$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \frac{1}{a} & -\frac{\bar{b}}{\bar{a}} \\ \frac{b}{a} & \frac{1}{\bar{a}} \end{pmatrix}$$

Dec 14. ~~Example~~

Review: Tree X, cosheaf M such that

$$\begin{tikzcd} C_1(X,M) \arrow[r, "d", "\sim"'] \arrow[d, equal] & C_0(X,M) \arrow[d, equal] \\ \bigoplus_{\{x,y\}} M_{\{x,y\}} \arrow[r] & \bigoplus_{x} M_x \end{tikzcd}$$

is acyclic.

canonical splitting $M_x = \bigoplus_{d_0\sigma = x} M_x^\sigma$ σ oriented edge with first vertex x.

$X^+_{\sigma=(x,y)} \subset X$

$X = X_\sigma^+ \cup X_\sigma^-$ $\qquad X_\sigma^+ \cap X_\sigma^- = \bar{\sigma}$

operator $M_x^\sigma \longrightarrow Z_1(X_\sigma^+, x; M)$

in words given $\xi \in M_x$ $\exists!$ 1-chain $d^{-1}(\xi)$ with $d(d^{-1}\xi) = \xi$, and $\xi \in M_x^\sigma \Leftrightarrow d^{-1}(\xi) \in C_1(X_\sigma^+)$

then

$$M_x^\sigma \xrightarrow{d^{-1}} M_\sigma \oplus C_1((X-\sigma)^+)$$

$$Z_1(X_\sigma^+, x) = M_\sigma \times_{M_y} Z_1($$

You need a notation, probably supports - you want to take chains with support in a set of 1 simplices such that d vanishes on a set of vertices

$M_x^\sigma \simeq$ 1 chains right of x
$d = 0$ at y and right of y.

~~[illegible]~~

$\xi \in M_x^\sigma$ yields $\alpha(\xi) \in M_\sigma$ and $\beta(\xi) \in \bigoplus_{\tau \neq \sigma} M_y^\tau$

can also look at $\eta \in M_\sigma$ such that $d_1 \eta \in$ ——

~~[illegible]~~ Call this $M_\sigma^+ \subset M_\sigma$. Then

$d_0 : M_\sigma^+ \xrightarrow{\sim} M_x^\sigma$ onto clear

$$\begin{tikzcd} M_x^\sigma & M_\sigma^+ \arrow[l, "\sim"'] \arrow[r, "d_1"] & \bigoplus_{\tau \neq \sigma} M_y^\tau \end{tikzcd}$$

Where are you? You have decomposed M_x into M_x^σ σ oriented edge with $d_0\sigma = x$ and M_σ into $M_\sigma^{d_0\sigma} \oplus M_\sigma^{d\sigma}$. Then ~~[illegible]~~

$$\begin{tikzcd} C_1(X, M) \arrow[r, "d"] \arrow[d, phantom, "\parallel"] & C_0(X, M) \arrow[d, phantom, "\parallel"] \\ \bigoplus_\sigma M_\sigma^{d_0\sigma} & \bigoplus_\sigma M_{d_0\sigma}^\sigma \end{tikzcd}$$

and d is the sum of

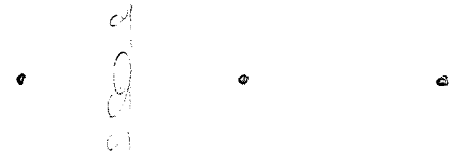
$$\begin{tikzcd} M_\sigma^{d_0\sigma} \arrow[r, "d_0", "\sim"'] \arrow[dr, "\nu"] & M_{d_0\sigma}^\sigma \\ & \bigoplus_{\tau \neq \sigma} M_{d_1\sigma}^\tau \end{tikzcd}$$

I don't quite understand the nilpotence yet, but it is getting clearer.

You need examples. Let a group act now.

Case ~~[illegible]~~ the tree $\mathbb{Z}$. l^2 versions?

Try to understand the case of $\mathbb{Z}/2 * \mathbb{Z}/2$



What is the structure you reach?
Does the cosheaf over the tree $\mathbb{Z}$ split? Yes.

In general. You decompose $C_i(X,M)$ $i=0,1$. according to ordered simplices. ~~[illegible]~~ so

$$C_\cdot(X,M) = \bigoplus_\sigma C_\cdot(X,M)_\sigma$$

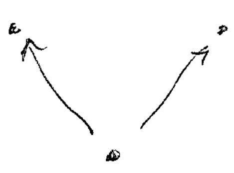
~~where?~~ ,

Look at $G = \mathbb{Z}/2 * \mathbb{Z}/2$. Here the tree is $\mathbb{Z}$

~~continued~~

~~[illegible]~~ The dihedral group acts ~~by~~ reflection at each vertex and translations by even integers. Splitting of the cosheaf ~~[illegible]~~ is ignorant of the action of G. It should reverse the left + right movers.

~~[illegible]~~



corres.

It seems that the category of nil systems (G-equivalent acyclic cosheaves ~~over~~ on the tree) is equivalent to $\mathbb{Z}_2$-graded modules equipped with nilpotent odd operator.

Try to do general case $G = G_A *_{G_C} G_B$

nil system = G equivariant acyclic cosheaf M over X.

$$M_A = \mathbb{Z}[G_A] \otimes_{G_C} M_A^{A,B} \qquad\qquad M_B = \mathbb{Z}[G_B] \otimes_{G_C} M_B^{B,A}$$

$$M_C = M_C^A \oplus M_C^B$$

$$M_A = \mathbb{Z}[G_A] \otimes_{G_C} M_A^{A,B} = M_A^+ \oplus \underbrace{\mathbb{Z}[G_A - G_C] \otimes_{G_C} M_A^+}_{M_A^-}$$

$$M_B = \underbrace{M_B^{B,A}}_{M_B^-} \oplus \underbrace{\mathbb{Z}[G_A - G_B] \otimes_{G_B} M_B^{B,A}}_{M_B^+}$$

Then have ~~isos~~.

$$\begin{tikzcd}
 & M_C^A \arrow[dl, "{\begin{pmatrix}\cong\\0\end{pmatrix}}"'] \arrow[dr, "{\begin{pmatrix}\cong\\0\end{pmatrix}}"] & \\
\boxed{M_A^+} \oplus M_A^- & & \boxed{M_B^-} \oplus M_B^+ \\
 & M_C^B \arrow[ul, "{\begin{pmatrix}0\\\cong\end{pmatrix}}"] \arrow[ur, "{\begin{pmatrix}\cong\\0\end{pmatrix}}"'] &
\end{tikzcd}$$

203 Looks like

$$\begin{array}{c} M_C^+ \\ \oplus \\ M_C^- \end{array} \xrightarrow{\begin{pmatrix} 1 \\ j \end{pmatrix}} \begin{array}{c} M_C^+ \oplus \mathbb{Z}[G_B-G_C]\otimes_{G_C..} M_B^{B,A} \\ \\ M_C^- \oplus \mathbb{Z}[G_B-G_C]\otimes_{G_C} \underbrace{M_A^{AB}}_{M_A^+} \end{array}$$

What can one expect? Answer - not much

Have G acting on tree with ~~basic~~ an edge AB for fundamental domain $\underset{A}{\cdot}$ $\underset{B}{\cdot}$ So all is determined by

$$\begin{tikzcd} M_A & & M_B. \\ & M_C \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Then have canonical splitting $M_C = M_{C,A} \oplus M_{C,B}$ G_C inv.

$$\begin{tikzcd} M_A & \mathbb{Z}[G_A]\otimes_{G_C} M_{C,A} \arrow[l, "\sim"'] & \\ & & M_{C,A} \oplus M_{C,B} \arrow[ul] \\ M_B & \mathbb{Z}[G_B]\otimes_{G_C} M_{C,B} \arrow[l, "\sim"'] & \end{tikzcd}$$

and then it's pretty hard.

K-theory analysis of Waldhausen as exposed by Thomason.

My original idea used filtrations which might also work here. ~~You have $M \in P(\mathbb{Z}[G])$~~

Given $A[t]$-module M choose gen. submodule F_0M_0, then define $F_pM = FM_0 + tFM_0 + \cdots + t^pFM_0$. Graded module $\oplus\, h^p F_pM$ over $\oplus\, h^p F_p A[t]. = A[h, ht]$.

So you have

$$\begin{tikzcd} 0 \arrow[r] & \begin{array}{c}\text{graded}\\ A[t_0,t_1] \text{ modules}\\ \text{locally killed by some}\\ \text{power of } t.\end{array} \arrow[r] & \begin{array}{c}\text{graded } A[t_0,t_1]\\ \text{modules}\end{array} \arrow[r] & \begin{array}{c}\text{graded } A[t_0,t_1][t_0^{-1}]\\ \text{modules}\\ \wr\!\wr \\ A[t]\text{-modules}\end{array} \arrow[r] & 0 \end{tikzcd}$$

abelian cat situation.

204 This has to be ~~amplified~~ supplemented by ~~exp~~ finiteness.

Maybe a bad idea to be prejudiced.

Instead tr ~~[illegible]~~

look at scattering. Start with 2 splittings

$$\begin{tikzcd} & U_1 & \\ T_1 & \oplus & U_0 \\ & T_0 & \end{tikzcd}$$

$$\begin{tikzcd} P \oplus Q & X \oplus Y \arrow[l, "\sim"'] \end{tikzcd}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{tikzcd} P \oplus Q \arrow[r, shift right, "\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}"'] & X \oplus Y \arrow[l, shift right, "\begin{pmatrix} a & b \\ c & d \end{pmatrix}"'] \end{tikzcd}$$

$$\begin{tikzcd} Q \arrow[r, shift left] \arrow[d, shift left] & X \arrow[l, shift left, "c"] \arrow[d, shift left, "a"] \\ Y \arrow[u, shift left, "d"] \arrow[r, shift left, "b"] & P \arrow[l, shift left] \arrow[u, shift left] \end{tikzcd}$$

This $SU(1,1) \hookrightarrow SU(2)$, is it a kind of Wick rotation.

Look at this algebra, assume $P, Q, X, Y = \mathbb{C}$ 1-dim'l.

~~Then~~ and matrices of det(1). $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$\begin{tikzcd} \mathbb{C} \arrow[r, shift left, "-b"] \arrow[d, shift left, "a"] & \mathbb{C} \arrow[l, shift left, "c"] \arrow[d, shift left, "a"] \\ \mathbb{C} \arrow[u, shift left, "d"] \arrow[r, shift left, "b"] & \mathbb{C} \arrow[l, shift left, "-c"] \arrow[u, shift left, "d"] \end{tikzcd}$$

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$$e^{ikx} \rightsquigarrow Ae^{ikx} + Be^{-ikx}$$

$$e^{-ikx} \rightsquigarrow Ce^{ikx} + De^{-ikx}$$

$$\frac{1}{D}e^{-ikx} \longleftrightarrow \frac{C}{D}e^{ikx} + e^{-ikx}$$

$$e^{i(kx-\omega t)}$$

$$e^{ikx} - \frac{B}{D}e^{-ikx} \longleftrightarrow \left(A - \frac{BC}{D}\right)e^{ikx}$$

$$\frac{1}{D}e^{-ikx} \longleftrightarrow \frac{C}{D}e^{ikx} + \boxed{e^{-ikx}}$$

$$\boxed{e^{ikx}} - \frac{B}{D}e^{-ikx} \longleftrightarrow \frac{1}{D}e^{ikx}$$

$$\begin{pmatrix} \frac{1}{D} & \frac{C}{D} \\ -\frac{B}{D} & \frac{1}{D} \end{pmatrix}$$

det is

$$\frac{1+BC}{D^2} = \frac{AD}{D^2} = \frac{A}{D}$$

Darey Evans $U(n,1)$ acting on $\mathcal{O}_n$.

Lie alg. $\left(\begin{array}{c|c} a & b \\ \hline b^* & c \end{array}\right)$ $a^* = -a$ $\bar{c} = -c$.

$$U(1,1) = \text{grp of } \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \quad |a|^2 - |b|^2 = 1.$$

$$|az_1 + bz_2|^2 - |\bar{b}z_1 + \bar{a}z_2|^2$$

$$= |a|^2|z_1|^2 + \cancel{az_1\bar{b}\bar{z}_2} + \cancel{\bar{a}\bar{z}_1 bz_2} + |b|^2|z_2|^2$$
$$- |b|^2|z_1|^2 - \cancel{\bar{b}z_1 a\bar{z}_2} - \cancel{b\bar{z}_1\bar{a}z_2} - |a|^2|z_2|^2$$

The real point in ~~defining~~ finding the action will be to do the case of the subgroup $\begin{pmatrix} 1 & 0 \\ 0 & U(1,1) \end{pmatrix}$.

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$$gW = \left\{ \begin{pmatrix} (a+bs)\xi \\ (c+ds)\xi \end{pmatrix} \,\middle|\, \xi \in H^n \right\}.$$

It should be true that $a+bs : H^n \to H$
$c+ds : H^n \to H$

are invertible. Check this. Use the polar decomp. of g

This is general for a $H = H^+ \oplus H^-$ polarization

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ polar decomp of g.

$$gH^+ = \left\{ \begin{pmatrix} a\xi \\ b\xi \end{pmatrix} \,\middle|\, \xi \in H^+ \right\}. = \{ \xi$$

$$g : H^+ \oplus H^- \xrightarrow{\sim} H^+ \oplus H^-$$

How do you write up an account.

First discuss $V = V^+ \oplus V^-$

Let $W \subset V$ be a subspace on which $\omega > 0$.

$$W \ni w \longmapsto w^+ \oplus w^-$$

$$0 < \|w^+\|^2 - \|w^-\|^2 \qquad w \neq 0.$$

$$\|w\|^2 = \|w^+\|^2 + \|w^-\|^2$$

$$\|w^+\|^2 \lesssim \|w\|^2 < 2\|w^+\|^2 \qquad \therefore\ w \mapsto w^+ \text{ inj.}$$

image is closed.

isomorphism. W maximal $\Longleftrightarrow W \xrightarrow{\sim} W^+$

Then $W = \Gamma_\alpha = \left\{ \begin{pmatrix} \xi_+ \\ \alpha\xi_+ \end{pmatrix} \,\middle|\, v_+ \in \xi \right\}$

$$gH^+ = \left\{ \begin{pmatrix} a\xi \\ b\xi \end{pmatrix} \,\middle|\, \xi \in V^+ \right\} = \Gamma_\alpha \qquad \alpha = ba^{-1}.$$

Suppose now have pol. $V = W^+ \oplus W^-$

Then $W^+ = \Gamma_\alpha \qquad \alpha : W^+ \to W^- \qquad \alpha^*\alpha < 1$

$W^- = \Gamma_\beta \qquad \beta : W^- \to W^+ \qquad \beta^*\beta < 1.$

$$\begin{pmatrix} \xi \\ \alpha\xi \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \beta\eta \\ \eta \end{pmatrix} \qquad (\xi, \beta\eta) = (\alpha\xi, \eta). \quad \therefore \beta = \alpha^*.$$

279 Get isom. $\underbrace{\begin{pmatrix}1 & \alpha^*\\ \alpha & 1\end{pmatrix}}$: $\begin{matrix}V_+\\ \oplus\\ V_-\end{matrix} \xrightarrow{\sim} \begin{matrix}W^+\\ \oplus\\ W_-\end{matrix}$

what you want is to show there is a g in $u(V,\omega)$ such that $g V_\pm = W_\pm$. You want to take α and write $\alpha = ba^{-1}$ where $a=a^*>0$

$$\left\{\begin{pmatrix}a\xi\\ c\xi\end{pmatrix}\right\} = \left\{\begin{pmatrix}\xi\\ \alpha\xi\end{pmatrix}\right\}.$$

$g = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in U(\omega) \iff g^{-1} = \begin{pmatrix}a^* & -c^*\\ -b^* & d^*\end{pmatrix}$

Lie alg $X = \begin{pmatrix}a & b\\ b^* & d\end{pmatrix}$ $a^*=-a$ $d^*=-d$

So tangent vector to the symm space is $\begin{pmatrix}0 & \alpha\\ \alpha^* & 0\end{pmatrix}$.

$g = \dfrac{1+X}{1-X}$ is this in $U(\omega)$. Yes. $g^*=g$ Yes

$g^* \varepsilon g = \frac{1+X}{1-X}\,\varepsilon\,\frac{1\mp X}{1+X}\,\varepsilon$

You want $g = \dfrac{1+X}{\sqrt{1-X^2}}$ $g^*=g$ $a^*b = c^*d$

$1-\alpha^*\alpha = 1 - c\underbrace{a^{-1}a^{*-1}}_{(a^*a)^{-1}}c^*$

$= 1 - c\dfrac{1}{1+c^*c}c^* = 1 - cc^*\dfrac{1}{1+cc^*} = \dfrac{1}{1+cc^*}$ $\boxed{bd^* = ac^*}$ $db^* = ca^*$

$\left\{\begin{pmatrix}a\xi\\ c\xi\end{pmatrix}\right\} = \left\{\begin{pmatrix}\xi\\ \alpha\xi\end{pmatrix}\right\}$ $\left\{\begin{pmatrix}b\eta\\ d\eta\end{pmatrix}\right\} = \left\{\begin{pmatrix}\alpha^*\eta\\ \eta\end{pmatrix}\right\}$.

$\alpha = ca^{-1}$ $\alpha^* = bd^{-1}$. $bd^{-1} = a^{*-1}c^*$

$\|a\xi\|^2 - \|c\xi\|^2 = \|\xi\|^2$ $a^*a - c^*c = 1.$ follows that a^*a

---

Look at $\left\{\begin{pmatrix}a\xi\\ c\xi\end{pmatrix} \middle| \xi\in H^+\right\} = W^+$

$\omega \longmapsto \omega_+, \omega_-$ $\|\omega_+\|^2 \le \|\omega\|^2 = \|\omega_+\|^2 + \|\omega_-\|^2 \le 2\|\omega_+\|^2$

∴

280 Continue to explain $U(n,1)$ action on $\mathcal{O}_n$.

$$V^+\otimes H \oplus V^-\otimes H = H^n \oplus H. \quad \left\{\begin{pmatrix}\xi \\ s\xi\end{pmatrix}, \xi\in H^n\right\}$$

You have $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in U(n,1)$ and $W = \Gamma_s = \Gamma_{s^*} = \left\{\begin{pmatrix} s^*\eta \\ \eta\end{pmatrix} \middle| \eta \in\right.$

The fact is that $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} 1 \\ s\end{pmatrix}H^n = \begin{pmatrix} a+bs \\ c+ds\end{pmatrix}H^n$ and the general theory says that $(a+bs)^{-1}$ exists and $g(s) = (c+ds)(a+bs)^{-1} : H^n \longrightarrow H$. The proof goes how. You want first to take the case dictated by the polar decomp. This means

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \frac{1+X}{\sqrt{1-X^2}} \quad \text{where} \quad X = \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0\end{pmatrix}$$

here $\alpha = (z_1, \dots, z_n)$

$\sum |z_i|^2 < 1.$

$$= \begin{pmatrix} \frac{1}{\sqrt{1-\alpha^*\alpha}} & \alpha^*\frac{1}{\sqrt{1-\alpha\alpha^*}} \\ \boxed{\alpha\frac{1}{\sqrt{1-\alpha^*\alpha}}} & \frac{1}{\sqrt{1-\alpha\alpha^*}}\end{pmatrix}$$

$$\frac{1}{\sqrt{1-\alpha\alpha^*}}\alpha$$

So can you make sense out of

$\alpha = (z_1, \dots, z_n)$ row vector

$$(c+ds)(a+bs)^{-1} = (z+s)\frac{1}{\sqrt{1-|z|^2}}\left(\frac{1}{\sqrt{1-z^*z}} + \frac{1}{\sqrt{1-z^*z}}z^*s\right)^{-1}$$

$$a = \frac{1}{\sqrt{1-\alpha^*\alpha}} \qquad b = \alpha^*\frac{1}{\sqrt{1-\alpha\alpha^*}}$$

$$c = \frac{1}{\sqrt{1-\alpha^*\alpha^*}}\alpha \qquad d = \frac{1}{\sqrt{1-\alpha\alpha^*}}$$

$$c+ds = \frac{1}{\sqrt{1-\alpha\alpha^*}}(\alpha+s)$$

$$a+bs = \frac{1}{\sqrt{1-\alpha^*\alpha}}(1+\alpha^*s)$$

$$(a+bs)^{-1} = (1+\alpha^*s)^{-1}\sqrt{1-\alpha^*\alpha}$$

$(c+ds)(a+bs)^{-1}$

$= \frac{1}{\sqrt{1-\alpha\alpha^*}}(\alpha+s)(1+\alpha^*s)^{-1}\sqrt{1-\alpha^*\alpha}$

very messy

28 What is $\frac{1}{\sqrt{1-\alpha^*\alpha}}$ $\alpha = (z_1, \dots, z_n)$ $\alpha^* = \begin{pmatrix} \bar z_1 \\ \bar z_n \end{pmatrix}$

$(\alpha\alpha^*)_{ij} = z_i \bar z_j$

$$(1-\alpha^*\alpha)^{-1/2} = \sum_n \frac{1\cdot 3 \cdots 2n-1}{n!} \underbrace{(\alpha^*\alpha)^n}_{\alpha^*|z|^{2n-2}\alpha}$$

$$= 1 +$$

Dec 22 Explain $U(n,1)$ action on $\mathcal{O}_n$.

background. $H = H^+ \oplus H^-$ $\omega(\xi) = \|\xi_+\|^2 - \|\xi_-\|^2$

~~[illegible]~~ describe maximal W closed $\subset H$ $\omega \geq 0$ on W

as $\Gamma_\alpha = \binom{1}{\alpha} H^+$ $1-\alpha^*\alpha \geq 0$.

describe max isotropic subspaces polarizations $H = W^+ \oplus W^-$

via $\frac{1+X}{\sqrt{1-X^2}}$ $X = \begin{pmatrix}0 & \alpha^* \\ \alpha & 0\end{pmatrix}$ $\alpha^*\alpha \leq 1-\varepsilon$

elements of $U(H,\omega)$ $g = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$ polar decomp.

$g = p_\alpha \begin{pmatrix} u_+ & 0 \\ 0 & u_-\end{pmatrix}$ $gg^* = p_\alpha p_\alpha^* = p_\alpha^2 = \frac{1+X}{1-X} = \frac{1+X^2+2X}{1-X^2}$

$g^*\begin{pmatrix}1&0\\0&-1\end{pmatrix} g = \begin{pmatrix}1&0\\0&1\end{pmatrix}$ $g = \begin{pmatrix}a&b\\c&d\end{pmatrix} \Rightarrow g^{-1} = \begin{pmatrix}a^* & -c^* \\ -b^* & d^*\end{pmatrix}$

$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}a^*&c^*\\b^*&d^*\end{pmatrix} = \begin{pmatrix}aa^*+bb^* \\ ca^*+db^*\end{pmatrix}$ $\alpha = ca^{-1}$

$\alpha^* = bd^{-1}$

$aa^*+bb^* = \frac{1+\alpha^*\alpha}{1-\alpha^*\alpha}$ | $\underbrace{ca^*+db^*}_{2ca^*} \overset{?}{=} \frac{2\alpha}{1-\alpha^*\alpha}$ $ca^* = db^*$

$\|$

$2ca^{-1}aa^*$ $\alpha^*\alpha = bd^{-1}ca^{-1}$

$aa^* - bb^* = 1$

$aa^* = \frac{1}{1-\alpha^*\alpha}$ ✓ action on max isot. subspaces $\binom{1}{u}$

$\begin{pmatrix}a&b\\c&d\end{pmatrix}\binom{1}{u} H^+ = \binom{a+bu}{c+du} H^+$ $(c+du)(a+bu)^{-1}$ is unitary $H^+ \to H^-$ ~~an isom~~.

Now take $H^+ = \mathbb{C}^n \otimes E$, $H^- = \mathbb{C}\otimes E$

a unitary $u: H^+ \to H^-$ same as $s_1, \dots, s_n : E \to E$

$\ni$ $s^*s = 1$ $ss^* = 1$.

So you have this action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(s) = (c+ds)(a+bs)^{-1}$$

It looks prettier if apply $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$

$$\begin{pmatrix} d & c \\ b & a \end{pmatrix}(s) = (b+as)(d+cs)^{-1} = (as+b)(cs+d)^{-1}.$$

But then you conflict with s being a row vector.

~~But here's a basic question.~~

Describe full isotropic subspaces as $\begin{pmatrix} s^* \\ 1 \end{pmatrix}E$?

$$H^+ = \mathbb{C}^n \otimes E = E^n \text{ col. v.}, \quad H^- = \mathbb{C}\otimes E = E$$

Take an unit isom

$$E^n \simeq E$$

If you want $s^*s = 1$ to mean $s_i^* s_j = \delta_{ij}$ then (s_j) must be a row v. and (s_i^*) must be column.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi \\ \xi_0 \end{pmatrix} = \begin{pmatrix} a\xi + b\xi_0 \\ c\xi + d\xi_0 \end{pmatrix}$$

So we get the action on

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} s^*\xi_0 \\ \xi_0 \end{pmatrix}\right\} = \left\{\begin{pmatrix} (as^*+b)\xi_0 \\ (cs^*+d)\xi_0 \end{pmatrix}\right\}$$

$$= \left\{\begin{pmatrix} \sum_j a_{ij}s_j^* + b_i \\ \underset{\text{invertible}}{\sum c_i s_i^* + d} \end{pmatrix}\xi_0\right\}$$

$$s_i'^* = \left(\sum_j a_{ij}s_j^* + b_i\right)\left(\sum c_i s_i^* + d\right)^{-1}$$

$$s'^*s' = (as^*+b)(cs^*+d)^{-1}\underbrace{(c+ds)(a+bs)^{-1}}_{(cs^*+d)s}$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u^* \\ 1 \end{pmatrix}H^+ = \begin{pmatrix} au^*+b \\ cu^*+d \end{pmatrix}H^+$$

$g(u^*) =$ ~~$(cu^*+d)(au^*+b)^{-1}$~~ $(au^*+b)(cu^*+d)^{-1}$

$$g(u) = (c+du)(a+bu)^{-1}$$

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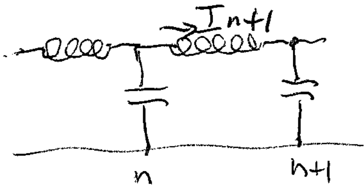
actual action $g \mapsto g\begin{pmatrix}1\\0\end{pmatrix}H^{+} = \begin{pmatrix}a\\c\end{pmatrix}H^{*} = \begin{pmatrix}1\\ca^{-1}\end{pmatrix}H^{*}$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix} = \begin{pmatrix}ac^{-1}\\1\end{pmatrix}$$

Plumbing with coaxial cable might be like surface theory.

Basic idea – do I start with transmission lines, or the moment problem, or what?

transmission lines.



$$E_{n+1} - E_n = -L\dot{I}_{n+1}$$
$$I_{n+1} - I_n = -C\dot{E}_n$$

do a continuous version 

E voltage depend on x, t.
I current.

rules $I = C\frac{dE}{dt}$ C capacitance

$E = -L\frac{dI}{dt}$ L inductance

transmission line

$$\partial_x E = -\ell\,\partial_t I$$
$$\partial_x I = -c\,\partial_t E$$

~~$\ell\partial_t I$~~

$$\partial_t I = -\ell^{-1}\partial_x E$$
$$\partial_t E = -c^{-1}\partial_x I$$

$$\partial_t^2 I = +\ell^{-1}c^{-1}\partial_x^2 I$$

Resistance circuits – recall

~~graph~~ 1-complex ~~[illegible]~~ each has a resistance. ~~[illegible]~~ Current is a 1 chain so having chosen ~~an~~ orientations for the 1-simp current becomes a number. ~~and~~, You have "internal" condition that $\sum$ currents entering an vertex = 0

~~[illegible]~~ Each edge has a resistance. ~~Voltage values~~ Voltage is a 0 chain which is specified at each external vertex.

1-complex, say conn.

I is a 1-chain

E is a 0-cochain modulo constants.

R collection of resistances numbers for each edge yields and inner product on 1-chains. Ohm's law says $(\delta E)_\sigma = R_\sigma I_\sigma$ probably ~~[illegible]~~ says

YES

$$\begin{tikzcd} C^0/\mathbb{C} \arrow[r, "\partial^*"] \arrow[d, equal] & C^1 \arrow[d, equal] \\ C_0 \supset \partial C_1 & C_1 \arrow[l, "\partial"'] \end{tikzcd}$$

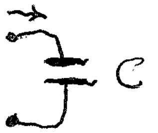
Crazy

$$\begin{tikzcd} E & \\ C^0/\mathbb{C} \arrow[r, "d"] \arrow[d, "\Delta"'] & C^1 \arrow[d, "R"] \\ \partial C_1 & C_1 \arrow[l, "\partial"'] \\ & I \end{tikzcd}$$

~~If~~ I take E_0 = applied voltage and I wish to extend E_0 to E such that the corresp. ΔE has the same support at E_0

285 Go back to capacitance + inductance.

~~and remember that~~ Complex inductance.



I 1-chain

E 0-chain? modulo constants

$$Q = CE$$

$$C\dot{E} = I$$

do L.T.

$$\cancel{\dot{E} = CI}$$

$$\cancel{s\hat{E} = C\hat{I}}$$

$$\cancel{\hat{E} = \frac{C}{s}\hat{I}}$$

$$\int_0^\infty e^{-st}\dot{E}\,dt = C\int_0^\infty e^{-st} I\,dt$$

$$\parallel$$

$$\left[e^{-st}E\right]_0^\infty - \int_0^\infty (-s)e^{-st}E\,dt$$

$$C\hat{I} = s\hat{E} - E(0)$$

$$\hat{E} = \frac{1}{Cs}\hat{I} + \frac{E(0)}{s}$$

other viewpoint

$$I = I_0 e^{-i\omega t}$$
$$E = E_0 e^{-i\omega t}$$

$$E_0(-i\omega)\cancel{e^{-i\omega t}} = C I_0 \cancel{e^{-i\omega t}}$$

$$\boxed{E_0 = \frac{C}{-i\omega} I_0}$$

$Z = \frac{1}{Cs}$ cap.

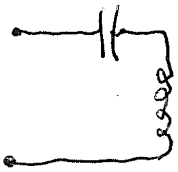
$= -Ls$ ind.

$$E = -L\dot{I}$$

$$E_0\cancel{e^{-i\omega t}} = -L I_0(-i\omega)\cancel{e^{-i\omega t}}$$

$$\boxed{E_0 = L(i\omega)\, I_0}$$

$$\hat{E} = -L\left(s\hat{I} - I(0)\right)$$



$Z = \frac{C}{s} - Ls$

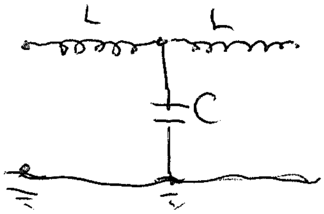
$Z = 0$ when $s = \sqrt{\frac{C}{L}}$

$Z = \frac{1}{\frac{s}{C} - \frac{1}{Ls}} =$

$= \frac{LsC}{Ls^2 - C}$

$Z = \infty$ when $s = \sqrt{\frac{C}{L}}$

what about something like.



$Z = -Ls + \frac{1}{\frac{s}{C} + \frac{1}{Z}}$

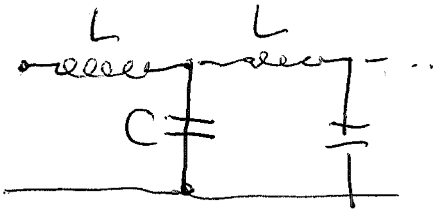
$Z = -Ls + \frac{CZ}{sZ + C}$

$(sZ+C)(Z+Ls) = CZ$

$sZ^2 + Ls^2Z + CLs = 0$

$Z^2 + LsZ + CL = 0$

$Z = \frac{-Ls \pm \sqrt{L^2s - 4CL}}{2}$



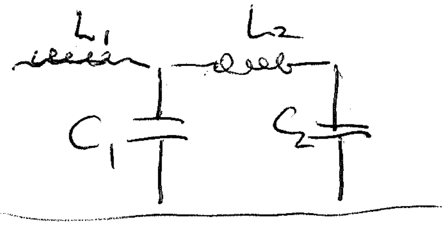
$Z = (-Ls) + \frac{1}{\frac{s}{C} + \frac{1}{Z}}$

$Z = (-Ls) + \frac{CZ}{sZ + C}$

$(Z+Ls)(sZ+C) = CZ$

$sZ^2 + Ls^2Z + \cancel{CZ} + sLC = \cancel{CZ}$

$Z^2 + LsZ + LC = 0$



Now instead of continued fractions use SL_2.

$$Z = -L_1 s + \cfrac{1}{C_1 s + \cfrac{1}{-L_2 s + \cfrac{1}{C_2 s + \cdot}}}$$

~~What you get is a~~

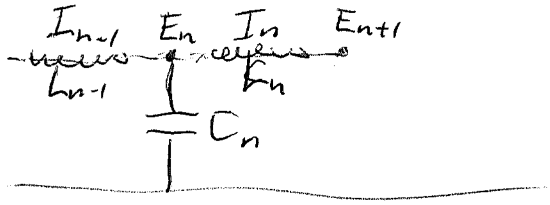
Aim to combine ~~electrical~~ resistance network idea with components whose impedance varies with s

now you need SL_2 in here. What is the variable s in all this? The basic construction above is

$$Z_n = -L_n s + \cfrac{1}{C_n s + \cfrac{1}{Z_{n+1}}}$$

What's becoming clear is that the resistance network argument is apt to be much better than simply SL_2.

Recall some ideas from before. transmission line equations.



$$\dot{E}_n = -C_n(I_n - I_{n-1}) \qquad \partial_t E = -C\partial_x I$$

$$E_{n+1} - E_n = -L_n(\dot{I}_n) \qquad \partial_x E = -L\partial_t I$$

$$\left(\begin{array}{l} \partial_t E = -C\partial_x I \\ \partial_t I = -L^{-1}\partial_x E \end{array} \right.$$

try $E = e^{i(kx-\omega t)}\hat{E}$ etc.

$$-i\omega\hat{E} = -Cik\hat{I}$$

$$-i\omega\hat{I} = -L^{-1}ik\hat{E}$$

$$+C\left(\frac{k}{\omega}\right)^2 L^{-1} = 1$$

$$\hat{E} = +C\frac{k}{\omega}\hat{I} \qquad \hat{I} = L^{-1}\frac{k}{\omega}\hat{E} \qquad \sqrt{CL^{-1}} = \frac{\omega}{k}$$

288 So how do I proceed. I think I want a 2 dimensional space at each edge.

~~I want V.S. at each stage~~

Dec 23. I need to amass examples. Some ideas analogy of Riemann surfaces and CFT to your electrical circuits | ~~analogy~~ you're trying to replace Diff S^1 by $SL_2(\mathbb{R}) = SU(1,1)$. + parameter s. ~~Res~~.

~~[illegible]~~ n-port ~~[illegible]~~ corresp to surface with n boundary components. ~~[illegible]~~ Resistance networks generalize to ~~[illegible]~~ impedance networks. Suppose you ~~[illegible]~~ analyze the sort of things you can construct on a graph. First problem: ~~Suppose~~ Make a network of capacitors + inductors and prove the basic existence result analogous to what happens on a network. ~~So~~ Should involve rational functions of s. Should get an idea of n-port. Next see if you can ~~expand gen~~ extend to edges with fancier impedance.

Start with resistance networks: ~~a~~ 1 - complex. has set X of vertices and edges - edge is a 2 elt subset of the vertices $E \subset (X \times X - \Delta X)_{\mathbb{Z}/2}$. Assume ~~[illegible]~~ X linearly ordered, where each ~~[illegible]~~ edge is oriented. Each edge ~~[illegible]~~ σ has a resistance $R_\sigma > 0$. Current is a 1-chain I, Voltage is a 0 ~~[illegible]~~ cochain mod constants. Fix two vertices b_0, b_1 and connect a battery to them. ~~[illegible]~~ Then current flows into the network. ~~[illegible]~~ Then you get a voltage function E and a current 1-chain I satisfying ~~certain~~ what equations? ∂I has support $\{b_0, b_1\}$ Of ~~first~~ course $\partial I_{b_0} + \partial I_{b_1} = 0$. $(dE)_{x_0x_1} = E_{x_1} - E_{x_0}$ $= R_{x_0x_1} I_{x_0x_1}$ $(dE)_\sigma = R_\sigma I_\sigma$ There's an initial condition that ~~[illegible]~~ $E_{b_1} - E_{b_0}$ = applied voltage

So how does the proof go? of what

You want to know that for every choice of external vertices there is a ~~unique~~ bijection between currents satisfying that-cycle condition on. Internal + External vertices.

S = set of external vertices. Look at I $\ni$ ∂I supported in S: $W_S = \{ I \in C_1 \mid \partial I = 0 \text{ on } X-S \}$.

and E prescribed on S. $C^0(S)/\mathbb{C}$ So what?

What can you say about the relation between Z_1 and $C^0/\mathbb{C}$. $Z_1 = H^1(X)$, so there are interesting cycles. Voltage function $= 1$ -coboundary

voltage space $=$ $dC^0 = B^1$

current space $=$ $\partial C_1 = Z_1$

You have

$$\begin{tikzcd}
dC^0 = B^1 \subset & C^1 \arrow[r] \arrow[d, "R"] & H^1 \arrow[r] & 0 \\
0 & \partial C_1 \arrow[l] & C_1 \arrow[l] & H_1 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

$$\boxed{\begin{tikzcd}
0 \arrow[r] & H^0 \arrow[r] & C^0 \arrow[r, "d"] \arrow[d, "\wr"] & C^1 \arrow[r] \arrow[d, "\wr"] & H^1 \arrow[r] & 0 \\
0 & H_0 \arrow[l] & C_0 \arrow[l] & C_1 \arrow[l, "\partial"] & H_1 \arrow[l] & 0 \arrow[l]
\end{tikzcd}}$$

these are dual. Have resistance metric $R: C^1 \xrightarrow{\sim} C_1$

So the picture is clear. You have a map going from ~~voltages~~ ~~the space~~ voltage functions ~~th~~ (0-chains mod constants) to currents at vertices (0-chains of avg 0). Laplacian map. Why is it an isomorphism. Weyl proof uses positivity. If you take 2 vertices you get an effective resistance. Can you see that this is positive? Take 3 vertices 2 dimensional space of input voltages and currents; some sort of positive quadratic form — power.

Return to

$$\begin{tikzcd}
0 \arrow[r] & C^0/H^0 \arrow[r, shift left, "d"] \arrow[d, phantom, "\wr"] & C^1 \arrow[l, shift left, "d^*"] \arrow[r] \arrow[d, "\wr"] & H^1 \arrow[r] & 0 \\
0 & (C^0/H^0)^* \arrow[l] \arrow[u] & C_1 \arrow[l, "d^t"'] & H_1 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

You are asking exactly the question when is d^*d an isomorphism. The standard reply is when the inner products are positive.

~~[illegible] Where the~~ Inner product comes from power. At the vertex end you have $(\partial I)_x$ the net current entering the vertex x and E_x the voltage at x so you have power $\sum_x E_x (\partial I)_x$ entering the network. This checks because the ~~boundary~~ $\sum (\partial I)_x = 0$ and E is determined up to a constant. ~~So you have~~

$$\langle E, \partial I \rangle = \underbrace{\langle dE, I \rangle}_{} \quad ?$$

power pairing, not the obvious duality

291

$$\langle E, \partial I \rangle = \sum_x E_x (\partial I)_x$$

$$(\partial I)_x = \sum_{(xy)} I_{xy}$$

$$\langle E, \partial I \rangle = \sum_x E_x \sum_{(xy)} I_{xy} = \sum_{\substack{(x,y)\\ \text{ordered}\\ \text{1-simplice}}} E_x I_{xy}$$

$$= \sum_{\substack{\text{unordered}\\ \text{1-simplices}\\ \{x,y\}}} \underbrace{(E_x - E_y)}_{R_{\{xy\}} I_{\{xy\}}} I_{\{xy\}} = \sum_\sigma R_\sigma I_\sigma^2$$

What do you need to know?
Suppose you have batteries volt E_x for each x negative terminals all conn. Then you get certain currents going in at each vertex. $(\partial I)_x$ [illegible]

Next you want to look at networks made of L_i, C_i. Now instead of R_σ you have an impedance $\frac{1}{Cs}$ or $+Ls$ $s = -i\omega$.

$Q = CE$ for a capacitance

$I = \dot{Q} = C\dot{E}$

$E = \hat{E} e^{st}$

$I = \hat{I} e^{st}$

$\hat{I} = Cs\hat{E}$

$\hat{E}/\hat{I} = \boxed{\frac{1}{Cs}}$

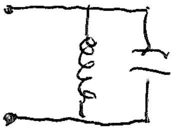
$E = +L\dot{I}$

$\hat{E} = \boxed{+Ls}\hat{I}$

Again we have a network say a 1-complex And I ~~need~~ want to know whether the Laplacean d^*d on voltage functions is invertible. Notice that if $s = -i\omega$, then $\frac{1}{Cs} = \frac{1}{C(-i\omega)} = i\frac{1}{\boxed{C\omega}}$ same sign as ω

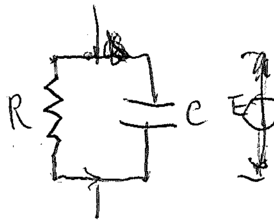
$-Ls = i\boxed{L\omega}$ same sign as ω, so that for ω real

292 we have i times a resistance ~~network~~ ———— for $\omega > 0$ and $-i$ ———— for $\omega < 0$. Something is wrong because you don't see resonance.



$$Z = \frac{1}{Cs + \frac{1}{-Ls}} = \frac{Ls}{CLs^2 - 1}$$

$$s^2 = \frac{1}{LC} \qquad -i\omega = \pm\frac{1}{\sqrt{LC}}$$

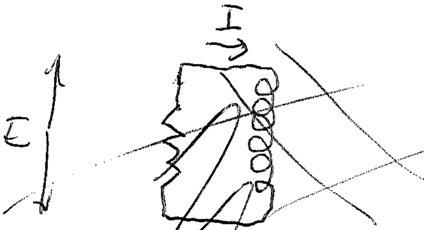


$$E = CQ = RI$$

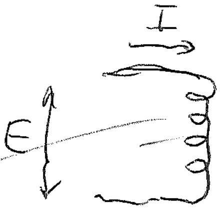
$$Z = \frac{1}{\frac{1}{R} + Cs} = \frac{R}{1 + RCs}$$

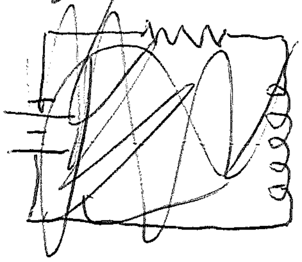
$$\frac{dE}{dt} = -CI \qquad E = RI$$

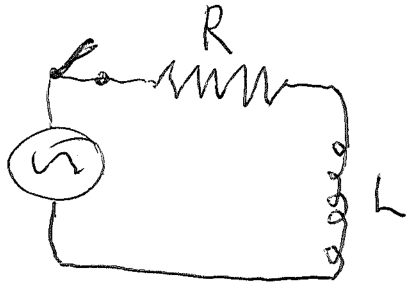
$$R\mathring{I} = -CI \qquad \mathring{I} = -\frac{c}{R}I \qquad I = I_0 e^{-\frac{c}{R}t}$$



~~$E = -L\frac{dI}{dt} = RI$~~

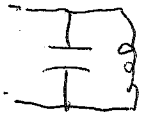




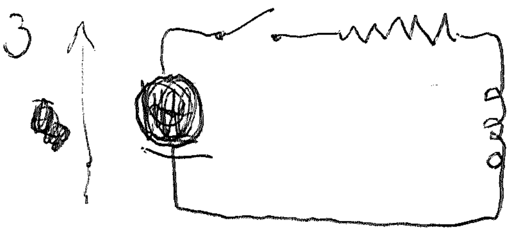


$$\hat{E} = (R \pm Ls)\hat{I}$$

~~$\frac{E_0}{s} =$~~ So the impedances are $\frac{1}{Cs}$ and Ls these preserve RHP

so for  get $Z = \frac{1}{Cs + \frac{1}{Ls}} = \frac{Ls}{CLs^2+1}$

which means oscillations at $\omega = \frac{1}{\sqrt{CL}}$



$$RI + L\frac{dI}{dt} = H(t) = \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$$

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{1}{L} \qquad t>0$$

$$I = \frac{1}{R} - \frac{1}{R}e^{-\frac{R}{L}t} = \frac{1-e^{-\frac{R}{L}t}}{R} \qquad t \geqslant 0$$

$$= 0 \qquad t \leq 0.$$

So if we use the L.T. the impedance is $R + Ls$

$$RI + \frac{1}{C}Q = H(t)$$

$$R\dot{I} + \frac{1}{c}I = \delta(t)$$

$$\dot{I} + \frac{1}{RC}I = \frac{1}{R}\delta(t)$$

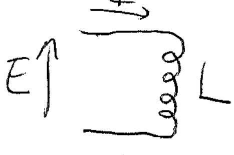
$I =$ [illegible]

$$= \frac{e^{-\frac{1}{RC}t}}{R} \qquad t>0$$

$$= 0 \qquad t<0$$

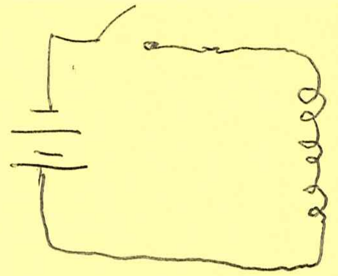
$$I = \begin{cases} \frac{1}{R}e^{-\frac{1}{RC}t} & t>0 \\ 0 & t<0 \end{cases}$$

$$\hat{E} = Ls\hat{I}$$

So it looks like 

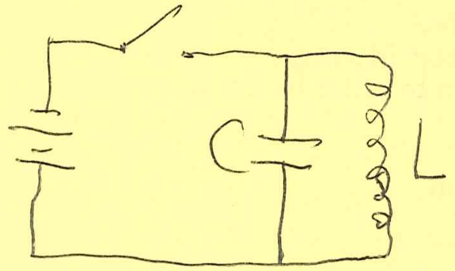
then $\boxed{E = L\frac{dI}{dt}}$

the voltage drop across the inductance is ~~big~~ when the current is increasing



$$L\frac{dI}{dt} = H(t)$$

$$I = \begin{cases} \frac{t}{L} & t>0 \\ 0 & t<0 \end{cases}$$



$$\frac{1}{s} = \hat{E} = \frac{1}{\frac{\ }{Cs} + \frac{1}{Ls}} \hat{I}$$

$$\hat{I} = \left(Cs + \frac{1}{Ls}\right)\frac{1}{s} = C + \frac{1}{Ls^2}$$

$$I(t) = C\delta(t) + \frac{1}{L}H(t)t$$

So now for the existence result. What happens is that for $s>0$ there's no problem because we have ~~an exact~~ the resistance network situation.

Vaughn Jones - Kadison stuff

$B \subset A \xrightarrow{\tau} B$ τ B-bimodule map.

get a dual pair over B $\left(A, A, \underset{(a_1,a_2)\mapsto \tau(a_1a_2)}{A\times A \longrightarrow B}\right)$

get $A\otimes_B A$ with $(a_1\otimes a_2)(a_3\otimes a_4) = a_1\tau(a_2a_3)\otimes a_4$

acting on A ~~both~~ sides. $a_0(a_1\otimes a_2) = \tau(a_0a_1)a_2$

$(a_1\otimes a_2)a = a_1\tau(a_2a)$

Suppose $\exists\, x_i\otimes y_i \in A\otimes_B A \ni a\sum x_i\otimes y_i = a$ $\sum\tau(ax_i)y_i = a$

$(\sum x_i\otimes y_i)a = a$ $\sum x_i\tau(y_ia) = a$

Then we know $A\otimes_B A \xrightarrow{\sim} \mathrm{Hom}_{B^{op}}(A,A)$

Claim can repeat and iterate. ~~Need to think of~~ Have

$A \longrightarrow \overbrace{A\otimes_B A}^{A_2}$.   Form $A_2\otimes_A A_2$ But you need

$a \longmapsto \sum ax_i\otimes y_i$ $\tau : A\otimes_B A \longrightarrow A$ A-bimod. map.

Wait $\mathrm{Hom}_{B^{op}}(A,A) = A \otimes_B A^\vee$

Do you actually assume $B \subset A$? ~~Just that~~

$A_2 = \mathrm{Hom}_{B^{op}}(A,A) = A \otimes_B A^\vee$

$$A_2 \otimes_A A_2 = A \otimes_B A^\vee \otimes_A A \otimes_B A^\vee$$

Wait: take $A_2 \otimes_A A_2 = A \otimes_B A \otimes_B A$ — so you get this incredible tower of algebras.

Wait: ~~Inside~~ $A \otimes_B A = A_2$ you have two canonical elements, namely $1 \otimes 1$ and $\sum x_i \otimes y_i$. The latter is the identity elt. the former is an idempotent

$$(1 \otimes 1)(1 \otimes 1) = \tau(1) \otimes 1 \quad \text{assuming } \tau(1) = 1.$$

Somehow there's a way to make sense out of $\underline{A^{(n)}}$ being a ring acting on $A^{(k)}$ on both sides.

$G \subset SU(d)$ acts on $\mathcal{O}_d$ obviously

fixpt subalg $\mathcal{O}_d^G$ $\qquad \mathcal{T}_d = \bigoplus_{p,q \geq 0} E^{\otimes p} \otimes E^{* \otimes q}$

$\mathcal{O}_d$ has a canonical endomorphism.

There is a faithful functor from the full ~~cat~~ of G representations consist of $E^{\otimes n}$ to

End$(\mathcal{O}_d)$ is a category, how?

Apparently End$(\mathcal{O}_d)$ contains a canonical element σ whose powers correspond somehow to the $E^{\otimes n}$, $n \geq 0$. representations of $SU(d)$. What is a map between endos.

$$\begin{tikzcd}
 & \mathcal{O}_d \arrow[dl, "\sigma^n"'] \arrow[dr, "\sigma^m"] & \\
\mathcal{O}_d \arrow[rr, dashed, "?"] & & \mathcal{O}_d
\end{tikzcd}$$

296 Take $G = SU(d)$ and calculate its fixed subalgebra in $\mathcal{O}_d$.

$$k$$

$$E \qquad E^*$$

$$E^{\otimes 2} \qquad E\otimes E^* \qquad E^{*\otimes 2}$$

What is $(\mathcal{T}_d)^{SU(d)}$? Apparently $\Lambda^d E$, $\Lambda^d E^*$ are in the fixed point subalgebra. One thing you can ~~do~~ examine is the ~~subs~~ degree 0 subalgebra.

What can you say about the cat of repns $E^{\otimes n}$ of $SU(d)$?

Suppose $G = 1$. Then you get category of v.s. $E^{\otimes n}$, $n \geq 0$. To each object you should get an endo of $\mathcal{O}_d$. How does $\otimes E^{\otimes n}$ give rise to an endo of $\mathcal{O}_d$.

$$A \otimes_B A \xrightarrow{\sim} A \otimes_B A^\vee \xrightarrow{\sim} \mathrm{Hom}_{B^{op}}(A,A)$$
$$a_1 \otimes a_2 \longmapsto a_1 \otimes \tau a_2$$

Is there a natural A-bimodule map $A\otimes_B A \longrightarrow A$? it's determined by its value on $1\otimes 1 \mapsto (a' \mapsto 1\cdot\tau(a'))$

Obvious choice is mult. $\tau_1 : a_1 \otimes a_2 \longmapsto a_1 a_2$

But then $\tau_1\left(\sum x_i \otimes y_i\right) = \sum x_i y_i$

December 24. Existence for C+L circuits.

Power $$\int_{-\infty}^{\infty} E(t)\, I(t)\, dt = \int_{-\infty}^{\infty} \int \hat{E}(\omega) e^{-i\omega t} \frac{d\omega}{2\pi} \int \hat{I}(\eta) e^{-i\eta t} \frac{d\eta}{2\pi}$$

$$= \int \hat{E}(\omega) \underbrace{\hat{I}(-\omega)}_{\overline{\hat{I}}(\omega)} \frac{d\omega}{2\pi}$$

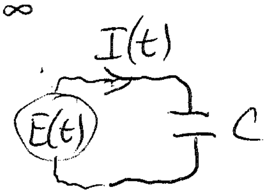
$$\int e^{-i(\omega+\eta)t}\, dt = 2\pi\, \delta(\omega+\eta)$$

What do I have to do? ~~I take~~

~~[illegible] compact support~~ I apply a voltage $E(t)$ and get a response $I(t)$ and calculate the power which will a quadratic form on ~~[illegible]~~ the space of $E(t)$. Say $E(t) \in C_0^\infty(\mathbb{R}$

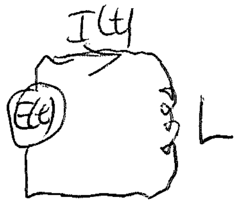
$$I(t) = \int_{-\infty}^{t} G(t-t')E(t')\,dt'$$

Example:



$$Q = CE$$
$$I = C\dot{E}$$

$$P = c^{-1}\int E(t)\dot{E}(t)\,dt = \frac{c^{-1}}{2}\left[E(t)^2\right]_{-\infty}^{\infty} = 0$$



$$E = L\dot{I}$$

$$P = \int L\dot{I}I\,dt = \frac{L}{2}\left[I^2\right]_{-\infty}^{\infty} = 0.$$

$$P = \int E I\,dt = \int \hat{E}(\omega)\,\overline{(G*E)\hat{}(\omega)}\,\frac{d\omega}{2\pi}$$

Power is a quadratic form on the real vector space of ~~of~~ real C_0^∞ fns $E(t)$. You can extend it to complex functions, how.

$$P = \int \hat{E}(\omega)\,\hat{G}(-\omega)\,\hat{E}(-\omega)\,\frac{d\omega}{2\pi}$$

~~$\int |\hat{E}(\omega)|^2\,\hat{G}(-\omega)\,\frac{d\omega}{2\pi}$~~

$$= \int \underbrace{\hat{E}(\omega)\hat{E}(-\omega)}_{|\hat{E}(\omega)|^2}\;\underbrace{\frac{\hat{G}(-\omega)+\hat{G}(\omega)}{2}}_{\mathrm{Re}(\hat{G}(\omega))}\;\frac{d\omega}{2\pi}$$

298 So how do I clean up this stuff.

You have a real vector space of $(E(t), I(t))$ and an energy bilinear form $\int E(t)I(t)\,dt$. You decompose. You have two real vector spaces $\{E(t)\}$ $\{I(t)\}$ and a pairing $\int E(t)I(t)\,dt$. You have the group of translation acting, and can decompose into irreducibles - these are real two planes corresp to each $\omega > 0$

Start again. You have some circuit

The work done is $\int E(t)I(t)\,dt$. The power at time t is $E(t)I(t)$

Fix $\omega > 0$ consider ~~$\{E(t)\} = \mathbb{C}\cos\omega t + \mathbb{C}\sin\omega t$~~

$$E(t) = \operatorname{Re}\left(Ae^{-i\omega t}\right)$$

A complex amplitude $A = a + ib$

$$= a\cos\omega t + b\sin\omega t$$

$$I(t) = \operatorname{Re}\left(Be^{-i\omega t}\right)$$

$B = c + id$

$$= c\cos\omega t + d\sin\omega t$$

$$E(t)I(t) = ac\cos^2\omega t + (ad+bc)\sin\omega t\cos\omega t + bd\sin^2\omega t.$$

$$\frac{1}{2N}\int_{-N}^{N} E(t)I(t)\,dt \Rightarrow \frac{ac+bd}{2} = \frac{1}{2}\operatorname{Re}\left(A\bar{B}\right)$$

~~Real part~~

$$d^t Z^{-1} dE$$

Introduce sesquilinear forms on $\bar{C}^0, C^1$

$$\langle \xi', \xi \rangle_0 = \sum_x \overline{\xi'_x}\,\xi_x$$

identifies $\bar{C}^0 \xrightarrow{\sim} \bar{C}_0$

$$\langle \xi', \xi \rangle_1 = \sum_\sigma \overline{\xi'_\sigma}\, Z_\sigma(s)^{-1}\xi_\sigma$$

299 Go back to cosheaf on a tree, say a correspondence $W \overset{a}{\underset{b}{\rightrightarrows}} V$, the tree being $\mathbb{R}$ with vertices $\mathbb{Z}$.

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & W \arrow[dl, "b"'] \arrow[dr, "az"] & & zW & \\
z^{-1}V & & V & & zV & & z^2V \\
\bullet & & \bullet & & \bullet & & \bullet \\
-1 & & 0 & & 1 & & 2
\end{tikzcd}$$

Before you looked at the case where $C_\cdot(X, F)$ is acyclic i.e. $az+b : W[z,z^{-1}] \xrightarrow{\sim} V[z,z^{-1}]$ Now you want variants. ~~Keep~~ take W, V finite dimensional over $\mathbb{C}$ This means $az+b$ invertible $\forall z \in \mathbb{C}$, $z \neq 0$. Now suppose $az+b$ is invertible for $z \notin S^1$. What happens? Again get splitting of $V = V^+ \oplus V^-$ where V. Wait $(az+b)^{-1}$ $\exists$ for $z \notin S^1$ means the complex of L^2 chains on the tree is acyclic. $v \in V^+$ means $v_+ = (az+b)\sum_{n\geq 0} w_n z^n$ In fact you take $(az+b)^{-1}v = \sum_{n\in\mathbb{Z}} w_n z^n$, then v_+ is and v_- is $(az+b)\sum_{n<0} w_n z^n$

$$v_- = \cdots + \overbrace{(aw_{-2}+bw_{-1})}^{0} + aw_{-1}$$

So we have $v_+ = bw_0 + \underbrace{(aw_0+bw_1)}_{0} z + \underbrace{(aw_1+bw_2)} z^2$

$w_0 = b^{-1}v_+$ in same sense

$w_1 = -b^{-1}aw_0 = -b^{-1}ab^{-1}v_+$

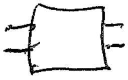
$w_2 = (-b^{-1}a)w_1 = b^{-1}ab^{-1}ab^{-1}v_+$

In some sense we have $\|b^{-1}a\| < 1$. on this part. ~~So lets see what we can do?~~

It seems there is no reflection coefficient

Let's try to set up something with a reflection coefficient.

300 Want $V = V^+ \oplus V^-$

~~Want W to be a correspondence~~ on V such that each element of V^+ determines an element of V^+ ~~[illegible]~~ (transmission ~~[illegible]~~) and an element of V^- (reflection). What you might do here is to iterate a correspondence resulting from a scattering situation. This means ~~[illegible]~~ connecting a 2-port  iteratively. But you first need to get 2-ports clear in your mind.

~~[illegible]~~ analogy with  Riemann surface 2-bdry components.

~~[illegible]~~ Back to graph ~~[illegible]~~ every edge has two vertices. (Lusztig's condition?). You distinguish current and voltage. a 1-port ~~[illegible]~~ has attached ~~[illegible]~~ two 1-diml. spaces: voltage (drops) and currents, ~~each of which~~ to trivialize you choose an orientation.

Look at an edge. ~~[illegible] edge~~

$$\boxed{(E_x - E_y) I_{xy}}$$

~~When you [illegible] fix a frequency ω > 0 this means~~

~~you allow~~ Wait: so far you have given a real 2 dimensional space split into lines and ~~a pa~~ a power pairing between the lines - so you have a real symplectic (also orthogonal hyperbolic) plane.

Now introduce time. You ~~[illegible]~~ consider applied voltage, response current. Translation invariance

301 leads to Fourier transform of $E(t), I(t)$.

Basically real functions $E(t)$ are decomposed into 2 dim'l subspaces $\cos(\omega t)$ $\sin(\omega t)$ $\forall \omega > 0$. Choose rep. $E = \mathrm{Re}\left(\hat{E}\, e^{-i\omega t}\right)$. What this does is to put a complex structure. Before you had a real line of possible voltages and a real line of possible currents and a pairing between them. Now you [redacted] replace the line of E by the 2 plane of periodic functions of t with frequency $\omega > 0$. Unique complex structure on this 2 plane such that time evolution is mult by $e^{-i\omega t}$. Next we get the impedance line: $\hat{E}(\omega) = Z(\omega)\hat{I}(\omega)$.

Now look at a network X of cap.+L's.

$$\begin{tikzcd}
\bar{C}^0(X,\mathbb{R}) \arrow[r, "d"] \arrow[d, phantom, "\times"] & C^1(X,\mathbb{R}) \arrow[d, phantom, "\times"] \arrow[d, "R^{-1}", shift left=4] \\
\bar{C}_0(X,\mathbb{R}) \arrow[d] & C_1(X,\mathbb{R}) \arrow[l, "\partial = d^t"'] \arrow[d] \\
\mathbb{R} & \mathbb{R}
\end{tikzcd}$$

$$E(t) = L\frac{dI(t)}{dt}$$

to show $\partial R^{-1} d$ is an isom. it suffices to show inj. Let $\partial R^{-1} dE = 0$. Then $0 = \langle E, \partial R^{-1} dE\rangle = \langle dE, R^{-1} dE\rangle$

$$= \sum_\sigma (dE_\sigma)^2 R_\sigma^{-1} \Rightarrow dE_\sigma = 0 \;\; \forall \sigma.$$

Next step introduce time, so $\mathbb{R}$ above is replaced by real fns of t $\mathbb{R}\{t\}$

~~smooth fns~~

So what's important ~~is~~ not $s=0$ behavior.

What we do: ~~$\bar{C}^0(X,\mathbb{R}) \xrightarrow{d} C^1(X,\mathbb{R})$~~.

examine periodic fns. freq. ω. $\mathbb{R}\cos(\omega t) + \mathbb{R}\sin\omega t$

$$\mathrm{Re}\left(Ae^{-i\omega t}\right)$$

$$\begin{tikzcd}
& \bar{C}^0(X,\mathbb{C}) \arrow[r, "d"] & C^1(X,\mathbb{C}) \arrow[dd, "Z^{-1}"] & \\
& \times & \times & \\
& \bar{C}_0(X,\mathbb{C}) \arrow[d] & C_1^{\#}(X,\mathbb{C}) \arrow[l, "\partial"'] \arrow[d] & \\
\sum \mathrm{Re}\left(\bar{\hat{E}}_x \hat{I}_x\right) \arrow[ur] & \mathbb{R} & \mathbb{R} & \sum_\sigma \mathrm{Re}\left(\bar{\hat{E}}_\sigma \hat{I}_\sigma\right) \arrow[ul]
\end{tikzcd}$$

to see $\partial Z^{-1} d$ is an isom, enough to check inj.

$$\left(E, \partial Z^{-1} d E\right) = \left(dE, Z^{-1} dE\right)$$

In general for $\{E_\sigma\} \in C^1$. we have

$$\left(E, \partial Z^{-1} d E\right) = \sum_\sigma \bar{E}_\sigma \; E_\sigma Z_\sigma^{-1} = \sum_\sigma |E_\sigma|^2 Z_\sigma^{-1}$$

It seems that you can use $\left\langle \sum \bar{E}_x I_x \right\rangle = \langle E, I \rangle$

You can use the sesquilinear form as well as its real part.

Now understand existence and in more generality because we can use more general $Z_\sigma(s)$ than Ls, $\frac{1}{Cs}$. Need $\mathrm{Re}(s) > 0 \Rightarrow \mathrm{Re}(Z(s)) > 0$, Now what sort of things?

Idea: How to get rid of singularities at $s=0$ maybe is to use Dirac ops.

Take a LC network, distinguish 2 vertices x,y and find relation between ~~the cur~~ applying a voltage drop $E_x - E_y$ ~~current~~ from x to y, and the current going in at x. Look other way: current 1 going in at x and coming out at y, then find $E_x - E_y$. Want the impedance $E_x - E_y = Z$ to have the ~~sig~~ expected properties $Re(Z(s)) > 0$ for $Re(s) > 0$.

The point is that when a subset S of nodes is selected (to apply voltages) ~~to~~ what? Consider / currents ~~I~~ · ∂I supported in S

respose $E^0 \longmapsto \partial Z^{-1} d E^0$ isom between $\bar{C}^0$ and $\bar{C}_0$

Choose S restrict to $E^0 \ni \partial Z^{-1} d E^0$ has supp in S.

Note $$\langle E^0, \partial Z^{-1} d E^0 \rangle = \sum_{x \in S} \overline{E^0_x} (\partial Z^{-1} d E^0)_x$$ power into network.

$$\shortparallel$$

$$\langle dE^0, Z^{-1} dE^0 \rangle = \sum_\sigma |(dE^0)_\sigma|^2 Z^{-1}_\sigma$$ has pos real part.

What are you trying to say? You have for each s an operator

You're trying to describe what you see at the set S. You have an operator from voltages on S (0 cochains on S mod. constants) to currents into S (0 chain of avg 0). This depends on s

Maybe you need to go the other way. Start with the fact that you have an invertible $\partial Z^{-1} d : \bar{C}^0 \to \bar{C}_0$

depending on s analytically in fact rationally, in fact $Z_\sigma(s)^{-1} = \frac{1}{Ls}$ or Cs in the ~~case~~ case I have in mind. Lets examine this situation. Basically you have something going on in degree 1, and you are compressing it to C^0. What is the ~~situation~~ in degree 1? A complex vector space [illegible] C^1 ~~[illegible]~~ ~~[illegible]~~ [illegible] with a basis, (assume simplices oriented), ~~[illegible]~~ another complex v.s. C_1 also with basis. Power form $\sum_\sigma E^1_\sigma \overline{I^1_\sigma}$

Let's start again with an edge, really a 1-port. ~~[illegible]~~ real line of voltage drops, real line of currents, once the edge is oriented these lines are trivialized, so there are natural metrics. Power $E \cdot I$. Now bring in time. Have $E(t)$ $I(t)$ work $\int E I \, dt$. Two types

$$Q = CE \qquad E = L\dot{I}$$
$$\boxed{I = C\dot{E}}$$

import restrict $E(t)$ $I(t)$ to periodic fns. of freq $\omega > 0$.

$$E(t) = \int \hat{E}(\omega) e^{-i\omega t} \frac{d\omega}{2\pi} \qquad I(t) = \int \hat{I}(\eta) e^{-i\eta t} \frac{d\eta}{2\pi}$$

$$\int E(t) I(t) \, dt = \int \hat{E}(\omega) \underbrace{\hat{I}(-\omega)}_{\overline{\hat{I}(\omega)}} \frac{d\omega}{2\pi}$$

~~[illegible]~~

$$E(t) = \int_0^\infty \frac{d\omega}{\pi} \operatorname{Re}\left(\hat{E}(\omega) e^{-i\omega t}\right) \qquad I(t) = \int_0^\infty \frac{d\omega}{\pi} \operatorname{Re}\left(\hat{I}(\omega) e^{-i\omega t}\right)$$

$$\int E(t) I(t) \, dt = \int_0^\infty \frac{d\omega}{\pi} \left(\underbrace{\frac{\hat{E}(\omega)\overline{\hat{I}(\omega)} + \hat{E}(-\omega)\overline{\hat{I}(-\omega)}}{2}}_{\operatorname{Re}\left(\hat{E}(\omega) \cdot \overline{\hat{I}(\omega)}\right)} \right)$$

305 So what structure do you have? An oriented edge gives a voltage amplitude function $\hat{E}(\omega)$ &, current amp $\hat{I}(\omega)$ related by $\hat{E}(\omega) = Z(\omega)\,\hat{I}(\omega)$.

Need L.T. - understand power in this picture

So use L.T.

$$E(t) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{st}\,\hat{E}(s)\,ds$$

$$\hat{E}(s) = \int_0^\infty e^{-st}E(t)\,dt.$$

$$E(t)\,I(t)\,\cancel{dt} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\,I(t)\,\hat{E}(s)\,ds$$

---

need abstract version. You have this complex $\bar{C}^0 \xhookrightarrow{d} C^1$

$$\begin{tikzcd}
\bar{C}^0 \arrow[r, hook, "d"] \arrow[d, phantom, "\times"] & C^1 \arrow[d, "Z^{-1}"] \\
\bar{C}_0 \arrow[d] & C_1 \arrow[l, two heads, "\partial"'] \arrow[d] \\
? & ?
\end{tikzcd}$$

and its dual $\bar{C}_0 \overset{\partial}{\twoheadleftarrow} C_1$ and an isom. $Z^{-1} : C^1 \to C_1$

To give such an isomorphism amounts to a pairing

You ~~so~~ consider $\bar{C}^0 \to \bar{C}_0$ $\boxed{\partial Z^{-1} d}$. To show this is an isom. ~~so~~ you should inj. So if $\partial Z^{-1} dE^0 = 0$, then $\langle E^0, \underset{\overset{\|}{0}}{\partial Z^{-1} dE^0}\rangle = \underbrace{\langle dE^0, Z^{-1} dE^0\rangle}_{\text{has real part } > 0.}$

So the, basic structure seems to be a complex, vector space C^1, some kind of dual C_1, some kind of isom ~~and an map~~ $C^1 \xrightarrow{Z^{-1}} C_1$ such that $\mathrm{Re}\langle \xi, Z^{-1}\xi\rangle > 0$. The argument, then gives a similar structure on any subspace $\bar{C}^0$ of C^1.

306 So you seem to have V a complex vector space $Z^{-1} : V \xrightarrow{\sim} V^*$ in your case this is a symmetric bilinear form. $\mathrm{Re}\langle \xi, Z^{-1}\xi \rangle > 0$.

I'm reminded of Siegel UHP complex symmetric matrices with pos. def. imaginary part.

Siegel. Let V be a real symplectic vector space, form ω, choose $(\ ,\ )$ pos. def. inner product, represent $\omega(x,y) = (x, Sy)$

$\omega(Sx,y) = -\omega(x,Sy)$
S invertible

Let $|S| = (-S^2)^{+1/2}$ $\quad J = \frac{S}{|S|}$

$$\omega(x,y) = \left(|S|x, \frac{S}{|S|}y\right)$$

So can suppose V complex with herm. scalar prod $\langle\ ,\ \rangle$ Re = real scalar prod. $\mathrm{Im} = \omega$. Define polar. of V to be $J \ni J^2 = -1$

$\omega(Jv_1, v_2) = -\omega(v_1, Jv_2)$
$= \omega(Jv_2, v_1)$
pos. scal. prod.

Dec 25. $V = \mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$ [illegible] $i = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} g^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = [illegible]\ g^{-1}$$

~~$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\ ) = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$~~

~~$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad g^{-1} = \begin{pmatrix} d & \\ -c & a \end{pmatrix}$~~

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$g^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a^t & c^t \\ b^t & d^t \end{pmatrix}\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -b^t & -d^t \\ a^t & c^t \end{pmatrix}\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} +d^t & -b^t \\ -c^t & a^t \end{pmatrix}$$

What is going on next?

$j = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad j^{-1} = \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix} = -j = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$

$d = -a^t$
$b^t = b$
$c^t = c$

307 So a polarization j has the form

$$j = \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} \quad \text{with} \quad b = b^t \quad c = c^t$$

and

$$j^2 = \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} = \begin{pmatrix} a^2+bc & ab-ba^t \\ ca-a^tc & cb+(a^t)^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Lie element $X = \begin{pmatrix} \dot a & \dot b \\ \dot c & -\dot d^t \end{pmatrix}$ $\quad -X = \begin{pmatrix} \dot d^t & -\dot b^t \\ -\dot c^t & \dot d^t \end{pmatrix}$ $\qquad \dot b, \dot c$ symm $\quad \dot d = -\dot a^t$

$$X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad -X = \begin{pmatrix} -\alpha & -\beta \\ -\gamma & -\delta \end{pmatrix} = \begin{pmatrix} \delta^t & -\beta^t \\ -\gamma^t & \alpha^t \end{pmatrix}$$

$\beta = \beta^t, \ \gamma = \gamma^t$
$\alpha = -\delta^t \quad \delta = -\alpha^t$

$\therefore X = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha^t \end{pmatrix}$ with β, γ symm.

Go back to ~~the~~ LC ~~system~~ networks.
try to write up the choices.

resistance theory. You have a conn. graph each edge~~s~~ has ~~distinct diff~~ two diff nodes. orient edges.
state at vertex^x is a pair of numbers E_x, I_x
state at edge$^\sigma$ is also a pair of numbers E_σ, I_σ
given power $E_x I_x$ going in at vertex $\quad E_\sigma I_\sigma$ being turned into heat
~~The~~ given resistance $R_\sigma > 0$ at each edge.
~~Basic~~ Focus on dim 1.
You have two real v.s. $C^1(X, \mathbb{R})$ $\quad C_1(X, \mathbb{R})$ in duality with power pairing. You have this resistance which is an isom. $R^{-1}: C^1(X,\mathbb{R}) \xrightarrow{\sim} C_1(X, R)$
i.e. you have a bilinear form on $C^1(X,\mathbb{R})$, in fact symmetric + pos. def. ~~Next you have~~
Next look at degree 0. $\bar C^0(X,\mathbb{R}) = C^0(X,\mathbb{R})/\mathbb{R}$ real v.s.
dual v.s. is $\bar C_0(X,\mathbb{R})$ = 0-chains of aug $= 0$. ~~have~~ Again you have two real v.s. in duality. ~~that~~ You have $d: \bar C^0 \hookrightarrow C^1$
transpose $\partial: C_1 \twoheadrightarrow \bar C_0$. You ~~wish~~ want to ~~solve~~ show ~~d ∂F~~
$\partial R^{-1} d : \underset{\text{applied volts}}{\bar C^0} \xrightarrow{\sim} \underset{\text{resulting currents}}{\bar C_0}$ is an isom. This is a map from the v.s. $\bar C^0$ to its dual bilinear form.

So the linear algebra problem you have is: Given a v.s. V, a n.d. bilinear form $R^{-1}: V \xrightarrow{\sim} V^*$, a subspace $W \overset{d}{\hookrightarrow} V$, you want the restriction of R^{-1} to W to be non degenerate. Sufficient: real vector spaces, symmetric pos. definite bilinear form. The actual proof proceeds as follows:, You wan $d^tR^{-1}d : W \to W^*$ to be an isom, enough to be injective, if $d^tR^{-1}d(w) = 0$, then apply this linear fnl to w ~~itself~~. $0 = \langle w, d^tR^{-1}dw \rangle$ $= \langle dw, R^{-1}dw \rangle$, > 0 if $dw \neq 0$, so $dw = 0 \Rightarrow w = 0$.

Note use only the symmetric part of the bilinear form.

Next restrict applied voltages to a subset S of vertices. Here you start with the space $W = \bar{C}^0$ and the non degenerate bilinear form $d^tR^{-1}d : \bar{C}^0 \longrightarrow \bar{C}_0 = W^*$. Then we are concerned with the subspace of $\bar{C}_0$ consisting of currents supported in S. and the quotient space of $\bar{C}^0$ where we ignore voltages at the vertices outside of S, i.e. we restrict $\bar{C}^0(X) \longrightarrow \bar{C}^0(S)$. So we have

$$\begin{tikzcd}
W \arrow[r, "T"', "\sim"] \arrow[d, two heads] & W^* \\
U & U^* \arrow[u, hook]
\end{tikzcd}$$

given

so you need to see that $U^* \hookrightarrow W^* \xrightarrow{T^{-1}} W \twoheadrightarrow U$ is an isom.

Go back to

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r, hook] \arrow[d, "s"'] & U \arrow[r] \arrow[d] & U/V \arrow[r] & 0 \\
0 & V^* \arrow[l] & U^* \arrow[l] & (U/V)^* \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

What I'm trying to say is that $\exists$ canonical splitting arising from opposite complementary filtrations.

Move on to complex case.

at edge have two real functions $E(t), I(t)$ say L^2

work pairing $\int_{-\infty}^{\infty} E(t)I(t)\,dt$, use F.T. $E(t)=\int_{-\infty}^{\infty}\hat{E}(\omega)e^{-i\omega t}\frac{d\omega}{2\pi}$

$$\|$$

$$\int_{-\infty}^{\infty} dt \int \hat{E}(\omega)e^{-i\omega t}\frac{d\omega}{2\pi}\,\hat{I}(\eta)e^{-i\eta t}\frac{d\eta}{2\pi}$$

$$\iint \frac{d\omega\,d\eta}{(2\pi)^2}\hat{E}(\omega)\hat{I}(\eta)\underbrace{\int_{-\infty}^{\infty} e^{-i(\omega+\eta)t}\,dt}_{2\pi\,\delta(\omega+\eta)}$$

$$\int \frac{d\omega}{2\pi}\hat{E}(\omega)\hat{I}(-\omega)$$

Break up according to freq $|\omega|$.

$$\int_0^{\infty}\left(\hat{E}(\omega)e^{-i\omega t}+\hat{E}(-\omega)e^{i\omega t}\right)\frac{d\omega}{2\pi}$$

$$2\,\mathrm{Re}\left(\hat{E}(\omega)e^{-i\omega t}\right).$$

Power $$\int_0^{\infty}\frac{d\omega}{2\pi}\underbrace{\hat{E}(\omega)\underline{\hat{I}(-\omega)}+\underline{\hat{E}(-\omega)\hat{I}(\omega)}}_{2\mathrm{Re}\left(\hat{E}(\omega)\overline{\hat{I}(\omega)}\right)}$$

Now fix $\omega>0$ get 2 complex nos. $\underline{\hat{E}(\omega)}$, $\hat{I}(\omega)$. and sesquilinear bilinear form $\mathrm{Re}\left(\hat{E}(\omega)\overline{\hat{I}(\omega)}\right)$ which is nondegenerate.

Consider V, W $\mathbb{C}$ v.s. and non deg $\{v,w\}$ non-degenerate. Get $\bar{V}\xrightarrow{\sim}W^*$. Suppose given impedance $V\xrightarrow[\sim]{Z'}W$. Then get $V^*\simeq\bar{V}$.

310 Let V, W be ℂ vector spaces let

~~$B(v, w)$ be a pairing of the underlying~~

$S(v, w)$ be a non-degenerate sequi-linear form, linear in W, antilinear in v. S same as a ℂ-linear isom $\bar{V} \xrightarrow{\sim} W^*$.

Put $\operatorname{Re} S(v,w) = S_r(v,w)$, then S_r is a bilinear form between the underlying real v.s.

$$S(v,iw) = iS(v,w) \quad \text{take Re} \quad S_r(v, iw) = -\operatorname{Im} S(v,w)$$

so $$S(v,w) = S_r(v,w) - i S_r(v, iw)$$

Conversely given $B(v,w)$ a real bilinear pairing $V \times W \to \mathbb{R}$,

put $$H(v,w) = B(v,w) - iB(v,iw)$$

then $$H(v,iw) = \underbrace{+iB(v,iiw)} + B(v,iw)$$

$$iB(v,w) + B(v,iw)$$

$$\|$$

$$iH(v,w) = iB(v,w) + B(v,iw)$$

Thus H is ℂ linear in w. Assume $B(iv,iw) = B(v,w)$

Then $$+iH(v,w) = B(v,\overset{①}{iw}) + iB(v,w)$$

$$H(iv,w) = B(iv,w) - i\underbrace{B(iv,iw)}_{B(v,w)}$$

$$\|$$

$$-B(iv, iiw)$$

$$\|$$

$$-B(v,\overset{①}{iw})$$

$$\therefore H(iv,w) = -iH(v,w)$$

H is ℂ anti-linear in v.

311 Point: Any $\overset{\mathbb{R}\text{-linear}}{}$ $f : V \to \mathbb{R}$ extends uniquely to a $\mathbb{C}$-linear $g : V \to \mathbb{C}$, namely

$$g(v) = f(v) - if(iv)$$

Given $B(v,w) : V \times W \to \mathbb{R}$ $\mathbb{R}$-bilinear

get ! extns $B(v,w) - iB(v,iw) : V \times W \to \mathbb{C}$

$$B(v,w) + iB(iv,w)$$

which are $\mathbb{C}$-linear in w, anti lin. in v resp.

these agree $\Leftrightarrow$ $B(iv,w) = B(v,-iw)$

$\Leftrightarrow$ $B(iv,iw) = B(v,w)$.

Back to edges. For each $\omega > 0$ get 2 cx. numbers – ~~skip you~~

You have $V = C^1(X,\mathbb{C})$, $W = C_1(X,\mathbb{C})$

nondegenerate sesquilinear pairing $H(I,E) = \sum_\sigma \bar{I}_\sigma E_\sigma$, whose real part is the power. Also have

$Z^{-1} : V \longrightarrow W$ $\qquad E \longmapsto Z^{-1}E = \{\sigma \mapsto Z_\sigma^{-1} E_\sigma\}$.

Get sesquilinear form on V:

$$H_Z(E',E) = \sum_\sigma \overline{Z_\sigma^{-1} E'_\sigma}\, E_\sigma$$

Let $S(v',v)$ be a non-degenerate sesquilinear for on V, $\mathbb{C}$-linear in v, anti-linear in v'.

3/12 $\quad S(v', v^{*}) = \sum_i \overline{v'_i}\, s_{ij}\, v_j$ $\qquad s_{ij}$ is invertible

$$s_{ij} = \frac{s_{ij} + \bar{s}_{ji}}{2} + \frac{s_{ij} - \bar{s}_{ji}}{2}$$

$$s = \frac{s+s^*}{2} + \frac{s-s^*}{2}$$

$$S(v', v) = v'^* s v = \boxed{v'^* \frac{s+s^*}{2} v} + \underset{\text{vanishes when } v = v'}{\boxed{v'^* \frac{s-\bar{s}^*}{2} v}}$$

You ~~[illegible]~~ plan to assume $\frac{s+s^*}{2}$ is >0. note then if $\frac{s+s^*}{2}$ ~~is invertible~~ >0 then s is injective

since $s(v) = 0 \implies 0 = v^* s v + (v^* s v)^* = v^* \underbrace{(s+s^*)}_{>0} v \Rightarrow v = 0$.

So at the moment we have $V \simeq \mathbb{C}^n$ equipped with a sequilinear form $S(v', v) = v'^*(s_{ij}) v$ whose symmetric part $S(v, v) = v^* \frac{s+s^*}{2} v$ ~~[illegible]~~ is pos. def. ~~Is there any point in assuming~~ ~~[illegible]~~ Notice that in this general context ~~[illegible]~~ it does not make sense to ask that ~~[illegible]~~ s be symmetric. need real condition.

In the ~~example~~ s case of one edge

$$Z^{-1} = Cs \text{ or } \frac{1}{Ls}$$

Situation $\quad V \simeq \mathbb{C}^n \qquad W = \overline{V^*} = \overline{\mathbb{C}^n}$

basic pairing is $\quad w, v \longmapsto w^* v = \sum \bar{w}_i v_i$

If you combine with $T: V \to W$ get

$$(v', v) \longmapsto (v', Tv) \longmapsto v'^* T v = \sum_{i,j} \bar{v}'_i T_{ij} v_j$$

and I can split T into hermitian + anti-herm. things.

313 What are the possibilities in dim 1.

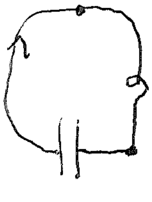
~~[illegible]~~ You are looking for matrix functions of **s** of some sort, analytic in the RHP. ~~[illegible]~~

An LC network is a kind of harmonic oscillator. How? Graph as before - giving you? ~~simplest~~ ~~How do~~ A harmonic oscillator is given by a real vector space equipped with pos. def scalar product and a non-degen. skew symm. form. What is phase space: $\bar{C}^0 \oplus \bar{C}_0$ real coeffs.

Have real structure around

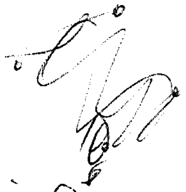
LC network has ~~following~~ phase space?

Each edge σ has voltage drop E_σ and a current I_σ linked by $I_\sigma(t) = C_\sigma \dot{E}_\sigma(t)$ for cap. or $E_\sigma(t) = L \dot{I}_\sigma(t)$ for ind. You need that at each ~~edge~~ vertex the ~~sum of the currents~~ net current leaving is 0 and that the E_σ form a 1-coboundary, i.e. ∃ voltage function on the vertices. It seems that the voltage space is



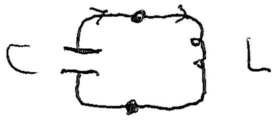
dim (C_1) = e no of edges

dim $(\bar{C}^0)$ = v-1



e = dim C_1 v-e = 1-l

v = dim C_0 2 2



$$E(t) = L\dot{I}(t)$$
$$C\dot{E}(t) = -\dot{I}(t)$$

state $E_x(t)$ ∀ vertex x mod constants

$I_\sigma(t)$ ∀ edge σ.

maybe you have a system with constraints

$$0 \to \bar{C}^0 \longrightarrow C^1 \to H^1 \to 0$$

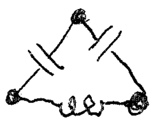
$$0 \leftarrow \bar{C}_0 \longleftarrow C_1 \leftarrow H^0 \leftarrow 0$$

~~Configuration~~ Phase space

You have an LC network - you want to find its oscillations. A state of the system is a pair (E, I) where $E \in \bar{C}^0(X, \mathbb{R})$ and $I \in Z_1(X, \mathbb{R})$. Dimension of state space is $v - 1 + l = e$. There should be a flow on state space $\frac{d}{dt}\begin{pmatrix} E \\ I \end{pmatrix} = \begin{pmatrix} \; ? \; \end{pmatrix}\begin{pmatrix} E \\ I \end{pmatrix}$ It may not take this form?

There is a $\overset{\text{first order}}{\text{DE}}$ for each edge. You have the same number of equations as unknowns, but the mixing is subtle because ~~the loops~~ you have to ~~express~~ choose a basis for Z_1. So this is a subtle question I didn't expect to encounter. Possible methods - Lagrange multipliers for handling constraints

315 Puzzle: Your state space has dimension $v-1+l = e$ (recall $v-e = 1-l$) which may be odd, so it can't be a symplectic ~~manifold~~ vector space.



Is it meaningful to look at the whole system where you have all possible voltage drops + currents, and then reduce ~~voltage~~ cochains to ~~conserva~~ 1-coboundaries and 1-chains to 1-cycles.

1 cochains + 1 chains

Another approach - look at ~~analytical~~ L.T. picture where we have an invertible matrix analytic function of s for $Re(s) > 0$, with singularities on $Re(s) = 0$. Use singularity to get normal modes. $\int_0^\infty f'(t)e^{-st}dt = \left[f(t)e^{-st}\right]_0^\infty - \int_0^\infty f(t)(-s)e^{-st}dt$

number of normal modes.

Back to L.T. picture. How to show that there is a flow on the state space $\bar{C}^0 \oplus Z_1$. Assume
$v-1 + l = e$
the flow exists and you try to find it. Initial values ~~[illegible]~~ is to be solved using L.T. ~~For~~ Assume given a solution $(E(t), I(t))$ for $t \geq 0$, of all the conditions

e.g.

$E_{d_0\sigma} - E_{d_1\sigma} = L_\sigma \dot{I}_\sigma$ if σ is an inductance

$C_\sigma(\dot{E}_{d_0\sigma} - \dot{E}_{d_1\sigma}) = I_\sigma$ if σ is a capacitance

$\partial I = 0$

Apply L.T. to get equations for $\hat{E}(s)$, $\hat{I}(s)$.

$\hat{E}_{d_0\sigma} - \hat{E}_{d_1\sigma} = L_\sigma(s\hat{I}_\sigma - I_\sigma(0))$ σ ind.

$C_\sigma(s(d\hat{E})_\sigma - (d\hat{E})_\sigma(0)) = \hat{I}_\sigma$ σ cap.

$\partial \hat{I} = 0$

basic eqns are ~~Subspace & Quotient space~~

$$(d\hat{E})_\sigma - (L_\sigma s)\hat{I}_\sigma$$

Q: What does this have to do with what you did before, namely, ~~produce an inverse for~~ to show $\partial Z^{-1} d : \bar{C}^0 \to \bar{C}_0$ is invertible

depends on s, but is invertible for $Re(s) > 0$; inverse should be a rational matrix function of s.

You want $\hat{E}(s) \in \bar{C}^0$ $\hat{I}(s) \in Z_1$

such that $Z^{-1} d\hat{E}$?

$$\left(d\hat{E}_\sigma\right) = \left(\frac{d\bar{E}_\sigma|_{t=0}}{s}\right) + \frac{\hat{I}_\sigma}{C_\sigma s} \quad \text{cap.}$$

$$d\hat{E}_\sigma = -L_\sigma I_\sigma|_{t=0} + Ls\hat{I}_\sigma \quad \text{ind.}$$

$$d\hat{E} - Z\hat{I} = \text{initial values essentially of } E \,\&\, I$$

$$\hat{I} = Z^{-1} d\hat{E} + \text{initial stuff}$$

$$0 = \partial\hat{I} = \partial Z^{-1} d\hat{E} + \partial(\qquad\qquad)$$

How to clear this up? You have

$$\begin{tikzcd}
\bar{C}^0 \arrow[r, "d"] & C^1 \arrow[r] \arrow[d, "Z^{-1}"] & {} \\
\bar{C}_0 & C_1 \arrow[l, "\partial"'] & Z_1 \arrow[l, hook']
\end{tikzcd}$$

317 Start with $E_0, I_0 \in \bar{C}^0$, Z_1 initial data

solve $d\mathring{E}_{t,\sigma} = \frac{1}{C_\sigma} \mathring{I}_{t,\sigma}$ or $d\mathring{E}_{t,\sigma} = L_\sigma \mathring{I}_{t,\sigma}$

use LT. $\mathcal{L}(\mathring{f}) = s\hat{f} - f_0$

$$\left(s\,d\hat{E}_{\omega,\sigma} - dE_{0,\sigma}\right) = \frac{1}{C_\sigma} \hat{I}_{\omega,\sigma}$$

$$d\hat{E}_{\omega,\sigma} - \frac{1}{C_\sigma s} \hat{I}_{\omega,\sigma} = dE_{0,\sigma} \qquad \sigma \text{ cap}$$

$$d\hat{E}_{\omega,\sigma} - L_\sigma s \hat{I}_{\omega,\sigma} = -L_\sigma I_{0,\sigma} \qquad \sigma \text{ ind}$$

$$\therefore \quad d\hat{E}_\omega - Z\hat{I}_\omega = \text{initial value eqs.} \in \bar{C}^0{}_+$$

[illegible]

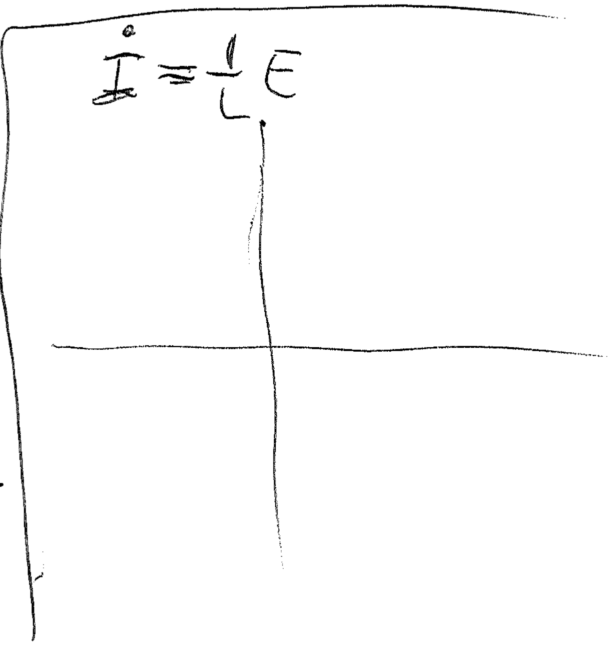
$$C_\sigma d\mathring{E}_{t,\sigma} \simeq \frac{1}{C_\sigma} I_{t,\sigma} \quad \text{or} \quad dE_{t,\sigma} = L_\sigma \mathring{I}_{t,\sigma}$$

$$C_\sigma\left(s\,d\hat{E}_{s,\sigma} - dE_{0,\sigma}\right) = \hat{I}_{s,\sigma} \qquad d\hat{E}_{s,\sigma} = L_\sigma\left(s\hat{I}_{s,\sigma} - I_{0,\sigma}\right)$$

$$C_\sigma s\,d\hat{E}_{s,\sigma} = I_{s,\sigma} + C_\sigma dE_{0,\sigma} \qquad (L_\sigma s)^{-1} d\hat{E}_{s,\sigma} = \hat{I}_{s,\sigma} - \left(\overset{s}{[illegible]}\right)^{-1} I_{0,\sigma}$$

$$Z_s^{-1} d\hat{E}_s = \hat{I}_s + \begin{cases} C_\sigma dE_{0,\sigma} \\ -\left(\overset{s}{[illegible]}\right)^{-1} I_{0,\sigma} \end{cases}$$

$$\partial Z_s^{-1} d\hat{E}_s = \overset{\overset{0}{\shortparallel}}{\partial \hat{I}_s} + \partial \begin{cases} C_\sigma dE_{0,\sigma} \\ -\left(\overset{s}{[illegible]}\right)^{-1} I_{0,\sigma} \end{cases}$$



$$\mathring{I} = \frac{1}{L} E$$