capuccino LHR £1.20 bus £8.00

I propose to review the proof of M-invariance of HC for h-unital rings, k flat. See if you can recall the main steps.

$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ s-firm assume A $l+r$ flat, B h-unital, k flat

A left flat firm $\Longrightarrow$ $P \otimes_A A = P$ is B-flat firm

rt flat firm $\Longrightarrow$ $A \otimes_A Q = Q$ is B^{op}-flat firm

then we know that if B k-flat then

P B-flat $\Longrightarrow$ $P = \varinjlim B^{n_\alpha}$

$\Longrightarrow$ P is k-flat. Sim. Q is k-flat.

finally $A = Q \otimes_B P = \varinjlim Q \otimes_B B^{n_\alpha} = \varinjlim Q^{n_\alpha}$ is k-flat.

$P \otimes_k V = P \otimes_B (B \otimes_k V)$

exact in V as B is k-flat

exact since P is B flat.

~~$V \otimes_k A = (V \otimes_k Q) \otimes_B P$~~

$A \otimes_k V = Q \otimes_B (P \otimes_k V)$

exact exact

2 basic isom in HH

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \qquad A \overset{L}{\otimes}_A = Q \overset{L}{\otimes}_B P \overset{L}{\otimes}_A$$ use Q_B or ${}_BP$ flat

$$= P \overset{L}{\otimes}_A Q \overset{L}{\otimes}_B$$ obv.

$$= B \overset{L}{\otimes}_B$$ B h-unital

need here $P \overset{L}{\otimes}_A Q \xrightarrow{\sim} B \iff B$ h-unital.

argument:

$$\begin{array}{ccc} B \overset{L}{\otimes}_B P \overset{L}{\otimes}_A Q & \longrightarrow & B \overset{L}{\otimes}_B B \\ \downarrow & & \downarrow \\ P \overset{L}{\otimes}_A Q & \longrightarrow & B \end{array}$$

always a quis since P is B-flat

need $P \overset{L}{\otimes}_A Q \to B$ B-nil-quis

also $B \overset{L}{\otimes}_B P \to P$ is A^{op}-nil-quis in general.

So OKAY and then you want to make it work ~~is good~~.

$$A \overset{L}{\otimes}_A = A \otimes_A E \otimes_A$$ ~~So these~~

$$\begin{array}{ccc} F \otimes_B P \otimes_A E \otimes_A Q & \longrightarrow & F \otimes_B B \\ \downarrow & & \downarrow \\ B \otimes_B P \otimes_A E \otimes_A Q & \longrightarrow & B \end{array}$$

$$= Q \otimes_B P \otimes_A E \otimes_A$$

$$= \underbrace{P \otimes_A E \otimes_A Q}\otimes_B$$

need to argue that this is a flat B-bimodule res. of B.

Try to link Morita invariance for K_1 with what you did for K_0.

Maybe an interesting project is to understand Bass FT. Can you get K_0 out of K_1. The first thing you have to do ?

What sort of example? What sort of example?

3 The first thing to understand probably is whether perfect complexes in some sense sit inside of K_1.

Unital theory from Bass. $K_1(R[t,t^{-1}])$

There are maps $K_1(R[t,t^{-1}]) \rightleftarrows K_0 R$.

~~splitting off K_0~~ making $K_0 R$ a retract of the former. How? an invertible $g \in GL_n(R[T])$ $T = \mathbb{Z}$ determines a v.b. on P^1_R by glueing, but $K_0(P^1_R) = K_0(R) \oplus K_0(R)$. So what happens???

What is the effect of the ideal A? OK

Some sort of compact support stuff on the complement. The picture might be to take $P^1_R \supset P^1_{R/A}$. All this looks stupid! You need some mechanism.

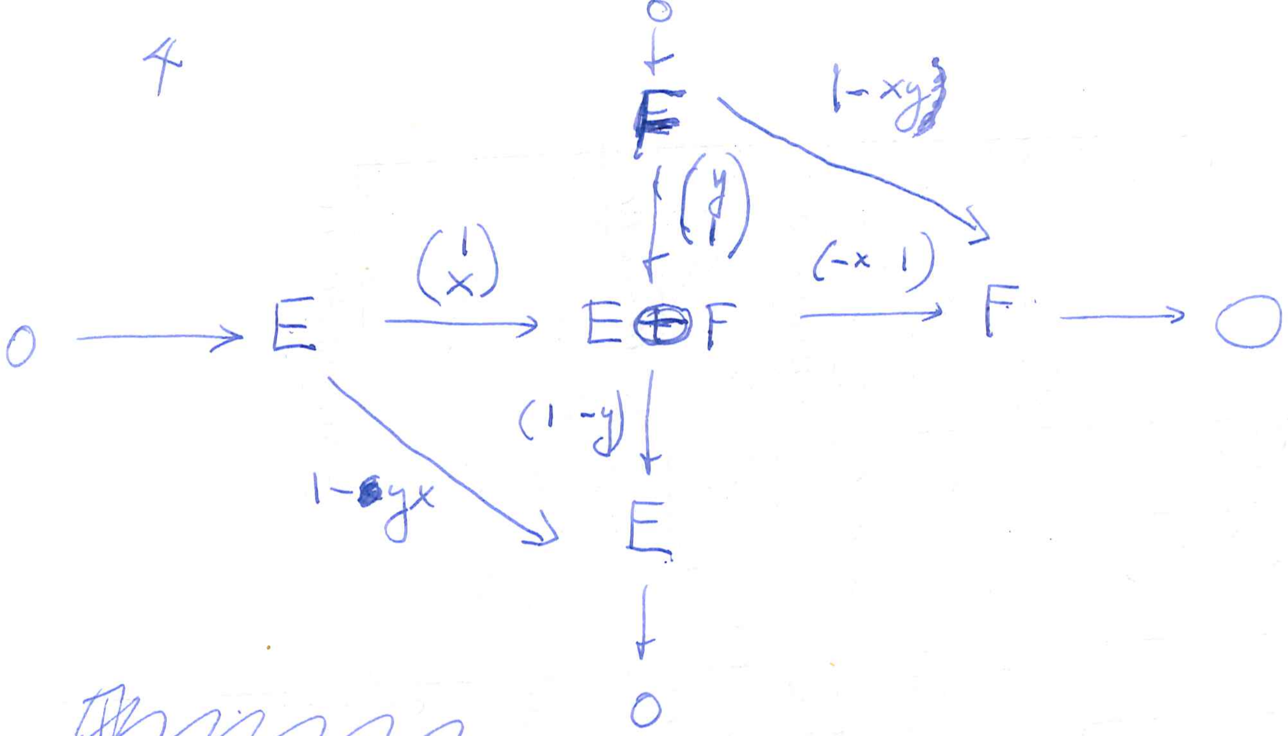
$$K_1 R \to K_1(R/A) \to K_0 A \to K_0(R) \to K_0(R/A)$$

some viewpts. I remember; Morita invariance of $K_1 A$ is related to defining determinant for 1+"trace class" operators

Review M invariance for K_1 of firm rings.
Vaserstein identity $1-xy$ invertible iff $1-yx$ is

$$(1-yx)^{-1} = 1 + yx + yxyx + \cdots \quad \text{formally}$$

$$= 1 + y(1 + xy + xyxy + \cdots)x = 1 + y(1-xy)^{-1}x$$



$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}$$

$$\parallel$$

$$\begin{pmatrix} 1 & 0 \\ y(1-xy)^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1-xy & x \\ -y & 1 \end{pmatrix}$$

$$\parallel$$

$$\begin{pmatrix} 1-xy & x \\ 0 & \underbrace{1+y(1-xy)^{-1}x}_{(1-yx)^{-1}} \end{pmatrix}\begin{pmatrix} 1 & -(1-xy)^{-1}x \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 1 & 0 \\ y(1-xy)^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}\begin{pmatrix} 1 & -(1-xy)^{-1}x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-xy & 0 \\ 0 & (1-yx)^{-1} \end{pmatrix}}$$

Anyway this identity tells me something like given $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ with $A^2=A$, $B^2=B$, $QP=A$, $PQ=B$

that $K_1(A) \longrightarrow K_1(C) \longleftarrow K_1(B)$ have the same image. ~~[illegible]~~ Idea ~~there~~ is assume A left + right flat, then P, Q, ~~[illegible]~~ B, B^{op} flat. This gives some sort of injectivity when A, B both ~~[illegible]~~ flat.

when $B = A/I$, ~~[illegible]~~ $A \to B$ neghom. iff $AIA = 0$. ~~The first thing to do.~~

Start with $B = B^2$. Choose $F \xrightarrow{\rho} B$ with F left B-flat. Define $A = F$ equipped with $a_1 a_2 = \rho(a_1) a_2$ $\quad A \longrightarrow B$ surj. version ~~analog~~ to inclusion of a left ideal gen. B.

$$BA = A \qquad AB = B$$

$$B \otimes_B A \xrightarrow{\sim} A \qquad A \otimes_A B \xrightarrow{\sim} B \qquad ?$$

In any case go back to $B = A/I$ where $IA = 0$, so that A becomes a left $B = A/I$ mod. ~~A~~ When is B a firm ring?

$$\begin{pmatrix} A & A/I \\ A & B \end{pmatrix}$$

$$\mathcal{M}(A) \qquad \mathcal{M}(B)$$

$$M \longmapsto A \otimes_A M = M$$

$$N = B \otimes_B N \longleftarrow N$$

$A \otimes_A M \xrightarrow{\sim} M$
$\implies IM = 0$
since $IA = 0$

So B firm as B-mod
$\implies$ B firm as A-mod

$I \hookrightarrow A \twoheadrightarrow B \qquad IA = 0. \quad \therefore I^2 = 0$

$M(I) \hookrightarrow GL(A) \twoheadrightarrow GL(B)$

$\underbrace{M(I)/[GL(A), M(I)]}_{} \longrightarrow GL(A)_{ab} \twoheadrightarrow GL_{ab}(B)$

you want this $= 0$. $\iff AI = I$.

Q: B firm $\iff AI = I$ (ass. A firm)

$$0 \to I \to A \to B \to 0$$

$$\begin{array}{ccccccccc}
 & & A \otimes_A I & \to & A \otimes_A A & \to & A \otimes_A B & \to & 0 \\
 & & \downarrow & & \downarrow \cong & & \downarrow & & \\
0 & \to & I & \to & A & \to & B & \to & 0
\end{array}$$

$$\begin{array}{ccccc}
A & \xrightarrow{\sim} & A * A' & \xleftarrow{\sim} & A' \\
\sim \downarrow & \searrow & \downarrow & & \downarrow \sim \\
B & & A * B' & \leftarrow & B'
\end{array}$$

map

Is $A * A' \longrightarrow A * B'$ surjective

$$\begin{array}{ccccc}
A & \longrightarrow & A * B & \longleftarrow & B \\
 & & \downarrow & & \downarrow w \\
 & & A * B' & \longleftarrow & B'
\end{array}$$

$IB = 0$

$B' = B/I$

$B' \otimes_B P \xrightarrow{\sim} P'$.

$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \qquad \begin{pmatrix} A & Q/QI \\ P & B/I \end{pmatrix}$

$(B/I) \otimes_B P = P$ since $IP = 0$.

$Q' = Q \otimes_B B/I = Q/QI$

7.

~~... suppose I consider A ... what I need to do is to handle coherence ... suppose $A \to A/I$ nilpotent extension ... which is a Morita equivalence~~

Suppose $A \longrightarrow A/I = B$ is a nilpotent extension: $I^n = 0$, and A is idempotent: $A = A^2$.

Then certainly possible for $I^3 \neq 0$; take A unital.
So a nilp extn need not be a meq. is not usually.

$A \longrightarrow A/I$ is meq $\iff$ $AIA = 0$

$AIA = 0 \implies I^3 = 0$. Does $\exists$ example with $I^2 \neq 0$? YES.

$$A = \begin{pmatrix} k & W \\ V & V\otimes W \end{pmatrix}$$

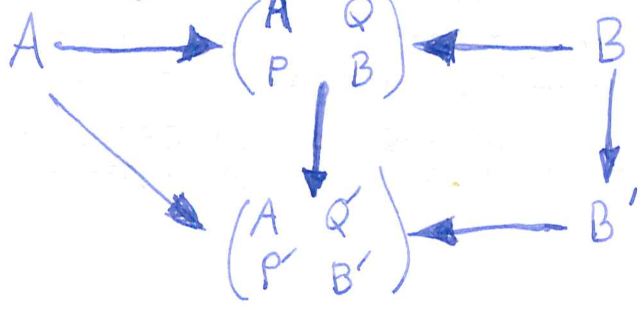
~~$A^2 = \ldots$~~

$$I = \begin{pmatrix} 0 & W \\ V & V\otimes W \end{pmatrix} \qquad I^2 = \begin{pmatrix} 0 & W \\ V & V\otimes W \end{pmatrix}\begin{pmatrix} 0 & W \\ V & V\otimes W \end{pmatrix}$$

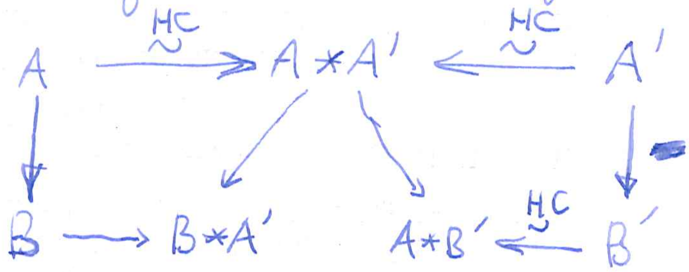
$$= \begin{pmatrix} 0 & 0 \\ 0 & V\otimes W \end{pmatrix} \neq 0$$

$$I^3 = \begin{pmatrix} 0 & W \\ V & V\otimes W \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & V\otimes W \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

I have a proposal to prove M-invariance for cyclic homology. Main point:



8. ~~What's the proof??~~ YES!!!! I want to argue that if B and B' are meq h-unital rings and we choose $A \twoheadrightarrow B$, $A' \twoheadrightarrow B'$ surjective meqs with A, A' flat then we have



$\overset{HC}{\sim}$ denotes HC isom. WAIT. Your basic argument was that was that $B \to B'$ meq hom of h-unital rings $\Rightarrow HC(B) \xrightarrow{\sim} HC(B')$. Proof proceeds by choosing auxiliary A and using canon. isom.

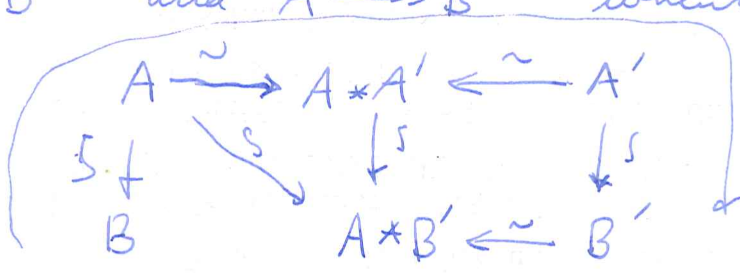
$$A \overset{L}{\otimes}_A = Q \overset{L}{\otimes}_B P \overset{L}{\otimes}_A = P \overset{L}{\otimes}_A Q \overset{L}{\otimes}_B = B \overset{L}{\otimes}_B$$

to get $HH(A) = HH(B)$

$$\begin{array}{ccc} HH(A) & = & HH(B) \\ & \searrow\!\!\!\!\!= & \swarrow \\ & HH(B') & \end{array}$$

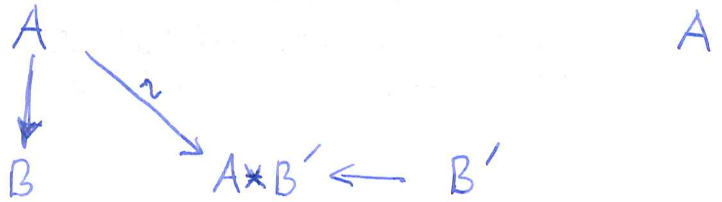
etc.

But then given B, B' meq h-unital you can pick $A \twoheadrightarrow B$ and $A' \twoheadrightarrow B'$ whence get iso etc.



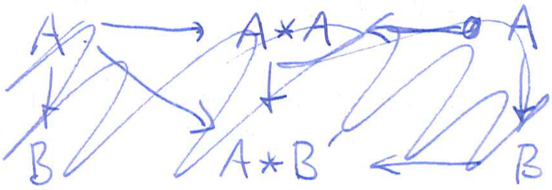
ind. of A'

similarly ind of choice of A.



Suppose have a weq. homom. $B \to B'$

Choose A, get



$$A \to A * A \leftarrow A \qquad A \rightrightarrows A * A \text{ equiv.}$$

seems to be the reason for canon. contr. arg.

$$A \to A * B \leftarrow B$$

Idea: Given B, B' you define $HC(B) \simeq HC(B')$ via

$$A \to A * B \leftarrow B$$

$$\searrow \quad A * B' \leftarrow B'$$

10 propose two things to work on.

1. M inv. of K_1 – go over all the steps.
2. to understand whether K_1 M-invariance my imply K_0 M-invariance.

First point: Given $B = B^2$ can find B-map $A \to B$ with A B-flat firm. Then $B = A/I$ where $IA = 0$

$$\begin{pmatrix} A & A/I \\ A & B \end{pmatrix} \qquad M(A) \qquad\qquad M(B)$$

$$M \qquad\qquad A \otimes_A M$$

A is A-flat $\iff$ A is B-flat. Then

B is A-firm $\iff$ B is B-firm

$$\begin{array}{ccccccccc} & & A \otimes_A I & \longrightarrow & A \otimes_A A & \longrightarrow & A \otimes_A B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow\wr & & \downarrow & & \\ 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

$IA = 0 \Rightarrow I^2 = 0$

$\therefore$ A/I is A-firm $\iff$ $AI = I$.

$$M_n(I)/[GL_n(A/I), M_n(I)] \longrightarrow GL_n(A)_{ab} \longrightarrow GL_n(A/I)_{ab} \longrightarrow 0$$

$$M_n(I) = [GL(A/I), M(I)] \iff I = AI.$$

So now consider $C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$, where B is a both left + right flat, firm M context.

Idea here is $Q = \varinjlim_\alpha F_\alpha$ $\quad F_\alpha$ f.t. free

$$Q \otimes P \longrightarrow A \qquad P \longrightarrow \mathrm{Hom}_A(F_\alpha, A) = F_\alpha^* A$$

$$C = \varinjlim \underbrace{\begin{pmatrix} A & AF_\alpha \\ P & P \otimes_A F_\alpha \end{pmatrix}} \qquad C_\alpha \longrightarrow \begin{pmatrix} A & AF_\alpha \\ F_\alpha^* A & F_\alpha^* A \otimes_A AF_\alpha \end{pmatrix} \simeq M_{n_\alpha+1}(A)$$

$\therefore$ $K_i(C) = \varinjlim K_i(C_\alpha)$ where $K_i(C_\alpha) \rightleftarrows K_i(A)$

11 So next comes? ~~Upperbound for $K_1 A$.~~

The above argument shows that $K_i A \hookrightarrow K_i C$

So how much done? So what seems to work?

A l+r. flat $\Leftrightarrow$ Q, P fl/B $\Rightarrow$ $K(B) \hookrightarrow K(C)$

But ~~these~~ Vaserstein game says $K(B) \to K(C) \leftarrow K(A)$ have same image

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Go over argument $C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$. Suppose A left flat. Then $P = P \otimes_A A$ is B flat. Then

$C = \lim C_\alpha$ ~~$C_\alpha = \begin{pmatrix} A & Q \\ F_\alpha A & F_\alpha Q \end{pmatrix}$~~ $C = \begin{pmatrix} Q \\ B \end{pmatrix} \otimes_B (P \ B)$

$$C_\alpha = \begin{pmatrix} Q \\ B \end{pmatrix} \otimes_B (BF_\alpha \ B) = \begin{pmatrix} Q \otimes_B F_\alpha & Q \\ BF_\alpha & B \end{pmatrix}$$

K-retracts to B. So we have injectivity

$K_i(B) \hookrightarrow K_i(C)$. Same should work for A right flat: $\Longrightarrow$ $A \otimes_A Q = Q$ is B^{op} flat $\Rightarrow Q = \varinjlim F_\alpha B$

$$C = \begin{pmatrix} Q \\ B \end{pmatrix} \otimes_B (P \ B) = \lim \begin{pmatrix} F_\alpha B \\ B \end{pmatrix} \otimes_B (P \ B)$$

Now ~~suppose~~ the Vaserstein argument says that inside $K_1(C)$, the images of $K_1 A$ $K_1 B$ agree. What does that say?

$$\begin{matrix} K_1(A) \to & K_1(C) & \hookleftarrow K_1(B) \\ & \downarrow & \downarrow \\ & K_1(C') & \hookleftarrow K_1(B') \end{matrix}$$

So if B, B' are flat then get

$$\begin{matrix} K_1(A) = & K_1(B) \\ \| & \downarrow \\ & K_1(B') \end{matrix}$$

So it's not finished yet, ~~[illegible]~~ But you ought to be able to ~~handle [illegible] simple~~ finish by using surjectivity in some form.

12 Suppose given pairing $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ with A flat

Now picks megham $A' \twoheadrightarrow B$ w A' flat,

$$\begin{array}{ccccc} A & \hookrightarrow & C' & \hookleftarrow & A' \\ & & \downarrow & & \downarrow \\ & & C & \hookleftarrow & B \end{array}$$

$$\begin{array}{ccccc} K_1 A & \xrightarrow{\sim} & K_1 C' & \xleftarrow{\sim} & K_1 A' \\ & & \downarrow & & \downarrow s \\ & & K_1 C & \hookleftarrow & K_1 \end{array}$$

This proves that $K_1 A \to K_1 C$ injective.

~~Make things clear So you~~

Return to talk on M-inv. of cyclic hom.

Main steps: Things to get straight. Some ideas worth mentioning.

Morita invariance of cyclic homology for h-unital rings

Absolute notion $A \otimes_A A \xrightarrow{\sim} A$, $\mathrm{Tor}_i^A(A,A) = 0$ $i \neq 0$.

L: If A h-unital, then X right bdd cx, $H_* X$ nil $\Rightarrow A \overset{L}{\otimes}_A X = 0$.

$$\begin{array}{ccc} B \overset{L}{\otimes}_B P \overset{L}{\otimes}_A Q & \longrightarrow & B \overset{L}{\otimes}_B B \\ \downarrow s & & \downarrow \\ P \overset{L}{\otimes}_A Q & \longrightarrow & B \end{array}$$

A is A-flat: $\Leftrightarrow$ $P = P \otimes_A A$ is B-flat

~~Anyway~~

Question: The key case to handle is when A, B both l. + r. flat. ~~Take the attitude that yields Basically you consider P, Q.~~ It seems that the critical case is $P = A^n$ $Q = A^m$ with an arbitrary pairing

J.E. Roos

organize talks

$m(A) \xrightarrow{\sim} \mathrm{mod}(\tilde{A})/\mathrm{mod}(\mathbb{Z})$

firm modules, nil modules

meq $m(A) \simeq m(B)$

$M \longmapsto P \otimes_A M$ P firm B, A bim.

$Q \otimes_B N \longleftarrow N$

Vaserstein identity.

$$\begin{pmatrix} 1 & 0 \\ y(1-xy)^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}\begin{pmatrix} 1 & -(1-xy)^{-1}x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-xy & 0 \\ 0 & (1-yx)^{-1} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ y(1-xy)^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1-xy & x \\ -y & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1-xy & x \\ 0 & (1-yx)^{-1} \end{pmatrix}\begin{pmatrix} 1 & -(1-xy)^{-1}x \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1-xy & 0 \\ 0 & (1-yx)^{-1} \end{pmatrix}$$

14. Think about

1) Def firm, nil

$A = A^2 \Longrightarrow \text{firm}(A) \xrightarrow{\sim} \text{mod}(\tilde{A}) / \text{nil}(A)$

$\text{nil}(A) = \bigcup_n \text{mod}(\tilde{A}/A^n)$

What to say about meq? Need to state theorem

Version 1. Let A, B be firm rings. Then up to canon isom any meq $\mathcal{M}(A) \simeq \mathcal{M}(B)$ is given by a Morita context $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ s firm $A \otimes_A A \to A \leftarrow Q \otimes_B P$ etc

Need to define sfirm M cont.

$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ 8 products a_1a_2, qp, aq, qb

pa, bp, b_1b_2, pq

$B \rightleftarrows P \otimes_A Q \qquad (p_1 \otimes q_1)(p_2 \otimes q_2) = p_1 \langle q_1, p_2 \rangle \otimes_A q_2$

A firm ring

Theorem: Equivalence of cat between

cat 1: objects is a triple $(P_A, {}_AQ, \langle , \rangle)$ P_A firm A^{op}-mod

${}_AQ$ firm A-module, $\langle\ \rangle : {}_AQ \otimes P_A \to A$ surjective A-bimodule map

morphisms obvious

cat 2: obj is a firm ring B together with a meq

$\mathcal{M}(A) \xrightarrow{F} \mathcal{M}(B)$

morphism $(B, F) \longrightarrow (B', F')$ is a homom.

$B \xrightarrow{w} B'$ together with an isom: $w_! F \xrightarrow{\sim} F'$

$\mathcal{M}(A)$, $F \swarrow$, $\searrow F'$, $\mathcal{M}(B) \xrightarrow{w_!} \mathcal{M}(B')$

$w_!(N) = B' \otimes_B N$

extra cat stuff which is not too interesting

What are the important points?

Part 1 $\quad mod(R)$ to be gen. to nonunital rings A

cat of A-modules $= mod(\tilde{A})$ $\qquad \tilde{A} = \mathbb{Z} \oplus A = \{n1 + a\}$

is too big: if A unital with 1 e

then

$$mod(\tilde{A}) \simeq mod(A) \times mod(\tilde{A}/A), \quad \tilde{A}/A \cong \mathbb{Z}$$

$$M = eM \oplus \{m - em \mid m \in M\}$$

Def: M firm when $A \otimes_A M \xrightarrow{\sim} M$

M nil when $\exists n$ $A^n M = 0$.

full subcats of $mod(\tilde{A})$: firm(A), $\bigcup_{n \geq 1} mod(\tilde{A}/A^n)$

Thm. 1. ~~[illegible]~~ Obvious fun.

$$firm(A) \longrightarrow mod(\tilde{A}) \Big/ \bigcup_n mod(\tilde{A}/A^n)$$

~~[illegible]~~ fully faithful

When $A = A^2$ it is an equivalence

From now on restrict to idempotent rings A,

put $\mathcal{M}(A) = firm(A)$.

(Add.) ~~[illegible]~~ if $M = AM \Rightarrow A \otimes_A M$ is firm

For any M $\quad A \otimes_A A \otimes_A M$ is firm

Def: say A is a firm ring when $A \otimes_A A \xrightarrow{\sim} A$. $\quad$ i.e. $A \in \mathcal{M}(A)$, $\in \mathcal{M}(A^{op})$

16 **Def** A, B meq when $\mathcal{M}(A) \simeq \mathcal{M}(B)$.

$F(\varinjlim M_i) \xleftarrow{\sim} \varinjlim F(M_i)$

Prop. If $F : \mathcal{M}(A) \to \mathcal{M}(B)$ is right cont. has form $F(M) \xleftarrow{\sim} P \otimes_A M$ where $P = F(A \otimes_A A)$ is a firm B,A-bimod: $P \otimes_A A \xrightarrow{\sim} P$, $B \otimes_B Q \xrightarrow{\sim} Q$.

e.g. $F(M) = M$ has form ~~$F(M) \simeq A$~~ $P = A \otimes_A A$.

A, B firm then any meq $\mathcal{M}(A) \simeq \mathcal{M}(B)$

$$M \longmapsto P \otimes_A M$$

$$Q \otimes_B N \longleftarrow N$$

$$P \otimes_A Q \otimes_B N \simeq N \implies P \otimes_A Q \simeq B$$

Morita context. $A, B, {}_B P_A, {}_A Q_B, Q \otimes_B P \to A, P \otimes_A Q \to B$

Df. A ~~2x2 matrix decomposition~~ Morita context is a ring C equipped with a decomp. $C = A \oplus P \oplus Q \oplus B$ into 4 abelian subgps such that if elements of C are written as 2×2 matrices $\begin{pmatrix} a & q \\ p & b \end{pmatrix}$, then the mult in C is cons. with matrix mult. $\implies$ A, B rings P (B,A)-bimod, Q (A,B)-bimod, $Q \otimes P \to A$, $P \otimes Q \to B$

$$C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

There are 8 products between the comps of C.

$a_1 a_2$, qp	aq, qb
pa, bp	$b_1 b_2$, pq

C is strictly firm: $A \otimes_A A \xrightarrow{\sim} A \xleftarrow{\sim} Q \otimes_B P$ ~~$A\otimes$~~

$$P \otimes_A A \xrightarrow{\sim} P \longleftarrow B \otimes_B P$$

$\therefore$ A, B firm rings, P Q firm bimods
$Q \otimes_B P \xrightarrow{\sim} A$, $P \otimes_A Q \xrightarrow{\sim} B$.

Define M-cont

sfirm M-cont.

Prop. [crossed out] Any A, B firm $\mathcal{M}(A) \simeq \mathcal{M}(B)$ given (up to canon. isom.) by unique sfirm M-cont.

[crossed out] Observation A sfirm $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ can be reconst. from A, P_A, ${}_AQ$, $Q \otimes_{\mathbb{Z}} P \xrightarrow{\langle , \rangle} A$ $\quad (p_1 \otimes q_1)(p_2 \otimes q_2) = p_1 \langle q_1, p_2 \rangle \otimes q_2$

Thm. A fixed firm ring. Then a firm ring B tog. w. an equiv. $\mathcal{M}(A) \simeq \mathcal{M}(B)$ is equivalent to a triple $(P_A, {}_AQ, \langle , \rangle : Q \otimes P \to A)$

P_A firm A^{op}-mod

${}_AQ$ firm A-mod

$\langle , \rangle$ surj A-bimod map.

Ex. A = field k $\begin{pmatrix} A & Q \\ P & P \otimes_A Q \end{pmatrix}$ [crossed out]

[crossed out]

$$HH_0(A) = A/[A,A] = A \otimes_A$$

L A-bimod $\quad L \otimes_A = L/\{la - al\}$

sfirm M cont. $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

$$A \otimes_A = Q \otimes_B P \otimes_A = P \otimes_A Q \otimes_B = B \otimes_B$$

$$\therefore HH_0(A) \simeq HH_0(B).$$

[crossed out] Assume rings flat over $\mathbb{Z}$ (i.e. torsion-free)

P_A right ${}_AQ$ left. $\quad H_n(P \overset{L}{\otimes}_A Q) = \mathrm{Tor}_n^A(P, Q).$

$$P \overset{L}{\otimes}_A Q = P \otimes_A E \otimes_A Q \qquad E \text{ any } \tilde{A}\text{-flat bimodule res. of } \tilde{A}$$

$$\tilde{A} \otimes A \otimes A \otimes \tilde{A} \longrightarrow \tilde{A} \otimes A \otimes \tilde{A} \longrightarrow \tilde{A} \otimes \tilde{A} \longrightarrow \tilde{A} \longrightarrow 0$$

1 $\qquad$ 0

A is h-unital when ~~Tor~~ $A \otimes_A A \xrightarrow{\sim} A$

$\mathrm{Tor}_n^A(A,A) = 0 \quad n \geq 1,$

~~$HB_n(A)$~~ $\quad HH_n(A) = H_n(A \overset{L}{\otimes}_A A)$

Thm. given B, B' ~~A, B~~ h-unital $\quad M(\ ) \xrightarrow{\sim} M(B')$

then there's a canon. iso. $\quad HH_*(B) \simeq HH_*(B')$

also for HC_*, HP^*, etc.

Prof: ~~Assume~~ $C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ s.t. A left + right flat

Then B is h-unital $\iff P \overset{L}{\otimes}_A Q \xrightarrow{\sim} B$ (i.e. $\mathrm{Tor}_n^A(P,Q) = 0$, $n \neq 0$)

~~and if so, then Q is h-unital~~ $\iff C$ is h-unital.

$$A \overset{L}{\otimes}_A = Q \otimes_B P \overset{L}{\otimes}_A$$

A is A-flat $\iff$ $P \otimes_A A = P$ is B-flat

$$\xleftarrow{\text{this}} Q \overset{L}{\otimes}_B P \overset{L}{\otimes}_A$$

$$= P \overset{L}{\otimes}_A Q \overset{L}{\otimes}_B \xrightarrow{\sim} B \overset{L}{\otimes}_B$$

~~Cor.~~

$$\begin{array}{ccccc} A & \hookrightarrow & \begin{pmatrix} A & Q \\ P & B \end{pmatrix} & \hookleftarrow & B \\ \| & & \downarrow & & \downarrow \\ A & \hookrightarrow & \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix} & \hookleftarrow & B' \end{array}$$

$$\begin{array}{ccc} HH(A) & \simeq & HH(B) \\ \| & & \downarrow \\ HH(A) & \simeq & HH(B') \end{array}$$

$B \qquad$ ~~(B B)~~ $\qquad B'$

9) Man and Beast
Man and animal have relied upon each other, for thousands of years. Create a piece of work which depicts a drama, or the closeness between humans and animals.
Refer to Stubbs, Marini, Frink, Greek freizes, Delacroix, Gericault.

~~Cohomology of discrete groups~~

~~[illegible]~~ Argument that worked for HC.
conically contractible ~~variety~~

$$\begin{array}{ccccc} K_1(A) & \xrightarrow{\sim} & \overline{K_1(C')} & \xleftarrow{\sim} & K_1(B') \\ & & \downarrow & & \downarrow \wr \\ & & \widetilde{K_1(C)} & \xleftarrow{\sim} & K_1(B) \end{array}$$

$\frac{1}{2}$ hour on McCarthy. He has an explicit way to see that the Hochschild complex is Morita invariant in the unital case. A critical case is just for $A \subset M_2 A$. I would to write out a clean version of his construction.

Anyway consider the case $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ where both A, B are both left and right flat

A is A-flat $\iff$ $P \otimes_A A = P$ is B flat

A is A^{op} flat $\iff$ $A \otimes_A Q = Q$ is B^{op} flat

B is B-flat $\iff$ $Q \otimes_B B = Q$ is B flat etc.

So I want ~~to~~ assume A is A-flat and A^{op}-flat then P is A^{op}-flat hence $P = \varinjlim E_\alpha$ $Q = \varinjlim F_\alpha$.

Let's concentrate on the limiting process.

Recall that since A is $l+r$ flat then B is h-unital iff $P \overset{L}{\otimes}_A Q \xrightarrow{\sim} B$. Logically there's a slight problem with the ~~cokernel~~ pairing being surjective.

~~A assume A ... A is l+r flat~~

A $l+r$ flat

$$\begin{array}{ccc} B \overset{L}{\otimes}_B P \overset{L}{\otimes}_A Q & \longrightarrow & B \overset{L}{\otimes}_B B \\ \downarrow \wr & & \downarrow \\ P \overset{L}{\otimes}_A Q & \longrightarrow & B \end{array}$$

What I might see is a weakening of the condition that $\langle Q, P\rangle = A$. No: You need $BP = P$ $PQP = P$

Try to find some general arguments. Yes ▦

What do we know. To each pairing triple $P, Q,$ $Q \otimes P \twoheadrightarrow A$ we get $K_n(P \otimes_A Q)$. We would be happy to ~~restrict~~ to P, Q flat over A.

Question: What happens if $QP \neq A$. We have

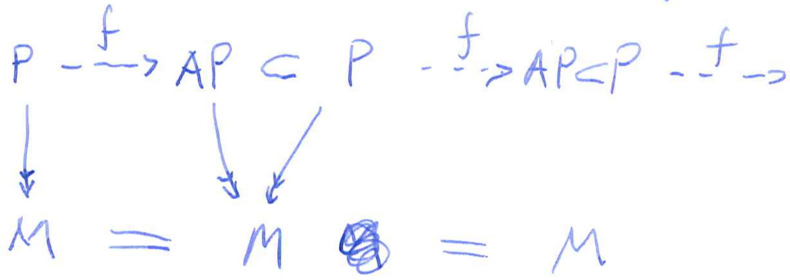
$QPQP =$ ~~QP~~ ?

Could what happens for additive cats be of interest? McCarthy's situation ?

Thm. K_1 is Morita invariant for firm rings.

$$\begin{pmatrix}1&0\\ y(1-xy)^{-1}&1\end{pmatrix}\begin{pmatrix}1&x\\ 0&1\end{pmatrix}\begin{pmatrix}1&0\\ -y&1\end{pmatrix}\begin{pmatrix}1&-(1-xy)^{-1}x\\ 0&1\end{pmatrix}=\begin{pmatrix}1-xy&\\ &(1-yx)^{-1}\end{pmatrix}$$

Prop. $M = AM \Rightarrow \exists$ firm flat A-module F and a surjection $F \to M$.



$$F = \varinjlim\left(P \xrightarrow{f} P \xrightarrow{f} \cdots\right) = \varinjlim\left(AP \xrightarrow{f} AP \xrightarrow{f} \cdots\right) = AF$$

Given $B = B^2$ choose $P \xrightarrow{f} B$ ($I \to P$) P firm fl/B.

make P ring $p_1 p_2 = f(p_1)p_2$ $IP = 0$

[illegible] $K_1 P \xrightarrow{\sim} K_1 B \iff$ [illegible] B firm.

$P \to B$ $P' \to B'$

reduce to a Map between ~~can assume~~ flat firm rings

$\begin{pmatrix}A & Q\\ P & B\end{pmatrix}$ B left flat $\iff$ Q A-flat.

$\begin{pmatrix}A & Q\\ P & B\end{pmatrix}$

expenses

tel.	3.36
bier	4.35
Xerox	1.00
	8.71

28 × 12 = 56 + 28 = 336

1.45 × 3 = 4.35

Θ

[redacted]

C_A Mcontexts sform $\begin{pmatrix} A & * \\ * & * \end{pmatrix}$

Cat of [redacted] (B, F)

---

bide comment ideal Suppose A given, consider its annihilator $\operatorname{Ker}(A \longrightarrow \operatorname{Hom}_{A^{op}}(A, A))$.

Wait, the question is whether I can construct a meghan $B \subset B'$ which eliminates part of the left + right annihilator. Example is to take $B = P \otimes_A Q$. Try to enlarge Q say to $Q \subset Q'$, for example adding something to Q which might pair better with P.

$$B = P \otimes_A Q \qquad \operatorname{Hom}_{B^{op}}(B, B) = \operatorname{Hom}_{A^{op}}(P, P)$$

$$\uparrow B$$

Keep P fixed, try enlarging Q.

$$Q \otimes P \xrightarrow{\phi} A$$

$$Q \longrightarrow A \otimes_A \operatorname{Hom}_{A^{op}}(P, A)$$

$$\cap \qquad \nearrow$$

$$Q'$$

[redacted]

There are questions, first whether you can find interesting maps $P \longrightarrow A$. ~~A is flat~~

$$\{a \mid aA = 0\} \longrightarrow A \longrightarrow \operatorname{Hom}_{A^{op}}(A, A) \qquad a \mapsto (a' \mapsto aa')$$

$$0 \to A \otimes_A \{a \mid aA = 0\} \longrightarrow A \longrightarrow A \otimes_A \operatorname{Hom}_{A^{op}}(A, A) \quad \text{if } A \text{ flat.}$$

Keep on trying 1/6 Answer

How to proceed? Consider

Let's go over again what I need to put into §27. You have Morbeam to discuss. converse direct part.

keep concrete.

1/20 so the question is whether this cat is fibred.

$$\begin{array}{ccc} A & \xrightarrow{u} & A' \\ \cup & & \cup \\ K & \longrightarrow & K' \end{array} \qquad u^*(K') = u^{-1}(K') \cap \lambda_g(A)$$

The fibre is poset of subgroups $K \subset \lambda_g(A) = \{a \mid Aa = aA = 0\}$ so it is fibred.

1/24 get cat stuff straight

$\mathcal{M}$ Roos category over k, so $k \to End(1_{\mathcal{M}}) = Hom_{A\text{-bimod}}(A^*, A)$

What should I think about? I want to explore invariant language.

$\mathcal{M}$ Roos categ. means $\exists$ equiv. $\mathcal{M} \simeq \mathcal{M}(B)$.

Such an equiv. given by an object $Q \in \mathcal{M}$, a ctcont. functor $P: \mathcal{M} \to ab$ and a surjective map of functors $Q \otimes_{\mathbb{Z}} P \to 1$. Finally $B = P(Q)$

Better language: $Q \in \mathcal{M}$ such that B acts on the right. $P: \mathcal{M} \to$

---

regularity. F regular t avoiding $Ass\,F$

$$0 \to F(-1) \xrightarrow{t} F \to F/tF \to 0$$

$$H^i(F(-i)) \to H^i(F/tF(-i)) \to H^{i+1}(F(-i-1))$$

shows F regular $\Rightarrow$ F/tF regular. Conversely

$$\to H^i(F(-i-1)) \xrightarrow{t} H^i(F(-i)) \to H^i(F/tF(-i))$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0$$

appears that $H^i(F(-i-1)) \twoheadrightarrow H^i(F(-i)) \xrightarrow{\sim} H^i(F(-i+1)) \xrightarrow{\sim}$

so conclude that if F/tF regular for some $t \neq 0$ on $Ass(F)$.

1/25 then F is regular. So tensor product should work by induction.

Regular sheaves

F coh. sheaf, $t \in H^0(\mathcal{O}(1))$ regular on F, $F/tF(-1)$ regular.

$$\longrightarrow H^i(F(-i-1)) \xrightarrow{t} H^i(F(-i)) \longrightarrow H^i((F/tF)(-i)) \rightarrow$$

Try to understand the situation.

$$0 \longrightarrow G(-1) \rightarrow \mathcal{O} \otimes H^0(F) \longrightarrow F \longrightarrow 0$$

$$H^i(\mathcal{O}(-i)) \overset{0}{\otimes} H^0(F) \rightarrow H^i(F(-i)) \rightarrow H^{i+1}(G(-i-1))$$

$$H^{i+1}(\mathcal{O}(-i)) \overset{0}{\otimes} H^0(F)$$

$g \subset E \otimes T^*$ prolongation.

2/2 I was trying to reconstruct the tentative approach to Waldhausen's amalgamated products theory I had many years ago (>20). The rough idea is to introduce a category of presentations and to localize (better might be: to take a quotient by a category of nil presentations). The building idea is the basic one, where you have a poset of nice presentations and the layers are "nil". I think I had a way of doing this for SV, which avoids R', and this was supposed to generalize. Back then Waldhausen had I think a perfect-complex picture, which is what I want now.

$\hat{\mathfrak{X}}$ 1/19 Take $j^*N \in \mathcal{M}(A)$ and describe all M in mod $(\tilde{A})$ ~~such that~~ equipped with $j^*M \cong j^*N$.

$$j_! j^* N \longrightarrow M \longrightarrow j_* j^* N$$

We are after all possible ways of factoring the canonical map $j_! \xrightarrow{\exists \text{ iso mod lisse subcat}} j_* i$. Why does this map $\exists$? $j_! \to j_* j^* j_! = j_*$. Best I can say seems to be that the image

$$\mathrm{Im}\left(j_! \xi \longrightarrow j_* \xi\right)$$

is the "minimal" possible M. nil-~~ff~~ free conil-free.

1/20 to describe the cat of idemp rings in terms of the cat of firm rings.

$$\mathrm{Hom}_{fr}(A, B^{(2)}) = \mathrm{Hom}_r(A, B).$$

so $B \mapsto B^{(2)}$ is right adjoint to inclusion

$$\{fr\} \underset{r}{\overset{i}{\rightleftarrows}} \{r\} \qquad ri \simeq 1 \qquad \alpha : ir \to 1$$

~~$\mathrm{Hom}_r(B_1, B_2) \longrightarrow \mathrm{Hom}_{fr}(B_1^{(2)}\ B)$~~

$$\mathrm{Hom}_{rgn}(B_1, B_2) \hookrightarrow \mathrm{Hom}_{frgn}(B_1^{(2)}, B_2^{(2)})$$

$$B_i = A_i / K_i \qquad A_i K_i = K_i A_i = 0.$$

~~[illegible]~~ B equiv. to (A, K)

~~[illegible]~~ form two cats. idrgns B pairs (A, K)

functors ~~A~~ $B \mapsto (B^{(2)}, \mathrm{Ker}\{B^{(2)} \to B\})$

$A/K \longleftarrow (A, K)$

A

OK what should have you said today?

Consider [illegible] $H \otimes V \to U$ with two 27
properties: surjective; $\forall v\ \exists h \ni h \otimes v \mapsto \neq 0$.

Example $V = H^0(F)$, $U = H^0(F(1))$. Then $\exists h : \mathcal{O} \to \mathcal{O}(1)$ such that $0 \to F \xrightarrow{h} F(1)$, whence $H^0(F) \hookrightarrow H^0(F(1))$. But actually you have stronger condition $\exists h \ni V \hookrightarrow U$.

So what happens is that

$$0 \to \mathcal{O}(-n-1) \to \mathcal{O}(-1) \otimes \mathbb{C}^{n+1} \to \mathcal{O} \otimes \mathbb{C}^n \to \mathcal{O}$$

$$\mathbb{C}^{n+1} \to H \otimes \mathbb{C}^n$$

What does Bass's fundamental thm. say for a firm ring

$$K_1(A[t,t^{-1}]) = \left(K_1(A[t]) \oplus K_1(A[t^{-1}])\right) / K_1 A \oplus K_0 A$$

What is basic? The clutching function idea. Basically you form a complex starting from $g \in GL(A[t,t^{-1}])$. Important idea from Ranicki, namely to allow t^q, $q \in \mathbb{Q}$. This kills the nil groups. Another idea is from Waldhausen's free products paper, namely running the arrows the other way. I need to work on this formalism for a few minutes.

Recall Waldhausen. You have unital A, B form $A * B$. This has a natural filtration.

$$\begin{array}{ccccccc} k & - & A & - & AB & - & ABA \\ & \diagdown & & \times & & \times & \\ & & B & - & BA & - & BAB \end{array}$$

You want to take apart, present, an $A*B$-module in terms of an A-module and a B-module.

Recall W considers analog products $A *_C B$. I recall looking at something

B

like M_A, M_B, M_C probably with maps Roughly the

$$M_A \longleftarrow M_C \longrightarrow M_B .$$

K-theory of these diagrams should be $KA \oplus KB \oplus KC$. The diagrams such that $M_A \xleftarrow{\sim} A \otimes_C M_C$, $B \otimes_C M_C \xrightarrow{\sim} M_B$ should yield $K(A *_C B)$. So something like what? You have this diag cat. $\mathcal{D}$ and $K\mathcal{D} = KA \oplus KB \oplus KC$. You have $K(A *_C B) \longrightarrow K\mathcal{D}/?$ defined by choosing a nice presentation. Maybe there's a localization sequence

$$\begin{array}{ccccc} \mathcal{S} & \longrightarrow & \mathcal{D} & \longrightarrow & A *_C B \\ & & \| & & \\ KC + KC & & KA \oplus KB \oplus KC & & \end{array} \qquad ?$$

yielding a MV sequence.

What happens in the case of a poly ring SV over a field k say? Also the noncomm poly ring TV. This I can handle by introducing the graded ring $\bigoplus_{n \geq 0} h^n F_n(SV)$. Filtered algebra situation that works for Ug. I seem to have a better notion

---

what happens? The graded module K-theory is $K(k)[t]$ or $K(k)[t, t^{-1}]$ depending on whether $\mathbb{N}$-graded or $\mathbb{Z}$-graded. But then have localization wrt h.

The approach

So I had some idea about Waldhausen's stuff I believe that generalizes what I did for polynomial rings. Better might be tensor algebra. Yes. There might be some way to avoid the of graded setup. Polynomial rings.

C

Anyway maybe ~~[crossed out]~~ I was mistaken. So what do you have? In the tensor ~~quotient~~ algebra[TV] case you have? It's not your abelian category localization thm., but Bass-Grayson's one with regular elements and the corresponding nil groups.

~~[crossed out]~~ You're stuck I think with filtration.

DM1307 PP.

food Husem. dinner	19.00
wurst	39.92
brot	14.90
mag	9.80
£320 =	752.00
200 Gld. =	181.20
roman =	3.00
hotel =	250.40
bus	8.20
	8,20
	2.00
mot coffee =	7.90

$$\begin{array}{r} 320 \\ \times\, 2\tfrac{1}{4} \\ \hline 640 \\ 80 \\ \hline 720 \end{array}$$

So let's try to continue

2/2 Am missing nice pen. over k

to consider $A = S(T)$. Aim to calculate $K_*(ST)$.

~~[crossed out]~~ Idea: you have a f.g. free ST-module M Choose ~~[crossed out]~~ generating subspaces. Idea if M is "compactified" ~~[crossed out]~~ in a sufficiently positive way, then you get ~~on~~ generating subspaces from global sections.

It's possible your picture is wrong, or doesn't work. In any case I remember the difference between my picture and Waldhausen's in the case $R = A *_C B$. I considered $M_A \leftarrow M_C \rightarrow M_B$ as a presentation of M_R, see exact ~~[crossed out]~~ sequence below, and he had arrows

$$0 \rightarrow M_R \rightarrow R \otimes_A M_A \oplus R \otimes_B M_B \rightarrow R \otimes_C M_C \rightarrow 0$$

D

$$0 \to R\otimes_C M_C \to R\otimes_A M_A \oplus R\otimes_B M_B \to M \to 0$$

It shouldn't make any difference on the level of f.g. proj complexes. So how does it proceed?

You are really after a good approach. Where might you begin? $R = k[t]$. Should you take M_R, choose a large generating subspace? I recall choosing $M_0 \underset{k}{\subset} M = M_R$ get

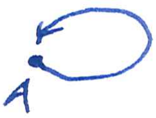
$$M_n = M_0 + tM_0 + \dots + t^n M.$$

$F_pM = M_0 + \dots + t^pM$, get filtered module over the filtered ring $R = k[t]$. Graded module

$$\bigoplus_{p\geq 0} F_pM/F_{p+1}M = M_0 \oplus (M_0 + tM_0)/M_0 \oplus \cdots$$

It's we can get any graded module gen. by degree zero.

So roughly what? You get this generating subspace idea motivated by the process of choosing a coherent subsheaf of j_*E, E the compactification. This idea you wanted to use to understand Waldhausen's theory. In W's situation the ring $A *_C B$ or $A_\alpha[t,t^{-1}]$ has a natural filtration. Geometrically these are fundl. groups. There are sort of trees around



Something new was Thomason–Trombaugh, who understood what Grothendieck's char. of perfect complexes means. This is again localization, i.e. describing things on an open subset. not

E the same as things with compact support.

~~to what happens there again~~

It seems that you might try to ~~see~~ think geometrically in the case of a free product. Assume W K-theory ~~these~~ involves perfect complexes over $\mathbb{Z}[\pi_1]$.

(Here ~~is~~ pops into mind Andrew's remark that Higman linearization is transversality in some interpretation. Is there some connection with canonical resolutions of regular sheaves.)

R unital $\qquad$ mod(R) = category of (left) unitary R-modules $1m = m$.

to <u>extend</u> $R \rightsquigarrow \text{mod}(R)$ to nonunital rings A.

category of A-modules = $\text{mod}(\tilde{A})$ $\qquad \tilde{A} = \mathbb{Z} \oplus A$, $\quad n1 + a$

this is <u>too big</u> e.g. if A is unital with identity elt e then

$$M = eM \oplus (1-e)M$$

$$\text{mod}(\tilde{A}) = \text{mod}(A) \times \text{mod}(\tilde{A}/A), \qquad \tilde{A}/A \cong \mathbb{Z}$$

Def: M an A-module

M is nil : $A^n M = 0$ $\qquad$ nil A-mods $\bigcup_n \text{mod}(\tilde{A}/A^n)$

M is firm : $A \otimes_A M \longrightarrow M$, $\; a \otimes m \longmapsto am$ is an isomorphism. firm(A).

Thm. 1) Obvious functor

$$\text{firm}(A) \longrightarrow \text{mod}(\tilde{A}) \Big/ \bigcup_n \text{mod}(\tilde{A}/A^n)$$

quot. ab. cat by a Serre subcat

is fully faithful. 2) When $A = A^2$ it's an equivalence.

~~mod(A) ... M(A) ≃ firm(A) when A = A²~~

Rmk. inverse functor induced by

$$\text{mod}(\tilde{A}) \longrightarrow \text{firm}(A)$$

$$M \longmapsto A \otimes_A A \otimes_A M \quad (\xrightarrow{\sim} A \otimes_A M \text{ when } M = AM)$$

Now on assume rings idemp + $\mathcal{M}(A) = \text{firm}(A)$

Def: Call A, B Morita equiv. when $\mathcal{M}(A)$, $\mathcal{M}(B)$ equiv.

Ex. $A \otimes_A A$ idem. ring $\quad (a_1 \otimes a_2)(a_3 \otimes a_4) = a_1 a_2 a_3 \otimes a_4$

$A \otimes_A A \longrightarrow A \quad (\mathcal{M}(A) = \mathcal{M}(A \otimes_A A)$ isom of cats.)

Def: Call A a <u>firm ring</u> when $A \otimes_A A \xrightarrow{\sim} A$, i.e. $A \in \mathcal{M}(A)$ or $A \in \mathcal{M}(A^{op})$

A *Morita context* is a ring $C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ equipped with a 2×2 matrix decomp.

There are 8 products $a_1 a_2, qp \in A$ $\quad aq, qb \in Q$

$pa, bp \in P$ $\quad b_1 b_2, pq \in B$.

A, B subrings ${}_B P_A$, ${}_A Q_B$ are bimodules, $P \otimes Q \to B$, $Q \otimes P \to A$

C is called *strictly firm* when

$$A \otimes_A A \xrightarrow{\sim} A \xleftarrow{\sim} Q \otimes_B P \qquad A \otimes_A Q \xrightarrow{\sim} Q \xleftarrow{\sim} Q \otimes_B B$$

$$P \otimes_A A \xrightarrow{\sim} P \xleftarrow{\sim} B \otimes_B P \qquad B \otimes_B B \xrightarrow{\sim} B \xleftarrow{\sim} P \otimes_A Q$$

$\Rightarrow QP = A$, $PQ = B$

Prop: a M cont $C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ yields an equiv.

$$\mathcal{M}(A) \xrightarrow{\sim} \mathcal{M}(B)$$
$$M \longmapsto P \otimes_A M$$
$$Q \otimes_B N \longleftarrow N$$

Conversely any M. equiv. $\mathcal{M}(A) \simeq \mathcal{M}(B)$ is given utci by a unique (utci) sfirm M. cont.

Observe $B \xleftarrow{\sim} P \otimes_A Q$ $\quad (p_1 q_1)(p_2 q_2) = p_1 (q_1 p_2) q_2$

Thm. A fixed firm ring. Then equivalence between:

- A firm ring B tog. with an equiv. $\mathcal{M}(A) \simeq \mathcal{M}(B)$
- a triple $(P, Q, \langle\,,\rangle)$ with P a firm A^{op}-mod, Q a firm A-mod and $\langle\ \rangle : Q \otimes_{\mathbb{Z}} P \to A$ a surj. A-bimod map.

$$(p_1 \otimes q_1)(p_2 \otimes q_2) = p_1 \langle q_1, p_2 \rangle \otimes q_2$$

Ex. A field $\begin{pmatrix} A & Q \\ P & P \otimes_A Q \end{pmatrix}$

3

$HH_0(A) = HC_0(A) = A/[A,A]$

If L A-bim put $L\otimes_A = L/\{la - al\}$

Then $HH_0(A) = A\otimes_A$.

M-inv. for HH_0 on firm rings:

$$A\otimes_A \simeq Q\otimes_B P\otimes_A = P\otimes_A Q\otimes_B = B\otimes_B$$

define $\overset{L}{\otimes}_A$. P A^{op}-mod, Q A-mod

$$P\overset{L}{\otimes}_A Q = P\otimes_A E\otimes_A Q \qquad E \text{ any } \tilde{A}\text{-flat bimod res. of } \tilde{A}$$

Assume all rings $\mathbb{Z}$-flat. Standard E

$$\to \tilde{A}\otimes A\otimes A\otimes\tilde{A} \longrightarrow \tilde{A}\otimes A\otimes\tilde{A} \longrightarrow \tilde{A}\otimes\tilde{A} \to \tilde{A} \to 0$$

$$H_n(P\overset{L}{\otimes}_A Q) = \mathrm{Tor}^A_n(P,Q).$$

<u>Def</u> A is h-unital when A is firm, $A\otimes_A A \xrightarrow{\sim} A$ and $\mathrm{Tor}^A_n(A,A) = 0$ $n \geq 1$.

<u>Thm.</u> Given B, B' h-unital and $\mathcal{M}(B) \simeq \mathcal{M}(B')$ then there are canon isos. $HH_*(B) = HH_*(B')$, and also for HC_*, HC^-_*, HP^*, ..

<u>Prop</u> $C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ s. firm, A left + right flat.

Then B is h-unital $\iff$ C is h-unital

$\iff \mathrm{Tor}^A_n(P,Q) = 0 \quad n > 0$

(i.e. $P\overset{L}{\otimes}_A Q \xrightarrow{\sim} B$ quis.)

$$A\overset{L}{\otimes}_A \simeq Q\otimes_B P\overset{L}{\otimes}_A \simeq Q\overset{L}{\otimes}_B P\overset{L}{\otimes}_A \quad \text{because } A \text{ } A\text{-flat} \iff P \text{ } B\text{-flat}$$

$$\simeq P\overset{L}{\otimes}_A Q\overset{L}{\otimes}_B \xrightarrow{\sim} B\overset{L}{\otimes}_B$$

Remaining steps

∃ nce of flat firm modules

$\forall\, B = B^2 \quad \exists\ A \twoheadrightarrow B$ surj hom. inducing a meq & A is left + right flat.

Prof. $A \to B$ as above $B \xrightarrow{w} B'$ homom. ind. meq with B, B' h-unital.

$$\begin{array}{ccccc} A & \hookrightarrow & \begin{pmatrix} A & Q \\ P & B \end{pmatrix} & \hookleftarrow & B \\ \| & & \downarrow & & \downarrow w \\ A & \hookrightarrow & \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix} & \hookleftarrow & B' \end{array}$$

A

$$\begin{array}{ccc} HH(A) & \simeq & HH(B) \\ \| & & \cong \downarrow w \\ HH(A) & \simeq & HH(B') \end{array}$$

⇓

w ind $\cong$ on HC_* etc.

$$HC(A) \xrightarrow{\sim} HC\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \xleftarrow{\sim} HC(B)$$

A Want to go back + understand Morita for K_1. The idea is Vaserstein's identity

$$\begin{pmatrix}1 & 0\\ y(1-xy)^{-1} & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -y & 1\end{pmatrix}\begin{pmatrix}1 & -(1-xy)^{-1}x\\ 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 0\\ y(1-xy)^{-1} & 1\end{pmatrix}\begin{pmatrix}1-xy & x\\ -y & 1\end{pmatrix} \qquad \begin{pmatrix}1 & x\\ y(1-xy)^{-1} & (1-xy)^{-1}\end{pmatrix}\begin{pmatrix}1 & -(1-xy)^{-1}x\\ -y & (1-yx)^{-1}\end{pmatrix}$$

$$\begin{pmatrix}1-xy & x\\ 0 & (1-yx)^{-1}\end{pmatrix}\begin{pmatrix}1 & -(1-xy)^{-1}x\\ 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}1-xy & 0\\ 0 & (1-yx)^{-1}\end{pmatrix}$$

$H^0(\mathcal{O}(n)) = 0 \quad n<0 \qquad H^1(\mathcal{O}(n)) = 0 \quad n \geq -1$

$H^0(\mathcal{O}(n)) \simeq \mathbb{C}^{n+1} \quad n \geq 0 \qquad H^1(\mathcal{O}(n)) = \mathbb{C}^{-n-2} \quad n \leq -2$

F reg : $H^1(F(-1)) = 0$ $\quad | \; F \twoheadrightarrow F''$ $\quad F$ reg $\Rightarrow F''$ reg.

Prop. $\begin{cases} F \text{ reg} \Rightarrow \mathcal{O}\otimes_{\mathbb{C}} H^0(F) \twoheadrightarrow F \\ H^0(F(-1)) = H^1(F(-1)) = 0 \Rightarrow \mathcal{O}\otimes_{\mathbb{C}} H^0(F) \xrightarrow{\sim} F \end{cases}$

canonical resolution of a reg sheaf

$0 \to \mathcal{O}(-1)\otimes_{\mathbb{C}} H^0(F(-1)) \to \mathcal{O}\otimes_{\mathbb{C}} H^0(F) \to F \to 0$

exact functor

B So Vaserstein's identity tells me that when $C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ ~~with~~ ~~and~~ strictly idempo the maps

$$K_1 A \longrightarrow K_1 C \longleftarrow K_1 B$$

have the same image.

So what is the argument? You need the injectivity. I guess then it's the argument which works in general. I recall

$$C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

Something is flat!! What? I recall choosing an A which is both left + right flat.

Go over the argument. If $C = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \quad Q)$ and if say Q is A-flat. Then $Q = \varinjlim F_\alpha = \varinjlim A F_\alpha$ so $C = \varinjlim_\alpha \begin{pmatrix} A & AF_\alpha \\ P & P\otimes_A AF_\alpha \end{pmatrix}$ and we have $F_\alpha^* A$ $\quad Q \otimes_{\mathbb{Z}} P \to A$

$$P \longrightarrow \operatorname{Hom}_A(Q, A) \longrightarrow \operatorname{Hom}_A(F_\alpha, A)$$

, so we have a map

$$C_\alpha = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \quad AF_\alpha) \longrightarrow \underbrace{\begin{pmatrix} A \\ F_\alpha^* A \end{pmatrix} \otimes (A \quad AF_\alpha)}$$

So we know $K_n A \to K_n C$ is injective $M_\alpha(A)$. when B is A-flat. How to finish?

You have B, B' nng

$$\begin{matrix} A & & A' \\ \downarrow & & \downarrow \\ B & & B' \end{matrix}$$

CYNTHIA QUILLEN

Point is ~~that~~

8

C

I need to review M inv for K_1 on firm rings. You know that

1) $C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ strictly idempotent $\Longrightarrow$ $K_1 A \to K_1 C \leftarrow K_1 B$ have same image

2) If $\left| \begin{array}{l} Q \text{ is } A\text{-flat} \\ P \text{ is } A^{op}\text{-flat} \end{array} \right.$ then $K_1 A \longrightarrow K_1 C$ injective

$\Updownarrow$

$\left(\begin{array}{l} B \text{ is } B\text{-flat} \\ \text{— } B^{op}\text{-flat} \end{array} \right.$

So you can argue as follows. Assume A is A-flat. Then choose $A' \twoheadrightarrow B$ with A' A'-flat

$$\begin{array}{ccccc} A & \hookrightarrow & C' & \hookleftarrow & A' \\ & & \downarrow & & \downarrow \\ & & C & \hookleftarrow & B \end{array} \qquad \begin{array}{ccccc} K_1 A & \xrightarrow{\sim} & K_1 C' & \xleftarrow{\sim} & K_1 A' \\ & & \downarrow & & \downarrow \wr \\ & & K_1 C & \xleftarrow{\sim} & K_1 B \end{array}$$

(Is it clear that C is firm?, i.e. strictly firm $\Rightarrow$ firm? $C = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \;\; Q)$

If A left and right flat, then $P \overset{L}{\otimes}_A Q \simeq P \otimes_A Q \iff B$ h-unital

$$\begin{array}{ccc} B \overset{L}{\otimes}_B P \overset{L}{\otimes}_A Q & \longrightarrow & B \overset{L}{\otimes}_B B \\ \wr \downarrow & & \downarrow \\ P \overset{L}{\otimes}_A Q & \longrightarrow & B \end{array}$$

OKAY.)

Let's look at Bass F.T.

If K commutative unital, A idempotent then $A \otimes_{\mathbb{Z}} K$ also idempotent.

D McCarthy's argument in the critical case of $eRe \subset R$.

If there are finitely many objects then it is just the cyclic bar complex of a ring. How does this go? Yes.

He chooses for each P a P' such that $P \oplus P'$ is free. So we have P given say $P = Re$, $Q = Re^{\perp}$ and then you need to choose Q' such that $Q \oplus Q' = (Re)^{\oplus n}$

One ring is $A = \mathrm{End}(P)$, the other is $C = \mathrm{End}(P \oplus Q)$. So you seem to get

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & \swarrow & \downarrow \\ M_n A & \longrightarrow & M_n C \end{array}$$

Suppose $n = 1$.

$$A \qquad \begin{pmatrix} A & eCe^{\perp} \\ e^{\perp}Ce & e^{\perp}Ce^{\perp} \end{pmatrix}$$

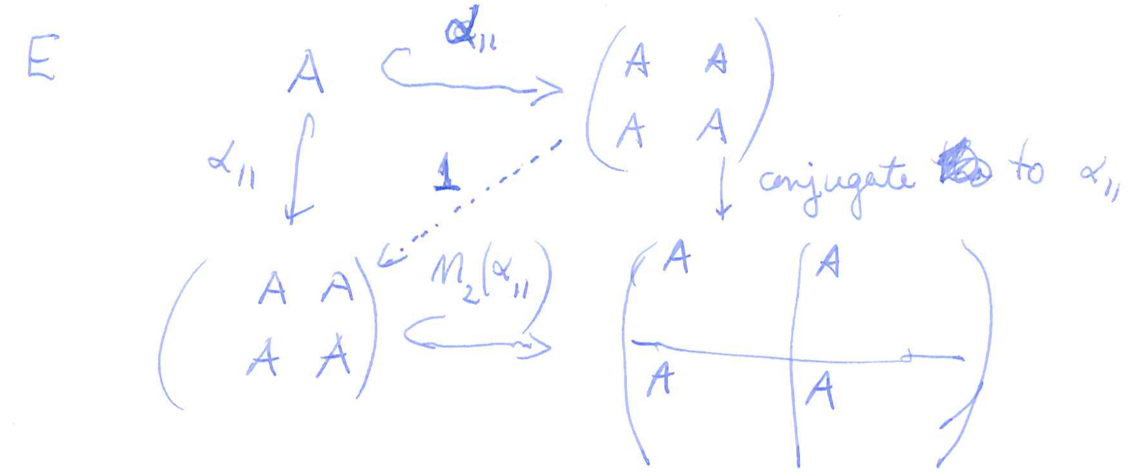
This seem to be clear enough

Set it up with fg proj modules Better - set it up with additive categories. Basically you have three objects P, Q, Q' such that $Q \oplus Q' = P^n$.

$$\begin{array}{c} A \\ \downarrow \\ M_n A \end{array}$$

First suppose $Q' = 0$.

E



$$\cdots \longrightarrow A \otimes A \xrightarrow{\partial} A \longrightarrow 0$$

$A\langle h \rangle \qquad \partial(a) = 0 \qquad \partial(h) = 1.$

$$\partial(h^2) = \partial h\, h - h\, \partial h = h - h = 0$$

What was I going to do.

$$A \otimes A$$

$$A \otimes_S A$$

Take simplest thing. So the real problem is what about two elements ξ, ξ' such that $\partial(\xi) = \partial(\xi') = 1$. So for example $1 - \xi = \partial(k)$. You have two homos. $\Gamma \rightrightarrows \Gamma$ their difference should be a derivation

$$(u - v)(xy) = (u(x) - v(x))\, u(y) + v(x)(u(y) - v(y))$$

$$\Delta(a_0 h a_1 h a_2) = a_0 k a_1 h a_2 - a_0 v(h) a_1 k a_2$$

F Let's see if I can get this derivation stuff to work. $\Gamma = A\langle h \rangle$. I have two DG homos. $u, v : \Gamma \longrightarrow R$. Assume these agree on A. $\Omega^1_A(\underbrace{A\langle h\rangle}_{T_A(A\otimes A)}) = \Gamma \otimes_A (A \otimes A) \otimes_A \Gamma = \Gamma \otimes \Gamma$

free Γ-bimodule on one generator, so any deriv. is specified by its value on h. So regard R as Γ bimodule v on left u on right.

$$(u-v)(xy) = \big(u(x)-v(x)\big)\,u(y) + v(x)\big(u(y)-v(y)\big)$$

$u-v$ determined by $(u-v)(h)$

$\partial(uh - vh) = u1 - v1 = 0$. Now define $d : \Gamma \to R$ of degree $+1$ such that

$$d(xy) = dx\,u(y) + (-1)^x v(x)\,dy$$

$$d(a) = 0 \qquad d(h) = k \quad \text{where}$$

$\partial(k) = uh - vh$. Assume $k\ \exists$. Then $[\partial, d]$ should be a derivation rel v, u, $= 0$ on a

$[\partial, d]h = \partial(k) + d1 = (u-v)(h)$.

So what does this argument amount to? So

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \otimes_S \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

$$\begin{pmatrix} A \\ P \end{pmatrix} \otimes_A \underbrace{(A \quad Q) \otimes_S \begin{pmatrix} A \\ P \end{pmatrix}} \otimes_A (A \quad Q)$$

$$\begin{pmatrix} A\otimes A \\ B\otimes_A Q\otimes P \end{pmatrix} \quad \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A \begin{pmatrix} A\otimes A & 0 \\ 0 & Q\otimes P \end{pmatrix} \otimes_A (A \quad Q) = \begin{pmatrix} A\otimes A & A\otimes Q \\ B\otimes P & B\otimes B \end{pmatrix}$$

G

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \otimes_S \begin{pmatrix} A\otimes A & \\ B\otimes P & \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

$$\begin{pmatrix} A \\ P \end{pmatrix} \otimes_A \begin{pmatrix} A & Q \end{pmatrix} \otimes_S \begin{pmatrix} A\otimes A \\ B\otimes P \end{pmatrix} = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A \begin{pmatrix} A\otimes A\otimes A & 0 \\ 0 & Q\otimes B\otimes P \end{pmatrix}$$

$$= \begin{pmatrix} A\otimes A\otimes A \\ B\otimes B\otimes P \end{pmatrix}$$

$C \otimes C$ has 16

$C \otimes_S C$ has 8 allowed

$C \otimes_S C \otimes_S$ has 4 allowed.

---

matrix case: Take $A = M_n B = S \otimes B$ $\quad S = M_n k$

Then

$$T_A(A\otimes A) \rightleftarrows T_A(A\otimes_S A)$$

In popular terms you have ~~...~~ pre cyclic objects

$$[(A\otimes A)\otimes_A]^{(n)} \longrightarrow [(A\otimes_S A)\otimes_A]^{(n)}$$

~~[illegible] does this!~~

$$A\otimes_S A\otimes_S \cdots A\otimes_S$$

$$S\otimes B\otimes_S S\otimes B\otimes_S \cdots \otimes_S S\otimes B\otimes_S = B^{\otimes n} \otimes S_4$$

So let's see that things works.

Rest of things:

A

So you have for R unital

$$0 \to K_1 R \to K_1 R[t] \oplus K_1 R[t^{-1}] \to K_1 R[t,t^{-1}] \to K_0 R \to 0$$

If you do this for $R = \tilde{A}$, $\mathbb{Z}$ you should get it for A. provided

$$K_1(\tilde{A}) = K_1(\mathbb{Z}) \oplus K_1 A$$

Also need maybe

$$K_1(\tilde{A} \otimes_{\mathbb{Z}} k) = K_1(k) \oplus K_1(A \otimes_{\mathbb{Z}} k)$$

Look at $k \oplus B$ semi direct

Then $K_1(k \oplus B) = K_1(k) \oplus K_1(B)$?

In def. of embedding of B as ideal in a unital ring? Is this always true?

$$GL(k \oplus B) = GL(k) \ltimes \underbrace{GL(B)}_{\mathrm{Ker}\{GL(\tilde{B}) \to GL(\mathbb{Z})\}}$$

I need to spend a little time on $W_1(A)$.

Thm: K_1 is Morita invariant for f. rings. In other words a M. eq $\mathcal{M}(A) \simeq \mathcal{M}(B)$ gives rise to a canon isom $K_1 A = K_1 B$

Conj. K_* is Morita invariant for h-unital rings (defined by $\mathrm{Tor}_n^{\tilde{A}}(\mathbb{Z}, A) = 0 \quad \forall n$).

Thm. Cyclic type homology: HH_*, HC_*, etc. is Morita invariant for h-unital algebras flat over a comm. unital ground ring.

I Let's try to understand excision a little.

What form should this take? You wish to avoid explicit relations. Ideas I like include trying to obtain periodicity out of the passing from finite GL_n to GL_∞. So the mechanism you need to understand? Goodwillie, McCarthy, Dundas type result. Compare a nilpotent extension of rings to the nilpotent upper triangular matrix groups. Use Volodin model.

Go over proof for K_1. $C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

Vaserstein identity $\operatorname{Im} K_1 A = \operatorname{Im} K_1 B$ in $K_1 C$.

Assume B is B-flat ($\Longleftrightarrow$ $Q \otimes_B B = Q$ is A-flat).

Q A-flat $\Rightarrow$ $Q = \varinjlim AF_\alpha = \varinjlim F_\alpha$ $F_\alpha = \tilde{A}^{n_\alpha}$

$$C = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \quad Q) = \varinjlim C_\alpha \text{ where } C_\alpha = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \quad AF_\alpha)$$

$$P \longrightarrow \operatorname{Hom}_A(Q, A) \longrightarrow \operatorname{Hom}_A(F_\alpha^*, A) = F_\alpha^* A$$

$\exists$ hom. $(P, AF_\alpha, \langle\, ,\rangle) \longrightarrow (F_\alpha^* A, AF_\alpha, \langle\, ,\rangle_{can})$

$$A \subset C_\alpha \longrightarrow M_{n_\alpha} A$$

$$K_n A \hookrightarrow K_n C_\alpha \longrightarrow K_n(M_{n_\alpha} A)$$

<u>Lemma:</u> $C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ s.t. B is B-flat (or B^{op}-flat).

then $K_n A \hookrightarrow K_n C$.

$$\begin{array}{ccccc} B_1 & \longrightarrow & B_1 \vee B_2 & \longleftarrow & B_2 \\ \downarrow & & & & \downarrow \\ A_1 & \longrightarrow & A_1 \vee A_2 & \longleftarrow & A_2 \end{array}$$

12/27 (A, P, Q, ϕ) 4 products given

assoc. given 5 A, P, Q

Question. Given $\mathcal{M}(A)$ can I find (P, Q, ϕ) firm flat triple such that $B = P \otimes_A Q$ acts faithfully on P and on Q?

Idea at the moment: If I choose P a generator, and take Q to be its "dual" $A \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$. Then I know something; namely the ring $B = P \otimes_A Q = P \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$ has $\mathrm{Mult}(B) \xrightarrow{\sim} \mathrm{Hom}_{B^{op}}(B, B)$

$\mathrm{Hom}_{A^{op}}(P, P)$ $\parallel$

12/28 Outline things again to find what to say about nonm. Mcontexts. What to say?

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

$$\begin{array}{ccc} B^{(2)} \otimes_B P \otimes_A A^{(2)} & \xrightarrow{\ \sim\ } & B^{(2)} \otimes_B P \\ \simeq \downarrow & & \downarrow \\ P \otimes_A A^{(2)} & \longrightarrow & P \end{array}$$

$$Q \otimes_B P \otimes_A A^{(2)} \xrightarrow{\ \sim\ } A^{(2)}$$

I need to know that if A firm then (P, Q, ϕ) firm $\Longrightarrow$ $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ sfirm

P A^{op}-firm $\Longrightarrow$ $P \otimes_A Q = B$ is B^{op} firm

$\Longrightarrow$ P firm bimod. $\therefore$ B firm ring.

Suppose A firm $\bar{A} = A/I$, $AIA = IA = 0$.

firm triples same for A and $\bar{A}$.

OKAY for P, Q. $Q \otimes P \longrightarrow A \longrightarrow \bar{A}$

(β) I am getting a new outline in mind for Ch 2.

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- rt cont fun.
- ring homom. + adj. fun.
- firm rings
- M contexts + meg th.

I have to carefully think about the organ.

meg thm: $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ surjective pairings yields meg.

Review ~~[illegible]~~ construction of flat modules

Given a sequence $a_1, a_2, \ldots \in A$ can form

$$F = \varinjlim \left(\tilde{A} \xrightarrow{\cdot a_1} \tilde{A} \xrightarrow{\cdot a_2} \tilde{A} \xrightarrow{a_3} \tilde{A} \longrightarrow \cdots \right)$$

and this gives a firm flat module, and I know you get enough of these to detect ~~non torsion [illegible]~~ non-torsion A^{op}-modules.

more general construction where you use matrices

$$\tilde{A}^{n_0} \xrightarrow{\cdot a_1} \tilde{A}^{n_1} \xrightarrow{\cdot a_2} \tilde{A}^{n_2} \xrightarrow{\cdot a_3} \cdots$$

These you know give generators. What I would like to do is to understand better whether I can construct a triple (P, Q, ϕ) such that $P \otimes_A Q$ acts faithfully on Q. You might hope to use the linear equations criterion for flatness, otherwise known as any map fin.pres $\longrightarrow$ flat **is** nuclear.

$$\mathrm{Hom}_A(Z, M) = \mathrm{Hom}_A(Z, A) \otimes_A M$$

if Z is f.p. + M flat. How might this works? I am thinking that I want to construct P, Q simultaneously.

5 to the triad (P, Q, ϕ) over A together with a left + right nil ideal I of $P \otimes_A Q$. When is $\sum p_i \otimes q_i$ in the left + right ann?

$$\sum p_i (q_i p) \otimes q \quad \forall p, q.$$

$$\Rightarrow \left(\sum p_i q_i p\right) A = 0.$$

Question when is $\sum p_i \otimes q_i \in B = P \otimes_A Q$ in the left and right annihilator of B? First look at ~~left~~ right annihilator.

$$\left(\sum p_i \otimes q_i\right)(p \otimes q) = \sum_i p_i q_i p \otimes q = 0 \quad \forall p, q.$$

$$\Longrightarrow \sum p_i q_i p q p' = 0 \quad \forall p, q, p'$$

$$\Longrightarrow \sum p_i q_i p = 0 \quad \forall p.$$

~~So it seems that~~ This I knew

The criterion I want is that I kills P, Q.

What are my plans? faithfully flat?

Do there exist faithfully flat firm modules which embed $\mathcal{M}(A)$ into Ab? Yes you can get them ~~as~~ $\varinjlim (\tilde{A} \xrightarrow{\cdot a_1} \tilde{A} \xrightarrow{\cdot a_2} \tilde{A} \xrightarrow{\cdot a_3} \dots)$ for all possible sequences. You might be able to do better for an idempotent ring. You need to review char of tors modules. $\mathcal{T}$ = Serre subcat of mod $(\tilde{A})$ closed under arb $\varinjlim$'s containing mod $(\tilde{A}/A)$. The torsion submodule of M is $\tau M = \{m \mid \forall (a_n)\ \exists n \ni a_n \cdots a_2 a_1 m = 0\}$.

submodule. ✓ Check $M/\tau M$ tors.-free. Suppose $\exists m$ $Am \subset \tau M$. Then $\forall (a_n)$ have $a_0 m \in \tau M$ so $\exists n$

E/ Idempotent case. $\tau M = \{m \mid \forall a,\ am = 0\}$.
~~OKAY Let us consider gilch~~ Suppose P faithfully flat. Can you say something about its annihilator?

---

consider $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ surj pairings

results.

Given (A, P, Q, ϕ) $B = P \otimes_A Q$ idemp. if $\phi: Q \otimes P \twoheadrightarrow A$ and either $PA = P$ or $AQ = Q$.

$P, Q, \phi \Longrightarrow \begin{pmatrix} A & Q \\ P & P \otimes_A Q \end{pmatrix}$

trad $\longmapsto$ Mcontext.

$P \otimes_A Q$ assoc. $((p_1 \otimes q_1)(p_2 \otimes q_2))(p_3 \otimes q_3)$

$= (p_1 \langle q_1 p_2 \rangle \otimes q_2)(p_3 \otimes q_3) = p_1 \langle q_1 p_2 \rangle \langle q_2 p_3 \rangle \otimes q_3$

other $(p_1 \otimes q_1)(p_2 \langle q_2 p_3 \rangle \otimes q_3) = p_1 (\langle q_1 | p_2 \langle q_2, p_3 \rangle \rangle) \otimes q_3$

$= p_1 (q_1 p_2)(q_2 p_3)$

apply to $P' = \begin{matrix} \tilde{A} \\ \oplus \\ P \end{matrix}$ $Q' = \tilde{A} \oplus Q$

$\left\langle (\tilde{a}_1, q) \,\middle|\, \begin{pmatrix} \tilde{a}_2 \\ p \end{pmatrix} \right\rangle = \tilde{a}_1 \tilde{a}_2 + \langle q | p \rangle$

$P' \otimes_A Q' = \begin{pmatrix} \tilde{A} \otimes_A A & \tilde{A} \otimes_A Q \\ A \otimes_A \tilde{A} & P \otimes_A Q \end{pmatrix} \xrightarrow{\simeq} \begin{pmatrix} \tilde{A} & Q \\ P & P \otimes_A Q \end{pmatrix}$

$(p_1 \otimes q_1) \cdot p = \begin{pmatrix} 0 & 0 \\ 0 & p_1 \otimes q_1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix} = \left(\begin{pmatrix} 0 \\ p_1 \end{pmatrix} \otimes (0 \ q_1) \right) \begin{pmatrix} 0 \\ p \end{pmatrix} \otimes (1 \ 0)$

$$\xi \checkmark = \begin{pmatrix} 0 \\ p_1\langle \xi_1|p\rangle \end{pmatrix} \otimes (1 \quad 0) \simeq \begin{pmatrix} 0 & 0 \\ p_1\langle \xi_1|p\rangle & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix} = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes (0 \quad \xi)\right)\left(\begin{pmatrix} 0 \\ p \end{pmatrix} \otimes (1 \quad 0)\right)$$

$$= \begin{pmatrix} 1\langle \xi|p\rangle \\ 0 \end{pmatrix} \otimes (1 \quad 0) = \begin{pmatrix} \langle \xi|p\rangle & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix}\begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} = \left(\begin{pmatrix} 0 \\ p \end{pmatrix} \otimes (1 \quad 0)\right)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes (0 \quad \xi)\right)$$

$$= \begin{pmatrix} 0 \\ p \end{pmatrix} \otimes (0 \quad \xi) = \begin{pmatrix} 0 & 0 \\ 0 & p\otimes\xi \end{pmatrix}$$

$$\xi(p_1 \otimes \xi_1) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes (0 \quad \xi)\right)\left(\begin{pmatrix} 0 \\ p_1 \end{pmatrix} \otimes (0 \quad \xi_1)\right) = \begin{pmatrix} \langle \xi|p_1\rangle \\ 0 \end{pmatrix} \otimes (0 \quad \xi_1)$$

$$= \langle \xi|p_1\rangle \xi_1 .$$

---

properties of Mcont with nry. pairings.

P firm braid $\iff$ P A^{op}-firm $\iff$ ~~P~~ B-firm

sim for Q.

if either P or Q is firm braid, then

$Q \otimes_B P \xrightarrow{\sim} A^{(2)}$ and ~~B~~ $P \otimes_A Q \xrightarrow{\sim} B^{(2)}$

Is it possible to construct ~~affect~~ an interesting flat triple as a simultaneous limit of fg free triples

□ τ □ $Q \xrightarrow{f} \!\!\!\!\!\to A$ $P \xrightarrow{\sim} \mathrm{Hom}_A(Q, A) \otimes_A A$

You would like to somehow arrange $B = P \otimes_A Q$ to embed in its mult alg.

$$B \longrightarrow \left(\mathrm{Hom}_{A^{op}}(P, P) \times \mathrm{Hom}_{A}(Q, Q)^{op} \right)^{\phi}$$

$$\parallel$$

$$B \longrightarrow \left(\mathrm{Hom}_{B^{op}}(B, B) \times \mathrm{Hom}_B(B, B)^{op} \right)^{\mu_B}$$

note that because P is the dual of Q in some sense, there should be a homom.

$$\mathrm{Hom}_A(Q, Q)^{op} \longrightarrow \mathrm{Hom}_{A^{op}}(P, P)$$

and with luck $B \to \mathrm{Mult}(B)$ should factor:

$$B \searrow \mathrm{Hom}_B(B, B)^{op} \nearrow \mathrm{Mult}(B)$$

Is it true that the right mult alg of B is a retract of $\mathrm{Mult}(B)$?

<u>Question</u> Suppose $B \xrightarrow{\sim} \mathrm{Hom}_B(B, B) \otimes_B B$. Does it follow that ~~there are ...~~ $\mathrm{Mult}(B) \xrightarrow{\sim} \mathrm{Hom}_B(B, B)^{op}$, or at least that this map admits a section?

---

anyway back to firm rings. Covering: $B \xrightarrow{f} \!\!\!\!\to A$ kernel $K \ni BK = KB = 0$. Then product in B descends to A

get $A \otimes_A A \longrightarrow\!\!\!\!\to B \longrightarrow A$

Better

$$\begin{array}{ccc} B \otimes_B B & \xrightarrow{\sim} & A \otimes_A A \\ \downarrow & \swarrow & \downarrow \\ B & \longrightarrow & A \end{array}$$

The product

$$\begin{array}{ccc} B \otimes_B B & \longrightarrow & B \\ S \downarrow & \nearrow & \downarrow \\ A \otimes_A A & \longrightarrow & A \end{array}$$

~~Suppose~~ Suppose $0 \longrightarrow K \longrightarrow B \xrightarrow{f} A \longrightarrow 0$

~~B ...~~ $KB = 0 \implies f$ becomes an A-mod. map

$BK = 0 \implies f$ ~~...~~ A-mod isom.

$\implies A \otimes_A B \xrightarrow{\sim} A \otimes_A A$. ~~...~~ I want to show that B firm over B iff B firm over A.

□ ∪ □

$$B \otimes_B B$$

"

$$K \otimes_A B \longrightarrow B \otimes_A B \longrightarrow A \otimes_A B \longrightarrow 0$$

"
0

---

list assertions.

1) $B = A^{(2)}$ is a firm ring

2) if $A = B/K$, $BK = KB = 0$, then $m(A) = m(B)$.

$AIA = 0$ $\begin{pmatrix} A & A/AI \\ A/IA & A/I \end{pmatrix} = \begin{pmatrix} A & A \\ A & A/I \end{pmatrix}$. when ~~A/AI~~ $AI = IA = 0$.

$$M \longmapsto A \otimes_A M = M.$$

Given $(P, Q, \phi : Q \otimes P \twoheadrightarrow A)$ firm triple over A and say $P \xrightarrow{\sim} \mathrm{Hom}_A(Q, A) \otimes_A A$ is the "dual" of Q. Then we have a homom.

$$\mathrm{Hom}_A(Q, Q)^{op} \longrightarrow \mathrm{Hom}_{A^{op}}(P, P)$$

which should be given by transposes. In any case let $x^r \in \mathrm{Hom}_A(Q, Q)$ x^l the image of x^r in $\mathrm{Hom}_{A^{op}}(P, P)$.

~~Thus $\langle g | x^l p \rangle$~~ Let $p = f \otimes a$ $f \in \mathrm{Hom}_A(Q, A)$.

then $x^l p = (g \longmapsto f(g x^r)) \otimes a$

Is it true that $\langle g | x^l p \rangle = \langle g x^r | p \rangle$.

$$\begin{matrix} \| & & \| \\ f(g x^r) a & & f(g x^r) a \end{matrix} \quad \text{Yes!}$$

Thus certainly we should have a lifting

$$\left(\mathrm{Hom}_{A^{op}}(P, P) \times \mathrm{Hom}_A(Q, Q)\right)^{op} \xrightarrow{pr_2} \mathrm{Hom}_A(Q, Q)^{op}$$

Is there any chance it could be an isom?

□ ϕ □ Suppose then (x^r, x^l) comp. with pairing, so

$$\begin{array}{ccc} P & \longrightarrow & \mathrm{Hom}_A(Q, A) \\ \downarrow x^r & & \downarrow (x^l)^t \\ P & \longrightarrow & \mathrm{Hom}_A(Q, A) \end{array}$$

commutes

But P is firm, so we get

$$\begin{array}{ccc} P & \xrightarrow{\sim} & \mathrm{Hom}_A(Q, A) \otimes_A A \\ \downarrow & & \downarrow (x^l)^t \otimes A \\ P & \xrightarrow{\sim} & \mathrm{Hom}_A(Q, A) \otimes_A A \end{array}$$

so x^r is determined by x^l.

~~Another idea is to consider~~ So the idea is to construct Q as flat firm generator so that $P \otimes_A Q = \mathrm{Hom}_A(Q, A) \otimes_A Q$ is faithful represented on Q. What would happen if

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12/25 I learned yesterday that given A firm, take $P = A$, $Q = A \otimes_A \mathrm{Hom}_{A^{op}}(A, A)$, then $B = P \otimes_A Q$ = Steffan's ring $A \otimes_A \mathrm{Hom}_{A^{op}}(A, A)$ should have the property: $\mathrm{Mult}(B) \xrightarrow{\sim} \mathrm{Hom}_{B^{op}}(B, B)$. This implies: $bB = 0 \Rightarrow Bb = 0$.

Goal: You want to construct P, Q flat firm generators with a pairing $\phi : Q \otimes P \twoheadrightarrow A$ so non degenerate that $B = P \otimes_A Q$ is faithfully represented on both Q and P. Then B would have some of the nice properties of a C^*-algebra, namely both left + right flat, injects into its left + right multiplier algebra.

$$Q \otimes P \twoheadrightarrow A^{(2)} \Longrightarrow Q \otimes P \to A$$

$\square \times \square$ if $M' \to M$ monom in $\mathrm{mod}(\tilde{A})$, then kernel of $A^{(2)} \otimes_A M' \longrightarrow A^{(2)} \otimes_A M$ is nil module.

$$\begin{array}{ccc} A^{(2)} \otimes_A M' & \longrightarrow & A^{(2)} \otimes_A M \\ \downarrow & & \downarrow \\ 0 \longrightarrow M' & \longrightarrow & M \end{array}$$

$$\therefore \quad P \otimes_A A^{(2)} \otimes_A M' \hookrightarrow P \otimes_A A^{(2)} \otimes_A M$$

Suppose you consider $A, P, Q, \phi : Q \otimes P \twoheadrightarrow A$

P given nice choice for Q is

$A \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$. In general the pairing ϕ gives a map $Q \longrightarrow$ ~~$A \otimes \mathrm{Hom}_{A^{op}}(P, A)$~~ $A \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$ and we are looking at the case where this is an isomorphism. Then should be true that the only pairs $(x^l, x^r) \in \mathrm{Hom}_{A^{op}}(P,P) \times \mathrm{Hom}_A(Q,Q)^{op}$ compatible $(qx^r)p = q(x^l p)$ are those where x^r is the transpose of x^l. ~~Suppose P = A, Then~~

~~$\begin{pmatrix} A & \mathrm{Hom}_{A^{op}}(A,A) \\ A & A \otimes_A \mathrm{Hom}_{A^{op}}(A,A) \end{pmatrix}$~~

~~Case where $B \longrightarrow \mathrm{Hom}($~~

What about $B = P \otimes_A Q = P \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$

~~Thus~~ It seems that since Q is finin dual of P we ~~know~~ should have $B \xrightarrow{\sim} B \otimes_B \mathrm{Hom}_{B^{op}}(B, B)$, and so $\mathrm{Mult}(B) \xrightarrow{\sim} \mathrm{Hom}_{B^{op}}(B, B)$. What seems interesting here is that in the case where we take $B = A$ then we have $\begin{pmatrix} A & A \\ A & A \end{pmatrix} \subset \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

□ ω □ so have a meg homom. ~~A → B~~

$$A \longrightarrow A \otimes_A \mathrm{Hom}_{A^{op}}(A, A) = B$$

such that ~~the~~

$$\mathrm{Ker}\,(B \longrightarrow \mathrm{mult}(B)) = \mathrm{Ker}(B \to \mathrm{Hom}_{B^{op}}(B, B))$$

$$\{b \mid Bb = bB = 0\} \qquad \{b \mid bB = 0\}$$

Now I don't know much about the kernel of $A \to B$.

Notice that $B \xrightarrow{\sim} B \otimes_B \mathrm{Hom}_{B^{op}}(B, B)$

$\Longleftrightarrow$ $B \longrightarrow \mathrm{Hom}_{B^{op}}(B, B)$ left B-nil-isom

$\therefore$ $B \longrightarrow \mathrm{Hom}_{B^{op}}(B, B)$ both left + right nil isom

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You should look at this Q. ~~I guess it's true the point~~ and see if it's intrinsic. You should learn the proof of Roos' theorem. Find out what's intrinsic, ~~and what~~.

12/27 $\mathrm{Hom}_A(M, A^{(2)} \otimes_A N) \cong \mathrm{Hom}_A(M, N)$

$$\xrightarrow{\sim} \mathrm{Hom}_A(M, \mathrm{Hom}_B(B, N))$$

$$\cong \mathrm{Hom}_B(B \otimes_A M, N)$$

Take $M = A^{(2)} \otimes_A N$, image of identity under is

$\alpha : w_! w^* \longrightarrow 1$

$$\alpha : B \otimes_A A^{(2)} \otimes_A N \longrightarrow N \qquad b \otimes a_1 \otimes a_2 \otimes n \mapsto b w(a_1 a_2) n$$

Claim other adj map $\beta : 1 \longrightarrow w^* w_!$

$$\beta : M = A^{(3)} \otimes_A M \longrightarrow A^{(2)} \otimes_A B \otimes_A M$$

$$\beta(a_1 a_2 a_3 m) = a_1 \otimes a_2 \otimes w(a_3) \otimes m$$

Check that if $N = B \otimes_A M$, then β maps to id of N.

$$(\alpha . w_!)(w_! . \beta) = 1$$

$$\begin{pmatrix} g & & \\ & 1 & \\ & & g^* \end{pmatrix}\begin{pmatrix} g & & \\ 0 & g & \\ 0 & & g^* \end{pmatrix}^{-1} = \begin{pmatrix} g & 0 & 0 \\ 0 & g^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} g_1 g_2 g_1^{-1} g_2^{-1} & & \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} g_1 & & \\ & g_1^{-1} & \\ & & 1 \end{pmatrix}\begin{pmatrix} g_2 & & \\ & 1 & \\ & & g_2^{-1} \end{pmatrix}\begin{pmatrix} g_1^{-1} & & \\ & g_1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} g_2^{-1} & & \\ & 1 & \\ & & g_2 \end{pmatrix}$$

$$(GL(A), GL(A)) \subset E(A) \qquad \therefore \; E(A) \triangleleft GL(A)$$

$$\begin{pmatrix} g_1 & & \\ & g_1^* & \\ & & 1 \end{pmatrix}\begin{pmatrix} g_2 & & \\ & 1 & \\ & & g_2^* \end{pmatrix}\begin{pmatrix} g_1 & & \\ & g_1^* & \\ & & 1 \end{pmatrix}^{-1}\begin{pmatrix} g_2 & & \\ & 1 & \\ & & g_2^* \end{pmatrix}^{-1} = \begin{pmatrix} g_1 g_2 g_1^{-1} g_2^{-1} & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

How about ~~the~~ Whitney sum? So what do we have so far? You need a more ~~intelligent~~ way to proceed. What method. Think about K_2. Ret

$$C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix} = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A \begin{pmatrix} A & Q \end{pmatrix}$$

$$P \to Hom_A(Q, A)$$

$$\downarrow$$

$$Hom_A(AF_\alpha, A) = F_\alpha^* A$$

$$C_\alpha = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A \begin{pmatrix} A & AF_\alpha \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} A \\ F_\alpha^* A \end{pmatrix} \otimes_A \begin{pmatrix} A & AF_\alpha \end{pmatrix}$$

Concentral upon the world

$$\begin{pmatrix} 1 & 0 \\ +y(1-xy)^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}\begin{pmatrix} 1 & -(1-xy)^{-1}x \\ 0 & 1 \end{pmatrix} = \Bigg($$

$$\begin{pmatrix} 1 & x \\ +y(1-xy)^{-1} & 1+y(1-xy)^{-1}x \end{pmatrix}\begin{pmatrix} 1 & -(1-xy)^{-1}x \\ -y & 1-y(1-xy)^{-1}x \end{pmatrix}$$

$$\Big(1-xy$$

$y(1-xy)^{-1} + (1+y(1-xy)^{-1}x)(-y)$. Where is the intelligent place to start. Review Vaserstein's argument. (Yes!!!)

~~Coherent sheaves~~ Check that $E(A)$ is normal.

Check that $E(A)$ is normal. Why

$\forall g_1 \ \exists g_1^*$ such that $\begin{pmatrix} g_1 & \\ & g_1^* \end{pmatrix} \in E(A)$.

Then

$$\begin{pmatrix} g_1 & \\ & g_1^* \\ & & 1 \end{pmatrix}\begin{pmatrix} g_2 & \\ & 1 \\ & & g_2^* \end{pmatrix}$$

$$\begin{pmatrix} g & & \\ & g^* & \\ & & g^* \end{pmatrix}\begin{pmatrix} 1 & & \\ & g^* & \\ & & g^* \end{pmatrix}^{-1} = \begin{pmatrix} g & & \\ & g^{-1} & \\ & & 1 \end{pmatrix}$$

$$\begin{pmatrix} g_1 & & \\ & 1 & \\ & & g_1^{-1} \end{pmatrix}\begin{pmatrix} 1 & & \\ & g_2 & \\ & & g_2^{-1} \end{pmatrix} = \begin{pmatrix} g_1 & & \\ & g_2 & \\ & & (g_2 g_1)^{-1} \end{pmatrix}$$

Outline:

§1. firm A-module: $A \otimes_A M \xrightarrow{\sim} M$

$\mathcal{M}(A)$ = cat of firm A-modules, $\mathcal{M}(A)$ abelian if $A = A^2$.

§2. Morita equivalence between A, B (idempotent): $\mathcal{M}(A) \simeq \mathcal{M}(B)$

e.g. A M. eq. to $\bigcup_n M_n A$

§3. Morita invariance of K_1 for firm rings ($A \otimes_A A \xrightarrow{\sim} A$):

If A, B firm, then $\mathcal{M}(A) \simeq \mathcal{M}(B) \implies K_1 A \simeq K_1 B$

R unital mod(R) = cat of (left) unitary R-modules

We want to <u>extend</u> $R \mapsto \mathrm{mod}(R)$ to nonunital rings A.

The cat of A-modules is $\mathrm{mod}(\tilde{A})$ $\tilde{A} = \{n1 + a \mid n \in \mathbb{Z}, a \in A\}$

too big e.g. if A has e

$M = e$

$$\mathrm{mod}(\tilde{A}) = \mathrm{mod}(A) \times \mathrm{mod}(\tilde{A}/A)$$

<u>Thm 1.</u> Obvious functor

$$\mathcal{M}(A) \longrightarrow \mathrm{mod}(\tilde{A}) \Big/ \overbrace{\bigcup_n \mathrm{mod}(\tilde{A}/A^n)}^{\text{A-modules } M \text{ s.t. } A^n M = 0 \text{ for some } n}$$

firm modules

a) This functor is fully faithful

b) If $A = A^2$ this functor is an equivalence.

Inverse is $M \mapsto A \otimes_A A \otimes_A M$ ($= A \otimes_A M$ if $M = AM$)

L. $M = AM \Rightarrow \exists\, F \twoheadrightarrow M$ with F A-flat + $AF = F$.

$$\begin{array}{ccccccccc} \tilde{A}^{(S)} \cong P & \overset{\exists f}{\dashrightarrow} AP \subset & P & \overset{f}{\longrightarrow} AP \subset & P & \cdots\cdots & \longrightarrow AF = F \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M & = & M & = & M & \cdots\cdots & M \end{array}$$

Ex. $0 \longrightarrow A \longrightarrow \tilde{A} \longrightarrow \mathbb{Z} \longrightarrow 0$, apply $- \otimes_A M$

$$0 \longrightarrow \mathrm{Tor}_1^{\tilde{A}}(\mathbb{Z}, M) \longrightarrow A \otimes_{\tilde{A}} M \longrightarrow \tilde{A} \otimes_{\tilde{A}} M \longrightarrow \mathbb{Z} \otimes_{\tilde{A}} M \longrightarrow 0$$

$$\tilde{A} \otimes_{\tilde{A}} M = M \longrightarrow M/AM = \mathbb{Z} \otimes_{\tilde{A}} M$$

M firm $\Leftrightarrow$ $M/AM = 0 = \mathrm{Tor}_1^{\tilde{A}}(\mathbb{Z}, M) = 0$

If M $\tilde{A}$-flat then M firm $\Leftrightarrow M = AM$.

$K \to Q \overset{f}{\twoheadrightarrow} A$ $\qquad q_1 q_2 = f(q_1) q_2$ $\qquad \mathrm{Ker}(f) Q = 0$

$K^2 = 0$

$$\begin{pmatrix} A & Q \\ A & Q \end{pmatrix}$$

Q flat firm over A
$\Rightarrow B = P \otimes_A Q = A \otimes_A Q = Q$
is flat firm over B.

$KA = 0$

$0 \longrightarrow M(K) \longrightarrow GL(Q) \longrightarrow GL(A) \longrightarrow 1$ $\qquad$ Lydakis Topol 34 (95)

$H_0(GL(A), M(K)) \longrightarrow H_1(GL(Q)) \longrightarrow H_1(GL(A)) \longrightarrow 1$

$M(K)/M(AK)$

Try to ~~eq~~ organize talk tomorrow

1. A idempotent ring. $A = A^2$ $\quad$ $\mathcal{M}(A)$ cat of firm modules "good" module category
2. theory of Morita equivalence
3. Morita invariance of K_1 for firm rings.

$$\mathcal{M}(A) \simeq \mathcal{M}(B)$$

$$M \longmapsto P \otimes_A M$$

$$Q \otimes_B N \longleftarrow N$$

${}_B P_A$ (B, A)-bimodule firm on both sides

${}_A Q_B$ (A, B)-bimod.

$$M \simeq (Q \otimes_B P) \otimes_A M$$

$\Longrightarrow$ $Q \otimes_B P \simeq A \otimes_A A$

$P \otimes_A Q \simeq B \otimes_B B$ $\left(= B \text{ if } B \text{ firm}\right)$

Thm: A fixed idemp. ring. One has an equivalence between

- firm rings B equipped with $\mathcal{M}(A) \simeq \mathcal{M}(B)$
- triples $(P, Q, \langle\, , \rangle)$ $\quad$ P firm A^{op}-module, Q — A-module

$q \otimes p \mapsto \langle q, p \rangle$ surjective bimod map

$Q \otimes_{\mathbb{Z}} P \longrightarrow A$ $\qquad \langle aq, p \rangle = a \langle q, p \rangle$

~~Eg~~ You've left out $\begin{pmatrix} A & Q \\ P & B \end{pmatrix} = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A \begin{pmatrix} A & Q \end{pmatrix}$

Ex: ~~$\left(\begin{array}{c|c} A & A^2 \\ \hline A^2 & M_2 A \end{array}\right)$~~ $\qquad \left(\begin{array}{c|c} A & M_{1n}(A) \\ \hline M_{n1}(A) & M_n(A) \end{array}\right)$

Outline

§1. cat $\mathcal{M}(A)$ of firm A-modules: $A \otimes_A M \xrightarrow{\sim} M$, $a\otimes m \mapsto am$
for $A = A^2$.

§2. Morita equivalence between A, B: $\mathcal{M}(A) \simeq \mathcal{M}(B)$

§3. Thm. A, B firm rings $(A \otimes_A A \xrightarrow{\sim} A)$, then
$\mathcal{M}(A) \simeq \mathcal{M}(B) \implies K_1 A \simeq K_1 B$

Talk about Morita equivalence for idempotent rings, including Morita invariance of HH, HC for h-unital rings,

§1. The category $\mathcal{M}(A)$.

If R is a unital ring, let $mod(R)$ be the category of (left) unitary R-modules, i.e. such that $1m = m$.

We want to extend $R \mapsto mod(R)$ to nonunital rings A.

Note the category of A-modules $= mod(\tilde{A})$, where $\tilde{A} = \mathbb{Z} \oplus A$ with elements $n1+a$ is the ring obtained by adjoining an identity.

This is too big e.g. if A happens to have an identity element e then any A-module splits

$$M = eM \oplus (1-e)M$$

$$mod(\tilde{A}) = mod(A) \times mod(\mathbb{Z})$$

Def: Call an A-module M *nil* when $A^n M = 0$ for some n, and *firm* when $A \otimes_A M \to M$, $a \otimes m \mapsto am$ is an isomorphism.

Thm: (a) The obvious functor

$$firm(A) \to mod(A) / \bigcup_n mod(\tilde{A}/A^n)$$

is fully faithful.

(b) If $A = A^2$ then this functor is an equivalence of categories. The inverse functor $mod(A) \to firm(A)$ sends M to $A \otimes_A A \otimes_A M$ (which $= A \otimes_A M$ when $M = AM$).

§ 2. Theory of Morita equivalence,

From now on assume rings are idempotent, and put $\mathcal{M}(A) = firm(A)$.

Def: By *Morita equivalence* between A, B we mean an equivalence $\mathcal{M}(A) \simeq \mathcal{M}(B)$.

Example: Let $A \otimes_A A$ be the idempotent ring with product $(a_1 \otimes a_2)(a_3 \otimes a_4) = a_1 a_2 a_3 \otimes a_4$. The surjection $A \otimes_A A \to A$, $a_1 \otimes a_2 \mapsto a_1 a_2$ has kernel K such that $KA = AK = 0$, so $KM = KAM = 0$ for any firm module M. This gives a Morita equivalence $\mathcal{M}(A) \simeq \mathcal{M}(A \otimes_A A)$.

Def: Say A is a *firm ring* when $A \otimes_A A \xrightarrow{\sim} A$, i.e. A is in $\mathcal{M}(A)$ and $\mathcal{M}(A^{op})$.

$A \otimes_A A$ is a firm ring, so for studying Morita equivalence we can restrict attention to firm rings. $A \otimes_A A \to A$ is analogous to the universal central extension of a perfect group.

Define *Morita context* to be a ring

$$C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

equipped with a 2×2 matrix decomposition. 8 products

$$\begin{matrix} a_1 a_2, qp \in A & aq, qb \in Q \\ pa, bp \in P & b_1 b_2, pq \in B \end{matrix}$$

16 associativity conditions which amount to having rings A, B, bimodules ${}_BP_A, {}_AQ_B$, and bimodule maps $Q \otimes_B P \to A$, $P \otimes_A Q \to B$, satisfying $(pq)p' = p(qp')$ and $(qp)q' = q(pq')$.

Say the Morita context C is *strictly firm* when the 8 products give isomorphisms $A \otimes_A A \xrightarrow{\sim} A$, $Q \otimes_B P \xrightarrow{\sim} A$, etc.

Prop: If $QP = A$, $PQ = B$ then we have a Morita equivalence $\mathcal{M}(A) \simeq \mathcal{M}(B)$ given by $M \mapsto P \otimes_A M$, $N \mapsto Q \otimes_B N$. If A, B firm Any Morita equivalence between A and B arises in this way by a unique (up to canonical isomorphism) strictly firm Morita context.

Observe that because $P \otimes_A Q \xrightarrow{\sim} B$ and $(p_1q_1)(p_2q_2) = p_1(q_1p_2)q_2$ a strictly firm Morita contex is determined by the modules P, Q over A and the pairing qp.

Thm. Let A be a fixed firm ring. Then we have an equivalence between:

- Firm rings B equipped with an equivalence $\mathcal{M}(A) \xrightarrow{\sim} \mathcal{M}(B)$.
- Triples $(P, Q, \langle -,- \rangle)$ with P in $firm(A^{op})$, Q in $firm(A)$, and $\langle -,- \rangle : Q \otimes_{\mathbb{Z}} P \to A$ a surjective A-bimodule map.

One has $B = P \otimes_A Q$ with product $(p_1 \otimes q_1)(p_2 \otimes q_2) = p_1 \langle q_1, p_2 \rangle \otimes q_2$

Example: When A is unital, can take any surjective pairing $\langle -,- \rangle$, where P, Q are unitary right and left modules over A.

$$\begin{pmatrix} 1 & 0 \\ y(1-xy)^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}\begin{pmatrix} 1 & -(1-xy)^{-1}x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-xy & 0 \\ 0 & (1-yx)^{-1} \end{pmatrix}$$

Θ

C_A Mcontexts s form $\begin{pmatrix} A & * \\ * & * \end{pmatrix}$

Cat of (B, F)

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bide comment ideal Suppose A given, consider its annihilator $\ker(A \longrightarrow \mathrm{Hom}_{A^{op}}(A, A))$. Wait, the question is whether I can construct a meghan $B \subset B'$ which eliminates part of the left + right annihilator. Example is to take $B = P \otimes_A Q$. Try to enlarge Q say to $Q \subset Q'$, for example adding something to Q which might pair better with P.

$$B = P \otimes_A Q \qquad \mathrm{Hom}_{B^{op}}(B, B) = \mathrm{Hom}_{A^{op}}(P, P)$$

$$\uparrow B$$

Keep P fixed, try enlarging Q. $\qquad Q \otimes P \xrightarrow{\phi} A$

$$Q \longrightarrow A \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$$

$$\cap \qquad \nearrow$$

$$Q'$$

There are questions, first whether you can find interesting maps $P \longrightarrow A$. ~~A is flat~~

$$\to \{a \mid aA = 0\} \longrightarrow A \longrightarrow \mathrm{Hom}_{A^{op}}(A, A) \qquad a \mapsto (a' \mapsto aa')$$

$$0 \to A \otimes_A \{a \mid aA = 0\} \longrightarrow A \longrightarrow A \otimes_A \mathrm{Hom}_{A^{op}}(A, A) \quad \text{if } A \text{ flat.}$$

Keep on trying 1/6 Answer
How to proceed? Consider

Let's go over again what I need to put into §27. You have Morbian to discuss. converse direct part.
Keep concrete.