318 Dec 26 Have puzzle

Finish off circuit theory. LC network (conn.) graph vertices x edges a yields exact seq

$$0 \longrightarrow \bar{C}^0 \xrightarrow{d} C^1 \longrightarrow H^1 \longrightarrow 0$$

and the dual exact sequence

$$0 \longleftarrow \bar{C}_0 \xleftarrow{\partial} C_1 \longleftarrow H_1 \longleftarrow 0$$

$\bar{C}^0$ voltage states $\qquad H_1$ current states

There's a flow on $\underbrace{\bar{C}^0 \oplus H_1}_{v-1\,+\,l\;=\;e} \simeq \mathbb{R}^e$ which has the form e^{tH}, H operator on $\bar{C}^0 \oplus H_1$.
If I restrict to LC networks the eigenvalues of H are purely imaginary and closed under $i\omega \mapsto -i\omega$.
Looks like oscillator except for $\omega = 0$.

IDEA: Analogy between circuits and surfaces, n-port corresponds to surface with n boundary components. Program to construct interesting self adjoint operators by gluing connecting ports together might have a surface analogue. ζ should enter via Kontsevich link - Euler characteristic of surface graphs given by ζ values.

Solve circuit flow by L.T. Everything is done in $C^1 \oplus C_1 = \bigoplus_\sigma \mathbb{C}^2$. For each σ you have either a capacitance or an inductance. Concentrate

C^1 1-voltages $\{E_\sigma\}$ $\qquad C_1$ 1-currents $\{I_\sigma\}$

$$R(s): C_1 \longrightarrow C^1 \qquad R(s)_\sigma = \begin{cases} L_\sigma s \\ (C_\sigma s)^{-1} \end{cases}$$

319 $\hat{E} = Ls\hat{I}$ or $\hat{I} = Cs\hat{E}$ at each σ

actually should be

$$\hat{E} = L(s\hat{I} - I_0) \quad \text{or} \quad \hat{I} = C(s\hat{E} - E_0)$$

It looks like you are going to split C^1 into 2 pieces. All right. ~~Now split~~ The dual situation is trivial. Things becomes interesting when you impose constraints.

$$\begin{array}{ccc} d\tilde{C}^0 & \subset & C^1 \\ & & \scriptstyle S\uparrow R_s \\ & & C_1 \supset Z_1 \end{array}$$

Note $Z_1 = (dC^0)^\perp$, so our state space is a maximal isotropic subspace of $C^1 \oplus C_1$

Is there a real skew adjoint operator around with the right eigenvalues?

Problem: How am I going to finish this off?

Return to yesterday. State space is $d\tilde{C}^0 \oplus Z_1$ real space of dim $v-1+\ell = e$ consisting of (E, I) where E is a potential function on vertices, I a current function on edges, such that $\partial I = 0$. There is a first order linear DE on $\tilde{C}^0 \oplus Z_1$ I want to construct.

$$(dE)^t_\sigma = L_\sigma \dot{I}^t_\sigma \qquad \text{ind edge}$$

$$I^t_\sigma = C_\sigma (d\dot{E})^t_\sigma \qquad \text{cap edge}$$

~~Let's think of dE as~~ Let's identify $E \in \tilde{C}^0$ with the 1-cobdy $dE \in C^1$. Then D.E.'s are

$$\dot{I}_\sigma = L_\sigma^{-1} E_\sigma \qquad \dot{E}_\sigma = C_\sigma^{-1} I_\sigma$$

with constraints $\quad E \in d\tilde{C}^0 \qquad \partial I = 0$

So what's going on is that we have an incomplete DE on $C^1 \oplus C_1$, actually only half of the $2e$ needed equations are given, but we are constrainted to a max. isotropic subspace $dC^0 \oplus Z_1$, so it is somehow OKAY.

In $C^1 \oplus C_1$ we have $dC^0 \oplus Z_1$ and $R_s : C_1 \xrightarrow{\sim} C^1$ whose graph might be relevant. What does solve mean? You will have $(E^0, I^0) \in dC^0 \oplus Z_1$ given. Then L.T. will make $(\hat{E}, \hat{I})$ out of (E^0, I^0) then the inverse L.T. will yield (E^t, I^t) for all $t \geq 0$.

What are the equations.

$$\mathcal{L}(\dot{f}) = \int_0^\infty e^{-st} \dot{f}\, dt = \left[e^{-st} f\right]_0^\infty - \int (-s) e^{-st} f \, dt$$

$$\boxed{\mathcal{L}(\dot{f}) = s\mathcal{L}(f) - f_0}$$

Equations (E^t, I^t) satisfy are

$$C\dot{E}^t = I^t \quad \text{for } C\text{-edges}$$
$$L\dot{I}^t = E^t \quad \text{for } L\text{-edges.}$$
$$E^t \in dC^0$$
$$I^t \in Z^1$$

Take LT.

$$C(s\hat{E} - E^0) = \hat{I} \qquad C \text{ edges}$$
$$L(s\hat{I} - I^0) = \hat{E} \qquad L \text{ edges}$$
$$\hat{E} \in dC^0$$
$$\hat{I} \in Z^1$$

321 $\hat{E} - R_s\hat{I} = \begin{cases} -LI^0 & L \text{ edges} \\ C^{-1}E^0 & C \text{ edges} \end{cases}$

Important is that the solution is unique, which means that the form on the right is not relevant to begin with. You first need to know that ~~there~~ there are no non-trivial solutions of $\hat{E} = R_s\hat{I}$ $(\hat{E},\hat{I}) \in dC^0 \oplus Z_1$.

This follows from inv. of $\partial R^{-1}d : \bar{C}^0 \to \bar{C}_0$.

So $\Gamma_{R_s} \cap dC^0 \oplus Z_1 = 0$ inside $C^1 \oplus C_1$

Since both have dim $= e$, these are transversal. So I can solve.

Nature of Γ_{R_s}: ~~You need now to~~ ~~understand~~ Is it isotropic? This probably means R_s is symmetric. I don't want to forget earlier analysis where C^1, C_1 are naturally conjugate dual.

Previous philosophy - response to applied potential, natural, nondegenerate sesquilinear pairing between C^1, C_1 - R_s yields non deg. sequilinear form on C^1 with pos. def. hermitian part. Induced system on any subquotient. ~~How to ... induced~~

Before trying to generalize, you should handle ~~...~~ completely the case of LC networks.

Questions: Nature of $s=0$ $(s=\infty?)$ You know eigenvalues on iR, enough eigenvectors?

322. ~~...~~ General LC network.
Applied Voltage, Response current analysis.

Here
$$\begin{array}{ccc} \bar{C}^0 & \xrightarrow{d} & C^1 \\ & & \uparrow{\scriptstyle s}\; R(s) \\ \bar{C}_0 & \xleftarrow{\partial} & C_1 \end{array}$$

Originally things are real. Each edge γ gives $(E_\gamma, I_\gamma) \in \mathbb{R} \oplus \mathbb{R}$. C_1, C^1 are naturally dual via pairing $(E, I) \mapsto \sum_\gamma E_\gamma I_\gamma$ = power. Then ~~...~~ complexify. For each γ have now $(E_\gamma, I_\gamma) \in \mathbb{C} \oplus \mathbb{C}$. To prove $\partial R(s)^{-1} d : \bar{C}^0 \to \underline{C}_0$ is invertible. Everything is $\mathbb{C}$-linear here, enough to do inj. ~~Take~~ Let $E \in \bar{C}^0$ sat $\partial R(s)^{-1} dE = 0$. What you do is to use that E gives rise to a linear functional on $\bar{C}_0$, denote it E^*. Then

$$0 = E^* \partial R^{-1} dE = (dE)^* R^{-1} dE = \sum_\gamma \overline{(dE)_\gamma}\, R_\gamma^{-1} (dE)_\gamma$$

contradicts ~~...~~ fact that R_γ^{-1} has pos. real part.

So the natural setting is that C^1, C_1 are conjugate dual, i.e. $\exists$ sequilinear nondegenerate pairing $E^* I$ $E \in C^1$, $I \in C_1$. Then $R^{-1} : C^1 \to C_1$ becomes a sesquilinear form. ~~...~~ So what?

~~...~~

Question: What structure do you have on $V \oplus \mathrm{Hom}_{\mathbb{C}}(\bar{V}, \mathbb{C})$? Real symplectic with ~~...~~ J

323. ~~[illegible]~~ Study pair (X,Y) consisting of a positive definite hermitian form and a skew hermitian form. The sum $X+tY$ for t real is non degenerate ~~[illegible]~~

Can do better. ~~[illegible]~~ Can take X as inner product, then Y is equiv. to a skew-adjoint operator B, ~~[illegible]~~ $tX+Y$ is equiv. to $t+B$ which is invertible for $t \notin i\mathbb{R}$. ~~[illegible]~~

~~[illegible]~~

If you take $V \oplus \bar{V}^{\vee}$ there ~~[illegible]~~ are two canonical sesquilinear forms $\begin{pmatrix} v \\ \lambda \end{pmatrix}, \begin{pmatrix} v' \\ \lambda' \end{pmatrix} \longmapsto \begin{matrix} \lambda'(v) \\ \lambda(v') \end{matrix}$?

maybe also the conjugates.

Later

~~Suppose you consider V equipped with sesquilinear form with~~

Go back to your graph. $V = C^1(\Gamma, \mathbb{C}) \ni (E_\gamma)$

$V^\dagger = C_1(\Gamma, \mathbb{C}) \quad (I_\gamma)$

$\langle I, E \rangle = \sum_\gamma \bar{I}_\gamma E_\gamma$ and you have $R(s) : V^\dagger \xrightarrow{\sim} V$

$(R(s)I)_\gamma = R_\gamma(s) I_\gamma = \begin{cases} L_\gamma s I_\gamma & \text{ind case} \\ (C_\gamma s)^{-1} I_\gamma & \text{cap} \ " \end{cases}$

$\langle I, RI \rangle = \sum_\gamma \bar{I}_\gamma R_\gamma I_\gamma = \sum_\gamma |I_\gamma|^2 R_\gamma$

 ~~Quite difficult~~

Wait : you have sesquilinear form $\langle I_1, R I_2\rangle$ on $C_1 \times C_1$

$$= \sum_\gamma \bar{I}_{1,\gamma} R_\gamma I_{2,\gamma}$$

You've been claiming that a sesquilinear form is the sum of a hermitian form and an anti-herm. form. Typical seq. form on $\mathbb{C}^n$ is

$$S(v,v') = \sum_{i,j} \bar{v}_i s_{ij} v'_j ,$$

$$S(v,v') = \sum_{i,j} S(\overset{\text{coeff}}{v_i} e_i, v'_j e_j) = \sum_{i,j} \bar{v}_i S(e_i,e_j) v'_j$$

$S(v,v') + \overline{S(v',v)}$ should be hermitian

$S(v,v') - \overline{S(v',v)}$ ——— anti

You're mistake is in thinking that the anti hermitian part vanishes when $v=v'$.

How am I to proceed further?

Keep on trying.

OKAY try to understand normal modes

State space is $dC^0 \oplus Z_1 \subset C^1 \oplus C_1$.

$$0 \longrightarrow d\bar{C}^0 \longrightarrow C^1 \longrightarrow H^1 \longrightarrow 0$$

$$\uparrow R$$

$$0 \longleftarrow \bar{C}_0 \longleftarrow C_1 \longleftarrow H_1 \longleftarrow 0$$

$$(dC^0 \oplus Z_1) \cap \Gamma_R = \left\{ (E,I) \;\middle|\; E = RI \quad \overset{d\xi}{\overset{\shortparallel}{E}} \in dC^0,\ I \in Z_1 \right\}$$

$\partial R^{-1} d\xi = \partial I = 0 \implies \xi = 0$.

Puzzle. You are given $E^0, I^0 \in dC^0 \oplus Z_1$

Need to solve

$$\hat{E} - R\hat{I} = \begin{cases} -LI^0 & \text{on } L \text{ edges} \\ C^{-1}E^0 & \text{on } C \text{ edges.} \end{cases}$$

with $(\hat{E}, \hat{I}) \in dC^0 \oplus Z_1$

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & H_1 \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & \bar{C}^0 \arrow[r] \arrow[dr] & C^1 \arrow[r] \arrow[d] & H^1 \arrow[r] & 0 \\
 & & \bar{C}_0 \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

Now you understand the L.T. solution, but you would the time evolution on $dC^0 \oplus Z^1$ if possible. This should be defined on the real chain + cochain spaces. I also want the singularities, characteristic equation

How does this work? $s = 0$.

Let's examine the response problem when $s = -i\omega$

Then Ls and $\frac{1}{Cs}$ are in $i\mathbb{R}$. So on the imag axis R is purely imag. So if we remove i it is as if we had a resistance network but with positive + negative resistances.

Let's work with real chains + cochains. Recall C^1 and C_1 are naturally dual we still have $R : C_1 \to C^1$ so R gives a bilinear form on C^1

It's actually symm. Both $\pm$

This seems to be an interesting situation ~~Namely, How [illegible]~~ You have split C' into two parts where $R = Ls$ on one and $(Cs)^{-1}$ on the other. $s = -i\omega$ so

$$iR = \begin{cases} Lis = L\omega \\ \dfrac{1}{Cis} = \dfrac{1}{C(-\omega)} = \dfrac{-1}{C\omega} \end{cases}.$$

So we have C' split into ~~pieces~~ two — so indefinite Lorentz type metric, each piece carries positive definite inner product, ω rescales the two emphasizing one over the other. This is the structure on C', now look at $dC^0 \subset C'$.

$$\tilde{C}^0 \xhookrightarrow{\ d\ } \begin{matrix} C'_+ \\ \oplus \\ C'_- \end{matrix} \qquad\qquad \begin{tikzcd}[row sep=small] & C'_+ \\ \tilde{C}^0 \arrow[ur, "d^+"] \arrow[dr, "d^-"'] & \\ & C'_- \end{tikzcd}$$

$$d\xi\, R^{-1} d\xi$$

Can you see why the eigenvalues are real?

C' splits into C'_+, C'_- where ?

First remark is that C', C_1 etc. have dist. real bases so that the structure of R is simple.

C', C_1 dual $R: C_1 \simeq C'$ is the same as a non-degenerate bilinear form on C' which happens to be symmetric, in fact diagonal. L, C real positive. ~~Critical thing is the~~ Splitting of C_1 into L and C edges. General R is $Ls \oplus C^{-1}s^{-1}$

327 $\bar{C}^0 \longrightarrow C^1 = C^1_L \oplus C^1_C$

$$\uparrow Ls \oplus C^{-1}s^{-1}$$

$\bar{C}_0 \longleftarrow C_1 = C_{1,L} \oplus C_{1,C}$

so it seems that the map $d^t R^{-1} d : \bar{C}^0 \longrightarrow \bar{C}_0$ is the sum of two pieces $s\underbrace{(d'^t L^{-1} d')}_{pos.} + s^{-1}\underbrace{(d''^t C d'')}_{pos.}$

So you get for $u \in \bar{C}^0$

$$(u, s(d'^t L^{-1} d')u) + (u, s^{-1}(d''^t C d'')u)$$

$$= s(d'u, L^{-1}d'u) + s^{-1}(d''u, C d''u)$$

This ~~doesn't~~ vanishes only for $s \in i\mathbb{R}$

What can I say about the structure?

A basic question is emerging, whether there is a skew adjoint operator naturally occurring which gives the eigenvalues.

So what happens roughly you have

$$\bar{C}^0 \xrightarrow{(a,b)} C^1_+ \oplus C^1_-$$

Euclidean spaces, and you are interested in $s\, a^t a + s^{-1} b^t b$ on $\bar{C}^0$ complexified

You will get a set of eigenvalues, ~~the~~ should be purely imaginary $\pm i\omega$. Enough eigenvectors.

$$s^2 a^t a + b^t b$$

Probably true. Obvious if $a^t a > 0$ change metric $|a^t a|^{1/2}$

Clear: arrange $a^t a + b^t b = 1$, then

 Return to graph Γ LC network.
Then everything can be handled in real terms I think. C^1 is a real vector space with an indef. quadratic form and polarization according to the L_s and C edges. Subspace $dC^0 \subset C^1 \to H^1$, use S to rescale the polarization. What do you want to understand? ~~You need~~ Applied voltage + Response currents, generalized reactance, need to see the ~~simple~~ matrices with S, S^{-1} get converted to more complicated rational functions. Hopefully you can say something about reconstruction - this might relate to GNS, compression etc. Also I want: normal modes for a circuit.

You have to review, understand better the normal modes. Recall state of circuit was (E, I) where $E \in \bar{C}^0$ and $I \in Z_1$ and there is it seems a time evolution on this space. Action of the add gp $\mathbb{R}$.
Let's put in now $C^1 = C^1_+ \oplus C^1_-$ From real viewpoint everything is simple. ~~C^1 comes with a~~

Go over response picture. C^1 polarized real Hilbert space

$$\bar{C}^0 \xrightarrow[d'+d'']{d} C^1 = C^1_+ \oplus C^1_-$$

basically

$$\omega \|d'u\|^2 - \omega^{-1}\|d''u\|^2$$

~~hermitian~~

hermitean form on $\bar{C}^0$ for any real ω which you can compare with $\|d'u\|^2 + \|d''u\|^2$. Take this to be the norm on $\bar{C}^0$. Then you ~~have~~ have ~~the oper~~

the ~~hermitian~~ selfadj operator $\omega\rho - \omega^{-1}(1-\rho)$ on $\bar{C}^0$ where $\rho = d'^{*}d'$ $0 \leq \rho \leq 1$. Spectral thm. apply to ρ. What is an eigenvector in $\bar{C}^0$? Say $\omega = 1$ is an eigenvalue. It means you have a ~~applied~~ voltage at frequency ω that produces zero net current at the vertices, i.e. a normal mode. What about $\omega = 0$ or ∞.

Recall that the eigenvectors are those of ρ. So you mean that the eigenvalue of ρ is 0 or 1. $\rho v = v \implies (\omega\rho - \omega^{-1}(1-\rho))v = \omega v$

Wait: You want ω, $v \neq 0$ such that

$$\omega\rho(v) - \omega^{-1}(1-\rho)v = 0$$

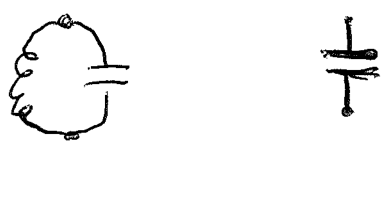
$$(\omega + \omega^{-1})\rho(v) = \omega^{-1} v \quad \text{or} \quad \rho(v) = \frac{\omega^{-1}}{\omega + \omega^{-1}} v$$

so $\lambda = \dfrac{1}{1+\omega^2}$ goes from 1 to 0

Clearly $\lambda = 1$ means $\omega = 0$
$\lambda = 0$ — $\omega = \infty$.

So an eigenvector for $\omega = 0$ means $\rho(v) = v$ i.e. $d''u = 0$. What I expect is that $\omega = 0$ makes the L edges have ~~0~~ are currents and the C edges have 0 currents.

So you would be looking at voltages ~~such that~~ constant on the L edges. Thus no $0, \infty$ modes if both the L edges and C edges yield connected graphs.



330 So what was happening with $\bar{C}^0 \oplus Z_1$?
You had some way to calculate normal modes.
Probably what you missed was the idea that
$\partial R^{-1} d u = 0 \implies u$ is a normal mode i.e.
the voltage u produces no response

How to organize the rest ??
Well, the first thing is to see how you get rational response functions. Keep things real.
You have $C^1 = C^1_+ \oplus C^1_-$ polarized $\mathbb{R}$-Hilbert space associated to the edges

Ribbon graph. Are the loops clear? Probably

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & Z_1 \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & \bar{C}^0 \arrow[r, "d"] \arrow[dr, "d^tR^{-1}d"'] & C^1 \arrow[r] \arrow[d, "d^tR^{-1}"] & H^1 \arrow[r] & 0 \\
 & & \bar{C}^0 \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

ω, ω^{-1}

Let's try just getting the response at a pair of vertices.

$$\begin{tikzcd}
\bar{C}^0 \arrow[r, hook] \arrow[d] & C^1 \\
\mathbb{C} & C_1 \arrow[d] \\
\mathbb{C} \arrow[r, hook] & \bar{C}_0
\end{tikzcd}$$

331 ~~This is fascinating~~

What do I need to restrict? We agree that if $H = H_+ \oplus H_-$ (polarized Hilb. space) and $V \subset H$ then get family $\omega\rho - \omega^{-1}(1-\rho)$ of s.a. ops on V; get a self adjoint contraction λ on on V, $0 \le \rho \le 1$.

$$(\omega+\omega^{-1})\overset{\lambda}{\rho} - \omega^{-1} \qquad \lambda = \frac{\omega^{-1}}{\omega+\omega^{-1}} = \frac{1}{1+\omega^2}$$

Question: Can I recover H from V, ρ. Yes because you can dilate a s.a. contraction to a s.a. involution in an essentially unique way.

Given (V,ρ) form $\sqrt{1-\rho}$ | $\begin{pmatrix} \rho & \sqrt{1-\rho^2} \\ \sqrt{1-\rho^2} & -\rho \end{pmatrix}$

~~You~~ are trying to do $-1 \le \rho \le 1$. not $0 \le \rho \le 1$

$$\frac{1}{2}\begin{pmatrix} 2\rho \cancel{-1} & 2\sqrt{\rho-\rho^2} \\ 2\sqrt{\rho-\rho^2} & -2\rho+2 \end{pmatrix} \qquad 1-(2\rho-1)^2 = 4(\rho-\rho^2)$$

$$\begin{pmatrix} \rho & \sqrt{\rho-\rho^2} \\ \sqrt{\rho-\rho^2} & 1-\rho \end{pmatrix}^2 = \begin{matrix} \rho^2+\rho-\cancel{\rho^2} & \cancel{\rho}\sqrt{\rho-\rho^2} + \cancel{\sqrt{\rho-\rho^2}(1-\rho)} \\ \sqrt{\rho-\rho^2}\,\cancel{\rho+(1-\rho)\sqrt{\rho-\rho^2}} & \rho-\rho^2+(1-\rho)^2 \end{matrix}$$

Dec 27 It seems that you have an easy way to explain the ~~eigenvalues~~ normal modes, namely compressing a polarization, but ~~things are not that~~ you need to understand how rational functions of ω arise.

Situation: $H = H^+ \oplus H^-$ real Hilbert space of 1-chains where an edge has ~~weight~~ length L, C^{-1}

$C^0(\Gamma, \mathbb{R})$ ~~norm of~~ $\|E\|^2 = \sum_\sigma E_\sigma^2/L_\sigma + \sum E_\sigma^2 C_\sigma$

$V \overset{d}{\subset} H$ get then $\underbrace{d'^*d'}_{0\le\rho\le 1} + \underbrace{d''^*d''}_{1-\rho} = 1$ on V

I want $d'^*d' - d''^*d'' = \iota^* \varepsilon \iota$ where $\varepsilon = \pm 1$ on $H^\pm$

 Now ~~[illegible]~~ you want to restrict ~~current~~ applied voltage to a subset of vertices and the response current to be supported on these vertices. A normal mode occurs when you find ~~a~~ a potential and frequency such that the response current is 0. So the ~~normal modes~~ frequencies don't change.

Actually ~~[illegible]~~ we are in the Grassmannian situation, namely $V \subset H = H^+ \oplus H^-$, $F = \pm 1$ on V resp $V^\perp$ and the eigenvalues are clear, as well as $0, \infty$.

Resistance network yields pos. def quad. form on C^1 $\sum E_\sigma^2 R_\sigma^{-1}$ ~~[illegible]~~ power consumed
Restrict to potentials $\tilde{C}^0$ ~~gets~~ pos. def quad form ~~[illegible]~~ - current response is the gradient of this quadratic form. If we now restrict applied volt. to a subset of vertices, then the potential at the remaining vertices is determined by minimizing. ~~[illegible]~~ This means ~~[illegible]~~ passing the q-form to ~~[illegible]~~ quotient.

LC network. | s.a. contraction

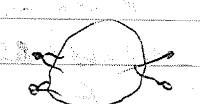
~~[illegible] current response~~

For a fixed ω you get a indef form on C^1_{ξ}, Restrict to $\tilde{C}^0$ if deg you get ~~normal~~ modes of osc. ~~[illegible]~~ Otherwise get current response. ~~Next take~~ ~~[illegible]~~ Next - subset of vertices, no reason to remain non-degenerate, so the ~~modes~~ frequencies change.
Question: Non deg quad form, how does it restrict to subquotients?

Resistance network. Γ connected graph
$R_\sigma > 0$ for each edge σ.
~~Of~~ State $E \in \tilde{C}^0(\Gamma, \mathbb{R})$ $\quad I \in C_1(\Gamma, \mathbb{R})$

$dE \in C^1(\Gamma, R)$ $\quad \boxed{dE = RI}$ e

mathematically a state is simply $E \in \tilde{C}^0(\Gamma, R)$
I given by Ohm's law. Get ~~correct~~ current
at vertices ∂I. $\quad E \longrightarrow \underline{\partial} R^{-1} dE$

~~You go on to conclude for any subset of~~
~~R/~~ S subset of vertices
~~Wrong~~ This doesn't cover a 2-port 
where you give voltage drops at either end.
2 dual space$_a$ and currents space (potential space)

Consider a LC network. Then C^1 is a
polarized Hilbert space $H_+ \oplus H_-$ and $\tilde{C}^0 \xrightarrow{d} C^1$
is a subspace which inherits a ~~inner product~~
Hilbert space structure and a hermitian
~~op~~ operator $d^* F d$ which is a contraction
$-1 \leq d^* F d \leqslant 1$. whose eigenvectors correspond to
normal modes of the circuit. Formulas

$$\mathrm{Ker}\left(\omega\, d_+^* d_+ - \omega^{-1} d_-^* d_-\right)$$

is the space of normal modes of frequency ω. Translates
~~Wo~~

~~[illegible]~~

$$\omega\left(\frac{1+\alpha}{2}\right) - \omega^{-1}\left(\frac{1-\alpha}{2}\right)$$

$$= \frac{\omega - \omega^{-1}}{2} + \frac{\omega + \omega^{-1}}{2}\alpha$$

a
normal mode of freq ω is thus an eigenvector of α
of eigenvalue $-\dfrac{\omega - \omega^{-1}}{\omega + \omega^{-1}}$ ~~[illegible]~~
~~exchanged~~ changed sign under $\omega \mapsto \omega^{-1}$

334 So, to clarify, on $\bar{C}^0$ we have a natural scalar product $\|d_+u\|^2 + \|d_-u\|^2 = \|u\|^2$ and a natural self-adjoint contraction $\alpha = d_+^*d_+ - d_-^*d_-$. One puzzle is the change $\omega \longmapsto -\frac{\omega-\omega^{-1}}{\omega+\omega^{-1}} = -\frac{\omega^2-1}{\omega^2+1} = -1+\frac{2}{\omega^2+1}$ between the frequency variable ω and the geometrically relevant variable describing the spectrum. This is probably the same as the description of pairs $z, \bar{z}$ on the unit circle.

Now fix 2 distinct vertices in the graph, and try to understand the response, for each ω such that ~~ω is not~~ $-\frac{\omega-\omega^{-1}}{\omega+\omega^{-1}}$ is not an eigenvalue of α you have $\omega d_+^*d_+ - \omega^{-1}d_-^*d_-$ invertible on $\bar{C}^0$. get response

So you have $\lambda + \alpha$ on $\bar{C}^0$ invertible for most λ. You have $\tilde{C}^0 \xrightarrow{j} \mathbb{C}$ $(\xi_x) \longmapsto \xi_0 - \xi_1$

You probably want $j\frac{1}{\lambda+\alpha}j^*$

We are nearing the end. You have the graph Γ with the $L+C$ edges which yields Hilbert space $H = C^1(\Gamma)$ and its polarizations. The rest involves subquotients of H.

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335. But stick to the case of $\bar{C}^0(\Gamma)$ and the 1-diml quotient arising from 2 vertices. $\bar{C}^0(\Gamma)$ is the space of vertex potentials. To each $\xi \in \bar{C}^0$ you get a vertex current which for generic frequency ω is bijective correspondence $\xi \longmapsto d_\omega^* d_\omega \xi$. I still think there is a point to separate chains + cochains, since there is no obvious inner product on $\bar{C}^0$. Not clear

Structure on C^1 is splitting in L, C parts + pos. def. forms on each: $\sum_\sigma E_\sigma^2 L_\sigma^{-1}$ or $\sum_\sigma E_\sigma^2 C_\sigma$. Then use $s = 1$ to put them together to get scalar product on C^1. Polarization

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Start again. Γ LC network.

$$C^1(\Gamma) = \underbrace{C^1(\Gamma)_+}_{\text{L-edges}} \oplus \underbrace{C^1(\Gamma)_-}_{\text{C-edges}} \qquad s = -i\omega$$

scalar product

$$\omega^{-1}\sum_\sigma E_\sigma^2 L_\sigma^{-1} - \omega \sum E_\sigma^2 C_\sigma$$

Real Hilbert with polarization!!

$$\bar{C}^0(\Gamma) \xrightarrow{d} C^1(\Gamma)$$

Question: Is there a conserved quantity like power in the LC case? Each (potential fn) $u \in \bar{C}^0$ gives rise to a vertex current response. Yes. In the case of a resistance network the vertex power = edge power, because the edge power is $\|du\|^2$ and the vertex power $= \langle u, d^*du \rangle$. In the LC case something similar holds I think.

 Ignore and concentrate the response to an applied EMF between two vertices. Do this in straightforward manner.

Given u in $\bar{C}^0$

$$\begin{tikzcd}
\bar{C}^0 \arrow[r, hook, "d"] \arrow[d, "d^t R_s^{-1} d"'] & C^1 \arrow[d, "R_s^{-1}"] & = & C^1_L \oplus C^1_C \arrow[d, "(Ls)^{-1}"'] \arrow[d, shift left=6, "Cs"] \\
\bar{C}_0 & C_1 \arrow[l, "d^t"'] & = & C_{1,L} \oplus C_{1,C}
\end{tikzcd}$$

Now assume known that $d^t R_s^{-1} d$ is invertible. Now look at two vertices (in order). $\begin{matrix} x \\ y \end{matrix}$ $u_x - u_y$ volt. drop.

Corresponding to the ~~voltage~~ applied EMF $u_x - u_y$ there is a response current. Find it. You restrict from $\bar{C}_0$ to vertex currents with support $\{x, y\}$.

You

$$\begin{tikzcd}
(\mathbb{C}\{x,y\}/\mathbb{C})^t \arrow[r, hook] \arrow[d] & \bar{C}_0 \arrow[d, "S \; (d^t R^{-1} d)^{-1}"] \\
\mathbb{C}\{x,y\}/\mathbb{C} & \bar{C}^0 \arrow[l]
\end{tikzcd}$$

What you know is that for each vertex current (of any $\circ$) $\exists!$ potential u mod constants producing it. YES.

Concentrate! In the resistance situation you take a vertex current with support $\{x, y\}$. ~~So what do we do?~~

What is critical is that if

$$\begin{tikzcd}
0 & \mathbb{C}\{x,y\} \arrow[l] & \bar{C}^0 \arrow[l, two heads] \arrow[d, "\sim"] & K \arrow[l] \arrow[d, "\wr"] & 0 \arrow[l] \\
0 \arrow[r] & C_0(\{x,y\}) \arrow[r, hook] \arrow[u, dotted] & \bar{C}_0 \arrow[r] & \mathrm{Cok} \arrow[r] & 0
\end{tikzcd}$$

is an isomorphism, then is an isomorphism and conversely. Now it's time to do this explicitly

337. [scribbled out] Here's how

$$
\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & D \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & A \arrow[r] \arrow[dr] & B \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & & E \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}
$$

There's some understanding here needed. But basically B has two filtrations: $0 \subset A \subset B$ and $0 \subset D \subset B$ and the assumption that ~~A~~ $A \xrightarrow{\sim} B/D$ says B splits: $B = A \oplus D$ ~~as well as~~ whence $D \xrightarrow{\sim} B/A = C$. ~~But~~

I feel however that $A \to E$ and $D \to C$ are quasi-isom.

$$
\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & D \arrow[d] \arrow[r] & D \arrow[d, equal] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] & E \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \times_C D \arrow[r] \arrow[d, "f"] & D \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}
$$

Natural quasi-isom.

$$\begin{tikzcd} D \arrow[r] & C \\ A\oplus D \arrow[u, "pr_2"] \arrow[r] \arrow[d, "pr_1"] & B \arrow[u] \arrow[d] \\ A \arrow[r] & E \end{tikzcd}$$

I would like to put my finger on the basic structure of a 1-port. OKAY.

$\frac{1}{2}$ hour on Cuntz alg and Dopplicher Roberts.

V f.d. Hilbert space have $\mathcal{O}_V$. concept of a Hilbert space inside a C^*-alg, namely a subspace V such that $v^*v \in \mathbb{C}1$. In $\mathcal{O}_V$ get $\sum v_i v_i^* = 1$.

$\mathcal{O}_V$ has a certain canonical endomorphism $\sigma : \mathcal{O}_V \to \mathcal{O}_V$ ~~non unital~~ unital

Yes $M_n(\mathcal{O}_V) \simeq \mathcal{O}_V$ $\qquad$ $V\otimes\mathcal{O}_V\otimes V^* \xrightarrow{\sim} \mathcal{O}_V$

$\mathcal{O}_V \nearrow$ $\qquad$ $s_i \otimes a \otimes s_j^* \longmapsto s_i a s_j^*$

$V\otimes \mathcal{O}_V \longrightarrow \mathcal{O}_V$

$s_i \otimes a \longmapsto s_i a$

Try $\quad a \xmapsto{\theta} \sum s_i a s_i^*$

$$\theta(a)\theta(a') = \sum_{i,j} s_i a \underbrace{s_i^* s_j}_{\delta_{ij}} a' s_j^* = \sum_i s_i a a' s_i^* .$$

other description Look at repns. $\quad H^n \xrightarrow{\sim} H$

$s = (s_1 \cdots s_n) : H^{\oplus n} \longrightarrow H.$

Dec 28 $\quad$ wave equation $\partial_t^2 u + d^*du = 0 \quad u = e^{-i\omega t}\hat{u}$

$(-\omega^2 + d^*d)\hat{u} = 0$, thus you are looking at the resolvent $(\omega^2 - d^*d)^{-1}$ of the Laplacean, something like ~~potential~~ theory with a spectral parameter. ~~Life~~ For an LC network slightly different, ~~more li~~ more like a Riemann surface where $d^*d = \partial^*\partial + \bar\partial^*\bar\partial$.

 You maybe should review Hodge theory notation

$$\bar{C}^0 \xrightarrow[d'+d'']{d} C^1 = C^{1L} \oplus C^{1C}$$

You understand the principles, ~~so~~ but you ~~[illegible]~~ need to find notation + assertions. Two viewpoints, space with quadratic form depending on ω, pair of spaces in duality (cochains and chains) and an operator depending on ω between them. They become the same if you replace quadratic form by bilinear form. Why do I prefer ~~the sp~~ dual pair approach? 1) More general because no specifying symmetry, e.g. skew-symmetric bilinear form gives 0 quadratic form. (Recall $V \oplus V^*$ can be viewed as either orthog or symp. space, max isotropics are resp. skew symm. (symmetric) bilinear forms). 2) Supports – when you restrict attention to a subset chains + cochains behave differently – chain supported in the subset ~~[illegible]~~, inclusion of chain groups, surjection of cochain groups, analogous to $\Gamma_c(U,-)$ vs $\Gamma(U,-)$. Describe basic LC networks ~~situation~~. C^1 splits $C^{1L} \oplus C^{1C}$ also $C_1 = C_1^L \oplus C_1^C$. Have $(Ls)^{-1} \oplus Cs : C^{1L} \oplus C^{1C} \xrightarrow{\sim} C_1^L \oplus C_1^C$, note L^{-1} is equiv. to the quad form $\sum_{\sigma\, L\, \text{edge}} (E_\sigma^L)^2 L_\sigma^{-1}$, note both $(Ls)^{-1} \oplus Cs$ and its inverse have the form $As + Bs^{-1}$ where A, B are ~~positive~~ nonnegative quadratic forms whose sum is positive def. ~~[illegible]~~. Picture

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bar{C}^0 & \xrightarrow{d} & C^1 & \xrightarrow{\pi} & H^1 & \longrightarrow & 0 \\ & & & & Z_s^{-1} \big\downarrow S & & & & \\ 0 & \longleftarrow & \bar{C}_0 & \xleftarrow{d^t} & C_1 & \xleftarrow{\pi^t} & H_1 & \longleftarrow & 0 \end{array}$$

so the $d^t Z_s^{-1} d$ and $\pi Z_s \pi^t$ have this $As + Bs^{-1}$ form. Of main interest: Quotient space of $\bar{C}^0$, say ~~the~~ quotient line $\bar{C}^0(\{x,y\})$ two vertices. ~~[illegible]~~ General picture: $H = H^L \oplus H^C$ polarized real Hilbert space. Consider a subquotient of H say V_2/V_1 where $0 \subset V_1 \subset V_2 \subset H$. Have $As + Bs^{-1} : H \to H^*$

generically isom, moreover this bilinear form when restricted to any subspace of H is nondegenerate for generic s. Thus get induced bilinear form on any V_2/V_1. Thus you should get a nondegenerate bilinear (symmetric) form which is a rational function of s.

Question: A rational function of s has partial fraction decomposition. Are the poles simple?

Discuss details: First $As + Bs^{-1} : V \to V^*$

Let $V \overset{(a)}{\underset{(b)}{\subset}} H = H^+ \oplus H^-$ equip V with induced norm so that $\|av\|^2 + \|bv\|^2 = \|v\|^2$

i.e. $\underbrace{a^*a}_{A} + \underbrace{b^*b}_{B} = 1$ on V. Pull back F on H to $a^*a - b^*b = \alpha$. Then $sA + s^{-1}B =$

$$s\frac{1+\alpha}{2} + s^{-1}\frac{1-\alpha}{2} = \left(\frac{s+s^{-1}}{2}\right) + \left(\frac{s-s^{-1}}{2}\right)\alpha.$$

Since α is $\underset{\text{s.a.}}{\text{contr.}}$

$sA + s^{-1}B$ is singular $\Rightarrow$ $\dfrac{s+s^{-1}}{s-s^{-1}} = -\dfrac{1+s^2}{1-s^2}$ $\underset{\text{real}}{|\;| \leq 1}$ $\Rightarrow s^2 \in \mathbb{R}_{\leq 0}$

other proof $(sA + s^{-1}B)v = 0 \Rightarrow s\underbrace{(v^*Av) + s^{-1}(v^*Bv)}_{\in \mathbb{R}_{\geq 0}} = 0$

Let's prove carefully everything we can about $V \overset{\iota}{\subset} H$.

$$\begin{tikzcd}
V \arrow[r, "\iota"] \arrow[d, "\iota^t Z_s^{-1} \iota"'] & H \arrow[r, two heads, "j"] & W \\
V^* & H^* \arrow[l, "\iota^t"'] \arrow[u, "s", "Z_s"'] & W^* \arrow[l, "j^t"'] \arrow[u, "j Z_s j^t"']
\end{tikzcd}$$

$Z_s = Ls + C^{-1}s^{-1} \qquad Z_s^{-1} = L^{-1}s^{-1} + Cs$

so $\iota^t Z_s^{-1} \iota = s^{-1}(\iota^t L^{-1} \iota) + s(\iota^t C \iota)$

$j Z_s j^t = s(j L j^t) + s^{-1}(j C^{-1} j^t)$

341 I know that $\iota^t Z_s^{-1} \iota$ is invertible except for a finite subset of $i\mathbb{R}$ stable under $i\omega \mapsto -i\omega$. So the inverse $(\iota^t Z_s^{-1}\iota)^{-1}$ is a rational matrix function of with these poles. In fact these poles should be simple, maybe with non negative residues. Another interesting point is that $(\iota^t Z_s^{-1}\iota)^{-1}$ and $(j Z_s j^t)^{-1}$ have the same poles. You ~~should~~ know that the whole situation $V \subset H^+ \oplus H^-$ is a direct sum of irreducible – this is the dihedral group F, ε – so the study is straight forward.

Now consider a subquotient of H. V_2/V_1

Two viewpoints

$$\begin{tikzcd}
V_2 \arrow[r, hook] \arrow[d] & H \arrow[d] \\
V_2/V_1 \arrow[r, hook] & H/V_1
\end{tikzcd}$$

$$\begin{tikzcd}
V_1 \arrow[r, hook] \arrow[d, "\wr"] & V_2 \arrow[r, hook] \arrow[d, "\wr"] & H \arrow[d] \\
V_1^* & V_2^* \arrow[l, two heads] & H^* \arrow[l, two heads]
\end{tikzcd}$$

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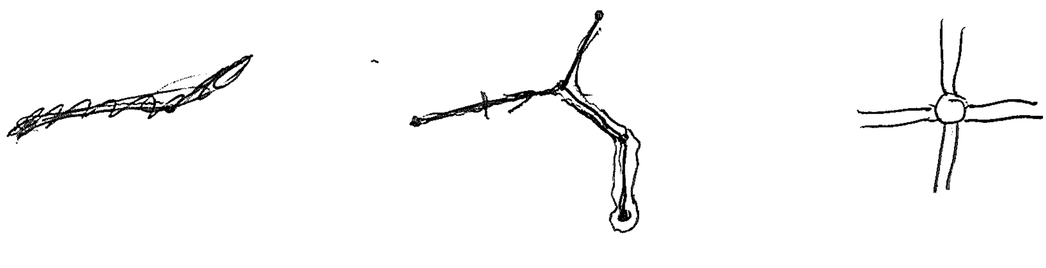
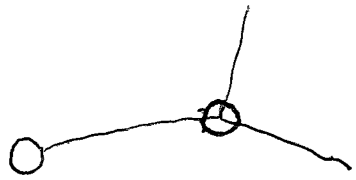
digress on ribbon graphs = graph with cyclic ordering at each vertex. Assume no valence 2 vertex.





Can you calculate the genus of the surface resulting from a ribbon graph.

Suppose the surface has genus g and $\overset{n}{\cancel{d}} \geq 1$ punctures. Then $h^0 = 1$, $h^1 = 2g - \overset{n}{\cancel{d}} + 1$, $h^2 = 0$

so $\chi = 1 - 2g + \overset{n}{\cancel{d}} - 1 = \boxed{-2g + n = v - e}$

$H = H^+ \oplus H^-$ polarized f.d Hilbert space

$V \overset{d}{\subset} H$ ~~closed~~ subspace. pull back

$s\|h_+\|^2 + s^{-1}\|h_-\|^2$

to get a sesquilinear ?

$\langle dv', \left(s\frac{1+\varepsilon}{2} + s^{-1}\frac{1-\varepsilon}{2}\right) dv \rangle$

$s\frac{1+\varepsilon}{2} + s^{-1}\frac{1-\varepsilon}{2}$

$\frac{s+s^{-1}}{2} + \frac{s-s^{-1}}{2}\varepsilon$

to simplify keep things real.

irreducibles. $V = \oplus \Gamma_T$ $\quad T: H^+ \to H^-$

can assume $H^+ = H^- = \mathbb{R}$ with $t > 0$.

$V = \mathbb{C}$. $\begin{pmatrix} \frac{1}{\sqrt{1+t^2}} \\ \frac{t}{\sqrt{1+t^2}} \end{pmatrix}^* \begin{pmatrix} \omega & 0 \\ 0 & -\omega^{-1} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+t^2}} \\ \frac{t}{\sqrt{1+t^2}} \end{pmatrix} = \frac{\omega + \omega^{-1}t^2}{1+t^2}$

It's inverse is $\frac{1+t^2}{\omega - \omega^{-1}t^2} = \frac{1}{-\omega^{-1}} + \frac{\frac{1+\omega^2}{-2\omega^{-1}\omega}}{t-\omega} + \frac{\frac{1+\omega^2}{(-2\omega^{-1})(-\omega)}}{\omega+t}$

$= -\omega + \frac{1+\omega^2}{2}\left\{\frac{+1}{\omega-t} + \frac{1}{\omega+t}\right\}$

$\frac{\overset{2\omega}{\cancel{+\omega+t} + \cancel{\omega-t}}}{\omega^2-t^2}$

$= -\omega + (1+\omega^2)\frac{\omega}{\omega^2-t^2} = \frac{1}{\omega^2-t^2}\left(-\omega(\omega^2-t^2) + \omega(1+\omega^2)\right)$

$= \frac{\omega(1+t^2)}{\omega^2-t^2}$

$V \subset H^+ \oplus H^-$ say $V = \begin{pmatrix} 1 \\ T \end{pmatrix} H^+$

then $V^\perp = \begin{pmatrix} -T^* \\ 1 \end{pmatrix} H^-$ and $F = \frac{1+X}{1-X}\varepsilon$ $X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$

basic operator on H is $\begin{pmatrix} -i\omega & 0 \\ 0 & i\omega^{-1} \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}$

The compression to V is

$$(1+T^*T)^{-1/2}\begin{pmatrix} 1 & T^* \end{pmatrix}\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}\begin{pmatrix} 1 \\ T \end{pmatrix}(1+T^*T)^{-1/2}$$

$$= \frac{s + s^{-1}T^*T}{1+T^*T}$$

I want the inverse $\frac{1+T^*T}{s+s^{-1}T^*T}$, split H^+ up into eigenspaces wrt T^*T, get the inverse expressed as sum of $\frac{1+\omega^2}{s+s^{-1}\omega^2}$ $0 \le t^2 < \infty$. We have split V into orthogonal lines on which the Greens operator has form $\frac{1+\omega^2}{s+s^{-1}\omega^2}$ – this is the inverse to $d^t Z_s^{-1} d$.

---

s is the variable here, the roots of denom are $s^2 = -\omega^2$ $s = \pm i\omega$.

$$\frac{s(1+\omega^2)}{s^2+\omega^2} = \left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right)\frac{1+\omega^2}{2}$$

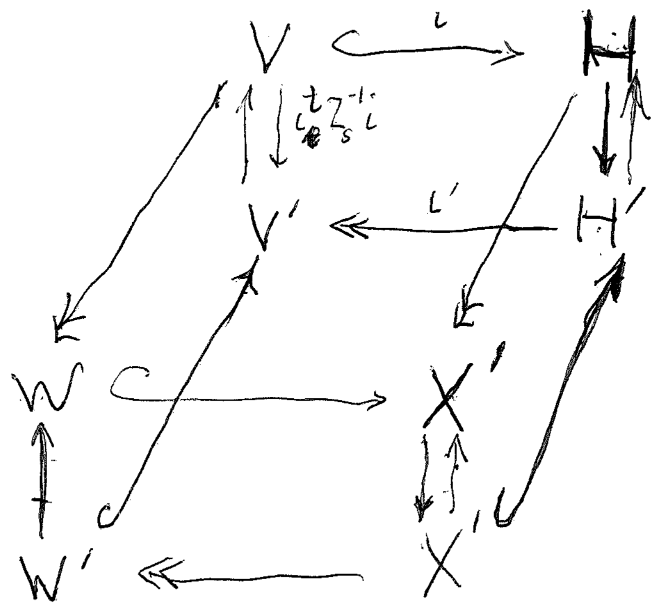
if $\omega = 0$ get $\frac{1}{s}$, if $\omega = \infty$ get s

---

So what does it mean to say that on a Hilbert space W we have an operator Z_s which is a rational function of s with simple poles whose residues are non negative?

Program: Take dim$(W) = 1$ first & completely understand then dim$(W) = 2$.

What sort of result would you like?



[redacted]

$B(v_1, v_2) = \langle Tv_1 | v_2 \rangle$

Given $V \xrightarrow[\sim]{T} V'$ nondegenerate bilinear form

Let $W \subset V$ be a subspace such that B rest. to W is nondeg. Meaning:

$$\begin{tikzcd} W \arrow[r, hook, "\iota"] \arrow[d, "s"] & V \arrow[r, "j"] \arrow[d, "T"] & V/ \arrow[d, dotted, "jT^{-1}j^t"] \\ W' & V' \arrow[l, two heads, "\iota^t"] & W^\perp \arrow[l, hook', "j^t"] \end{tikzcd}$$

$\iota^t T \iota$ invertible

Then I know $jT^{-1}j^t$ inv.

$T(W)$ is a complement to $W^\perp$

and $T^{-1}(W^\perp)$ ———— W

What's the general framework? It might be true that the appropriate class of rational matrix functions is stable under inverses. need to understand better. Go over. $V \subset H^+ \oplus H^-$

$F = +1$ on V $\quad F = -1$ on $V^\perp$. $\quad g = F\varepsilon$. Consider the case $g = \pm 1$. $\quad g = +1$ means $F = \varepsilon$, i.e. $V \subset H^+$. $V^\perp \subset H^-$

In this case $i^*\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} i = s$, $g = -1$ means $F = -\varepsilon$ i.e. $V = H^-$, $V^\perp = H^+$. Then $i^*\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} i = s^{-1}$.

What happens in general is that the situation splits into $H^+ \cap V \oplus H^- \cap V \oplus \Gamma_T$

345 So $V \xrightarrow{(a,b)} H^+, H^-$ $\qquad a^*a + b^*b = 1$

$\mathrm{Ker}(b) = V \cap H^+ \qquad \mathrm{Ker}(a) = V \cap H^-$. If you remove these subspaces that.

Wait Given F, ε look at the $g = 1$ eigenspace K stable under ε, so its $K = K^+ \oplus K^-$ where

K^+ is where $F = \varepsilon = 1$ i.e. $H^+ \cap V$

K^- ——— $F = \varepsilon = -1$ i.e. $H^- \cap V^\perp$

You are confused. Start again with $H = H^+ \oplus H^-$ a polarized Hilbert space. Let $V \subset H$, F be the inv. $= +1$ on V, -1 on $V^\perp$, $g = F\varepsilon$. $\varepsilon g \varepsilon^{-1} = g^{-1}$ so eigenvalues of g closed under $\zeta \mapsto \bar\zeta = \zeta^{-1}$. Poss. are $-1 \le \cos\theta \le +1$.

Look at ~~cos w~~ case where eigenvalues are $e^{\pm i\theta}$ $\quad 0 < \theta < \pi$.

Then H splits ~~[illegible]~~ into $e^{i\theta}$ eigenspace and $e^{-i\theta}$ eigenspace

~~Let $W = \{h \mid F\varepsilon h = e^{i\theta} h\}$~~

$g = \frac{1+X}{1-X} = -1 + \frac{2}{1-X}$ $\qquad \cancel{\varepsilon h = e^{i\theta} F h} \qquad \cancel{Fh} = e^{-i\theta}\varepsilon h$

$g + 1 = \frac{2}{1-X}$ $\qquad \cancel{F(h_+, h_-) = e^{-i\theta} h_+, e}$

$g - 1 = \frac{2X}{1-X}$

~~[illegible]~~ Assume ~~[illegible]~~ $g+1$ non-singular

then $X = \frac{g-1}{g+1}$ is skew adjoint $X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$.

and $V = \Gamma_T =$ [redacted] $\begin{pmatrix} 1 \\ T \end{pmatrix} H^+ \qquad V^\perp = \begin{pmatrix} -T^* \\ 1 \end{pmatrix} H^-$

You're interested in the operator bilinear form.

$$\langle v, s v_+ + s^{-1} v_- \rangle = s\|v_+\|^2 + s^{-1}\|v_-\|^2$$

on V.

$\cancel{v = \begin{pmatrix} 1 \\ T \end{pmatrix} (1 + T^*T)^{-1/2} h_+}$ $\qquad v = \begin{pmatrix} 1 \\ T \end{pmatrix} h_+$

$\cancel{s v_+ + s^{-1} v_- = \begin{pmatrix} s \\ s^{-1} T \end{pmatrix} (1 + T^*T)^{-1/2} h_+}$ $\qquad (v, s v_+ + s^{-1} v_-)$

$\cancel{(v, s v_+ + s^{-1} v_-) = h_+, (1 + T^*T)^{-1/2} s}$ $\qquad = (h_+, s h_+) + (T h_+, s^{-1} T h_+)$

$= (h_+, (s + s^{-1} T^*T) h_+) \quad = s\|h_+\|^2 + s^{-1}\|T h_+\|^2$

Back to maps from V to V^*

Let V be a Hilbert space. Consider operators on V depending on a parameter s which arise the following way.

Consider
$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r, "i"] \arrow[d] & H^+ \oplus H^- \arrow[r, "j"] \arrow[d, "\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}"] & V^\perp \arrow[r] & 0 \\
0 \arrow[r] & V & H^+ \oplus H \arrow[l, "i^*"'] & V^\perp \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

$$i = \begin{pmatrix} 1 \\ T \end{pmatrix} (1+T^*T)^{-1/2} \qquad \text{if} \quad V = \Gamma_T$$

$$(1+T^*T)^{-1/2} \begin{pmatrix} 1 \\ T \end{pmatrix}^* \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ T \end{pmatrix} (1+T^*T)^{-1/2}$$

$$= \frac{s + s^{-1}T^*T}{1+T^*T}$$

Dec 29.

Let H be a Hilbert space with polarization ε, let $W \subset V$ be subspaces. On H you have a 1-parameter family of sesquilinear forms $\langle \xi', s\xi_+ + s^{-1}\xi_- \rangle$. operator $s\frac{1+\varepsilon}{2} + s^{-1}\frac{1-\varepsilon}{2} = \frac{s+s^{-1}}{2} + \frac{s-s^{-1}}{2}\varepsilon$ family of invertible operators rational function of s, singular at $s=0,\infty$.

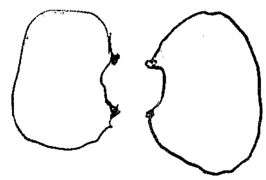
Forget the Hilbert stuff and concentrate on the algebra. You start with a vector space V and a rational family of linear ops $sa+s^{-1}b : V \to V'$ whose inverse has the same form. Focus on a family of maps T_s from H to H' generically invertible rational family of non degenerate bilinear forms on H.

347 Start again. You need to check your conclusions yesterday. You start with $H = H^+ \oplus H^-$ and a bilinear form on H, i.e. map $H \xrightarrow{T_s} H^*$ depending rationally on s and generically ~~nondeg~~ invertible. ~~Bu~~
Again. NO

You should stick to your example LC circuit - not Hilbert spaces. For networks the ~~...~~ basic spaces are of the form $V \oplus V^*$ V real V is the space of potentials V^* the space of currents and there is a natural power pairing. Important case: ~~a 1~~ a 1-port. This is an LC network with an ordered pair of vertices given. Response function Apply voltage $\mathrm{Re}(E_\omega e^{-i\omega t})$ get current $\mathrm{Re}(I_\omega e^{-i\omega t})$, response is $\frac{E_\omega}{I_\omega} = Z_\omega$

Properties: rational function of ω purely imaginary for ω real etc. Actually Z_ω is what? It is a complex line in the complexification of $V \oplus V^*$. Leads to a real rational function $i^{-i}Z_\omega$ of ω, which maps UHP into the UHP. This means $V \oplus V^*$ should be viewed as symplectic + not orthogonal.

Connecting two 1-ports.



~~...~~ There are two ways to connect, corresponds to signs of current, you need to identify the voltage spaces.

oscillator - an LC circuit is a harmonic oscillator whose configuration space is $B^1 \oplus Z_1$ and whose phase space is $C^1 \oplus C_1$. Put $W = B^1 \oplus Z_1$. This is maximal isotropic in $V = C^1 \oplus C_1$. Is there a natural complement? You want to choose complements for B^1 in C^1 and Z_1 in C_1. Somehow

$$0 \to B^1 \to C^1 \to H^1 \to 0$$

$$C_0 \leftarrow C_1 \leftarrow H_1$$

Wait: This is not clear because ?

348 An LC circuit should be a harmonic NO oscillator ~~somehow~~, although it is not as obvious as I thought. ~~[illegible]~~ One has ~~[illegible]~~ $C^1 \oplus C_1$ with natural symplectic structure and $W = B^1 \oplus Z_1$ is a natural maximal isotropic subspace. ~~[illegible]~~ Is there a natural pos. def quadratic form on $C^1 \oplus C_1$? Apparently one has a degenerate oscillator ~~[illegible]~~ with phase space $C^1 \oplus C_1$. Look at ~~[illegible]~~ an edge say $E = L\dot{I}$ flow on $(E,C) \in \mathbb{R} \oplus \mathbb{R}$

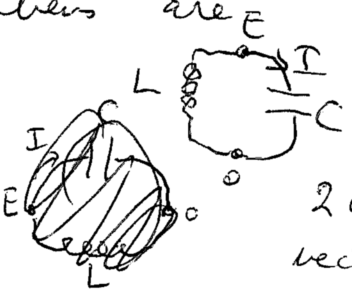
~~[illegible]~~ $H(E,C) = EC$. $\qquad H(q,p) = \frac{1}{2}p^2$

$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q} \qquad \dot{q} = p \qquad \dot{p} = 0.$ So

on $C^1 \oplus C_1$ you have the degenerate form.

$$\frac{1}{2}\sum_{ind} E_\sigma^2 L_\sigma^{-1} \quad \frac{1}{2}\sum_{cap} I_\sigma^2 C_\sigma^{-1} = H(E_\sigma, I_\sigma)$$

except the equations are $E = L\dot{I}$, $I = C\dot{E}$ so there are sign problems.



Example 2 dim'l ~~[illegible]~~ state space $(E,I) \in \mathbb{R}^2$ vector field $\dot{I} = CE \quad \dot{E} = -LI$

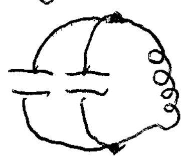
~~[illegible]~~ What's the induced flow on $\Lambda^2\mathbb{R} = \mathbb{R}$ $\quad (E_1,I_1)\wedge(E_2,I_2) = E_1 I_2 - E_2 I_1$

$$\frac{d}{dt}\left(E_1 I_2 - E_2 I_1\right) = -\cancel{L I_1 I_2} + \cancel{E_1 C E_2} + \cancel{L I_2 I_1} - \cancel{E_2 C E_1}$$
$$= 0$$

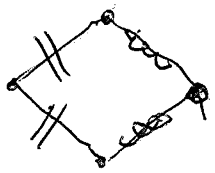
~~Energy~~ $\qquad \frac{1}{2}CE^2 + \frac{1}{2}I^2 L = H(E,I)$

$$\dot{I} = \frac{\partial H}{\partial E} = CE \quad \Big| \quad -\frac{\partial H}{\partial I} = -IL = \dot{E}$$

349 Still very confused. But I know now that a general LC circuit is not a harmonic oscillator. Because state space $B^1 \oplus Z_1$ has dim $v-1+l = e$ can be odd. Does this imply that you have a mode of frequency 0 or ∞?

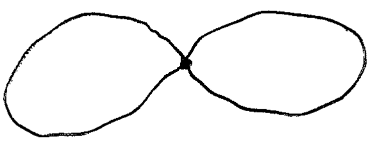


Set $\omega = 0$



OKAY so we learn that it's not good to think of an LC network as a harmonic oscillator, because of $\omega = 0, \infty$ modes. At $\omega = 0$, L edges become wires and C edges are removed. Possible voltages are $\tilde{H}^0$ of resulting graph = functions constant on each component and possible currents are H_1 of resulting graph. At $\omega = \infty$ L edges are removed and C-edges become wires.

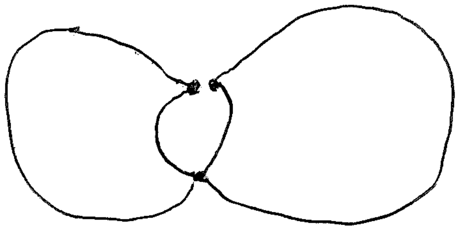
How about gluing networks.



A connected sum of has same state space as the disjoint union

note a 0-cochain on $X \vee Y$ mod constants is same as 0 coch. vanishing at basepoint $\therefore$ $\tilde{C}^0(X \vee Y) = \tilde{C}^0(X) \oplus \tilde{C}^0(Y)$.

Next two other vertices.



Better: take a connected network and identify two vertices. Then $\tilde{C}^0$ changes into the codim 1 subspace of potentials with equal values at the two points, and there is an extra 1-cycle rep. by a path joining the two pts.

What can I say about a tree? $Z_1 = 0$

so $\bar{C}^0 \xrightarrow{\sim} C^1$ and the basis response $\xi \longmapsto \partial Z_s^{-1} d\xi$ is just Z_s^{-1}. There are only modes for $\omega = 0$ and $\omega = \infty$.

edge with one vertex:  $=$ ~~has no effect on the modes since~~ for $\omega = 0$ no current
$\omega = \infty$ yes current.

Back to yesterday's analysis. Go over until clean.

You have LC network connected.

$$\begin{tikzcd} 0 \arrow[r] & \bar{C}^0 \arrow[r, "d"] & C^1 \arrow[r] \arrow[d, shift left, "Z_s^{-1}"] & H^1 \arrow[r] & 0 \\ 0 & \bar{C}_0 \arrow[l] & C_1 \arrow[l, "\partial = d^t"'] \arrow[u, shift left, "Z_s"] & H_1 \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

$$(Z_s I)_\sigma = L_\sigma s I_\sigma \quad \text{or} \quad (C_\sigma s)^{-1} I_\sigma$$

Get $\partial Z_s^{-1} d : \bar{C}^0 \longrightarrow \bar{C}_0$ invertible for generic s.

It has the form $s^{-1} \partial_L L^{-1} d_L + s\, \partial_C C d_C$. Its inverse is a rational function of s which we claim has a special form, namely simple poles purely imag. This rational matrix function has a partial fraction expansion, which I now describe.

Recall C^1 splits $C^{1,L} \oplus C^{1,C}$, so $d\bar{C}^0 = B^1$ is a subspace of $C^{1,L} \oplus C^{1,C}$. $C^{1,L}$ has inner product

$$(\xi', \xi)_L = \langle \xi', L^{-1}\xi \rangle = \sum_{\sigma\ L\text{-edge}} \xi'_\sigma L_\sigma^{-1} \xi_\sigma$$

sim.

$$(\xi', \xi)_C = \sum \xi'_\sigma C_\sigma^{-1} \xi_\sigma$$

pos. def, so can add to make $C^1 = C^{1,L} \oplus C^{1,C}$ into a Real polarized Hilb. space. $\bar{C}^0$ inherits an ~~becomes a~~

inner product via d namely $(\xi', \xi) =$

$$(d\xi', d\xi) = (d_L\xi', d_L\xi) + (d_C\xi', d_C\xi)$$

$$= \langle d_L\xi', L^{-1} d_L \xi\rangle + \langle d_C\xi', C d_C \xi\rangle$$

$$= \langle \xi', (\partial L^{-1} d_L + \partial C d_L)\xi \rangle.$$

Notation terrible. You have splitting $C^1 = C^{1L} \oplus C^{1C}$ a corresp splitting of the dual space $C_1 = C_1^L \oplus C_1^C$ and $Z_s^{-1} : C^1 \to C_1$ is the direct sum $s^{-1}L^{-1} \oplus sC$.

Repeat.

$$\begin{array}{ccc} \bar{C}^0 & \xrightarrow{d} & C^1 = C^{1L} \oplus C^{1C} \\ & & \Big\downarrow (Ls)^{-1} \oplus Cs = Z_s^{-1} \\ \bar{C}_0 & \xleftarrow{d^t} & C_1 = C_1^L \oplus C_1^C \end{array}$$

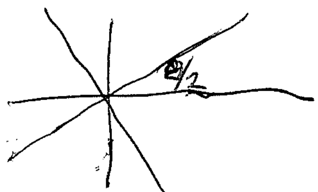
so you have $d^t Z_s^{-1} d : \bar{C}^0 \to \bar{C}_0 = (\bar{C}^0)^*$. But let me use $Z_1^{-1} = L^{-1} \oplus C$ to ~~define~~ define inner prod $\|\ \|^2$ on C^1 and then take $\|d\xi\|^2$ to be the inner prod on $\bar{C}^0$ $= (\xi, d^* d \xi)$ then $d^* d = 1$ because d is an isometry. ~~[illegible]~~ Identifying $\bar{C}^0 \simeq \bar{C}_0$ via this inner product we get $d^t = d^*$. Then $\bar{C}^0$ is a Hilbert space and $d^t Z_s^{-1} d = s^{-1} d_L^* d_L + s\, d_C^* d_C$.

Get natural ≥ 0 s.a. ops. $d_L^* d_L$, $d_C^* d_C$.

The logic here is that given a subspace V of a polarized Hilbert space $H^+ \oplus H^-$, then this situation admits a spectral decomposition

OKAY. Next. On V have quad. forms $A, B \geq 0$ $\ni A + B > 0$. Consider $sA + s^{-1}B$

Actually it should be no surprise that the inverses you need have a partial fraction representation, so all this stuff should be basically trivial. So let's go over it until it becomes clear. So begin with V subspace of $H^+ \oplus H^-$, let $F = \pm 1$ on V resp $V^\perp$, get representation of $\langle F, \varepsilon \rangle = \varepsilon \ltimes \mathbb{Z}$, split H up into irred repns of this dihedral group. Picture of an irred



$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad g = \begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix} \qquad g = F\varepsilon$$

$$F = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \qquad \frac{1+F}{2} = \begin{pmatrix} \cos^2\frac{\theta}{2} & \sin\theta/2 \cos\theta/2 \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\theta/2 \end{pmatrix}$$

eigen vector is $\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}$ $\qquad X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$

alternative $\quad V = \Gamma_T = \begin{pmatrix} 1 \\ T \end{pmatrix} H^+$, then

$$F = \frac{1+X}{1-X}\varepsilon = \frac{1+X^2+2X}{1-X^2}\varepsilon = \begin{pmatrix} \frac{1-T^*T}{1+T^*T} & \frac{+2T^*}{1+TT^*} \\ \frac{2T}{1+T^*T} & -\frac{(1-TT^*)}{1+TT^*} \end{pmatrix}$$

How does this help? What should happen?

Go back to $V \overset{d}{\subset} H^+ \oplus H^- = H$ and put operator $s\left(\frac{1+\varepsilon}{2}\right) + s^{-1}\left(\frac{1-\varepsilon}{2}\right)$ on H. Then get $s\, d_+^* d_+ + s^{-1} d_-^* d_- = T_s$

$d = \begin{pmatrix} d_+ \\ d_- \end{pmatrix} : V \longrightarrow \begin{matrix} H^+ \\ \oplus \\ H^- \end{matrix}$ $\qquad$ So what do I know about T_s^{-1}?

e.g. if $V = \Gamma_T$ $\qquad d_+ = (1+T^*T)^{-1/2}$ $\qquad$ Then
$\qquad d_- = T(1+T^*T)^{-1/2}$

$$s\, d_+^* d_+ + s^{-1} d_-^* d_- = \frac{s + s^{-1}T^*T}{1+T^*T} \quad \text{so}$$

$$\left(s d_+^* d_+ + s^{-1} d_-^* d_-\right)^{-1} = \frac{1+T^*T}{s + s^{-1}T^*T} = \frac{s(1+T^*T)}{s^2+T^*T}$$

is nicely direct sum of 2-dim ops. $\qquad \dfrac{s(1+\omega^2)}{s^2+\omega^2}$

for $\omega > 0$. $\qquad = \left(\dfrac{1+\omega^2}{2}\right)\left(\dfrac{1}{s-i\omega} + \dfrac{1}{s+i\omega}\right)$

353 Dec 30. Review. $H = H^+ \oplus H^-$ $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ a polarized Hilbert space $V \subset H^+ \oplus H^-$ a closed subspace, Let $F = \pm 1$ on $V, V^\perp$ resp. ~~Then~~ $g = F\varepsilon$ is unitary and $\varepsilon g \varepsilon^{-1} = g^{-1}$. Split H into $g = -1$ eigenspace and ~~perpendicular~~ orth complement. ~~Examine $g=-1$~~

Better: let $d_+, d_- : V \longrightarrow H^\pm$ be the proj so that $\|v\|^2 = \|d_+ v\|^2 + \|d_- v\|^2$ or $1 = d_+^* d_+ + d_-^* d_-$.

Examine $s d_+^* d_+ + s^{-1} d_-^* d_-$. This is invertible for $Re(s) \neq 0$, since $Re\left(v, (s d_+^* d_+ + s^{-1} d_-^* d_-)v\right) = Re(s)\|d_+ v\|^2 + Re(s^{-1})\|d_- v\|^2 \geq \varepsilon \|v\|^2$ with $\varepsilon > 0$. ~~Better: Stick to finite dim. first~~

Study inverse, a kind of resolvent.

Split H into -1 eigenspace W of $g = F\varepsilon$ and orth. comp.

$g = -1 \Rightarrow F = -\varepsilon$ so $W = V \cap H^- \oplus V^\perp \cap H^+$

Examine $\varepsilon = 1, F = -1$ i.e. $V^\perp \cap H^+$. Here have situation where $F = -1$ so $V = 0$, nothing to do other case $\varepsilon = -1, F = 1$, here $V \subset H^-$ so $s d_+^* d_+ + s^{-1} d_-^* d_- = s^{-1}$.

Next assume $\operatorname{Ker}(g+1) \neq 0$, i.e. $\underline{V \cap H^- = 0}$ and $\underline{V^\perp \cap H^+ = 0}$

$d_+ : V \longrightarrow H^+$ is injective, so $V = \begin{pmatrix} 1 \\ T \end{pmatrix} \mathcal{D}_T$ is the graph of a ~~partially~~ densely defined op T. But in f.d. $\mathcal{D}_T = H^+$. $H^+ \xrightarrow{\sim} V$ $\begin{pmatrix} 1 \\ T \end{pmatrix}(1+T^*T)^{-1/2}$

$$d_+ = (1+T^*T)^{-1/2} \qquad d_- = T(1+T^*T)^{-1/2}$$

$$s d_+^* d_+ + s^{-1} d_-^* d_- = \frac{s + s^{-1} T^*T}{1 + T^*T}.$$

Alternative would be to use $\rho = d_+^* d_+$, get $s\rho + s^{-1}(1-\rho)$. But T is relevant, because the eigenvalues are of the form $s = \mp i\omega$ ω char. value of T. So what is going on? ~~[illegible]~~

Point: $X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$ has eigenvalues $\pm i\omega$ with $\omega \geq 0$. ~~Put together that~~

$$\frac{1+\omega^2}{s + s^{-1}\omega^2} = \left(\frac{1+\omega^2}{2}\right)\left(\frac{1}{s - i\omega} + \frac{1}{s + i\omega}\right)$$

 Recap – You have redone what you did before with ~~quadratic for~~ maps $V \to V'$. Before you used non-degenerate bilinear forms, i.e. isom. $T: V \to V'$, and you became involved with filtrations of V and the corresp. filtration of V'. The basic statement should be that if I give $sa + s^{-1}b : H \to H'$ satisfying suitable conditions, then ~~on~~ any subquotient V/W of H there's an induced family T_s such that T_s and T_s^{-1} are rational in s with simple poles and non negative residues.

Now all this should look simpler in the Hilbert space setup. But first you need to check things carefully to avoid a mistake.

So consider $W \subset V \subset H = H^+ \oplus H^-$

$$s\left(\frac{1+\varepsilon}{2}\right) + s^{-1}\left(\frac{1-\varepsilon}{2}\right)$$

So want do you ~~want~~ to do?

~~First take something~~ You have to be careful in the Hilbert space picture, since ~~it seems~~ you must distinguish between the ~~subquotient~~ V/W of H and the orthogonal complement $V \ominus W$.

First check what happens in the duality situation.

How to go. Check argument before trying to find a new one. Claim that the bilinear form $T_s : H \xrightarrow{\sim} H'$ induces a form on V/W in two ways. Wait forget s. ~~You have $T: H \xrightarrow{\sim} H'$~~

$$\begin{array}{ccc} V & \xrightarrow{i} & H \\ {\scriptstyle j}\downarrow & & \downarrow \\ V/W & \hookrightarrow & H/W \end{array}$$

$T: H \xrightarrow{\sim} H'$ yields $i^t T i : V \to V'$ which we assume invertible. then get $j(i^t T i)^{-1} j^t : V/W \to (V/W)'$

$$\begin{tikzcd}
V \arrow[rrr, hook, "\iota"] \arrow[dr, "\iota^t T \iota"] \arrow[ddd, "j"'] & & & H \arrow[dl, "T"'] \arrow[ddd, "j_1"] \\
& V' & H' \arrow[l, "\iota^t"'] & \\
& (V/W)' \arrow[u, "j^t"] \arrow[dl, shift left] & (H/W)' \arrow[l, "\iota_1^t"] \arrow[u, "j_1^t"'] \arrow[dr, "j_1 T^{-1} j_1^t"] & \\
V/W \arrow[ur, shift left] \arrow[rrr, hook, "\iota_1"'] & & & H/W
\end{tikzcd}$$

I want to compare $j(\iota^t T \iota)^{-1} j^t$ and $\iota_1^t (j_1 T^{-1} j_1^t)^{-1} \iota_1$

The claim is that these are inverse because they both describe the non degenerate bilinear form on V/W induced by T on H. Given $T : H \to H'$, the induced form on V is given by $\iota^t T \iota : V \to V'$, the induced form on H/W ________ $j_1 T^{-1} j_1^t : (H/W)' \to (H/W)$, then to get the induced form on V/W you can use either

$j(\iota^t T \iota)^{-1} j^t : (V/W)' \to V/W$ or

$\iota_1^t (j_1 T^{-1} j_1^t)^{-1} \iota_1 : (V/W) \to (V/W)'$

$\iota_1^t (j_1 T^{-1} j_1^t)^{-1} \underbrace{\iota_1 j}_{j_1 \iota} (\iota^t T \iota)^{-1} j^t$ doesn't work.

What should work is to use the fact that there is a canonical splitting of H such that all maps are diagonal,

$$H = \underbrace{W \oplus W_1}_{V} \oplus W_2$$

$$H' = W' \oplus W_1' \oplus W_2'$$

 So now what? What am I doing? Still caught between Hilbert space and quadratic form viewpoints. Point 1. Main tool is nondegeneracy

Discuss what to do. First understand LC networks completely, avoid trying to generalize. Try to find the right category, class of objects arising from LC networks. First they are real vector spaces, in fact, probably even integral - No, L_i C_i are arb real nos. Real spaces carrying a quadratic form depending on s rationally and nondegenerate. Partial fraction decomposition - ask for simple poles $s \in i\mathbb{R} \cup \infty$ symm. under $-$. residues ≥ 0. Thus have

$$\sum_\omega$$

Try the converse. Suppose we have a real vector space X together with a quadratic form depending rationally on s having a partial fraction exp. sum of terms

$$A_\infty s \ , \ A_0 s^{-1}, \ A_\omega \frac{(1+\omega^2)/2}{s^2+\omega^2} \quad 0<\omega<\infty$$

where A_∞, A_0, A_ω are nonnegative quadratic forms on X.

Assume $$A_\infty + A_0 + \sum_{0<\omega<\infty} A_\omega > 0$$

Then you can find Euclidean spaces V_ω and maps $f_\omega : X \longrightarrow V_\omega$ such that $f_\omega^* f_\omega = A_\omega$

Then go back to your quotient of H.

---

given $\xi \in V^\perp$ to find $v \in V$ such that $T_s(\xi + v) \in V^\perp$
$T_s(\xi) + T_s(v) \in V^\perp$. So you proje $T_s(V)$ onto V and

So it seems we are consider a real v.s. V equipped with a sequence of non negative quadratic forms A_ω, $0 \le \omega \le \infty$ whose sum is positive definite ~~nondegenerate~~. To this we associate the quadratic form

$$\sum_{0 \le \omega \le \infty} A_\omega \frac{(1+\omega^2)s}{s^2+\omega^2}$$

depending rationally on s. Note that this form has positive definite imaginary part for $Re(s) > 0$.
negative ———————— $Re(s) < 0$.

Now let's see if we can dilate in a reverse subquotient way to a polarized Hilbert space.

Let's go backwards. $0 \to V \to \overbrace{H^+ \oplus H^-}^{H} \to V^\perp \to 0$

Take quadratic form $s\|h_+\|^2 + s^{-1}\|h_-\|^2$ on H.
~~This is nondegen~~ Key $s > 0$, so this form is positive def and so is nondegenerate when restricted to any subspace. Let's calculate the induced quadratic form on the quotient space $H/V \xrightarrow{\sim} V^\perp$.

Take $\xi \in V^\perp$ and minimize $s\|h_+\|^2 + s^{-1}\|h_-\|^2$ on the coset $\xi + V$. $(h, (sp_+ + s^{-1}p_-)h)$. ~~So proceed~~

Here $(\ ,\ )$ is the pos. def scalar prod on H.

Let $T_s = sp_+ + s^{-1}p_-$ ~~Then if $(\xi+v, T_s(\xi+v))$~~

Point is that $(h, T_s h)$ has a unique minimum on the coset $\xi + V$. Suppose h is the min. pt. Then $\forall v \in V$

$$(h+v, T_s(h+v)) = \underbrace{(v, T_s h) + (h, T_s v)}_{0} + O(v^2)$$

$$\| $$

$$2(v, T_s h).$$

So the min occurs when $T_s h \in V^\perp$. ~~So your form~~ What are you asking?

358 ~~So~~ given

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r, "\iota"] \arrow[d] & H \arrow[r, "j"] \arrow[d, "s"', "T_s"] & V^\perp \arrow[r] & 0 \\ 0 & V \arrow[l] & H \arrow[l, "\iota^*"'] & V^\perp \arrow[l, "j^*"'] & 0 \arrow[l] \end{tikzcd}$$

$\xi \in V^\perp$

Start with $j^*\xi \in V^\perp$, seek $j^*\xi + \iota v$ such that $T_s(j^*\xi + \iota v) \in V^\perp$

i.e. $\iota^* T_s(j^*\xi + \iota v) = 0$, or $\iota^* T_s j^*\xi + \iota^* T_s \iota v = 0$

Thus the formula is $\xi \longmapsto$

$(j^*\xi + \iota v, T_s(j^*\xi + \iota v))$ v min. pt. means $T_s(j^*\xi + \iota v) \in j^* V^\perp$

or $\iota^* T_s j^* \xi + \iota^* T_s \iota v = 0$

so $v = -(\iota^* T_s \iota)^{-1} \iota^* T_s j^* \xi$

and $\boxed{j^*\xi + \iota v = j^*\xi - \iota(\iota^* T_s \iota)^{-1} \iota^* T_s j^* \xi}$

$(j^*\xi + \iota v, T_s(j^*\xi + \iota v)) =$

$\langle j^*\xi + \iota v, T_s j^*\xi - T_s \iota (\iota^* T_s \iota)^{-1} \iota^* T_s j^* \xi \rangle$

$(\xi, \underbrace{j T_s j^* \xi - j T_s \iota (\underbrace{\iota^* T_s \iota})^{-1} \iota^* T_s j^* \xi})$ not clear.

This is the formula for the quad form on $V^\perp$. It's hard to see ~~...~~ partial frac decomp.

359 So what next? Reconstruction. Maybe this will clarify things. First start with V/W of dim 1. Then you have a rational function determined up to squares

$$\sum_{0 \le \omega \le \infty} a_\omega \frac{(1+\omega^2)s}{s^2+\omega^2} \quad \text{where } a_\omega > 0.$$

If you expect this to arise from a subquotient line of $H^+ \oplus H^-$, then you know something about the size of V.

Let's go. $V \xrightarrow{\binom{a}{b}} H^+ \oplus H^-$ $\qquad a^*a + b^*b = 1$.

$$T_s = s a^*a + s^{-1} b^*b$$

Use spectral theorem for a^*a to split V into lines of various slope. What's happening in the end is ~~a spectral decomposition~~ that you get for T_s^{-1} a pf. exp.

$$\sum_{0 \le \omega \le \infty} A_\omega \frac{(1+\omega^2)s}{s^2+\omega^2}$$

where the A_ω is an orthogonal decomp of V. Precisely the A_ω are quad. forms on V ≥ 0, $\sum A_\omega > 0$ etc. So if I ~~start~~ with T_s, say with V/W of dim 1, then I get a V of dimension n with a linear functional Ker W.

Suppose given $\sum_{0 \le \omega \le \infty} A_\omega \frac{(1+\omega^2)s}{s^2+\omega^2}$ $\qquad A_\omega \ge 0 \quad \sum_\omega A_\omega > 0$

Somehow you need to separate the A_ω, take $\sqrt{A_\omega}$. Point A_ω has a support which is a subspace V_ω. Then take $\oplus V_\omega = V$, and you get a minimal choice for V. Then have to double except at $0, \infty$. So now what? What happens with $\omega = 0, \infty$. $V \subset H^+ \oplus H^-$ $\qquad g = F\varepsilon \qquad g = -1$

then $V = \underbrace{(H^+ \cap V^+)}_{} \oplus \underbrace{(H^- \cap V)}_{\text{should contr.}}$ $\qquad V \longrightarrow H^+ \oplus H^-$

$s^{-1}\|v_-\|^2$. ? $\qquad$ need inverse op. for pf. dec.

$V \subset H^+ \oplus H^-$ $\qquad V = \binom{1}{1} H^+$

$$\frac{s + s^{-1}T^*T}{1 + T^*T} \qquad \frac{(1+T^*T)s}{s^2 + T^*T}$$

~~Try following: $V \subset H^+ \oplus H^- \to V^\perp$ NO~~

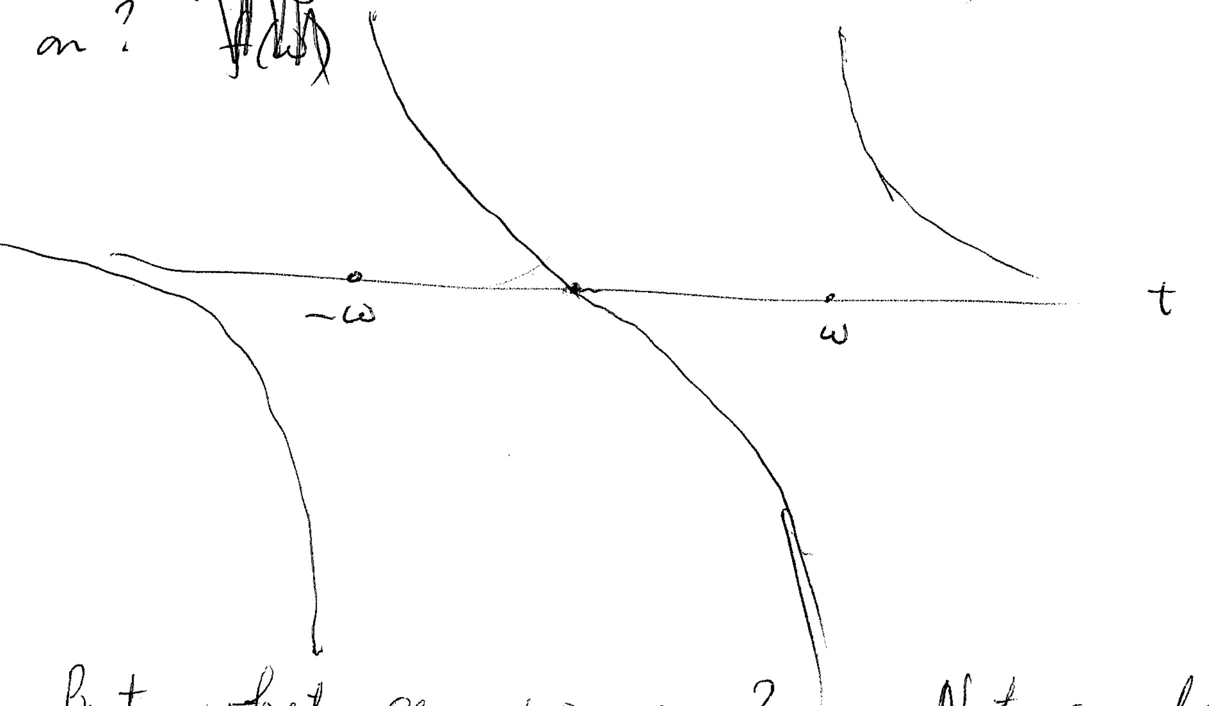
So it remains to write this up in a convenient form, but basically I think you understand pretty well. ~~Study 2-ports next~~

Study 1-port. real rational function $\sum_{0 \le \omega \le \infty} a_\omega \frac{(1+\omega^2)s}{s^2+\omega^2} = f(s)$ where $a_\omega \geq 0$ and $\sum a_\omega = 1$.

Now plot this $s = -it$ $\quad a_\omega \frac{1+\omega^2}{2}\left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right)$

$$a_\omega \frac{1+\omega^2}{2}\left(\frac{1}{-it-i\omega} + \frac{1}{-it+i\omega}\right) = i a_\omega \frac{1+\omega^2}{2}\left(\frac{1}{t+\omega} + \frac{1}{t-\omega}\right)$$

Look at $\operatorname{Im} f(-it)$. What going on?



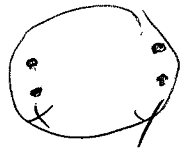
But what can you say? Not a lot about the actual zeros.

a

So basically you end up with response function

$$\sum A_\omega \frac{(1+\omega^2)s}{s^2+\omega^2} \qquad A_\omega \geq 0 \qquad \sum A_\omega = 1$$

361 Now look ~~at~~ a 2-port.



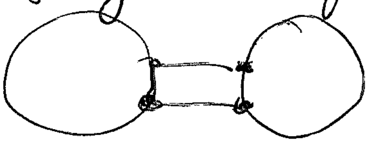
LC circuit with 4 vertices
2 terminals. Restricting to ~~two~~
$\bar{C}^0 \longrightarrow\!\!\!\!\rightarrow \bar{C}^0(X) \times \bar{C}^0(Y)$. Here's one idea - Wick rotation? Real s from Purely imag.

Let's examine mapping properties!! ~~What next?~~ Go back to $f(s) = \sum a_\omega \underbrace{\frac{(1+\omega^2)s}{s^2+\omega^2}}$ $\quad a_\omega \geq 0$, $\sum a_\omega = 1$.

$$\left(\frac{1+\omega^2}{2}\right)\left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right) \quad \begin{matrix} \nearrow s^{-1} & \omega = 0 \\ \searrow s & \omega = \infty. \end{matrix}$$

meromorphic in s analytic for $Re(s) > 0$.
Moreover $Re(s) > 0 \implies Re(f(s)) > 0$. So we have a real rational function of s which preserves the line $Re(s) = 0$ and maps the RHP to itself. You can shift the RHP to the unit circle. ~~Works~~

Port picture. Make a category out of ports. E.g. if you connect

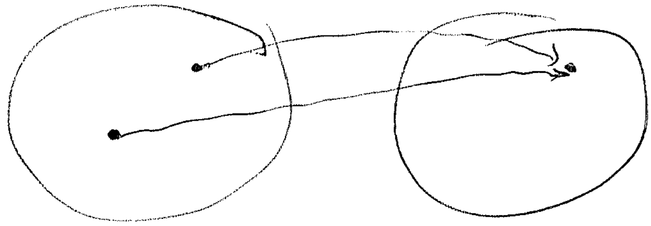


and you know the resp. functions, can you find the frequencies of the combination.

Dec 31. Develop the port picture
Suppose 2 1-ports connected, you know both impedance fns., how do you find the frequencies
Need basic glueing ~~structure~~ process.
Yesterday led to a picture of ~~response~~ LC impedance namely a subquotient of a polarized ~~real~~ Hilbert space. ~~with~~
Two sets of frequencies, which are interlaced, one ~~are Next viewpoint~~ the poles the other the zeroes of the impedances form. Dealing with quad forms leads to surgery? Direct sum + dividing by an

 isotropic subspace. Witt group is formed from quad forms using $\oplus$ and killing hyperbolics. This is the framework for quad forms. Can I adapt it to ~~[illegible]~~ ports? ~~[illegible] Structure very easy to understand~~ Adapt to ports? This should be easy.

You take direct sum of the vector spaces - this like connected sum of the two LC networks. ~~The~~ How do you connect two networks?



So the basic operation is ~~direct sum~~ connected sum and then identifying two points.

$$\boxed{\text{number of modes} = \sum \text{rank}(A_\omega) = \dim H?}$$

~~At the basic ape~~

Step toward writing up. Take $H^+ \oplus H^- \twoheadrightarrow W$ and descend the q form $s\|h_+\|^2 + s^{-1}\|h_-\|^2$. Suppose $H^- \xrightarrow{\sim} W$ ~~so that~~ so that we can take $W = H^-$ and Assume the surj is $\begin{matrix} H^+ \\ \oplus \\ H^- \end{matrix} \xrightarrow{(T \;\; 1)} H^-$

Now descend $s\|h_+\|^2 + s^{-1}\|h_-\|^2$, $\quad \text{Ker}(T \;\; 1) = \text{Im}\begin{pmatrix} 1 \\ -T \end{pmatrix}$

Orth comp. of $\text{Im}\begin{pmatrix} 1 \\ -T \end{pmatrix}$ for s-qf. ~~is?~~

Assume $\begin{pmatrix} h_+ \\ h_- \end{pmatrix}^* \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \begin{pmatrix} \xi \\ -T\xi \end{pmatrix} = 0 \quad \forall\, \xi \in H^-$

$$s(h_+^* \xi) - s^{-1} h_-^* T\xi = 0 \quad \therefore\; s h_+ = s^{-1} T^* h_-$$

$$h_+ = s^{-2} T^* h_-$$

363 So $\underbrace{\begin{pmatrix} s^{-2}T^* \\ 1 \end{pmatrix} h_-}_{} \xmapsto{(T\ 1)} (1+s^{-2}TT^*)h_-$

has s norm^2 $\quad (s^{-2}T^*h_-)^* s (s^{-2}T^*h_-) + s^{-1}h_-^*h_-$

$$= h_-^* T s^{-3} T^* h_- + s^{-1} h_-^* h_-$$

$$= s^{-3}\ h_-^* (s^2 + TT^*) h_-.$$

so h_- has descended norm^2.

$$h_-^* s^{-3} (1+s^{-2}TT^*)^{-1} (s^2+TT^*) (1+s^{-2}TT^*)^{-1} h_-$$

$$h_-^* s^{-1}(1+s^{-2}TT^*)^{-1} h_- = h_-^* \frac{s}{s^2+T^*T} h_-$$

another way to understand denominator is that

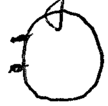
$$\begin{pmatrix} h^+ \\ -Th^+ \end{pmatrix}^* \begin{pmatrix} s & \\ & s^{-1} \end{pmatrix} \begin{pmatrix} h_+ \\ -Th_+ \end{pmatrix} = h^{+*} \Big(\underbrace{s + T^*Ts^{-2}}_{\frac{s^2+T^*T}{s}} \Big) h_+$$

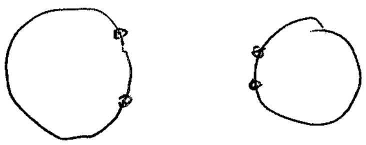
~~Compare~~

Back to connecting. You have described a "category" of some sort. Idea: Think of the symplectic space $V \oplus V^*$. Then connecting corresponds to direct sum followed by ~~...~~ a symplectic sort of shrinking, specifically, ~~...~~ $V \oplus V^* \mapsto V/W \oplus W^0$

These are not all possible symplectic reductions, but the framework looks good; namely, a quadratic form on V yields a maximal isotropic subspace Γ_T, so we find a bigger "category"

364 Now so far you could think of $V \oplus V^*$ **also** as hyperbolic quadratic space. OKAY what next.

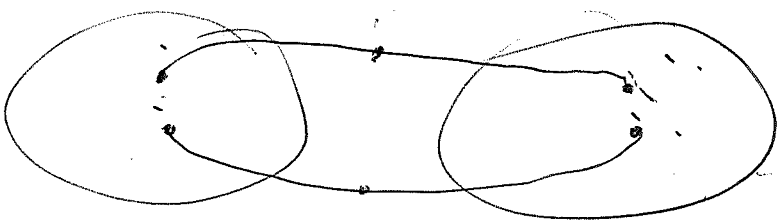
Before you generalize study examples. look at 1-port  . Response fn.

$f(s) = \sum a_\omega \frac{(1+\omega^2)s}{s^2+\omega^2}$ $a_\omega \geq 0$ $\sum a_\omega > 0$. Given two 1-ports  , there are 2-ways to connect, actually there are $\binom{4}{2} = \frac{4 \cdot 3}{2} = 6$ 2-element subsets, $\therefore$ 6 ways to connect ~~and~~ wire

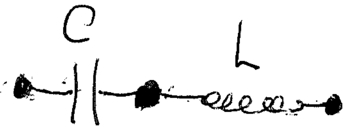
Look at  ~~for each real ω~~

For each s have Z_s : current

Learned: Identifying two vertices (connecting by a wire) leads to a subspace of $\bar{C}^0$. It still amounts to a subquotient of $\mathbf{C}'$ = polarized Hilbert space.



$E = L\dot{I}$



$v-1 = 2$
$\ell = 0$
$e = 2$

$v-1 = 1$
$\ell = 1$
$e = 2$

$$\frac{1}{\frac{1}{Cs} + Ls} = \frac{Cs}{1+LCs^2}$$

$$\frac{1}{Cs} + Ls$$

At some point you need to understand 2-ports. ~~What~~ First connect two 1-ports, and ask about the modes. The first gives $V_1 \xrightarrow{Z_s^{-1}} I_1$ and similarly for the second. What do you mean by a connection?? Look at it this way. Each port occurs as a 1-diml subquotient of a polarized Euclidean space, so can take direct sum to get a 2-diml subquotient of a polarized Euc. space. Presumably connection is frequency independent. The connection leaves us with a subspace of V. You end with a graph having two edges described by Z_1, Z_2 and you need to give the subspace $\bar{Z}^0$.

graphs with 2 edges.

4 vertices | 3 vertices | 2 vertices | 1 vertex

disc. no conn. | series conn

this is a tree hence $\bar{C}^0 \simeq C^1$ so the modes are what occur separately

Here are 2 possibilities for a 1-port. Either short or leave terminals open. Former is $\underbrace{V=0,\ I \text{ arb.}}_{Z_\omega = 0}$. Latter is $\underbrace{V \text{ arb},\ I=0}_{Z_\omega^{-1} = 0}$.

Picture I guess is that the rational function Z_ω^{-1} maps $P_1(\mathbb{R}) \to P_1(\mathbb{R})$ actually its almost a function of ω^2. $\sum a_\omega \frac{(1+\omega^2)s}{s^2+\omega^2}$

 To make further progress you should avoid the details of connecting edges, and instead think symplectically. Starting point is the observation that ~~the space of~~ quadratic forms on V is the set of a max. isot. subspaces of $V \oplus V^*$ which are graphs.

$\binom{1}{T} V$

$$\begin{pmatrix} v_1 \\ Tv_1 \end{pmatrix}^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_2 \\ Tv_2 \end{pmatrix} = -v_1^t T v_2 + (Tv_1)^t v_2$$
$$= v_1^t(-T + T^t) v_2$$

$\therefore \binom{1}{T} V$ is isotropic iff $T = T^t$.

So we are dealing with rational positive maps from $P_1\mathbb{R}$ to the Grass of max isot. subspaces.

Cayley transform game?

~~Rational~~ Analog of modular functor category - oriented 1-manifolds and cobordisms. Objects are closed oriented 1-manifolds, maps are cobordisms, so get some sort of 2-category. To an oriented 1-manifold have attached a loop group and irred repn., tensor product over components. Given R.S. M with ∂M, get vector in ~~representation~~ $V(\partial M)$.

Your analog. instead of oriented 1-manifolds you have real fin. dim. vector spaces V, have direct sum operation. Given V, the analog of a surface with boundary V is a q.f. $V \to V^*$ depending rationally on s of the type above. Rational map simple poles non-negative residues.

Unlike oriented 1-manifolds where you have signed charges, these V seem to be positive, unless you make something out of $0, \infty$. ~~not clear. No.~~ ~~Not yet. Things are very hard.~~ Recall $W \hookrightarrow H^+ \oplus H^-$. On W get $s\|d_+w\|^2 + s^{-1}\|d_-w\|^2$

Split off $g = -1 = F\varepsilon$ where $\underset{H}{\bullet} = \underset{\bar{}}{H^- \cap W} \oplus \underset{V}{H^+ \cap W^\perp}$

367 Unfortunately it seems that our setup does not corresp very well with R.S. case, but maybe with ribbon graphs?

Let's try to remove the reality condition. You recall connection with deficiency indices.

Wait before leaving LC picture ~~what ab~~ look at harm. oscillators. Similar but 0, ∞ maybe do not occur. Connection? Take W symplectic give positive def. quad form on W. Then get skew-symm nondeg. op. which you can convert to a complex structure. This is your analogy of polarization. What's needed now is dilation - might be symplectic reduction of a complex structure.

To handle ∫ you might need something related to renormalization that shifts Re(s) to $\frac{1}{2}$.

So now it's time to work out polarizations in the symplectic case.

Given a harmonic osc.: symp. v.s. V & pos. def Ham. and a symplectic quotient W^0/W of V, then H induces a pos. def form on W^0/W. Does this corresp to adding constraints to an oscillator? What is the mechanism for handling constraints? Lagrange multipliers. Do it variationally: looking at

$\int (p\dot{q} - H)\,dt$? Yuck.

I should understand pols. in the symp. case. You should start with a polarized symp. v.space, i.e. complex Hilbert space. Try to describe iso. subspaces. These are real subspaces L such L, iL are ⊥.

So my guess appears wrong.

$$g^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} g^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = g^{-1}$$

$$\text{"} \begin{pmatrix} a^t & c^t \\ b^t & d^t \end{pmatrix} \text{"}$$

Lie conditions

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c^t & -a^t \\ d^t & -b^t \end{pmatrix} = \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix} \doteq \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

so $b = +b^t$, $c = c^t$, $a^t = -d$

$$\dim Sp_{2n} = n^2 + 2\frac{n^2+n}{2} = 2n^2 + n$$

Count max no. subspaces $\quad 2n + 2n - 1 + \cdots + n+1 = ~~\ \ ~~ \frac{3n+1}{2} n - n^2 = \frac{n^2+n}{2}$

dim stabilizer $\quad n^2 + \frac{n^2+n}{2}$

dim symm space $= n^2 + n^2 + n - n^2 = n^2 + n$

~~So~~ what?

Let J be symplectic and $J^2 = -1$ i.e.

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

b, c symm & $a^t = -d$.

Look: You have $V + iV$. I want to describe complex structures. Complex structure is ~~a~~ max isotropic subspace of $V \otimes_{\mathbb{R}} \mathbb{C}$ nothing real.

$$\operatorname{Re}(v_1, iv_2) = \operatorname{Re}((v_1, v_2)i) = -\operatorname{Im}(v_1, v_2)$$

$$\therefore \operatorname{Re}(L, iL) = 0 \iff L \text{ isotropic}$$