\omega_0^2$

$$T = \frac{1}{2}\dot{x}^2 \qquad V = \frac{1}{2}\omega_0^2x^2 - Fx \qquad F=F(t)$$

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega_0^2x^2 + F(t)x$$

$$\frac{\partial L}{\partial \dot{x}} = \dot{x} \qquad \frac{\partial L}{\partial x} = -\omega_0^2 x + F(t)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \qquad \ddot{x} = -\omega_0^2 x + F(t)$$

$$\ddot{x} + \omega_0^2 x = +F(t)$$

How do I set this up? ~~simple harmonic oscillator~~ approach

Look at Hamiltonian

$$H = \dot{x}\frac{\partial L}{\partial \dot{x}} - L = \dot{x}^2 - \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega_0^2 x^2 - F(t)x$$

$$H = \frac{1}{2}p^2 + \frac{\omega_0^2}{2}q^2 - F(t)q$$

$$\boxed{\dot{q} = \frac{\partial H}{\partial p} = p \quad \Bigg| \quad \dot{p} = -\frac{\partial H}{\partial q} = -\omega_0^2 q + F(t)}$$

$$\ddot{x} + \omega_0^2 x = F(t).$$

$$\frac{d}{dt}(H) = p\left(-\omega_0^2 q + F(t)\right) + \left(\omega_0^2 q - F(t)\right)$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial t}$$

$$= \cancel{\left(\omega_0^2 q - F(t)\right)p + p\left(-\omega_0^2 q + F(t)\right)} - F'(t)q$$

$$\tilde{H} = \frac{p^2}{2} + \frac{\omega_0^2}{2}q^2 + f(t)p + g(t)q$$

$$\dot{q} = \frac{\partial \tilde{H}}{\partial p} = p + f(t) \qquad \dot{p} = -\frac{\partial \tilde{H}}{\partial q} = -\omega_0^2 q - g(t)$$

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} f(t) \\ -g(t) \end{pmatrix}$$

Assume

$$\begin{pmatrix} f(t) \\ -g(t) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} e^{i\omega t}$$

$$\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{i\omega t}$$

$$i\omega\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\partial_t\psi = \begin{pmatrix}\partial_x & i\\ i & -\partial_x\end{pmatrix}\psi \qquad \omega\psi = \begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix}\psi \qquad \begin{matrix}(\omega-k)\psi^1=\psi^2\\ (\omega+k)\psi^2=\psi^1\end{matrix}$$ 2882

$$\begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix}\begin{pmatrix}1 & -1\\ \omega-k & \omega+k\end{pmatrix} = \begin{pmatrix}\omega & +\omega\\ \begin{matrix}1-k\omega+k^2\\ \omega^2-k\omega\end{matrix} & \begin{matrix}-1-k\omega+k^2\\ -k\omega-\omega^2\end{matrix}\end{pmatrix}$$

$$= \begin{pmatrix}1 & -1\\ \omega-k & \omega+k\end{pmatrix}\begin{pmatrix}\omega & 0\\ 0 & -\omega\end{pmatrix}$$

$$\cancel{\begin{pmatrix}\omega+k & 1\\ -\omega+k & 1\end{pmatrix}\begin{pmatrix}e^{i\omega t} & 0\\ 0 & e^{-i\omega t}\end{pmatrix}\begin{pmatrix}1 & -1\\ \omega-k & \omega+k\end{pmatrix}\frac{1}{2\omega} = e^{i\begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix}t}}$$

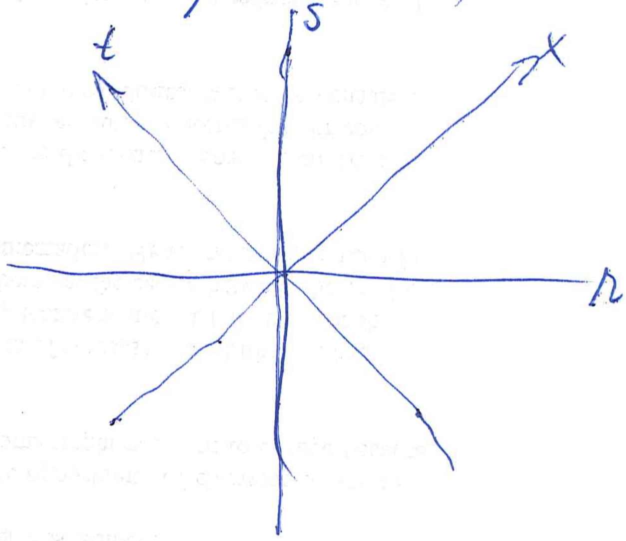
gen. solution

$$\psi(x,t) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\, e^{i\begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix}t}\, e^{ikx}\begin{pmatrix}A(k)\\ B(k)\end{pmatrix}$$

~~Let~~

$$\exp\left(i\begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix}t\right) = \begin{pmatrix}1 & -1\\ \omega-k & \omega+k\end{pmatrix}\begin{pmatrix}e^{i\omega t} & 0\\ 0 & e^{-i\omega t}\end{pmatrix}\begin{pmatrix}\omega+k & 1\\ -\omega+k & 1\end{pmatrix}\frac{1}{2\omega}$$

Go back to
$$(\partial_t-\partial_x)\psi^1 = i\psi^2$$
$$(\partial_t+\partial_x)\psi^2 = i\psi^1$$



$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\underbrace{\frac{\partial x}{\partial r}}_{1} + \frac{\partial f}{\partial t}\underbrace{\frac{\partial t}{\partial r}}_{-1}$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\underbrace{\frac{\partial x}{\partial s}}_{1} + \frac{\partial f}{\partial t}\underbrace{\frac{\partial t}{\partial s}}_{1}$$

$$x = r+s$$
$$t = -r+s$$
$$r = \frac{x-t}{2} \qquad s = \frac{x+t}{2}$$

$$\partial_r = -\partial_t + \partial_x \qquad -\tfrac{1}{i}\partial_r \psi^1 = \psi^2 \qquad -\rho\psi^1 = \psi^2$$
$$\partial_s = \partial_t + \partial_x \qquad \tfrac{1}{i}\partial_s \psi^2 = \psi^1 \qquad \sigma\psi^2 = \psi^1$$

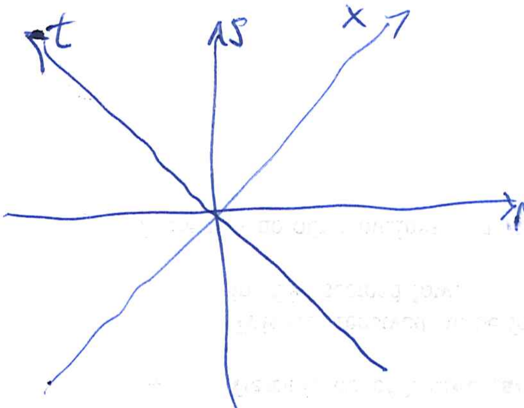
So you end up with spectrum $-\rho\sigma = 1$.

$$\psi(r,s) = \int_{-\infty}^{\infty} e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)$$

$$r\rho - s\rho^{-1} = \frac{x-t}{2}\rho - \frac{x+t}{2}\rho^{-1} = x\underbrace{\left(\frac{\rho-\rho^{-1}}{2}\right)}_{k} - t\underbrace{\left(\frac{\rho+\rho^{-1}}{2}\right)}_{\omega}$$

Important thing here is to calculate IH.

Concentrate
$$\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}\psi \qquad \begin{aligned}(\partial_t - \partial_x)\psi^1 &= i\psi^2 \\ (\partial_t + \partial_x)\psi^2 &= i\psi^1\end{aligned}$$



$$r = \frac{-t+x}{2} \qquad t = -r+s$$
$$s = \frac{t+x}{2} \qquad x = r+s$$

$$\partial_r = \partial_x \frac{\partial x}{\partial r} + \partial_t \frac{\partial t}{\partial r} = \partial_x - \partial_t$$
$$\partial_s = \partial_x \frac{\partial x}{\partial s} + \partial_t \frac{\partial t}{\partial s} = \partial_x + \partial_t$$

$$-\partial_r \psi^1 = i\psi^2 \qquad -\rho\psi^1 = \psi^2 \qquad \sigma = -\rho^{-1}$$
$$\partial_s \psi^2 = i\psi^1 \qquad \sigma\psi^2 = \psi^1$$

$$\psi = \int_{-\infty}^{\infty} e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} \psi^1 \, d\rho$$

$$r\rho - s\rho^{-1} = \frac{x-t}{2}\rho - \frac{x+t}{2}\rho^{-1} = x\overbrace{\left(\frac{\rho-\rho^{-1}}{2}\right)}^{k} - t\overbrace{\left(\frac{\rho+\rho^{-1}}{2}\right)}^{\omega}$$

$$\psi = \int_{-\infty}^{\infty} e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)\, d\rho =$$

$$\omega = \frac{\rho+\rho^{-1}}{2} \qquad \rho = \omega + k$$
$$k = \frac{\rho-\rho^{-1}}{2} \qquad \rho^{-1} = \omega - k$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} \left(e^{-i\omega t} \begin{pmatrix} 1 \\ -\omega-k \end{pmatrix} f(k) + e^{i\omega t} \begin{pmatrix} 1 \\ \omega-k \end{pmatrix} g(k) \right)$$

here $\omega = +\sqrt{1+k^2}$

Question: What is the grid space? Simplest answer would be to give a basis, rather to give the transform between the grid space and functions ~~one~~ resulting from this basis. First this coming to mind is the Cauchy data, along $t=0$, - this corresponds to ascending staircase. σ and the Wronskian might be easy to understand in this picture

The space of Cauchy data = space of two functions of k.

$$\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix} \begin{pmatrix} 1 \\ \omega-k \end{pmatrix} = \begin{pmatrix} k+\omega-k \\ 1-k\omega+k^2 \end{pmatrix} = \begin{pmatrix} \omega \\ (\omega-k)\omega \end{pmatrix}$$

$$\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix} \begin{pmatrix} -1 \\ \omega+k \end{pmatrix} = \begin{pmatrix} -k+\omega+k \\ -1-k^2-k\omega \end{pmatrix} = \begin{pmatrix} \omega \\ -\omega^2-k\omega \end{pmatrix} = \begin{pmatrix} -1 \\ \omega+k \end{pmatrix}(-\omega)$$

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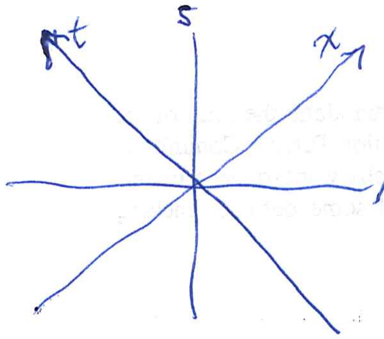
Review: Consider $\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix} \psi$, $(\partial_t - \partial_x)\psi^1 = i\psi^2$, $(\partial_t + \partial_x)\psi^2 = i\psi^1$



$$\partial_r = -\partial_t + \partial_x \qquad \partial_r f = \frac{\partial f}{\partial t}\frac{\partial t}{\partial r} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial r}$$

$$\partial_s = \partial_t + \partial_x \qquad \partial_s f = \frac{\partial f}{\partial t}\frac{\partial t}{\partial s} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial s}$$

$$t = -r + s \qquad -\partial_r \psi^1 = i\psi^2 \qquad -\rho\psi^1 = \psi^2$$

$$x = r + s \qquad \partial_s \psi^2 = i\psi^1 \qquad \sigma\psi^2 = \psi^1$$

$$\psi(r,s) = \int_{-\infty}^{\infty} e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)\, d\rho \qquad r = \frac{x-t}{2}, \quad s = \frac{x+t}{2}$$

$$r\rho - s\rho^{-1} = \frac{x-t}{2}\rho - \frac{x+t}{2}\rho^{-1} = x\underbrace{\left(\frac{\rho-\rho^{-1}}{2}\right)}_{k} - t\underbrace{\left(\frac{\rho+\rho^{-1}}{2}\right)}_{\omega}$$

$$\boxed{\psi(x,t) = \int_{-\infty}^{\infty} e^{ix\left(\frac{\rho-\rho^{-1}}{2}\right) - it\left(\frac{\rho+\rho^{-1}}{2}\right)} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)\, d\rho}$$

describes general solutions of $\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix} \psi$ in terms of an arbitrary function, distribution $f(\rho)$ on $\mathbb{R}$.

You also can describe solutions by Cauchy data

$$\psi(x,0) = \int_{-\infty}^{\infty} e^{ix\left(\frac{\rho-\rho^{-1}}{2}\right)} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)\, d\rho \qquad k = \frac{\rho-\rho^{-1}}{2}, \ \omega = \frac{\rho+\rho^{-1}}{2}, \ \rho = \omega + k, \ \rho^{-1} = \omega - k$$

You want to write this as

$$\psi(x,0) = \int_{-\infty}^{\infty} e^{ikx} \begin{pmatrix} 1 \\ -\omega-k \end{pmatrix} f(k) \quad ?$$

Somehow you want to organize $\{\rho \mid \frac{\rho-\rho^{-1}}{2} = k\}$.

two roots $\overset{\omega+k}{\rho}$ and $-\rho^{-1} = -\omega + k$. Then get

$$\boxed{\psi(x,0) = \int_{-\infty}^{\infty} e^{ikx}\left(\begin{pmatrix} 1 \\ -\omega-k \end{pmatrix} f(k) + \begin{pmatrix} 1 \\ +\omega-k \end{pmatrix} g(k)\right) dk}$$

Repeat what you [redacted] have found.
You are studying [redacted] the appropriate space of solutions of the massive Dirac. eqn $\partial_t\psi = \begin{pmatrix}\partial_x & i\\ i & -\partial_x\end{pmatrix}\psi$.
This space is the grid space (still needs a precise definition).

The obvious thing (from the wave equation viewpoint) (also from the Hilbert space picture – finite energy solutions) (also increasing staircase orthonormal basis) is Cauchy data picture at $t=0$. This means you look at the $\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ which are L^2 with the time evolution given by the skew-adj op $\begin{pmatrix}\partial_x & i\\ i & -\partial_x\end{pmatrix}$, natural to do F.T., to look at $\psi(x)=\int e^{ikx}\hat{\psi}(k)$ with the skew-adj op $i\begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix}$. This seems new to me, to have for each $k\in\mathbb{R}$ a 2-diml Hilbert space for the values of $\hat{\psi}$ at k.

Maybe better to consider more general equation $\partial_t\psi = \begin{pmatrix}\partial_x & ih\\ i\bar{h} & -\partial_x\end{pmatrix}\psi$, where $h=h(x)$. Again have increasing staircase orthonormal basis along $t=0$, which means simply the Hilbert space of L^2 functions $\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ of x, equipped with skew-adjoint $\begin{pmatrix}\partial_x & ih\\ i\bar{h} & -\partial_x\end{pmatrix}$. What does spectral theory say? Look at space of eigenfunctions V_ω of this operator for a (real) frequency ω.

Review - ~~think~~ $\partial_t \psi = \underbrace{\begin{pmatrix} \partial_x & ih \\ i\bar{h} & -\partial_x \end{pmatrix}}_{X} \psi$, X is a skew adjoint operator on the Hilbert space of $\psi \in \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$. ~~[illegible]~~ there is a spectral decomposition, abstract one involving a family of projectors, but also a concrete one à la Titchmarsh, which should come from contour integration of the Green's function. Yes, this is the missing point, the link between the ~~[illegible]~~ 2 diml spaces $V_\omega = \{\psi(x) \mid X\psi = i\omega\psi\}$

~~[illegible]~~

$$i\omega\psi = \begin{pmatrix} \partial_x & ih \\ i\bar{h} & -\partial_x \end{pmatrix}\psi \qquad \begin{pmatrix} \frac{1}{i}\partial_x - \omega & h \\ -\bar{h} & +\frac{1}{i}\partial_x + \omega \end{pmatrix}\psi = 0$$

$$\frac{1}{i}\partial_x \psi = \begin{pmatrix} \omega & -h \\ \bar{h} & -\omega \end{pmatrix}\psi \qquad ?$$

$$\omega\psi = \begin{pmatrix} \frac{1}{i}\partial_x & h \\ \bar{h} & -\frac{1}{i}\partial_x \end{pmatrix}\psi \qquad \begin{pmatrix} \frac{1}{i}\partial_x - \omega & h \\ -\bar{h} & +\frac{1}{i}\partial_x + \omega \end{pmatrix}\psi = 0$$

$$\frac{1}{i}\partial_x\psi = \begin{pmatrix} \omega & -h \\ +\bar{h} & -\omega \end{pmatrix}\psi \qquad \partial_x\psi = \begin{pmatrix} i\omega & -ih \\ i\bar{h} & -i\omega \end{pmatrix}\psi$$

Lie Alg $su(1,1)$ $\begin{pmatrix} 1+a\varepsilon & b\varepsilon \\ \bar{b}\varepsilon & 1+\bar{a}\varepsilon \end{pmatrix}$, $1 + (a+\bar{a})\varepsilon = 1$ $\begin{pmatrix} i\alpha & b \\ \bar{b} & i\alpha \end{pmatrix}$

$\therefore a + \bar{a} = 0.$

~~[illegible]~~ When you convert the DE one the line, $X\psi = i\omega\psi$, to ~~transfer form~~ propagating form $\partial_x \psi = \begin{pmatrix} i\omega & b \\ \bar{b} & -i\omega \end{pmatrix}\psi$, you are effectively working in ~~[illegible]~~ $su(1,1)$ picture — the IH picture

~~(ii) should then be possible to handle~~

To study $\partial_x \psi = \begin{pmatrix} i\omega & \overset{\bar m}{-ih} \\ \underset{m}{ih} & -i\omega \end{pmatrix}\psi$

2 dim'l space V_ω of solutions, splits according to boundary conditions. First get forms of this straight.

$$i\omega\psi = \begin{pmatrix} \partial_x & i\bar m \\ i m & -\partial_x \end{pmatrix}\psi \qquad \boxed{\omega\psi = \begin{pmatrix} \frac{1}{i}\partial_x & \bar m \\ m & -\frac{1}{i}\partial_x \end{pmatrix}\psi}$$

$$0 = \begin{pmatrix} \frac{1}{i}\partial_x - \omega & \bar m \\ -m & +\frac{1}{i}\partial_x + \omega \end{pmatrix}\psi \qquad \boxed{\partial_x\psi = \begin{pmatrix} i\omega & -i\bar m \\ i m & -i\omega \end{pmatrix}\psi}$$

?

$$\boxed{\frac{1}{i}\partial_x\psi = \begin{pmatrix} \omega & -\bar m \\ m & -\omega \end{pmatrix}\psi}$$

Start again with $\quad i\omega\psi = \begin{pmatrix} \partial_x & i\bar m \\ i m & -\partial_x \end{pmatrix}\psi \qquad \boxed{\omega\psi = \begin{pmatrix} \frac{1}{i}\partial_x & \bar m \\ m & -\frac{1}{i}\partial_x \end{pmatrix}}$

$$0 = \begin{pmatrix} \frac{1}{i}\partial_x - \omega & \bar m \\ -m & +\frac{1}{i}\partial_x + \omega \end{pmatrix}\psi \qquad \boxed{\frac{1}{i}\partial_x\psi = \begin{pmatrix} \omega & -\bar m \\ m & -\omega \end{pmatrix}\psi}$$

Now analyze $\begin{vmatrix} \omega-\lambda & -\bar m \\ m & -\omega-\lambda \end{vmatrix} = -\omega^2 + \lambda^2 + |m|^2$

$$\lambda^2 = \omega^2 - \underbrace{|m|^2}_{1}$$

eigenvalues are $k = \pm\sqrt{\omega^2 - 1}$

$e^{ikx} = e^{\pm i\sqrt{\omega^2-1}\,x}$ if $|\omega| > 1$ get something oscillatory, but if $|\omega| < 1$

Formulate, discuss the problems, how to proceed? Begin with eigenfunction equation which you can write in various forms, but in the end you have a 2-dim'l space V_ω of solutions for each complex number $\omega \in \mathbb{C}$, maybe $\omega = \infty$ can also be included.

How do you locate the spectrum? Method (Titmarsh) form resolvent look at singularities which lie on real ω-axis, since you are dealing with a self-adjoint operator $A = \begin{pmatrix} \frac{1}{i}\partial_x & 1 \\ 1 & -\frac{1}{i}\partial_x \end{pmatrix}$. Where is $\omega - A$ invertible on L^2? Away from $\omega = \pm\sqrt{k^2+1}$ $k \in \mathbb{R}$.

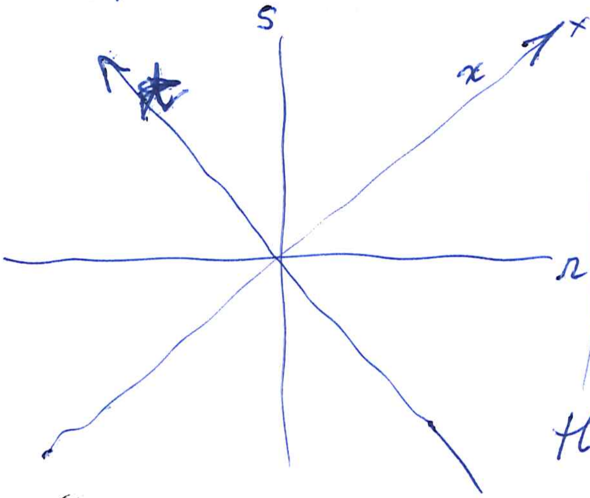
Repeat: $\partial_t \psi = \begin{pmatrix} \partial_x & im \\ im & -\partial_x \end{pmatrix} \psi$ wave equation

What are the basic problems? Already for $m=1$. What is (a good candidate) for the grid space? How to calculate IH?

Structure of grid space: 1-parameter group e^{∂_t} i.e. a module for the additive group $\mathbb{R}$, some kind of module over the group ring, it should be universal for solutions of the wave equation should map to others i.e. have ind. limit type, like compact support, Pass to char. coords.

$$-\partial_r \psi^1 = (\partial_t - \partial_x)\psi^1 = i\psi^2$$
$$\partial_s \psi^2 = (\partial_t + \partial_x)\psi^2 = i\psi^1$$

very simple type of PDE in the r,s plane



$t = -r+s \qquad r = \frac{x-t}{2}$

$x = r+s \qquad s = \frac{x+t}{2}$

Because of translation invariance it should be easy to describe grid space as module over the group ring of $\mathbb{R}+\mathbb{R}$. What's the answer? Use F.T.

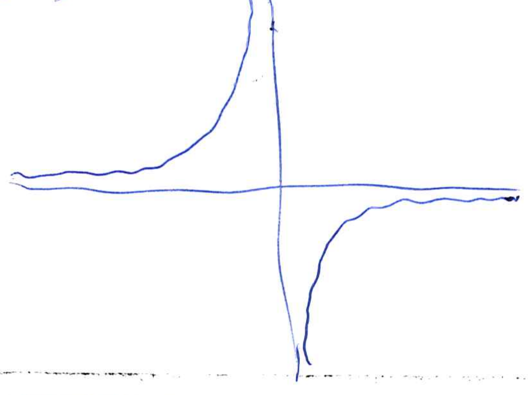
$$-\rho\psi^1 = \psi^2 \qquad \psi(r,s) = \int_{-\infty}^{\infty} e^{i(r\rho - s\rho^{-1})}\binom{1}{-\rho} f(\rho)\,d\rho$$
$$\sigma\psi^2 = \psi^1$$

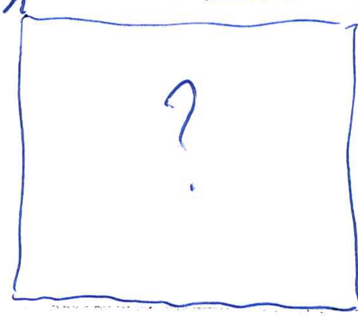
So grid space should admit a picture as functions (maybe distributions) of ρ with translation action

$$\psi(r+\varepsilon, s+\eta) = \int e^{i(r\rho - s\rho^{-1})}\binom{1}{-\rho}\, e^{i(\varepsilon\rho - \eta\rho^{-1})} f(\rho)\,d\rho$$

i.e. $\partial_r = +i\rho$, $\partial_s = -i\rho^{-1}$. However something else is happening, namely, you should consider the spectrum of the module over $\mathbb{R}^2$. This lies in the dual space of $\{(\rho,\sigma)\in\mathbb{R}^2\}$, and $\{(\rho,\sigma)\in\mathbb{R}^2 \mid \rho\sigma = -1\}$.

Here is what you can do. Form $(\mathbb{R}\cup\infty)^2 \simeq S^1\times S^1$ and look inside at the curve $\rho\sigma = -1$, which should be a smooth circle cutting the axes nicely in 2 points?





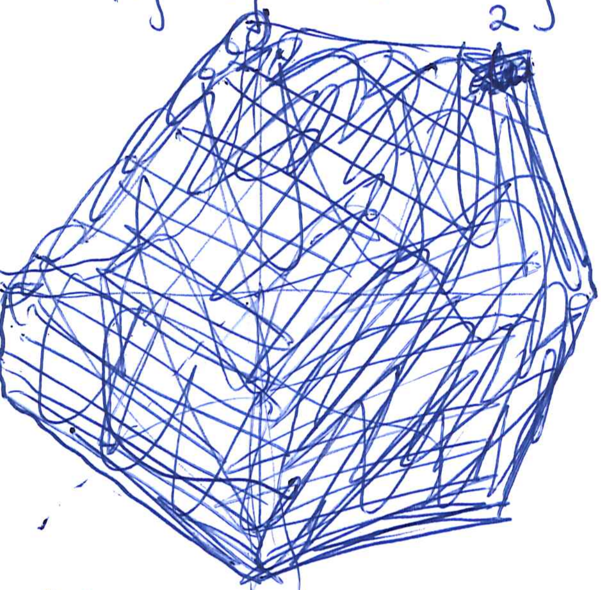
It looks like you want to take $f(\rho,\sigma)\in\mathcal{S}(\mathbb{R}^2)$ and consider

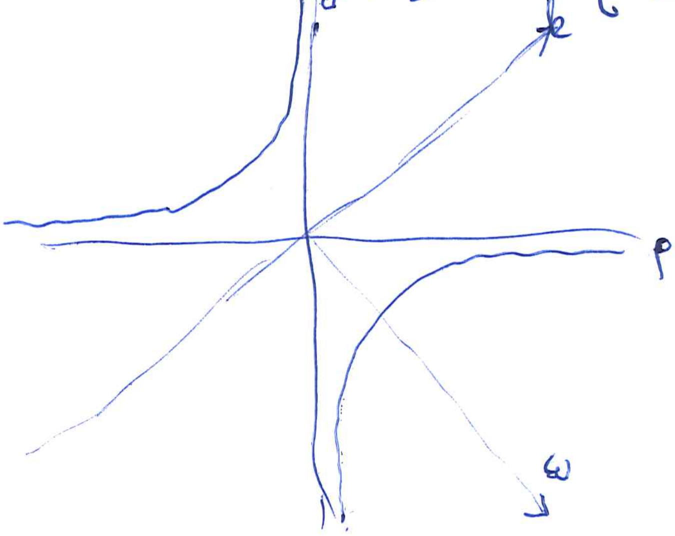
$$\psi(r,s) = \int e^{i(r\rho - s\rho^{-1})}\begin{pmatrix}1\\-\rho\end{pmatrix} f(\rho,-\rho^{-1})\,d\rho$$

What else happens?

You want to ~~look at the curve~~ consider the Cauchy problem, with initial data on $t=0$, means looking at $\{(\rho,\sigma)\mid \rho\sigma=-1\}$ "over" the ω axis where $\omega = \frac{\rho+\rho^{-1}}{2}$, $k=\frac{\rho-\rho^{-1}}{2}$. Check

$$r\rho - s\rho^{-1} = \frac{x-t}{2}\rho - \frac{x+t}{2}\rho^{-1} = x\left(\frac{\rho-\rho^{-1}}{2}\right) - t\left(\frac{\rho+\rho^{-1}}{2}\right)$$



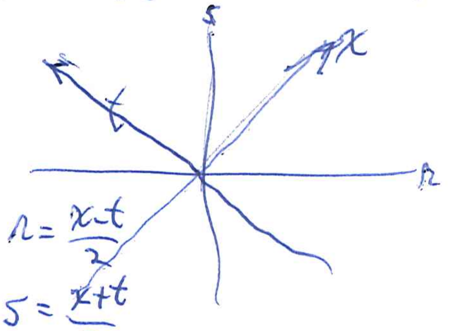


What are you trying to do? To represent grid space via Cauchy data:

$$\psi(x,0) = \int_{-\infty}^{\infty} e^{i\left(\frac{\rho-\rho^{-1}}{2}\right)x}\begin{pmatrix}1\\-\rho\end{pmatrix} f(\rho)\,d\rho$$

get two functions of k.

Repeat: Studying $\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix} \psi$ or



$$(\partial_t - \partial_x)\psi^1 = i\psi^2$$
$$(\partial_t + \partial_x)\psi^2 = i\psi^1$$

$$\partial_r = -\partial_t + \partial_x$$
$$\partial_s = \partial_t + \partial_x$$

$$-\partial_r \psi^1 = i\psi^2 \qquad -\rho\psi^1 = \psi^2$$
$$\partial_s \psi^2 = i\psi^1 \qquad \sigma\psi^2 = \psi^1$$

$$t = -r + s \qquad x = r + s$$

$$r = \frac{x-t}{2} \qquad s = \frac{x+t}{2}$$

$$\psi(x,t) = \int_{-\infty}^{\infty} e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)\, d\rho$$

$$r\rho - s\rho^{-1} = \left(\frac{x-t}{2}\right)\rho - \left(\frac{x+t}{2}\right)\rho^{-1} = x\underbrace{\left(\frac{\rho-\rho^{-1}}{2}\right)}_{k} - t\underbrace{\left(\frac{\rho+\rho^{-1}}{2}\right)}_{\omega}$$

$$\omega + k = \rho$$
$$\omega - k = \rho^{-1}$$

$$\psi(x,0) = \int_{-\infty}^{\infty} e^{i\overbrace{\left(\frac{\rho-\rho^{-1}}{2}\right)}^{}\underbrace{}_{k} x} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)\, d\rho$$

For each k have two values: $\rho = \omega + k$, $-\rho^{-1} = -\omega + k$

$$= \int_0^{\infty} e^{i\left(\frac{\rho-\rho^{-1}}{2}\right)x} \left(\begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho) + \begin{pmatrix} 1 \\ \rho^{-1} \end{pmatrix} f(-\rho^{-1}) \right) d\rho$$

This is not very clear, you need a better way to proceed.

You want to handle $\partial_t \psi = \begin{pmatrix} \partial_x & ih \\ ih & -\partial_x \end{pmatrix} \psi$ where $h = h(x)$. Ignore fine stuff – grid space, univ. soln. Instead describe all solutions (reasonable). Reasonable should mean can understand time flow via F.T. (maybe LT). So you go from to

$$\omega \hat\psi = \begin{pmatrix} \partial_x & ih \\ ih & -\partial_x \end{pmatrix} \hat\psi$$

where $\hat\psi = \hat\psi(x,\omega)$. So now you have replaced solutions $\psi(x,t)$ by sections $\omega \mapsto \psi(x,\omega)$ of the bundle of eigenfunctions. $\omega \mapsto V_\omega$

Analyze ~~$\omega\psi = \begin{pmatrix} \frac{1}{i}\partial_x - \omega & h \\ \bar{h} & -\frac{1}{i}\partial_x - \omega \end{pmatrix}\psi = 0$~~

$$0 = \begin{pmatrix} \frac{1}{i}\partial_x - \omega & h \\ -\bar{h} & +\frac{1}{i}\partial_x + \omega \end{pmatrix}\psi \qquad \boxed{\frac{1}{i}\partial_x\psi = \begin{pmatrix} \omega & -h \\ \bar{h} & -\omega \end{pmatrix}\psi}$$

This DE has $su(1,1)$ propagation, so the space of solutions V_ω should have σ, vol.

$$\sigma\left(\frac{1}{i}\partial_x\begin{pmatrix}\psi^1 \\ \psi^2\end{pmatrix}\right) = +i\,\partial_x\begin{pmatrix}\overline{\psi^2} \\ \overline{\psi^1}\end{pmatrix}$$

$$\sigma\left(\begin{pmatrix}\omega & -h \\ \bar{h} & -\omega\end{pmatrix}\begin{pmatrix}\psi^1 \\ \psi^2\end{pmatrix}\right) = \sigma\begin{pmatrix}\omega\psi^1 - h\psi^2 \\ \bar{h}\psi^1 - \omega\psi^2\end{pmatrix} =$$

$$= \begin{pmatrix}h\overline{\psi^1} - \omega\overline{\psi^2} \\ \omega\overline{\psi^1} - \bar{h}\overline{\psi^2}\end{pmatrix} = \begin{pmatrix}-\omega & h \\ -\bar{h} & \omega\end{pmatrix}$$

~~$\begin{pmatrix}\omega & -\bar{h} \\ h & -\omega\end{pmatrix}\begin{pmatrix}\overline{\psi^2} \\ \overline{\psi^1}\end{pmatrix} = \begin{pmatrix}\omega\overline{\psi^2} - \bar{h}\overline{\psi^1} \\ h\overline{\psi^2} - \omega\overline{\psi^1}\end{pmatrix}$~~

$$\partial_x\psi = \begin{pmatrix} i\omega & -ih \\ i\bar{h} & -i\omega \end{pmatrix}\psi \qquad \begin{pmatrix} -i\omega & i\bar{h} \\ -ih & i\omega \end{pmatrix} \rightsquigarrow \begin{pmatrix} i\omega & -ih \\ i\bar{h} & -i\omega \end{pmatrix}$$

$$\begin{pmatrix}\psi^1(x) \\ \psi^2(x)\end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{SU(1,1)}\begin{pmatrix}\psi^1(y) \\ \psi^2(y)\end{pmatrix}$$

$$\begin{pmatrix}\overline{\psi^2(x)} \\ \overline{\psi^1(x)}\end{pmatrix} = \begin{pmatrix} \bar{d} & \bar{c} \\ \bar{b} & \bar{a} \end{pmatrix}\begin{pmatrix}\overline{\psi^2(y)} \\ \overline{\psi^1(y)}\end{pmatrix}$$

$$\partial_x \begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix} = \begin{pmatrix} i\omega & -ih \\ i\bar h & -i\omega\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix} = \begin{pmatrix} i\omega\psi^1 - ih\psi^2 \\ i\bar h\psi^1 - i\omega\psi^2\end{pmatrix}$$

$$\partial_x \begin{pmatrix}\bar\psi^2\\ \bar\psi^1\end{pmatrix} = \begin{pmatrix} -ih\bar\psi^1 + i\omega\bar\psi^2 \\ -i\omega\bar\psi^1 + i\bar h\bar\psi^2\end{pmatrix} = \begin{pmatrix} i\omega & -ih\\ i\bar h & -i\omega\end{pmatrix}\begin{pmatrix}\bar\psi^2\\ \bar\psi^1\end{pmatrix}$$

Thus you get a conjugation σ on V_ω, ω real defined by $\sigma\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix} = \begin{pmatrix}\bar\psi^2\\ \bar\psi^1\end{pmatrix}$

Review example. $V = \mathbb{C}^2$ $\quad \sigma\begin{pmatrix}z_1\\ z_2\end{pmatrix} = \begin{pmatrix}\bar z_2\\ \bar z_1\end{pmatrix}$

$$\omega\left(\begin{pmatrix}z_1\\ z_2\end{pmatrix}, \begin{pmatrix}z_1'\\ z_2'\end{pmatrix}\right) = \begin{vmatrix} z_1 & z_1' \\ z_2 & z_2'\end{vmatrix}$$

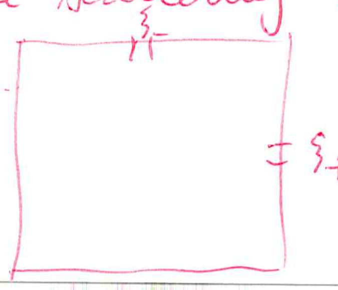
$$\overline{\omega\left(\begin{pmatrix}z_1\\ z_2\end{pmatrix}, \begin{pmatrix}z_1'\\ z_2'\end{pmatrix}\right)} = \begin{vmatrix} \bar z_1 & \bar z_1' \\ \bar z_2 & \bar z_2'\end{vmatrix} = -\begin{vmatrix} \bar z_2 & \bar z_2' \\ \bar z_1 & \bar z_1'\end{vmatrix} = -\omega\left(\sigma\begin{pmatrix}z_1\\ z_2\end{pmatrix}, \sigma\begin{pmatrix}z_1'\\ z_2'\end{pmatrix}\right)$$

$$\omega\left(\sigma\begin{pmatrix}z_1\\ z_2\end{pmatrix}, \begin{pmatrix}z_1'\\ z_2'\end{pmatrix}\right) = \begin{vmatrix} \bar z_2 & z_1' \\ \bar z_1 & z_2'\end{vmatrix} = \bar z_2 z_2' - \bar z_1 z_1'$$

$$H(v, v) = |z_2|^2 - |z_1|^2.$$

Now what are you hoping to get.

Begin again - discuss Wronskian. Let's start with the scattering setup.



$$\xi_+ L^2 \oplus \xi_- L^2 = E \qquad \begin{pmatrix}\xi_+\\ \xi_+'\end{pmatrix} = \begin{pmatrix}\delta & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\xi_-'\\ \xi_-\end{pmatrix}$$

$$\|\xi_+ f + \xi_- g\|^2 = \int \begin{pmatrix}f\\ g\end{pmatrix}^*\begin{pmatrix}1 & \bar\beta\\ \beta & 1\end{pmatrix}\begin{pmatrix}f\\ g\end{pmatrix}$$

$$\|(\xi_-'\delta + \xi_-\beta) f + \xi_- g\|^2 = \|\delta f\|^2 + \|\beta f + g\|^2$$

$IH(\xi_+ f + \xi_- g) = \|f\|^2 - \|g\|^2$. Define

$$\sigma(\xi_+ f + \xi_- g) = \xi_+ \bar{g} + \xi_- \bar{f}$$

$$\|\sigma(\xi_+ f + \xi_- g)\|^2 = \| \text{———} \|^2 = \int \begin{pmatrix} \bar{g} \\ \bar{f} \end{pmatrix}^* \begin{pmatrix} 1 & \bar{\beta} \\ \beta & 1 \end{pmatrix} \begin{pmatrix} \bar{g} \\ \bar{f} \end{pmatrix}$$

$$= \int \underbrace{\begin{pmatrix} \bar{g} \\ \bar{f} \end{pmatrix}^t}_{\begin{pmatrix} g \\ f \end{pmatrix}^*} \begin{pmatrix} 1 & \beta \\ \bar{\beta} & 1 \end{pmatrix} \begin{pmatrix} g \\ f \end{pmatrix} = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar{\beta} \\ \beta & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

$$\| \qquad \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ \bar{\beta} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

$$IH(\sigma(\xi_+ f + \xi_- g)) = IH(\xi_+ \bar{g} + \xi_- \bar{f}) = \|\bar{g}\|^2 - \|\bar{f}\|^2 = \|g\|^2 - \|f\|^2$$

$$\sigma(\xi_+ f + \xi_- g) \wedge (\xi_+ f + \xi_- g) = (\xi_+ \bar{g} + \xi_- \bar{f}) \wedge (\xi_+ f + \xi_- g)$$

$$= (|g|^2 - |f|^2)\, \xi_+ \wedge \xi_-$$

So now return to

$$\begin{vmatrix} \omega - k & -1 \\ 1 & -\omega - k \end{vmatrix} = -\omega^2 + k^2 + 1$$

$$\frac{1}{i} \partial_x \psi = \begin{pmatrix} \omega & h \\ \bar{h} & -\omega \end{pmatrix} \psi$$

~~[illegible]~~. Stick with scattering situation, but try to use the fact that IH is essentially integrating ~~the~~ over the circle of the local hermitian form — the Wronskian of $\sigma(\psi) \wedge \psi'$

You ought to be able to see what's going on pointwise. What do you mean by $\xi_+ f + \xi_- g$? since you haven't explained the grid space?

Let's make an attempt.

Start with the ~~[illegible]~~ wave equation

$$\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix} \psi .$$

Using FT in time you can replace this by

$$\omega \psi = \begin{pmatrix} \frac{1}{i}\partial_x & 1 \\ 1 & -\frac{1}{i}\partial_x \end{pmatrix} \psi$$

or

$$\frac{1}{i}\partial_x \psi = \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix} \psi .$$

By replace you mean that the FT sets ~~[illegible]~~ a good correspondence between solutions of the wave eqn and ~~[illegible]~~ sections over real ω-line of the vector bundle with fibre V_ω = solutions of the ODE ~~for~~ eigenvalue ω.

Go back to scattering situation

$$\frac{1}{i}\partial_x \psi = \begin{pmatrix} \omega & -h \\ \bar{h} & -\omega \end{pmatrix} \psi$$

where $h(x)$ decays as $|x| \to \infty$.

Then you see some structure on V_ω namely the four elements $\xi_\pm, \xi'_\pm$, related by the transfer or scattering matrices evaluated at ω. These should yield the boundary conditions for the Green's fn. at ω

$$\left(-\omega + \tfrac{1}{i}X\right)\psi = 0$$

Repeat: $\partial_t \psi = \begin{pmatrix} \partial_x & ih \\ i\bar{h} & -\partial_x \end{pmatrix} \psi$ $\quad X$

$$\cancel{\omega\psi =} \begin{pmatrix} \frac{1}{i}\partial_x - \omega & h \\ -\bar{h} & +\frac{1}{i}\partial_x + \omega \end{pmatrix} \psi = 0$$

$$E_\omega = \left\{ \psi(x) \,\middle|\, \frac{1}{i}\partial_x \psi = \begin{pmatrix} \omega & -h \\ \bar{h} & -\omega \end{pmatrix} \psi \right\}$$

~~eigenfns~~

Green's fn. You have to ~~[illegible]~~ express, study, make clear, how the boundary conditions enter to make the Green's function, whose singularities yield the spectrum

$$\partial_t\psi = \underbrace{\begin{pmatrix}\partial_x & ih \\ i\bar{h} & -\partial_x\end{pmatrix}}_{X}\psi, \quad \text{solutions of wave eqn.}$$

$= \text{sections of bundle } \{E_\omega\}$

$$E_\omega = \{\psi(x) \mid (i\omega - X)\psi = 0\}.$$

On E_ω have Wronskian skew form, ~~which gen~~ namely, given $\begin{matrix}(i\omega - X)\phi = 0 \\ (i\omega - X)\psi = 0\end{matrix}$ form $\phi\wedge\psi = \begin{vmatrix}\phi^1 & \psi^1 \\ \phi^2 & \psi^2\end{vmatrix}$.

roughly, which will be constant. Write $(i\omega - X)\phi = 0$ as $\partial_x\phi = i\underbrace{\begin{pmatrix}\omega & -h \\ \bar{h} & -\omega\end{pmatrix}}_{L}\phi$, Then

$$\partial_x(\phi\wedge\psi) = L\phi\wedge\psi + \phi\wedge L\psi$$

this should be $(\text{trace } L)(\phi\wedge\psi)$.

$$\begin{vmatrix}a\phi^1 + b\phi^2 & \psi^1 \\ c\phi^1 + d\phi^2 & \psi^2\end{vmatrix} + \begin{vmatrix}\phi^1 & a\psi^1 + b\psi^2 \\ \phi^2 & c\psi^1 + d\psi^2\end{vmatrix}$$

$$(a + d)\phi^1\psi^2 + (b - b)\phi^2\psi^2$$
$$+ (-c + c)\phi^1\psi^1 + (-d - a)\phi^2\psi^1$$

OKAY.

~~so that~~ You also have conjugation σ when ω is real, probably in general from $E_\omega \to E_{\bar{\omega}}$.

$$E_\omega = \left\{\psi(x) \,\middle|\, \partial_x\psi = i\begin{pmatrix}\omega & -h \\ \bar{h} & -\omega\end{pmatrix}\psi\right\}$$

$$\sigma\left\{i\begin{pmatrix}\omega\psi^1 - h\psi^2 \\ \bar{h}\psi^1 - \omega\psi^2\end{pmatrix}\right\} = -i\begin{pmatrix}h\bar{\psi}^1 - \bar{\omega}\bar{\psi}^2 \\ \bar{\omega}\bar{\psi}^1 - \bar{h}\bar{\psi}^2\end{pmatrix} = i\begin{pmatrix}\bar{\omega} & -h \\ \bar{h} & -\bar{\omega}\end{pmatrix}\begin{pmatrix}\bar{\psi}^2 \\ \bar{\psi}^1\end{pmatrix}$$

$$\sigma(\partial_x\psi) = \partial_x(\sigma\psi) = \partial_x\begin{pmatrix}\bar{\psi}^2 \\ \bar{\psi}^1\end{pmatrix}$$

YES.

Suppose $\phi \in E_\omega \quad \psi \in E_\xi$

$$\partial_x(\phi \wedge \psi) = L_\omega \phi \wedge \psi + \phi \wedge L_\xi \psi$$

$$i\begin{vmatrix} \begin{pmatrix} \omega & -h \\ \bar{h} & -\omega \end{pmatrix}\begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} & \begin{matrix} \psi^1 \\ \psi^2 \end{matrix} \end{vmatrix} + i\begin{vmatrix} \begin{matrix} \phi^1 \\ \phi^2 \end{matrix} & \begin{pmatrix} \xi & -h \\ \bar{h} & -\xi \end{pmatrix}\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \end{vmatrix}$$

$$= i\omega \begin{vmatrix} \phi^1 & \psi^1 \\ -\phi^2 & \psi^2 \end{vmatrix} + i\xi \begin{vmatrix} \phi^1 & \psi^1 \\ \phi^2 & -\psi^2 \end{vmatrix}$$

$$\boxed{\partial_x(\phi_\omega \wedge \psi_\xi) = i(\omega - \xi)(\phi^1\psi^2 + \phi^2\psi^1)}$$

$$\psi = \sigma\phi \in E_{\bar{\omega}} \qquad \sigma\phi = \begin{pmatrix} \overline{\phi^2} \\ \overline{\phi^1} \end{pmatrix} = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$$

$$\boxed{\partial_x(\phi_\omega \wedge \sigma(\phi_\omega)) = i(\omega - \bar{\omega})\left(|\phi_\omega^1|^2 + |\phi_\omega^2|^2\right)}$$

All this is familiar, it suggests you are on the right track. to where?

So what have you done? Replace wave equation by the Dirac equation spectral decomp., which you are in the process of constructing via Titchmarsh method. This means constructing the Green's function. ~~to~~

Want Green's function for $\partial_x \psi = \begin{pmatrix} \omega & -h \\ \bar{h} & -\omega \end{pmatrix}\psi$

$$G_\omega(x, x') = \begin{cases} \psi_\omega^>(x)(\quad) & x > x' \\ \psi_\omega^<(x)(\quad) & x < x' \end{cases}$$

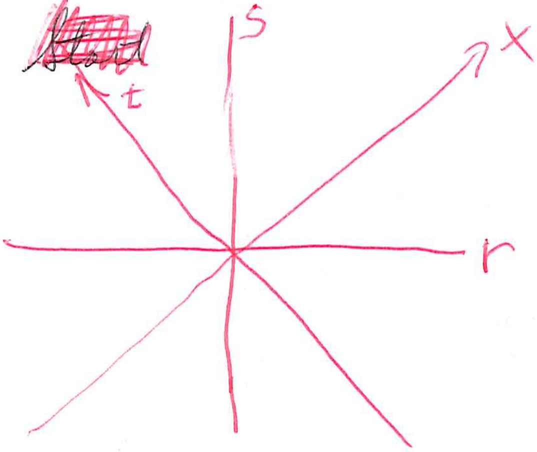
where $\psi_\omega^>$ resp $\psi_\omega^<$ sat left + right b.c.

Repeat $\partial_t\psi = \begin{pmatrix}\partial_x & ih\\ i\bar h & -\partial_x\end{pmatrix}\psi$ $\qquad i\omega\psi = \underbrace{\begin{pmatrix}\partial_x & ih\\ i\bar h & -\partial_x\end{pmatrix}}_{X}\psi$

$(i\omega - X)^{-1}$ on $L^2(\mathbb{R},dx)^{\oplus 2}$ $\qquad (i\omega - X)G(x,x') = I\,\delta(x-x')$

$G_\omega(x,x') = (x|\frac{1}{i\omega - X}|x')$

maybe you should calculate $G_\omega(x,x')$ for $h=1$.



$\partial_r = -\partial_t + \partial_x$ $\qquad t = -r+s$ $\qquad r = \frac{x-t}{2}$

$\partial_s = \partial_t + \partial_x$ $\qquad x = r+s$ $\qquad s = \frac{x+t}{2}$

$\partial_t\psi = \begin{pmatrix}\partial_x & i\\ i & -\partial_x\end{pmatrix}\psi$ $\qquad (\partial_t-\partial_x)\psi^1 = i\psi^2$

$(\partial_t+\partial_x)\psi^2 = i\psi^1$

$-\partial_r\psi^1 = i\psi^2$ $\qquad -\rho\psi^1 = \psi^2$ $\qquad \rho\sigma = -1$

$\partial_x\psi^2 = i\psi^1$ $\qquad \sigma\psi^2 = \psi^1$

$$r\rho - s\rho^{-1} = \frac{x-t}{2}\rho - \frac{x+t}{2}\rho^{-1} = x\underbrace{\left(\frac{\rho-\rho^{-1}}{2}\right)}_{k} - t\underbrace{\left(\frac{\rho+\rho^{-1}}{2}\right)}_{\omega}$$

$\rho = \omega + k$

$\rho^{-1} = \omega - k$

Repeat $\partial_t\psi = \begin{pmatrix}\partial_x & ih\\ i\bar h & -\partial_x\end{pmatrix}\psi$, solution $\psi(x,t)$ of this wave equ corresp via FT to a $\psi(x,\omega)\in E_\omega$,

$$E_\omega = \mathrm{Ker}(i\omega\psi - X) = \mathrm{Ker}\left(\partial_x - i\begin{pmatrix}\omega & -h\\ \bar h & -\omega\end{pmatrix}\right).$$

Note $\sigma : E_\omega \xrightarrow{\sim} E_{\bar\omega}$ since

$$\sigma\,\frac{1}{i}\begin{pmatrix}\omega & -h\\ \bar h & -\omega\end{pmatrix}\sigma^{-1} = +i\begin{pmatrix}-\bar\omega & h\\ -\bar h & \bar\omega\end{pmatrix} = \frac{1}{i}\begin{pmatrix}\bar\omega & -h\\ \bar h & -\bar\omega\end{pmatrix}$$

Also $Wr(\phi,\psi)(x) = \begin{vmatrix}\phi_1(x) & \psi_1(x)\\ \phi_2(x) & \psi_2(x)\end{vmatrix}$ is ind of x for

$\phi,\psi\in E_\omega$ as $\partial_x = i\begin{pmatrix}\omega & -h\\ \bar h & -\omega\end{pmatrix}$ has trace 0.

Combine to get $Wr(\sigma\phi,\psi) = \begin{vmatrix}\bar\phi_2 & \psi_1\\ \bar\phi_1 & \psi_2\end{vmatrix} = \bar\phi_2\psi_2 - \bar\phi_1\psi_1$

hermitean form on E_ω. NO ???

Let $\phi, \psi \in E_\omega$ ω real $\Rightarrow \sigma\phi \in E_\omega$

so $Wr(\sigma\phi, \psi)$ is independent of x for $\phi, \psi \in E_\omega$ ω real.

Repeat: $i\omega\psi = \begin{pmatrix} \partial_x & ih \\ i\bar{h} & -\partial_x \end{pmatrix}\psi$ $\quad \begin{pmatrix} \frac{1}{i}\partial_x - \omega & h \\ -\bar{h} & +\frac{1}{i}\partial_x + \omega \end{pmatrix}(\psi) = 0$

$$E_\omega = \left\{ \psi(x) \,\middle|\, \frac{1}{i}\partial_x \psi = \begin{pmatrix} \omega & -h \\ \bar{h} & -\omega \end{pmatrix}\psi \right\}$$

$$-i\,\partial_x(\sigma\psi) = \begin{pmatrix} +\bar{\omega} & -h \\ +\bar{h} & -\bar{\omega} \end{pmatrix}(\sigma\psi) \quad \therefore \boxed{\sigma E_\omega = E_{\bar{\omega}}}$$

Let $\phi \in E_\xi$, $\psi \in E_\omega$

$$\frac{1}{i}\partial_x \begin{vmatrix} \phi^1 & \psi^1 \\ \phi^2 & \psi^2 \end{vmatrix} = \left| \begin{pmatrix} \xi & -h \\ \bar{h} & -\xi \end{pmatrix} \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} \quad \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \right.$$

$$\frac{1}{i}\partial_x \left[\begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} \wedge \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \right] = \begin{pmatrix} \xi\phi^1 - \cancel{h\phi^2} \\ \cancel{\bar{h}\phi^1} - \xi\phi^2 \end{pmatrix} \wedge \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} + \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} \wedge \begin{pmatrix} \omega\psi^1 - \cancel{h\psi^2} \\ \cancel{\bar{h}\psi^1} - \omega\psi^2 \end{pmatrix}$$

$$= \xi(\phi^1\psi^2 + \phi^2\psi^1) - \omega(\phi^1\psi^2 + \phi^2\psi^1)$$

$$\phi \in E_\xi, \psi \in E_\omega \Longrightarrow \frac{1}{i}\partial_x \begin{vmatrix} \phi^1 & \psi^1 \\ \phi^2 & \psi^2 \end{vmatrix} = (\xi - \omega)(\phi^1\psi^2 + \phi^2\psi^1)$$

$\phi = \sigma\psi$

$$\frac{1}{i}\partial_x \underbrace{\begin{vmatrix} \overline{\psi^2} & \psi^1 \\ \overline{\psi^1} & \psi^2 \end{vmatrix}}_{\text{independence of } x \text{ for } \omega \in \mathbb{R}} = (\bar{\omega} - \omega)(|\psi^2|^2 + |\psi^1|^2)$$

What do you know?

Anyway $Wr(\sigma\psi,\psi)(x) = \begin{vmatrix} \bar\psi^2 & \psi^1 \\ \bar\psi^1 & \psi^2 \end{vmatrix}(x)$ is a hermitian form on E_ω for ~~…~~ any ω, x but it depends on x unless $\omega \in \mathbb{R}$. ~~…~~

The formula $$\boxed{\frac{1}{i}\partial_x \begin{vmatrix} \bar\psi^2 & \psi^1 \\ \bar\psi^1 & \psi^2 \end{vmatrix} = (\bar\omega - \omega)\left(|\psi^2|^2 + |\psi^1|^2\right)}$$ is some sort of power energy relation. Essentially the same result as.

$$\partial_t\left(\tfrac{1}{2}(E^2 + I^2)\right) = E\dot E + I\dot I = -E\partial_x I - I\partial_x E$$
$$= -\partial_x(EI).$$

So it should have a direct derivation from

$$\partial_t\psi = \underbrace{\begin{pmatrix} \partial_x & ih \\ ih & -\partial_x \end{pmatrix}}_{X}\psi.$$

$$\partial_t\left(\bar\psi^1\psi^1 + \bar\psi^2\psi^2\right) = \psi^*\partial_t\psi + (\partial_t\psi)^*\psi$$
$$= \psi^* X\psi + (X\psi)^*\psi$$
$$= \psi^*(\varepsilon\partial_x\psi) + (\partial_x\psi)^*\varepsilon\psi$$

$$\boxed{\partial_t(\psi^*\psi) = \partial_x(\psi^*\varepsilon\psi)}$$

suppose $\psi(x,t) = e^{i\omega t}\psi_\omega(x)$

$$\partial_t\left(e^{-i\bar\omega t}e^{i\omega t}\right) = \partial_t\left(e^{i(\omega-\bar\omega)t}\right)$$

$$\boxed{i(\omega - \bar\omega)(\psi_\omega^*\psi_\omega) = \partial_x(\psi_\omega^*\varepsilon\psi_\omega)}$$

~~GOOD ST~~

SO FAR SO GOOD. Is it possible now to

determine IH? In the case $h=1$?
Maybe scattering case first. This should be easy and provide insight for dealing with E_ω.

$$\begin{pmatrix} e^{\omega x} & 0 \\ 0 & e^{-\omega x}\end{pmatrix}\begin{pmatrix} \\ \end{pmatrix} \longleftarrow \psi_\omega(x) \longrightarrow \begin{pmatrix} e^{\omega x} & 0 \\ 0 & e^{-\omega x}\end{pmatrix}\begin{pmatrix} \\ \end{pmatrix}$$

$$\begin{pmatrix} \\ \end{pmatrix} \xleftarrow{x \to -\infty} \begin{pmatrix} e^{-\omega x} & 0 \\ 0 & e^{\omega x}\end{pmatrix}\psi_\omega(x) \xrightarrow{x\to+\infty} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\partial_t \psi = \begin{pmatrix} \partial_x & ih \\ i\bar{h} & -\partial_x \end{pmatrix}\psi \qquad \partial_t(\psi^*\psi) = [(\varepsilon\partial_x + A)\psi]^*\psi + \psi^*[(\varepsilon\partial_x + A)\psi]$$
$$= \partial_x(\psi^*\varepsilon\psi).$$

$$\underset{\sim}{\mathrm{IH}}(\psi) = -\mathrm{Wr}(\sigma\psi, \psi) = -\begin{vmatrix} \bar\psi^2 & \psi^1 \\ \bar\psi^1 & \psi^2 \end{vmatrix} = \psi^*\varepsilon\psi$$

check
$$H(v,v) = -\begin{vmatrix} \bar v_2 & v_1 \\ \bar v_1 & v_2 \end{vmatrix} = |v_1|^2 - |v_2|^2.$$

$$\omega\psi_\omega = \begin{pmatrix} \frac{1}{i}\partial_x & 1 \\ 1 & -\frac{1}{i}\partial_x \end{pmatrix}\psi_\omega \qquad \omega\psi = \begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}\psi \qquad \begin{vmatrix} k-\omega & 1 \\ 1 & -k-\omega \end{vmatrix}$$
$$= \omega^2 - k^2 - 1$$

$$\begin{pmatrix} k-\omega & 1 \\ 1 & -k-\omega \end{pmatrix}\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = 0 \qquad \begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}\begin{pmatrix} 1 \\ \omega-k \end{pmatrix} = \begin{pmatrix} \omega \\ \omega^2 - k\omega \end{pmatrix} = \begin{pmatrix} 1 \\ \omega-k \end{pmatrix}\omega$$

$$\begin{pmatrix} -k & 1 \\ 1 & +k \end{pmatrix}\begin{pmatrix} 1 \\ +\omega+k \end{pmatrix} = \begin{pmatrix} +\omega \\ \omega^2 + k\omega \end{pmatrix} = \begin{pmatrix} 1 \\ +\omega+k \end{pmatrix}(+\omega)$$

$$\psi_\omega = \begin{pmatrix} 1 \\ \omega-k \end{pmatrix}A + \begin{pmatrix} 1 \\ +\omega+k \end{pmatrix}B \qquad ?$$

$$\psi_\omega(x) = e^{ikx}\begin{pmatrix} 1 \\ \omega-k \end{pmatrix}A + \underbrace{e^{-ikx}\begin{pmatrix} 1 \\ \omega+k \end{pmatrix}B}_{\begin{pmatrix} \omega-k \\ 1 \end{pmatrix}(\omega+k)B}$$

$$\psi_\omega(x) = e^{ikx}\begin{pmatrix}1\\ \omega-k\end{pmatrix}A + e^{-ikx}\begin{pmatrix}\omega-k\\ 1\end{pmatrix}B$$

This checks that the eigenspace is closed under σ

$$\left(e^{ikx}\begin{pmatrix}1\\ \omega-k\end{pmatrix}\right)^*\begin{pmatrix}1&0\\0&-1\end{pmatrix}\left(e^{ikx}\begin{pmatrix}1\\ \omega-k\end{pmatrix}\right)$$

$$= \begin{pmatrix}1\\ \omega-k\end{pmatrix}^*\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\ \omega-k\end{pmatrix} = 1-|\omega-k|^2$$

$$= 1-(\underbrace{\omega^2}_{1+k^2}-2k\omega+k^2) = 2k\omega - 2k^2 = \quad \text{if } k \text{ real}$$

$$= 2k(\omega-k) \quad \text{i.e. } |\omega|\geq 1$$

$k\in i\mathbb{R}$ i.e. $|\omega|<1$. $\quad 1-\omega^2-(\omega^2-1)$?

$$\psi_\omega(x) = e^{ikx}\begin{pmatrix}1\\ \omega-k\end{pmatrix}$$

$$-W(\sigma\psi_\omega,\psi_\omega) = |e^{ikx}|^2 - |\omega-k|^2|e^{ikx}|^2$$

$$= (1-|\omega-k|^2)|e^{ikx}|^2$$

$\omega^2 = k^2+1$. Assume $|\omega|<1$. $\quad k^2 = -(1-\omega^2)$

$k = \pm i(1-\omega^2)^{1/2}$

$$\omega - k = \omega \pm i(1-\omega^2)^{1/2}$$

$$|\omega-k|^2 = \omega^2 + 1-\omega^2 = 1 \quad \therefore \boxed{1-|\omega-k|^2 = 0}$$

another way to see this is to note for ω real that $-W(\sigma\psi_\omega,\psi_\omega)$ is ind. of x, so k has to be real

Assume $|\omega|>1$, then $-W(\sigma\psi_\omega,\psi_\omega) = 1-(\omega-k)^2$

$$= \underbrace{1-\omega^2}_{-k^2} + 2k\omega - k^2 = 2k\omega-2k^2 = 2k(\omega-k)$$

[redacted] 904

Begin again. Consider solutions of $\partial_t\psi = \begin{pmatrix}\partial_x & ih\\ i\bar{h} & -\partial_x\end{pmatrix}\psi$
where $\psi(x,t)\in\mathbb{C}^2$; e.g. $\psi(x,t)=e^{i\omega t}\psi(x)$, where

$$\omega\psi = \begin{pmatrix}\frac{1}{i}\partial_x & h\\ \bar{h} & -\frac{1}{i}\partial_x\end{pmatrix}\psi \qquad \begin{pmatrix}\frac{1}{i}\partial_x-\omega & h\\ -\bar{h} & +\frac{1}{i}\partial_x+\omega\end{pmatrix}\psi = 0 \qquad \boxed{\frac{1}{i}\partial_x\psi = \begin{pmatrix}\omega & -h\\ \bar{h} & -\omega\end{pmatrix}\psi}$$

Given a solution $\psi(x,t)$ of the wave equation then $\sigma(\psi(x,-t))$ is also a solution

$$\partial_t\,\sigma(\psi(x,t)) = \begin{pmatrix}-\partial_x & -ih\\ -i\bar{h} & \partial_x\end{pmatrix}\sigma(\psi(x,t))$$

$$\partial_t\,\sigma(\psi(x,-t)) = \begin{pmatrix}\partial_x & ih\\ i\bar{h} & -\partial_x\end{pmatrix}\sigma(\psi(x,-t))$$

Also if $\psi(x,t) = \int e^{i\omega t}\,\hat{\psi}(x,\omega)$

$$\sigma(\psi(x,-t)) = \int e^{i\bar{\omega}t}\,\sigma(\hat{\psi}(x,\omega))$$

which is consistent with $\sigma E_\omega = E_{\bar{\omega}}$.

Check $\frac{1}{i}\partial_x\psi = \begin{pmatrix}\omega & -h\\ \bar{h} & -\omega\end{pmatrix}\psi$

$$+\frac{1}{i}\partial_x\,\sigma(\psi) = \begin{pmatrix}+\bar{\omega} & -h\\ +h & -\bar{\omega}\end{pmatrix}\sigma(\psi)$$

Guess that $IH(\psi)$, $\psi=\psi(x,t)$ solution of wave equation involves the local expression

$$\psi^*\varepsilon\psi = \begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}^*\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix} = |\psi^1|^2-|\psi^2|^2$$

$$\partial_t(\psi^*\psi) = \psi^*X\psi + (X\psi)^*\psi = \partial_x(\psi^*\varepsilon\psi).$$

e.g. $\psi(x,t) = e^{i\omega t}\psi_\omega(x)$ $\qquad \psi_\varepsilon \in E_\omega$

$$\psi^*\varepsilon\psi = \psi_\omega^*\varepsilon\psi_\omega = |\psi_\omega^1(x)|^2 - |\psi_\omega^2(x)|^2$$

$$\partial_t(\psi^*\psi) = \partial_t\left(e^{i(\omega-\bar\omega)t}\psi_\omega^*\psi_\omega\right) = i(\omega-\bar\omega)\,\psi^*\psi$$

$$\therefore \boxed{i(\omega-\bar\omega)\,\psi_\omega^*\psi_\omega = \partial_x\left(|\psi_\omega^1|^2 - |\psi_\omega^2|^2\right)}$$

Where are you at the moment? Given a solution $\psi(x,t)$ of the wave equation, say $\psi(x,t) = \int_{-\infty}^{\infty} e^{i\omega t}\psi_\omega(x)\,\frac{d\omega}{2\pi}$

where $\psi_\omega(x) \in E_\omega$, i.e. $\frac{1}{i}\partial_x\psi_\omega = \begin{pmatrix}\omega & -h\\ h & -\omega\end{pmatrix}\psi_\omega$, here $|\omega| \geq 1$, then ψ has

$$\text{energy} = \int_{-\infty}^{\infty}\psi^*\psi\,dx$$

$$IH = \int_{-\infty}^{\infty}\psi^*\varepsilon\psi\,dx$$

At the moment you are missing the eigenfn. transform.

$$\psi(x) = \int \underbrace{K(x,\omega)}_{\text{kernel}}\,\hat\psi(\omega)$$

kernel $\ni K(-,\omega) \in E_\omega$

Construct the eigenfunction expansion via Titchmarsh method, singularities of the Green's function, Green's function requires boundary conditions, obvious L^2 ones for ω not real

First for scattering situation $X = \begin{pmatrix} \partial_x & ih \\ ih & -\partial_x \end{pmatrix}$

$h(x)$ decays far out. $i\omega\psi = X\psi$, $i\omega - X$ is invertible for $\omega \notin \mathbb{R}$ either $\underline{\operatorname{Im}(\omega) > 0}$ or < 0.

Suppose $h = 0$. $\psi = \begin{pmatrix} e^{i\omega x} & \alpha \\ e^{-i\omega x} & \beta \end{pmatrix}$

$\operatorname{Im}(\omega) > 0 \implies e^{i\omega x}$ decays as $x \mapsto +\infty$

$e^{-i\omega x}$ —————— $x \mapsto -\infty$

Simple example $\partial_t u = \partial_x u$, $-i\omega\hat{u} = \partial_x \hat{u}$

$\hat{u} = c e^{-i\omega x} = c e^{sx}$ $\qquad (\partial_x - s)^{-1}(x, x') = H(x-$?

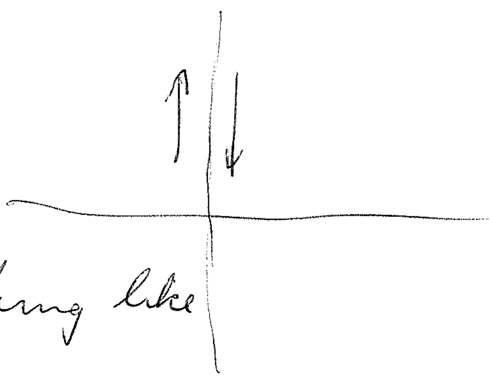
$$(\partial_x - s)\, G_s(x - x') = \delta(x - x')$$

$$G_s(x - x') = \begin{cases} e^{s(x-x')} & x > x' \\ 0 & x < x' \end{cases} \qquad \operatorname{Re}(s) < 0$$

$$G_s(x) = H(x)e^{sx} \qquad \operatorname{Re}(s) < 0$$

Check. $G_s(x) = -H(-x)e^{sx}$ $\qquad \operatorname{Re}(s) > 0$

$$\frac{1}{2\pi i}\int G_s(x)\, ds$$



so you end up with something like

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{s(x-x')}\, ds = \delta(x - x').$$

$\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}\psi$, if $\overbrace{e^{+i\omega t}\psi(x)}^{\psi(x,t)}$, then

$$0 = \begin{pmatrix} \frac{1}{i}\partial_x - \omega & 1 \\ -1 & +\frac{1}{i}\partial_x - \omega \end{pmatrix}\psi_\omega \qquad \frac{1}{i}\partial_x \psi_\omega = \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}\psi_\omega$$

You need Green's fn., not necessarily, you need the eigenfunction expansion for the operator $\frac{1}{i}X = \begin{pmatrix} \partial_x & 1 \\ 1 & -\frac{1}{i}\partial_x \end{pmatrix}$

Let's go over the ideas carefully. You start with the wave equation $\partial_t \psi = X\psi$, $\frac{1}{i}X = \begin{pmatrix} \frac{1}{i}\partial_x & 1 \\ 1 & -\frac{1}{i}\partial_x \end{pmatrix}$.

The basic object is the space Ω of its nice solutions, where nice means the time evolution is given by real frequencies, i.e. analyzable via F.T.

This space Ω can be described as the space of Cauchy data i.e. $\psi(x) = \begin{pmatrix} \psi^1(x) \\ \psi^2(x) \end{pmatrix}$ or by eigenfunction of $\frac{1}{i}X$. Energy norm seems to be $\int \psi^* \psi \, dx$, since time evolution is given there should be a symplectic form around

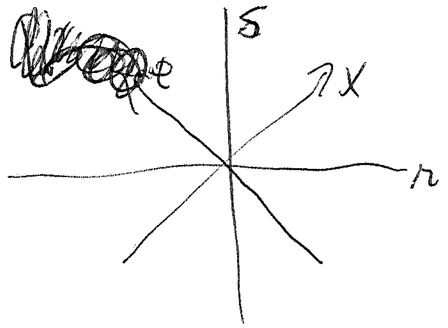
Eigenfunction expansion. Given ω real with $|\omega| > 1$ there are two values of k with $\omega^2 = k^2 + 1$.

Idea: Treat x, t symmetrically

$$0 = \begin{pmatrix} \partial_x - \partial_t & i \\ -i & +\partial_x + \partial_t \end{pmatrix}\psi \qquad \boxed{\partial_x \psi = \begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix}\psi}$$

$$\boxed{\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}\psi}$$

There should be an equivalance between $t=0$ and $x=0$ Cauchy data



$$\partial_r = -\partial_t + \partial_x \qquad t = -r + s \qquad r = \frac{x-t}{2}$$

$$\partial_s = \partial_t + \partial_x \qquad x = r + s \qquad s = \frac{x+t}{2}$$

$$-\psi^1 = \psi^2$$

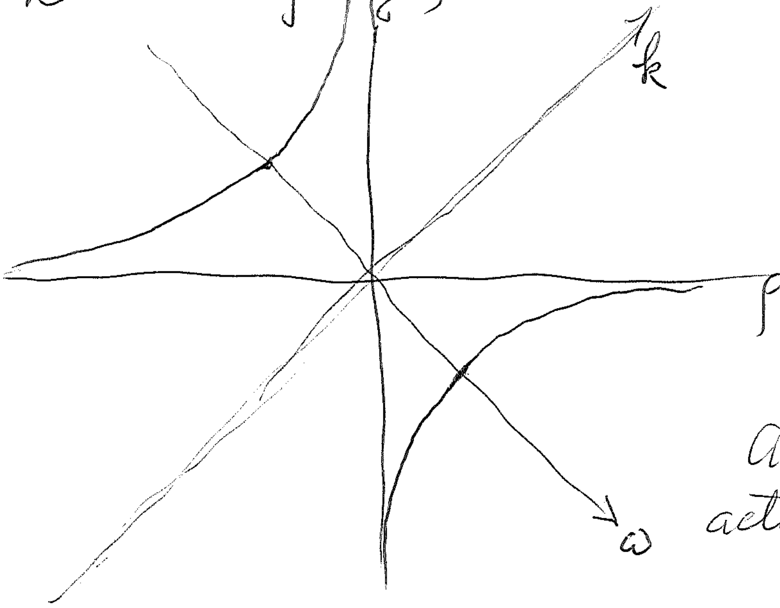
$$\psi^2 = \psi^1$$

$$\rho r - \rho^{-1} s = \rho\left(\frac{x-t}{2}\right) - \rho^{-1}\left(\frac{x+t}{2}\right)$$

$$= \underbrace{\left(\frac{\rho - \rho^{-1}}{2}\right)}_{k} x - \underbrace{\left(\frac{\rho+\rho^{-1}}{2}\right)}_{\omega} t$$

$$\psi = \int e^{i(kx - \omega t)} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)\, d\rho \qquad \rho = \omega + k \qquad \rho^{-1} = \omega - k$$

Your aim. This formula parametrizes solution space as F.T. of tempered distribution supported in the ~~curve~~ $-\rho\sigma = 1$, or $\omega^2 = k^2 + 1$.



Solution space can also be described Cauchy data on $t=0$ and on $x = 0$. So you want the explicit transform between $x=0$ and $t=0$.

An interesting point is the action of Lorentz transf.

Start with

$$\psi(x, 0) = \int e^{ikx}\left(\begin{pmatrix} 1 \\ -\omega - k \end{pmatrix} \overset{f_1(k)}{\cancel{f(\omega+k)}} + \begin{pmatrix} 1 \\ \omega - k \end{pmatrix} \overset{f_2(k)}{\cancel{f(-\omega+k)}}\right)$$

~~the~~ $\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix} \psi$

Anyway, ~~where to next?~~

$$\begin{pmatrix} \partial_x - \partial_t & i \\ -i & +\partial_x + \partial_t \end{pmatrix}$$

Think, Concentrate,

$$\partial_x \psi \equiv \begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix} \psi \qquad \frac{1}{i}\partial_x \psi = \begin{pmatrix} \frac{1}{i}\partial_t & -1 \\ 1 & -\frac{1}{i}\partial_t \end{pmatrix} \psi$$

Solutions of $\frac{1}{i}\partial_x\psi = \begin{pmatrix} \frac{1}{i}\partial_t & -1 \\ 1 & -\frac{1}{i}\partial_t \end{pmatrix}\psi$ same as $\psi(k,t)$

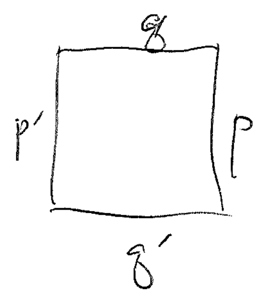
Sat

$$k\psi(k,t) = \begin{pmatrix} \frac{1}{i}\partial_t & -1 \\ 1 & -\frac{1}{i}\partial_t \end{pmatrix}\psi(k,t)$$

You want control of the eigenfunctions of $\begin{pmatrix} \frac{1}{i}\partial_t & -1 \\ 1 & -\frac{1}{i}\partial_t \end{pmatrix}$

really control ~~of~~ the eigenfunction expansion.

Continuous grid eqns. $-\partial_r p = ihq$



$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 & hq \\ \bar{h}\varepsilon & 1 \end{pmatrix}\begin{pmatrix} p' \\ q' \end{pmatrix}$$

$$\partial_r p = hq \qquad \partial_s q = \bar{h}p$$

this is not working. Why? You are pretty sure that the continuous case is

$$\boxed{\begin{aligned} -\partial_r\psi^1 &= ih\psi^2 \\ \partial_s\psi^2 &= i\bar{h}\psi^1 \end{aligned}}$$

i.e. $(+\partial_t - \partial_x)\psi^1 = ih\psi^2$, $(\partial_t + \partial_x)\psi^2 = i\bar{h}\psi^1$ $\quad \partial_t\psi = \begin{pmatrix} \partial_x & ih \\ i\bar{h} & -\partial_x \end{pmatrix}\psi$

~~[illegible]~~ Adopt the time-like hypersurface viewpoint. ~~[illegible]~~, i.e. noncharacteristic ~~[illegible]~~ curve should be the analog of ascending + descending staircases.

What should ~~[illegible]~~ happen is that $\psi^*\psi$, $\psi^*\varepsilon\psi$ are components of a closed 1-form.

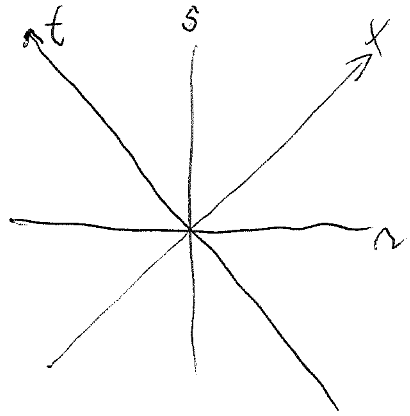
You know $\partial_t(\psi^*\psi) = (X\psi)^*\psi + \psi^*(X\psi)$ 910

where $X = \varepsilon\partial_x + iA$ 2??

~~so~~ $\cancel{\psi^*\begin{pmatrix}0 & ih\\ i\bar{h} & 0\end{pmatrix}\psi = \begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}^*\begin{pmatrix}0 & ih\\ i\bar{h} & 0\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}}$

But $(iA\psi)^*\psi + \psi^*(iA\psi)$

$$= -i\psi^*A\psi + i\psi^*A\psi = 0.$$



Thus $\boxed{\partial_t(\psi^*\psi) = \partial_x(\psi^*\varepsilon\psi)}$

1-form $\psi^*\psi\,dx + \psi^*\varepsilon\psi\,dt$

which is closed, So its integral over an exhaustive space-like curve, (resp. time-like), should be indep of the choice of the curve. This seems to be the way to define two hermitian forms.

$\rho = \omega + k$

$\rho^{-1} = \omega - k$

Consider case $h = 1$.

$$\psi(x,t) = \int_{-\infty}^{\infty} e^{i\left(\left(\frac{\rho-\rho^{-1}}{2}\right)x - \left(\frac{\rho+\rho^{-1}}{2}\right)t\right)}\begin{pmatrix}1\\ -\rho\end{pmatrix} f(\rho)\,d\rho$$

$$\psi(x,0) = \int_{-\infty}^{\infty} e^{ikx}\left\{\begin{pmatrix}1\\ \underbrace{-\omega-k}_{-\rho}\end{pmatrix} f(\overbrace{\omega+k}^{\rho}) + \begin{pmatrix}1\\ \underbrace{+\omega-k}_{\rho^{-1}}\end{pmatrix} f(\overbrace{-\omega+k}^{-\rho^{-1}})\right\} dk$$

$\rho = \sqrt{k^2+1} + k$ $\quad \frac{d\rho}{dk} = \frac{1}{2}(k^2+1)^{-1/2}2k + 1 = \frac{k}{\sqrt{\ }} + 1 = \frac{\omega+k}{\omega}$

$$d\rho = \frac{\omega+k}{\omega}\,dk$$