α   04/05/97   1036

Back to mathematics.

Review Morita invariance of K_*.

~~[illegible]~~ Alternative - stability for a field.

I want to review what I learned, and make a new attempt at Suslin's results via flatness.

$$K_*(A) \overset{\text{defn}}{=} \operatorname{Ker}\{K_*(\tilde A) \to K_*(\mathbb{Z})\}.$$

Main construction: to $(P, Q, Q\otimes P \xrightarrow{\langle\,,\,\rangle} A)$ arbitrary dual p over A with P $\tilde A^{op}$ flat we assoc. a trace map

$$tr^P : K_*(P\otimes_A Q) \longrightarrow K_*(A)$$

by ~~colim~~ filtered ~~colim~~ limits enough to do when $P \in \mathcal{P}(\tilde A^{op})$ + naturality. Defn.

$$K_*(P\otimes_A Q) \longrightarrow K_*(P \otimes_A \operatorname{Hom}_{A^{op}}(P, A)) \quad \left(= A\otimes_A \check P\right)$$

Point is that we have a rep of $P\otimes_A Q$ ~~on~~ on P ~~[illegible]~~ over A

$$\begin{array}{ccccc} \therefore \text{ canon. map } K_*(P\otimes_A Q) & \longrightarrow & K_*(\operatorname{Hom}_{A^{op}}(P,P)) & \longrightarrow & K_*(\operatorname{Hom}_{\mathbb{Z}}(P/PA, P/PA)) \\ & & \downarrow \text{can} & & \downarrow \text{can.} \\ & & K_*(\tilde A) & \longrightarrow & K_*(\mathbb{Z}) \end{array}$$

naturality. $(P, Q) \longrightarrow (P', Q')$ factors; $(P, Q) \to (P', Q^{\cdot}) \xrightarrow{\text{ok.}} (P', Q')$

~~$P \otimes (P, Q)$~~

$$\begin{array}{ccc} P\otimes_A Q & \longrightarrow & P'\otimes_A Q' \\ \downarrow & & \downarrow \\ P\otimes_A P'^* & \longrightarrow & P'\otimes_A P'^* \\ \downarrow & & \wr \\ P\otimes_A P^* & \rightsquigarrow & \end{array}$$

$$\begin{array}{ccccc} P\otimes_A Q & \longrightarrow & P\otimes_A P'^{*} & \longrightarrow & P'\otimes_A P'^* \\ \downarrow & \swarrow & & & \\ P\otimes_A P^* & & & & \end{array} \qquad P\otimes_A Q \longrightarrow P'\otimes_A Q'$$

β So what am I doing? Arguing that for $\forall\, P \in \mathcal{P}(\tilde{A}^{op})$ have canon. map

$$tr^P : K_*(P \otimes_A P^*) \longrightarrow K_*(A)$$

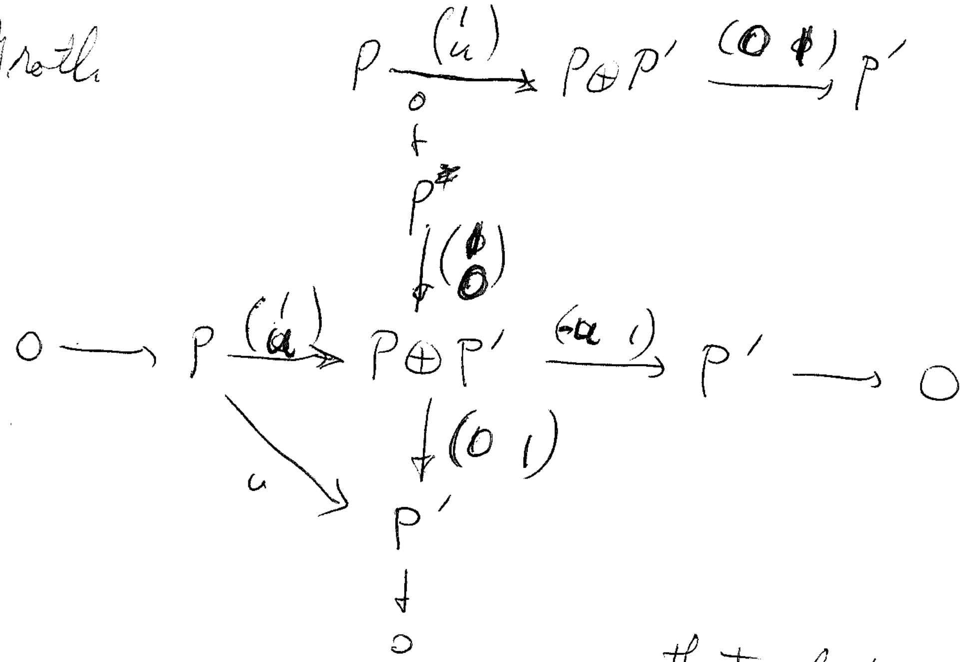
such that $\forall$ pair P, P' in $\mathcal{P}(\tilde{A}^{op})$ you have commutativity or compatibility. ~~general map~~

For any $u : P \to P'$ you want

$$\begin{array}{ccc} K_*(P \otimes_A P'^*) & \longrightarrow & K_*(P' \otimes_A P'^*) \\ \downarrow & & \| \text{can} \\ K_*(P \otimes P^*) & \overset{\text{can}}{=} & K_*(A) \end{array}$$

to commute

Groth



To what extent can I assume that u factor $P \to P'_A \subset P'$

Consider (nonunital) category of $P \to M$ where maps are $P \xrightarrow{u} P'$ over M with u matrix over A

$$\begin{array}{ccc} P & \xrightarrow{u} & P' \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

Is there a relation between Vaserstein's Whitehead lemma and writing E as a filtered colim of f.g. frees.

8

Problem: Central problem for me is to show that two flat Morita equiv. firm rings have same $H_*(BGL)$. Simp. gp argument reduces to <u>left Morita equivalence.</u>

1600. Simplicial group argument. Given B idemp. say h-central. Then $\exists$ simp resolution of B by flat firm B-modules

$$A_2 \equiv A_1 \rightrightarrows A_0 \to B$$

Interpret as s. rings

Look at simp gp $GL(A_\bullet)$ resolving $GL(B)$.

Left flat thm. should say that $H_*(GL(A_p))$ (loc.) constant functor, so you get $H_*(GL(A_0)) \xrightarrow{\sim} H_*(GL(B))$.

Is it possible to use this construction inductively to say something about $H_*(GL(A_0)) \longrightarrow H_*(GL(B))$, when B is left flat? Independence of the flat resolution.

Start with the cat of $A \twoheadrightarrow B$. What do we know if B is left flat.

$$\begin{pmatrix} A & B \\ A & B \end{pmatrix}$$

B is left flat $\Longleftrightarrow$ $Q \otimes_B B = B$ is A-flat

A is left flat $\Longleftrightarrow$ $P \otimes_A A = A \otimes_A A = A$ is B-flat,

Assuming the result I am after holds, $H_*(GL(B))$ should be a summand of $H_*(GL(A))$ in general, because can pick $\bar{A} \longrightarrow A \twoheadrightarrow B$ with $\bar{A}$ flat.

8 Can we define $H_*(GL(B)) \to H_*(GL(A))$. ~~[illegible]~~ ~~Such~~ idea is that B is B-flat and A rt acts on B?

$$\begin{pmatrix} A & B \\ A & B \end{pmatrix}$$

B is B-flat ?

Wrong way. You want to have a rep of B over A in order to have $K_*(B) \to K_*(A)$. So the variance?

Assume B left flat, equiv: B is A-flat

Know A left flat $\iff$ A is B-flat

Thus choosing $A' \twoheadrightarrow A$ with A' B-flat we get

$K_*(A') \longrightarrow K_*(A) \longrightarrow K_*(B)$.

So why if A' is B-flat should there be a map $K_*(B) \to K_*(A')$

~~Maybe because A is A-flat and~~ Point is that B is A'-flat and B rt acts on itself. ~~[illegible]~~

But then ~~[illegible]~~

$$A \otimes_{A'} B \simeq A \otimes_A B = B \quad \text{is } A\text{-flat.} \qquad \text{STUPID}$$

Go over it again. B left flat $\iff$ $\overset{B}{\overset{\shortparallel}{Q}} \otimes_B B = B$ is A-flat

So B is A-flat, and B rt acts on itself, so we have a flat rep. of B over A

04/06/97 Let's take the special case where B is unital i.e $A \in \mathcal{P}(A^{op})$ and $B = \mathrm{Hom}_{A^{op}}(A, A)$. Here B is both left and right flat, so ~~B is A-flat~~

$Q \otimes_B B = B$ is A-flat and

$B \otimes_B P = A$ is A^{op}-flat

ε If I can't handle this case then I can't do the general case. ~~[illegible]~~

There should be two ways to map $K_*(B)$ to $K_*(A)$

B ~~left~~ right acts on B which is A-flat

B left acts on A which is A^{op}-pr

First case A is a unitary B-mod mapping onto B so $A = B \oplus I$ where $I \in Mod(B)$ and $IB = 0$. This should be the same as a ring with left identity $A = eA$ $e^2 = e$. $B = Ae$. Now you need to understand this case. What tools? You have homos. $B \longrightarrow A \longrightarrow B$ composition 1. So what remains? What simplicial possibilities are there? ~~[illegible]~~ We can vary I ~~[illegible]~~. Take B to be a field

$$H_p\left(GL(B), H_q(M(I)) \right)$$

rationally $H_q(M(I)) = \Lambda^q(M(I))$

1. April 23, 97 1656

Spend a few minutes on mathematics – how about stability for $GL_n(F)$, F field

~~[illegible]~~ symmetric groups Σ_n.
All this stuff about ~~[illegible]~~ buildings.
Basic idea I think ~~is concerns~~ is about a group acting on a simplicial complex of high connectivity such as a building.

Some examples. ~~[illegible]~~ V vector space

Take a simplicial complex of frames. ~~[illegible]~~ |
Vertices = $v \neq 0$ r ~~[illegible]~~ simplex $\{v_0, .., v_r\}$ ind.
Over an infinite field you find ~~that there are~~ this ~~has no homology [illegible]~~ is a bouquet of spheres.
Now can ~~[illegible]~~ I analyze this?

~~[illegible]~~ $N = \dim(V)$. $G = GL(V)$.

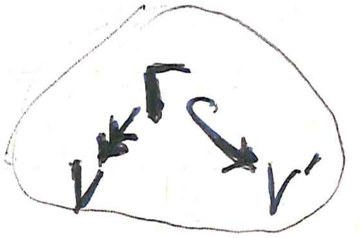
Let's use semi-simplicial. I guess we get a complex of chains.

$$0 \to M \to C_N \to \cdots\cdots \to C_1 \to C_0 \to \mathbb{Z} \to 0$$

You've forgotten all of this. Is there something you can do with the Q construction. Filter ~~[illegible]~~ the Q cat of vector spaces. There is some sort of spectral sequence arising which involves $H_*(GL_p, St_p)$.
How can this work? Let us see what happens? Ideas. You have a filt.

$$F_0 \subset F_1 \subset F_2 < \cdots$$

of the Q-cat.   Q cat cons. of V



There is a point here. Namely $P \xrightarrow{u} P'$ is a complex on which B acts. Somehow showing that up to exact sequences (Δ's) it's equivalent to Point. Each $b \in B$ on this complex is ~ 0.

So maybe it's a DG module over $\left(B \xrightarrow{1} B \right)$

define action of $\underset{\underset{B \oplus Bh}{\shortparallel}}{B[h]}$ $\quad |h| = 1$, $[d, h] = 1$, $h^2 = 0$

~~[illegible]~~ $\quad h(pq \otimes (p')) = p\, v(q) p'$

~~[illegible]~~

$$\begin{array}{ccc} \mathrm{Hom}(P', P) & \longrightarrow & \mathrm{Hom}(P,P) \oplus \mathrm{Hom}(P',P') \\ \uparrow & & \uparrow \\ B & \xrightarrow{\ 1\ } & B \end{array}$$

$h : B \longrightarrow \mathrm{Hom}(P', P)$

$\quad pq \longmapsto (p' \longmapsto p\, v(q) p')$

$dh : pq \longmapsto (p' \longmapsto u(p\, v(q) p') = w(pq) p')$

$hd : pq \longmapsto ($

Condition I used before, namely, U/AU acyclic

2 Review: To set up equiv. between the cat of fdp /A and the cat of f rings B equipped with meq to A. ~~Assoc to P, Q~~

First cat is obvious: objects $\left(\underbrace{{}_A P, Q_A}_{\text{firm}}, \langle\ \rangle : Q \otimes P \twoheadrightarrow A \right)$

maps $P, Q \xrightarrow{u,v} P', Q'$ $\ni \langle v(q), u(p) \rangle = \langle q, p \rangle$.

Equivalent category: firm M cont. $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ with A fixed

map $\begin{pmatrix} 1 & v \\ u & w \end{pmatrix} : \begin{pmatrix} A & Q \\ P & B \end{pmatrix} \longrightarrow \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$.

2nd cat: Obj B firm r. with $F : \mathcal{M}(A) \xrightarrow{\sim} \mathcal{M}(B)$,

Map $(B, F) \longrightarrow (B', F')$ cons of $w : B \to B'$ and $\theta : w_! F \xrightarrow{\sim} F'$.

$(B, F) \xrightarrow{(w,\theta)} (B', F') \xrightarrow{(w', \theta')} (B'', F'')$

comp. is $w'w : B \to B''$ and $(w'w)_! F = w'_! w_! F \xrightarrow{w'_! \theta} w'_! F' \xrightarrow{\theta'} F''$

Equiv. cat cons. of B, P where P is a firm invertible B, A-bimodule. $B, P \xrightarrow{w,\theta} B', P'$ $\theta : B' \otimes_B P \xrightarrow{\sim} P'$.

$$B' \otimes_B P \xrightarrow{\sim} P' \iff Q' \otimes_B B \xrightarrow{\sim} Q \quad (\phi)$$

$$q' \otimes pq \longmapsto q' u(p) q$$

$$\begin{array}{ccc} Q' \otimes_B B & \xrightarrow{\sim} & Q \\ \downarrow & & \\ Q' & & \end{array}$$

$$B' \otimes_B P \xrightarrow{\sim} P' \implies Q' \otimes_B P \otimes_A Q \xrightarrow{\sim} Q' \otimes_B P' \otimes_A Q$$

$$\begin{array}{ccc} Q' \otimes_B P \otimes_A Q & \xrightarrow{\sim} & Q' \otimes_B P' \otimes_A Q \\ \| & & \| \\ Q' \otimes_B B & \xrightarrow{\sim} & Q \\ q' \otimes pq & \longmapsto & q' u(p) q \end{array}$$

$$Q' \otimes_B B \xrightarrow{\sim} Q \implies P' \otimes_A Q' \otimes_B P \xrightarrow{\sim} P' \otimes_A Q \otimes_B P$$

$$\begin{array}{ccc} P' \otimes_A Q' \otimes_B P & \xrightarrow{\sim} & P' \otimes_A Q \otimes_B P \\ \| & & \| \\ B' \otimes_B P & \simeq & P' \end{array}$$

$\Updownarrow$

$v : Q \longrightarrow Q'$ B^{op}-mod isom.

$$Q' \otimes_B P \simeq A$$

1) · cut and paste.

Given $\begin{pmatrix} 1 & v \\ u & w \end{pmatrix} : \begin{pmatrix} A & Q \\ P & B \end{pmatrix} \longrightarrow \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$ hom. of firm M cont.

get isos $\theta : B' \otimes_B P \xrightarrow{\sim} P'$ $\quad \xi : Q \otimes_B B' \xrightarrow{\sim} Q'$

$\theta(b' \otimes p) = b' u(p)$ $\quad \xi(q \otimes b') = v(q) b'$

$\theta^{-1}(p' q p) = p' v(q) \otimes p$ $\quad \xi^{-1}(q p q') = q \otimes u(p) q'$

θ (is ~~[scribbled out]~~ bimod iso corresp. to / can be identified with an isom.) $w_! F \xrightarrow{\sim} F'$

hence $w_!$ quasi-invertible, w is meg hom.

ξ (is the bimodule iso corresp to the θ-ind iso $G w^* \xrightarrow{\sim} G'$ / can be ident with the iso of q inv. funs $G w^* \xrightarrow{\sim} G'$ corresp to θ.)

· get iso of M cont

$$\begin{pmatrix} 1 & \xi \\ \theta & w \end{pmatrix} : \begin{pmatrix} A & Q \otimes_B B' \\ B' \otimes_B P & B' \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$$

Converse: given $w : B \longrightarrow B'$, $\theta : B' \otimes_B P \xrightarrow{\sim} P'$

know w is meg hem, know $\exists\ \xi : Q \otimes_B B' \xrightarrow{\sim} Q'$

$\ni (\theta, \xi) : (w_! F, ~~G~~ w^*) \xrightarrow{\sim} (F', G')$. iso of pqifs

canon. $\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \xrightarrow{\gamma} \begin{pmatrix} A & Q \otimes_B B' \\ B' \otimes_B P & B' \end{pmatrix}$ $\quad \gamma(bp) = w(b) \otimes p$, $\quad \gamma(q b_1) = q \otimes w(b_1)$

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what sort of things

2). Given $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ form and $\omega : B \to B'$ meg hom w B' form

get $\mathcal{M}(A) \underset{(F,G)}{\simeq} \mathcal{M}(B) \underset{(\omega_!, \omega^*)}{\simeq} \mathcal{M}(B')$ comp. meg

$$\begin{pmatrix} A & Q \otimes_B B' \\ B' \otimes_B P & B' \end{pmatrix}$$

$$(b_1' \otimes p)(q \otimes b_2') = \alpha(b_1' \otimes pq \otimes b_2') = b_1' \omega(pq) b_2'$$

$$(q \otimes b_2')(b_1' \otimes p) = \beta^{-1}(q \otimes b_2' b_1' \otimes p)$$

$$(qb_2 \otimes b_2')(b_1' \otimes b_1 p) = qbp$$

where $\omega(b) = \omega(b_2) b_2' b_1' \omega(b_1)$

Note $\exists$ canon. hom $\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & \xi_\omega \\ \theta_\omega & \omega \end{pmatrix}} \begin{pmatrix} A & Q \otimes_B B' \\ B' \otimes_B P & B' \end{pmatrix}$

$\theta_\omega(bp) = \omega(b) \otimes p$, $\xi_\omega(qb_2) = q \otimes \omega(b_2)$

~~Now suppose given $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ $\begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$ forms and an isom~~

$$(q \otimes b_2')(b_1' \otimes p) = \beta^{-1}(q \otimes b_2' \otimes b_1' \otimes p)$$

$$= \sum_{i,j} q_i b_{ij} p_j$$

where $q = \sum q_i b_i$, $p = \sum b_j p_j$

$\omega(b_{ij}) = \omega(b_i) b_2' b_1' \omega(b_j)$

difficulty apparently is moving B' ~~to~~ the right side. From $B'\otimes_B P \simeq P'$ you can get

$$B'\otimes_B B \simeq P'\otimes_A Q$$

$$Q'\otimes_B B \simeq Q$$

but the prime is always on the left. So you need an arg using that $B'\otimes_B B$, $B\otimes_B B'$ are "adjoint", $B'\otimes_B B$ is invertible, hence ~~these~~ the adjunction maps are isos.

$$\alpha : FG \longrightarrow 1 \qquad \beta : 1 \longrightarrow GF$$

$F \xrightarrow{F.\beta} FGF \xrightarrow{\alpha.F} F$ is the identity

$G \xrightarrow{\beta.G} GFG \xrightarrow{G.\alpha} G$ is id.

~~[illegible]~~.

$$1 \xrightarrow{\beta} GF \xrightarrow{F^{-1}.\alpha.F} 1$$

$F \longrightarrow F'$ $\qquad G' \xrightarrow{\beta.G'} GFG' \xrightarrow{G.\theta.G'} GF'G' \xrightarrow{G.\alpha} G.$

~~to adjust our pricing policies to the changes in the environment.~~
~~How will the homepage be edited?~~
~~I propose that the homepage will be edited by a group of topologists representing the various publications mentioned above.~~

4)

Notation. (P, Q) firm dual pair over A,
$B = P \otimes_A Q$ corresp firm ring

We have a meq $\mathcal{M}(A) \simeq \mathcal{M}(B)$ given by
$F = P \otimes_A -$, $G = Q \otimes_B -$.

(P', Q') another fdp, B', F', G' sim defd.

$(u,v) : (P, Q) \to (P', Q')$ a map of dual pairs

$w : B \to B'$, $w(pq) = u(p) v(q)$ corresp. homom.

Real logic

1) Given (P, Q) $w : B \to B'$ get $(B' \otimes_B P, Q \otimes_B B')$
describing $\mathcal{M}(A) \simeq \mathcal{M}(B) \simeq \mathcal{M}(B')$

2) Given $(u,v) : (P, Q) \to (P', Q')$ get $w : B \to B'$ and $(\theta, \xi) : (B' \otimes_B P, Q \otimes_B B') \xrightarrow{\sim} (P', Q')$
i.e. a map of fdps yields a map of f rings equipped with meq to A.

3) Conversely given (P, Q), (P', Q') $w : B \to B'$ and
isom $(\theta, \xi) : (B' \otimes_B P, Q \otimes_B B') \simeq (P', Q')$, then get $u : P \to P'$
$v : Q \to Q'$ from canonical $P \to B' \otimes_B P$, $Q \to Q \otimes_B B'$
$bp \mapsto w(b) \otimes p$ $qb \mapsto q \otimes w(b)$.

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2 So what happens?

04/24 0623. I'd like to reconstruct the stability arguments I found years ago. These were based on Stieffel manifolds analogs made out of unimodular sequences. Review

E vector bundles over X, when can you split off a trivial line bundle? Method: Choose $\theta \otimes V \twoheadrightarrow E$. Use Sard's thm. Mainly

Suppose you want to show $BG_n \to BG_{n+1}$ is a hom. isom. in a certain range. Put $Y = BG_{n+1}$ $X = EG_{n+1} \times_{G_{n+1}} (G_{n+1}/G_n)$, so you have a fibration $X \xrightarrow{f} Y$, then analyze à la Groth $\overrightarrow{\overrightarrow{\rightarrow}} X \times_Y X \rightrightarrows X \to Y$, spec. seq.

Pattern here; s-G_{n+1} set $\overrightarrow{\overrightarrow{\rightarrow}} (G_{n+1}/G_n)^2 \rightrightarrows (G_{n+1}/G_n)$ contr.

So what's going on is you have $G = G_{n+1}$ acting on a simplicial set which is acyclic such that the G set of vertices is G_n.

Example: $G_n = Aut(V_n)$ unimodule

First work over a field. G acts on $V - 0$ trans. with stabilizer G_{n-1} at least ignoring Δ nilp. There's a simplicial set consisting of frames $v_0, \ldots)$

$|GL_2(\mathbb{F}_2)| = 3 \cdot 2 = 6$

$|GL_3(\mathbb{F}_2)| = 7 \cdot 6 \cdot 4 = 168$

$|GL_2(\mathbb{F}_3)| = 8 \cdot 6$ SL

$|SL_2(\mathbb{F}_5)| = \frac{24 \cdot 20}{4} = 120$ $SL_2(\mathbb{F}_5) = \tilde{A}_5$

$|GL_2(\mathbb{F}_4)| = 15 \cdot 12$

a April 27, 1997 1546

I have done little mathematics since March 23. Only a few pages on April 5. Tomorrow I think I start some lectures, talk on June 20 looms ahead.

How to get started? Lecture?

Instead look at stability for a field and see if you can work out your old result as well as Suslin's. What should be the basic idea? First handle mod p homology p invertible in F. The key is to consider the simplicial set consisting of indep. sequences $(v_0, \dots, v_p)$

0-simp. $X_0 = V - \{0\}$

1-simple X_1 ord. pairs of ind. vectors. $X_1 \subset X_0 \times X_0$

In general $X_n \subset X_0^{n+1}$. We get an acyc. cx.

$$\begin{array}{ccccccc} \Rrightarrow & \mathbb{Z}[X_0 \times X_0] & \rightrightarrows & \mathbb{Z}[X_0] & \longrightarrow & \mathbb{Z} & \longrightarrow 0 \\ & \cup & & \| & & \| & \\ \Rrightarrow & \mathbb{Z}[X_1] & \rightrightarrows & \mathbb{Z}[X_0] & \longrightarrow & \mathbb{Z} & \longrightarrow 0 \end{array}$$

Notice no degeneracies. Can make a simplicial set by allowing repetitions. What sort of relations arise? Inside V. Say $V = F^3$

$$\mathbb{Z}[X_3] \longrightarrow \mathbb{Z}[X_2] \longrightarrow \mathbb{Z}[X_1] \longrightarrow \mathbb{Z}[X_0] \longrightarrow \mathbb{Z} \longrightarrow 0$$

This complex should be acyclic by general position arguments in degrees < 3. If true, then what sort of result do we get? G acts trans. on X_p

$p \leq 3$ Say $X_p = G/G_p$