 December 14

learn periodicity thm. proof of Atiyah-Bott, also the Atiyah proof with Fredholm operators. Start with the latter. I recall the structure

$X = \mathbb{C}\times Y$. Define $\pi_*: K^0_*(\mathbb{C}\times Y) \to K^0_*(Y)$ compatible with pull back, and such that $\pi_*\pi^* = 1$. Then in

$$\begin{array}{ccc} X\times_Y X & \xrightarrow{pr_2} & X \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\pi} & Y \end{array} \quad ?$$

you need take $X \xrightarrow{\xi} \text{Fred}$ and define $\pi_*(\xi)$ this is a kind of Kasparov product. Over $\mathbb{C}$ you have something simple, shift operator on Hardy ~~space~~?

exercise. Start with a clutching function $f \in (A[t,t^{-1}])^\times$ How do you replace it by a linear clutching function? Or, Kronecker module.

f defines a v.b. over $\mathbb{P}^1$

$$\Gamma(U_0,\mathcal{E}_f) = A[t] \xrightarrow{\cdot f^{-1}} \Gamma(U_0\cap U_\infty,\mathcal{E}_f) = A[t,t^{-1}]$$
$$\Gamma(U_\infty,\mathcal{E}_f) = A[t^{-1}] \xrightarrow{1} \Gamma(U_0\cap U_\infty,\mathcal{E}_f)$$

$$H^0(\mathbb{P}^1,\mathcal{E}_f) = f(t)^{-1}A[t] \cap A[t^{-1}]$$
$$f(t)^{-1}A[t] \supset A[t] \quad \text{if } f \text{ is a poly in } t.$$

$$H^1(\mathbb{P}^1,\mathcal{E}_f) = A[t,t^{-1}] \big/ f^{-1}A[t] + A[t^{-1}]$$

$$H^0(\mathbb{P}^1,\mathcal{E}_{t^{-1}f}) = t\,f(t)^{-1}A[t] \cap A[t^{-1}]$$
$$\simeq A[t] \cap f(t)\,t^{-1}A[t^{-1}]$$

 So what. You want to see the exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \otimes H^0(\mathcal{E}(-1)) \longrightarrow \mathcal{O} \otimes H^0(\mathcal{E}) \longrightarrow \mathcal{E} \longrightarrow 0$$

But wait. If $f(t)$ is a poly in t, then we should have $\mathcal{O} \hookrightarrow \mathcal{E}$ where the cokernel is supp. at ∞. (killed by power of t^{-1}).

think carefully.

$$H^0(\mathcal{E}_f) = A[t] \cap f(t) A[t^{-1}].$$

$$H^0(\mathcal{E}_{t^{-1}f}) = A[t] \cap f(t) t^{-1} A[t^{-1}]$$

You ought to be able to work out the details.

Suppose you have ~~$H^0(\mathcal{E}$~~

Put $V = A[t] \cap f(t) A[t^{-1}]$

$W = A[t] \cap f(t) t^{-1} A[t^{-1}]$

Then you have two A-modules maps $W \overset{1}{\underset{t}{\rightrightarrows}} V$

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & W \arrow[r] \arrow[d, shift right, "1"'] \arrow[d, shift left, "t"] & A[t] \oplus f(t) t^{-1} A[t^{-1}] \arrow[r] \arrow[d, shift right] \arrow[d, shift left, "\begin{pmatrix} t & t \end{pmatrix}"] & A[t, t^{-1}] \arrow[r] \arrow[d, equal] \arrow[d, shift left=3, "\cong t"] & 0 \\
0 \arrow[r] & V \arrow[r] & A[t] \oplus f(t) A[t^{-1}] \arrow[r] \arrow[d] & A[t, t^{-1}]. \arrow[r] & 0 \\
 & & A \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

Notice that $A[t] \oplus f(t) t^{-1} A[t^{-1}]$

$$\begin{tikzcd}
0 \arrow[r] & W \arrow[r] & f(t)A[t] \oplus f(t)t^{-1}A[t^{-1}] \arrow[r] \arrow[d, hook] & A[t,t^{-1}] \arrow[r] \arrow[d, equal] & 0 \\
& W \arrow[r] & A[t] \oplus t^{-1}A[t^{-1}] \arrow[r] \arrow[d] & A[t,t^{-1}] \arrow[r] & 0 \\
& & A[t]/f(t)A[t] \arrow[d] & & \\
& & 0 & &
\end{tikzcd}$$

So $W = A[t]/f(t)A[t] \underset{t}{\overset{1}{\rightrightarrows}} V = A[t]/f(t)t\,A[t]$

You're missing an organizing principle — namely invertible matrices over $A[t,t^{-1}]$ determines v.b.'s over $\mathbb{P}^1_A$. Compatible with $\oplus$. Elementary?

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notice $\quad f \in GL_r(A[t,t^{-1}]) \qquad GL_n(A[t,t^{-1}])/GL_n(A[t^{-1}])$

given an f you get $\quad f(A[t]^n)\, L_0$.

Fix L_0, then $f \longmapsto f(L_0)$. This should be commensurable with L_0. We know

$f = t^{-N} \underbrace{t^N f}_{\in A[t]}$.

L_0

$t^{-N}L_0$

Wait. Assume $f \in A[t]$

$L_0 \qquad f(L_0)$

125 Consider $A[t,t^{-1}]^\times$ acting on $A[t,t^{-1}]^\times / A[t^{-1}]^\times$

Fix a basis point i.e. $A[t^{-1}]^\times$. Think of lattices. Given $g \in A[t,t^{-1}]$ have gL_0.

Assume $g \in A[t]$, $\exists\, g'$ $gg' = g'g = t^N$.

Maybe it's easier to use $GL(A[t,t^{-1}])/GL(A[t])$

So let $g \in A[t,t^{-1}]^\times$, assume first $g \in A[t]$.

$gg' = g'g = t^N$.

$$\begin{tikzcd} A[t] \\ g'A[t] \arrow[u, hook] \\ {} \arrow[u, hook] \end{tikzcd} \qquad \begin{array}{ccc} gA[t] & \subset & A[t] \\ \cup & & \cup \\ t^N g'g A[t] & \subset & g'A[t] \end{array}$$

$$\begin{tikzcd} 0 \arrow[r] & gA[t]/t^N A[t] \arrow[r] & A[t]/t^N A[t] \arrow[r] & A[t]/gA[t] \arrow[r] & 0 \\ & A[t]/g'A[t] \arrow[u, "\wr"] \end{tikzcd}$$

$$0 \longrightarrow A[t] \xrightarrow{g} A[t] \longrightarrow A[t]/gA[t] \longrightarrow 0$$

$$A[t]/gA[t] \in \mathcal{P}^1(A)$$

Try again: ~~A A[t,t^{-1}]~~ Given $f \in A[t,t^{-1}]^\times$ you get v.b. $E_f(-1)$ over $\mathbb{P}^1$ given by

$$A[t] \hookrightarrow A[t,t^{-1}] \hookleftarrow ft^{-1}A[t^{-1}]$$

Good case when $f \in A[t]$. Then $A[t] + ft^{-1}A[t] \supset$

$$fA[t] + ft^{-1}A[t^{-1}] = fA[t,t^{-1}] = A[t,t^{-1}]$$

moreover

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
 & A[t] \oplus t^{-1}A[t^{-1}] \arrow[r, phantom, "="] \arrow[d, "f \oplus 1"] & A[t,t^{-1}] \arrow[d, "f", "\wr"'] & \\
0 \arrow[r, hook] & A[t] \cap ft^{-1}A[t^{-1}] \arrow[r] \arrow[dr, "\sim"] & A[t] \oplus ft^{-1}A[t^{-1}] \arrow[r] \arrow[d] & A[t,t^{-1}] \arrow[r] & 0 \\
 & & A[t]/fA[t] \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

shows that $H^0(E_f(-1)) \simeq A[t]/fA[t]$. Try to understand. You have $A[t,t^{-1}]^\times$ acting on $A[t^{-1}]$-lattices inside $A[t,t^{-1}]$. ~~Confused~~ $A[t,t^{-1}]$ acts on certain $A[t^{-1}]$ submodules of $A[t,t^{-1}]$. Then you take zilch.

Keep on ~~trying~~. What can you say about sending $f \in GL_\bullet(A[t,t^{-1}])$ to the difference

$$\left[A[t]/t^s f A[t]\right] - \left[A[t]/t^s A[t]\right] \in K_0(A).$$

s large enough so that $t^s f \in A[t]$. ~~Not Confused~~

Fibring over a circle. Start with a finite $\overset{\text{conn.}}{\text{complex}}$ X, assume given $\alpha: \pi_1(X) \twoheadrightarrow \mathbb{Z}$, $\tilde{X}$ corres Galois covering. If $\exists$ fibring $X \xrightarrow{f} S^1$ inducing α, then $\tilde{X} \sim$ fibre of f, so $\tilde{X}$ htpic to a finite cx. ~~Look at $\mathbb{Z}[\tilde{X}]$ complex of free~~ Assume $\pi_1(X) \xrightarrow{\sim} \mathbb{Z}$

$\mathbb{Z}[\tilde{X}]$ f. free complex over $\mathbb{Z}[t,t^{-1}]$?

 Spend time until dinner trying to understand Bass FT. Describe $K_1(A[t,t^{-1}])$. How much do I understand so far? I recall Andrew saying that ~~...~~ replacing a clutching function by a linear clutching function is an ~~...~~ instance of transversality. Why? Maybe something similar happens in Waldhausen's free product theory. Review your view point previously. ~~...~~

$R =$ ~~...~~ $A \underset{C}{*} B$. I remember filtered modules.

R has an obvious filtration

$$\begin{tikzcd} & A \arrow[r, dash] \arrow[dr, dash] & AB & ABA \\ C \arrow[ur, hook] \arrow[dr, hook] & & & \\ & B \arrow[r, dash] \arrow[uur, dash] & BA & BAB \end{tikzcd}$$

assoc. graded.

$$\begin{matrix} & \bar{A} & \bar{A}\otimes_C \bar{B} \\ C & & \\ & \bar{B} & \bar{B}\otimes_C \bar{A} \end{matrix}$$

~~...~~

I also remember using systems $M_A \leftarrow M_C \rightarrow M_B$ to describe an R-module. Associate to such a system the pushout

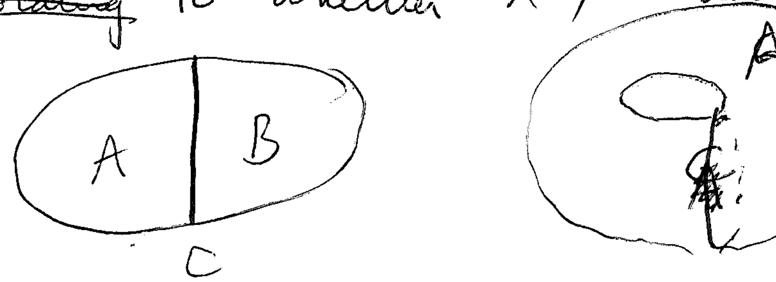
$$\begin{tikzcd} R\otimes_C M_C \arrow[r] \arrow[d] & R\otimes_B M_B \arrow[d] \\ R\otimes_A M_A \arrow[r] & M \end{tikzcd}$$

I remember Waldhausen running the arrows backwards, i.e. using M_A, M_B, M_C together with $M_A \to A\otimes_C M_C$ and $M_B \to B\otimes M_C$. ~~...~~ To this system you can assoc. the pushback:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & R\otimes_B M_B \arrow[r] & R\otimes_B B\otimes_C M_C \arrow[d, equal] \\ R\otimes_A M_A \arrow[r] & R\otimes_A A\otimes_C M_C = & R\otimes_C M_C \end{tikzcd}$$

128 Roughly W describes R-modules as these systems modulo nil systems.

Another case was $A_\alpha[t, t^{-1}]$ twisted Laurent polys. Important: Geometry of an oriented hypersurface in a manifold. X closed manifold Y oriented codim 1 submanifold. ∃ Two cases as ~~according~~ to whether $X - Y$ has 1 or 2 components.



$C \rightrightarrows A$

In the geometry the rings are group rings of π_1's.
van Kampen tells you that $\mathbb{Z}[\pi_1(X_A \amalg_{X_C} X_B)] = A *_C B$

So what is the ~~general~~ situation? Your idea was to consider filtered objects. ~~Best example might be~~
Example. $S(V)$-modules. Given M choose F_0 generating M, then put $F_p = S_{\le p}(V) M_0$ i.e. $F_p = F_{p-1} + V F_{p-1}$.
get graded modules over $\bigoplus_{p \ge 0} S_p(V)$. Nice class

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12/05/97 Can we prove Morita invariance of K_0?

$C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ write out all details

can I derive ~~to prove~~ $PQ = B \implies K_0(A) \xrightarrow{\sim} K_0(C)$. from the result $\begin{matrix} PQ = B \\ QP = A \end{matrix} \implies K_0(A) = K_0(B)$.

$$\begin{matrix} A & \tilde{A} & Q \\ A & A & AQ \\ P & P & B \end{matrix}$$

so find $K_0(A) = K_0 \begin{pmatrix} A & AQ \\ P & B \end{pmatrix}$

~~mistake~~ contradiction here, ~~so~~ or maybe ~~rather~~ some interesting new phenomenon.

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To show $C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$, $PQ = B \Rightarrow K_0(A) \xrightarrow{\sim} K_0(C)$.

$$\begin{tikzcd}
\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \arrow[r, phantom, "\subset"] & \begin{pmatrix} \tilde{A} & Q \\ P & B \end{pmatrix} \arrow[r] & \mathbb{Z} \\
A \arrow[u, hook] \arrow[r] & \tilde{A} \arrow[u, hook] \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

OKAY

$$\begin{matrix} \tilde{A} & A \\ A & A^2 \end{matrix}$$

$$\begin{tikzcd}
K_0 A \arrow[r, "\text{upper left}", "\sim"'] \arrow[d, equal, "\text{upper left}"] \arrow[dr, "\sim"] & K_0 \begin{pmatrix} A & \tilde{A} & Q \\ A & A & AQ \\ P & P & B \end{pmatrix} & K_0 \begin{pmatrix} A & AQ \\ P & B \end{pmatrix} \arrow[l, "\sim"'] \\
K_0(A) \arrow[r, "u.l.", "\sim"'] \arrow[d, hook] & K_0 \begin{pmatrix} A & \tilde{A} \\ A & \tilde{A} \end{pmatrix} \arrow[u, "\subseteq"] \arrow[d] & K_0(A) \arrow[l, "\sim", "\text{bottom right}"'] \arrow[u, "\text{upper left}"'] \arrow[d, hook] \\
K_0(\tilde{A}) \arrow[r, "\sim"] & K_0 \begin{pmatrix} \tilde{A} & \tilde{A} \\ \tilde{A} & \tilde{A} \end{pmatrix} & K_0(\tilde{A}) \arrow[l, "\sim"']
\end{tikzcd}$$

Is $\begin{pmatrix} A & AQ \\ P & B \end{pmatrix}$ an ideal in $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}^2 = \begin{pmatrix} A^2 + QP & AQ + QB \\ PA + BP & B^2 + PQ \end{pmatrix} \subset \begin{pmatrix} A & AQ \\ P & B \end{pmatrix}$$

$$QB = QPQ \subset AQ$$

Wait

If $PQ = B$ then $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}^2 = \begin{pmatrix} A^2 + QP & AQ \\ PA & B \end{pmatrix}$

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}\begin{pmatrix} A^2+QP & AQ \\ PA & B \end{pmatrix} = \begin{pmatrix} A^3+AQP+QPA & A^2Q+QB \\ PA^2+PQP+BPA & PAQ+B^2 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix} \qquad X^2 = \begin{pmatrix} QP & 0 \\ 0 & PQ \end{pmatrix}$$

$$(X+X^2)^2 = \begin{pmatrix} QP & Q \\ P & PQ \end{pmatrix}$$

$$(X+X^2)^3 = \begin{pmatrix} QPQP & QPQ \\ PQP & PQPQ \end{pmatrix}$$

So the argument above $\begin{pmatrix} A & AQ \\ P & B \end{pmatrix} \supset \begin{pmatrix} A & AQ \\ P & PAQ \end{pmatrix}$

Anyway what happens?

$$\begin{pmatrix} A & Q \\ P & PQ \end{pmatrix} \supset \begin{pmatrix} A & AQ \\ P & PQ \end{pmatrix} \supset \begin{pmatrix} A & Q \\ P & PQ \end{pmatrix}^2$$

$$\begin{pmatrix} A & Q \\ P & PQ \end{pmatrix}^2 = \begin{pmatrix} A^2+QP & AQ \\ PA & PQPQ \end{pmatrix}$$

$$K_0\begin{pmatrix} A & Q \\ P & PQ \end{pmatrix} \leftarrow K_0\begin{pmatrix} A & AQ \\ P & PQ \end{pmatrix} \leftarrow K_0\begin{pmatrix} A & AQ \\ P & PAQ \end{pmatrix}$$

All the rings you write down are ideals of C contained ~~between~~ $C \supset D \supset C^2$

so $K_0(D) \to K_0(C) \to K_0(C/D)$

$$C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix} \supset D = \begin{pmatrix} A & AQ \\ P & B \end{pmatrix} \supset C^2 = \begin{pmatrix} A^2+QP & AQ \\ PA & B \end{pmatrix}$$

$$D = \begin{pmatrix} A & AQ \\ P & B \end{pmatrix} \supset \begin{pmatrix} A & AQ \\ P & PAQ \end{pmatrix} \supset D^2$$

stupid but OKAY. probably

Start with $U_1 \underset{d}{\overset{\exists h}{\rightleftarrows}} U_0$ $\quad \exists h \ni 1-[d,h]$ nuclear

Form $P\otimes_A U_1 \xrightarrow{d} P\otimes_A U_0$ $\quad$ First assume $U_i \in \mathcal{P}(\tilde{A})$

$$1-[d,h] \in \cancel{\mathrm{Hom}_B(P\otimes_A U, P\otimes_A U)}$$

$$\mathrm{Hom}_A(U,U) = \mathrm{Hom}_A(U,\tilde{A}) \otimes_A U \quad \text{as } U\in\mathcal{P}(\tilde{A})$$

$$\mathrm{Hom}_A(U,AU) = \mathrm{Hom}_A(U,\tilde{A}) \otimes_A A \otimes_A U.$$

So basically $1-[d,h]$ factors $U \longrightarrow AT \subset T \longrightarrow U$

but I am assuming $U \in \mathcal{P}(\tilde{A})$. Look at

$P\otimes_A U$, $U^\vee \otimes_A Q$. What do we know?

$P\otimes_A U$ is a complex. with diffl $1\otimes d$ and htpy $1\otimes h$

$1-[1\otimes d, 1\otimes h] = 1\otimes f$. $\qquad P\otimes_A U \xrightarrow{1\otimes f}$ [redacted] $.P\otimes_A U$

so $f : U \longrightarrow A\otimes U = QPU$

I want a nuclear map on $P\otimes_A U$. You have

$1-dh : U_0 \longrightarrow AU_0 = A\otimes_A U_0$ $\qquad$ So what?

Start again: You have $U_1 \xrightarrow{d} U_0$ over A such that $\exists h$ with $1-[d,h]$ A-nuclear. To show $P\otimes_A U_1 \longrightarrow P\otimes_A U_0$ has a htp of $h \ni 1-[d,h]$ is B-nuclear. $1-[d,h] \in \mathrm{Im}\left\{ \mathrm{Hom}(U,A)\otimes_A U \longrightarrow \mathrm{Hom}_A(U,U) \right\}$

 Assume $U \in \mathcal{P}(\tilde{A})$. Then

$$
\begin{tikzcd}
0 \arrow[r] & U^\vee \otimes_A A \otimes_A U \arrow[r, phantom, "\subset"] & U^\vee \otimes_A U \arrow[r] & \operatorname{Hom}_{\mathbb{Z}}(U^\vee/U^\vee \tilde{A}, U/\tilde{A}U) \arrow[r] & 0 \\
 & (U^\vee \otimes_A Q)\otimes_B (P \otimes_A U) \arrow[u] & & &
\end{tikzcd}
$$

So it's clear. The step seems to be this:

Start with $U = \{U_1 \xrightarrow{d} U_0\}$ over A such that 1_U can be deformed to an A-nuclear map.

i.e. $\exists\, h \ni 1-[d,h]$ is A-nuclear. Use $\tilde{A}$-nuclear

$1-[d,h]$: $\operatorname{Im}\{\operatorname{Hom}_A(U,\tilde{A}) \otimes_A U \longrightarrow \operatorname{Hom}_A(U,U)\}$

Then get homotopy equivalence of U with a f proj cx T.

$$
\begin{tikzcd}
T_1 \arrow[r] \arrow[d] & T_0 \arrow[d] \\
U_1 \arrow[r] \arrow[u, phantom, "\oplus"] & U_0
\end{tikzcd}
$$

It should follow that T/AT is contractible.

$$
\begin{tikzcd}[column sep=huge]
T_1 \arrow[r, shift right, "\begin{pmatrix} j_1 \\ -d \end{pmatrix}"'] & U_1 \oplus T_0 \arrow[l, shift right, "(i_1\ \ -h)"'] \arrow[r, shift right, "(d\ \ j_0)"'] & U_0 \arrow[l, shift right, "\begin{pmatrix} h \\ i_0 \end{pmatrix}"']
\end{tikzcd}
$$

If you reduce mod A, then $j_0 i_0 = 0$.

Something is curious. Assume $\underline{d^U h^U = 1}$

$$
\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & T_0 \arrow[d, "\begin{pmatrix}0\\1\end{pmatrix}"] \arrow[dr, "-h"] & & \\
0 \arrow[r] & U_0 \arrow[r, "\begin{pmatrix}h\\\iota_0\end{pmatrix}"] \arrow[dr, "h"'] & U_1 \oplus T_0 \arrow[r, "(d_1\ -h)"] \arrow[d, "(1\ 0)"] & T_1 \arrow[r] & 0 \\
 & & U_1 \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & U_1 \arrow[d, "\begin{pmatrix}1\\0\end{pmatrix}"] \arrow[dr, "d"] & & \\
0 \arrow[r] & T_1 \arrow[r, "\begin{pmatrix}j_1\\-d\end{pmatrix}"'] \arrow[dr, "-d"'] & U_1 \oplus T_0 \arrow[r, "(d\ \ j_0)"'] \arrow[d, "(0\ 1)"] & U_0 \arrow[r] & 0 \\
 & & T_0 \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}
$$

d^T isom $\iff$ d^U isom.
h^T — '' — h^U —

Assume d^U, h^U inverses

$$
\begin{tikzcd}
T_1 \arrow[r, "\sim"] \arrow[d, "j_1"] & T_0 \arrow[d, "j_0"] \\
U_1 \arrow[r, "\sim"] & U_0
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & U_0 \arrow[d, "\begin{pmatrix}1\\0\end{pmatrix}"] \arrow[dr, "1"] & & \\
0 \arrow[r] & T_0 \arrow[r, "(j_0\ -1)"] \arrow[dr, "-1"'] & U_0 \oplus T_0 \arrow[r, "(1\ j_0)"'] \arrow[d, "(0\ 1)"] & U_0 \arrow[r] & 0 \\
 & & T_0 & &
\end{tikzcd}
$$

134. I guess it's now easy enough to check Morita invariance modulo homotopy — same as in your paper. ~~Objects are length 1 complexes~~ There seems to be an equivalence of homotopy categories of these complexes.

Basically you do this. Given $U_1 \xrightarrow{d} U_0$ you know it is heq a perfect T: $T_1 \xrightarrow{d} T_0$. ~~If you can argue as follows.~~ Proceed as follows. ~~[illegible]~~

Given $U_1 \xrightarrow{d} U_0$ & $\exists h$ with $1-[d,h]$ nuclear you factor $1-dh = j_0 i_0 : U_0 \to T_0 \to U_0$, then define T_1 by fibre ~~[illegible]~~ product, etc. Alt. version:

~~Given [illegible] U_1, U_0, α You choose T_0~~

Given A-modules U_1, U_0 and $\alpha : U_1/AU_1 \xrightarrow{\sim} U_0/AU_0$ assume you can lift α, α^{-1} to d, h such that $1-dh : U_0 \to U_0$ and $1-hd : U_1 \to U_1$ are ~~all~~ nuclear. ~~Then get A-htpy~~

~~Go from $U_1 \rightleftarrows U_0$~~

Start with $(U_1, U_0, \alpha : U_1/AU_1 \simeq U_0/AU_0)$ where $U_i \in P(\tilde{A})$. Wait. Suppose we begin with a general pair U_1, U_0 and such α. Assume α, α^{-1} can be lifted to d, h such that $1-[d,h]$ is nuclear. Writing $1 - dh = j_0 i_0 : U_0 \longrightarrow T_0 \longrightarrow U_0$ with $T_0 \in P(\tilde{A})$ get

$$
\begin{tikzcd}
0 \arrow[r] & T_1 \arrow[r, shift right, "\binom{j_1}{-d}"'] & U_1 \oplus T_0 \arrow[l, shift right, "(i_1\ -h)"'] \arrow[r, shift right, "(d\ \ j_0)"'] & U_0 \arrow[l, shift right, "\binom{h}{i_0}"'] \arrow[r] & 0
\end{tikzcd}
$$

homot. equiv. $T \underset{j_*}{\overset{i}{\rightleftarrows}} U$ of complexes with $T_i \in P(\tilde{A})$. Reduce mod A to see T/AT contractible

135 Let's begin with the ~~definition~~

Start again. Given U_1 U_0 d, h $\ni$ $1-[d,h]$ nuc.

One question: Given a nuclear $f \in U^\vee \otimes_A U$ we know how writing $f = \sum \lambda_i \otimes u_i$ is equiv. to a fact. $U \longrightarrow \tilde{A}^m \longrightarrow U$. What is the arbitrariness? Ask about $U \xrightarrow{i} \tilde{A}^m \xrightarrow{j} U$ being zero. This is nil stuff of some sort. What would be a meaningful statement? Look at the operator $ij : \tilde{A}^m \longrightarrow U \longrightarrow \tilde{A}^m$. This is a nilpotent operator on a f.p. module.

Maybe however you should just take $U : U_1 \xrightarrow{d} U_0$ and ask about hcg's $T \xrightarrow{j} U$. You know that such a T is determined by a map $T_0 \longrightarrow U_0$ ~~with~~ which is transversal to $d : U_1 \to U_0$

I don't see what this means, but continue with the program of M equiv for K_0.

Start with (U_1, U_0, α) $U_i \in P(\tilde{A})$, choose liftings of α, α^{-1} to d, h. Then $1-[d,h] : U \longrightarrow AU$.

Better $f = 1-[d,h] \in U^\vee \otimes_A A \otimes_A U$ $\qquad U^\vee = Hom_A(U, \tilde{A})$

Have $(U^\vee \otimes_A Q) \otimes_B (P \otimes_A U) \longrightarrow U^\vee \otimes_A A \otimes_A U$, so it seems that $1-[d,h]$ on $P \otimes_A U$ is nuclear.

So you get.

$$\begin{tikzcd}
T_1 \arrow[r, "d"] \arrow[d, "j_1"'] & T_0 = \tilde{B}^{\oplus N} \arrow[d, shift right, "j_0"'] \\
P \otimes_A U_1 \arrow[r, shift left, "d"] & P \otimes_A U_0 \arrow[l, shift left, bend right, "h"'] \arrow[u, shift right, "i_0"']
\end{tikzcd}$$

$[d, j] = 0$

$[h, i] = 0$

OKAY.

What to ask? You make these choices: d, h, i_0, j_0 to get $(T_1, T_0, d \bmod B)$ and you want to know the arbitrariness. I know that the important choice is d lifting α, that any two choices of $j_0 T_0$ lead to homotopy equivalent T

(Digress: given $\dot{U}_1 \xrightarrow{d} \dot{U}_0 \ni \exists h$ ~~with~~ $1-[d,h]$ nuc.
given any $T_0 \to \dot{U}_0$ transversal to d i.e. $U_1 \oplus T_0 \twoheadrightarrow U_0$.
Then this surjection has a section. ~~So~~ Because
you already have get it straight.

$$\begin{tikzcd} & \dot{U}_1 \oplus T_0 \arrow[d] \\ \dot{U}_1 \oplus \dot{T}_0 \arrow[ur, "1\oplus"] \arrow[r, shift left] & \dot{U}_0 \arrow[l, shift left] \end{tikzcd}$$
)

Start with U_1, U_0, α (in $P(\tilde{A})$) choose $d, h \ni 1-[d,h]$ nuc.

Look at $P \otimes_A U_1 \underset{d'}{\overset{h'}{\rightleftarrows}} P \otimes_A U_0$

$$1-[d',h'] = 1 \otimes \underbrace{(1-[d,h])}_{U \longrightarrow AU} : P \otimes_A U \longrightarrow P \otimes_A AU$$

$$P \otimes_A AU = PA \otimes_A U \qquad PA = PQP \subset BP.$$

so modulo B the complex $P \otimes_A U$ is contractible.
even better d', h' are inverses modulo B.
So apparently we get a well-defined element of $K_0(B)$.

So far it seems that given $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ with $QP = A$ I might get a well-defined map $K_0 A \to K_0 B$. Let's examine whether this is consistent. I know that
$C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ with $QP = B \implies K_0 B \xrightarrow{\sim} K_0 C$ and since we have obviously $A \to C$ we get $K_0 A \to K_0 C \xleftarrow{\sim} K_0 B$.
I think I can also show that, when $QP = A$ and $PQ = B$, then $K_0 A \to K_0 B \to K_0 A$ are inverses. What you need is an M-cont. linking B and C.

$$\begin{matrix} A & Q & Q \\ P & B & B \\ P & B & B \end{matrix}$$

reverse ~~Addn~~

$$\begin{array}{c|cc} & A & Q \\ \hline A & A^2 & AQ \\ P & PA & B \end{array}$$

137 Dec 6. Try again

idea: extend trace map $K_*(P\otimes_A Q) \to K_*(A)$ where (P,Q) is super. Analog for complex of P flat.

$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ Assume $QP = A$.

Take $\xi \in K_0 A$.

Represent by triple (U_1, U_0, α) with $U_i \in \mathcal{P}(A)$, $\alpha : U_1/AU_1 \xrightarrow{\sim} U_0/AU_0$. Choose d, h lifting α, α^{-1}. Then $U_1 \xrightarrow{d} U_0$ is a f.proj. complex over A ~~admitting~~ acyclic mod A. So I know that $P\otimes_A U_0 \to P\otimes_A U_1$ admits parametrix. ?

Given ξ, choose (U_1, U_0, α), choose d lifting α, get complex $U_1 \xrightarrow{d} U_0$, send to $P\otimes_A U_1 \to P\otimes_A U_0$, can choose a f.proj. T acyclic mod B and a heq $T \to P\otimes_A U$. Then $T: T_1 \xrightarrow{d} T_0$ yields $(T_1, T_0, d: T_1/BT_1 \xrightarrow{\sim} T_0/BT_0)$ hence an element of $K_0 B$. To check this is a well defined map $K_0 A \to K_0 B$. I already know it is well-defined $K_0' A \to K_0' B$.

The kernel of $K_0'A \to K_0 A$ is gen by elts of form $L \xrightarrow{1+a} L$ L free over A. But actually $d: U_1 \to U_0$ is unique up to a map $g \in U_1 \to AU_0$, which is nuclear and 0 mod A. Does this ~~become~~ B-nuclear $P\otimes_A U_1 \to P\otimes_A U_0$?

$$g \in U_1^\vee \otimes_A A \otimes_A U_0$$
$$\uparrow$$
$$(U_1^\vee \otimes_A Q)\otimes_B (P\otimes_A U_0) \longrightarrow (P\otimes_A U_1)^* \otimes_B (P \otimes U_0)$$

~~idea $(P\otimes_A U_0)\otimes_{\mathbb{Z}} (U_0^\vee \otimes_A Q) \to P \otimes_A \tilde{A} \otimes_A Q$~~

$$(P\otimes_A U_1)\otimes_{\mathbb{Z}} (U_1^\vee \otimes_A Q) \to P\otimes_A \tilde{A} \otimes_A Q \longrightarrow B$$

$$\therefore\ U_1^\vee \otimes_A Q \longrightarrow \mathrm{Hom}_B(P\otimes_A U_1, B) = (P\otimes_A U_1)^*$$

$$\begin{tikzcd} & T_0 \arrow[d, "j_0"] \\ U_1 \arrow[r, "d"'] & U_0 \end{tikzcd} \qquad 1 - dh = j_0 \iota_0$$

$$T_1 \xrightarrow{\begin{pmatrix} j_1 \\ -d \end{pmatrix}} \begin{matrix} U_1 \\ \oplus \\ T_0 \end{matrix} \underset{(d\ \ j_0)}{\overset{\begin{pmatrix} h \\ \iota_0 \end{pmatrix}}{\rightleftarrows}} U_0$$

so I ask whether the Δh I envisage comes from $\mathrm{Hom}(U_0, T_1)$

$$\begin{tikzcd} \mathrm{Hom}(U_0,T_1) \arrow[r] & \mathrm{Hom}(U_0,U_1) \\ & \mathrm{Hom}(U_0 A U_1) \arrow[u] \end{tikzcd}$$

Here we have to be careful. Start with

$$\tilde{A} \xrightarrow{\cdot(1+a)} \tilde{A}$$

$d = 1 - a$
$h = 1$

$$P \underset{\cdot(1-a)}{\overset{1}{\rightleftarrows}} P \qquad 1 - dh = a \in QP$$

assume $a = gp$. Then get $1 = dh + gp$

Prove Milnor thm.

$$\begin{tikzcd} & R \arrow[d, "p"] \\ R' \arrow[r, "p_0"'] & R/I \end{tikzcd} \qquad \mathcal{P}(R' \times_{R/I} R) = \mathcal{P}(R') \times_{\mathcal{P}(R/I)} \mathcal{P}(R)$$

given (P', P, α)

$$P'^{\vee} \otimes_{R'} P' \longrightarrow p_*(P')^{\vee} \otimes_{R/I} p_*(P) \twoheadleftarrow P^{\vee} \otimes_R P$$

$$\sum \lambda_i' \otimes p_i' \longmapsto \qquad \sum$$

In this way you construct

$$P' \times_\alpha P \longrightarrow \left(R' \times_{R/I} R\right)^N \longrightarrow P' \times_\alpha P$$

composition is the identity on P', diff. from 1 is a map $P \to IP$ which you can separately factor.

139 Go back to ~~a~~ problem: Suppose you have two diffls

$U_1 \overset{d}{\underset{d'}{\rightrightarrows}} U_0$ which are congruent mod A.

Assume $U_i \in \mathcal{P}(\tilde{A})$ and $U_1/AU_1 \overset{d\ d'}{\cong} U_0/AU_0$ so

you have an $h: U_0 \to U_1$ such that $1-[d,h]: U \to AU$

~~[illegible]~~ Pick $T_0 \xrightarrow{j_0} U_0$ transversal to d

and form the fibre product

$$\begin{array}{ccc} T_1 & \xrightarrow{d} & T_0 \\ \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_0} \\ U_1 & \xrightarrow{d} & U_0 \end{array}$$

Can I arrange d'^U to be covered by a d'^T? My rough idea. What happens when A is unital?

Start again. You have a complex $U_1 \xrightarrow{d} U_0$ with a parametrix $h: U_0 \to U_1$, i.e. $1-dh$, $1-dh$ are A-nuclear. You know $U \sim T$ with 1_T nuclear. Aim to vary d by an A-nuclear map $U_1 \xrightarrow{\delta} U_0$.

$$1 - d'h = 1 - dh - \delta h : U_0 \longrightarrow U_0.$$

Let us change notation and think of varying h. Suppose we are given $U_1 \xrightarrow{d} U_0$ admitting a parametrix h. This means $1-dh: U_0 \to U_0$ is A-nuclear, ~~so that~~ so $\exists\ j_0: T_0 \to U_0$ covering $\mathrm{Coker}(d)$. Form

$$\begin{array}{ccc} T_1 & \xrightarrow{d} & T_0 \\ \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_0} \\ U_1 & \xrightarrow{d} & U_0 \end{array}$$

 This gives basic exact seq.

$$0 \to T_1 \xrightarrow{\binom{+j_1}{-d}} \begin{matrix} U_1 \\ \oplus \\ T_0 \end{matrix} \xrightarrow{(d\ j_0)} U_0 \to 0$$

we are concerned with splittings of this sequence. We know $\exists$ at least one ~~...~~ given by $\binom{h}{i_0} : U_0 \to \begin{matrix} U_1 \\ \oplus \\ T_0 \end{matrix}$ satisfying $(d\ j_0)\binom{h}{i_0} = dh + j_0 i_0 = 1$. Any other splitting differs by a map $U_0 \xrightarrow{f} T_1$. Then

$$\delta \binom{h}{i_0} = \binom{j_1}{-d} f$$

So I need only solve $\delta h = j_1 f$

~~...~~ Now let's d, h again.

This time you have $U_1 \underset{d}{\overset{h}{\rightleftarrows}} U_0$ with $1-dh$, $1-hd$ A-nuclear. This ~~says~~ we can factor $1-d\cancel{h}h$ into $U_0 \xrightarrow{i_0} AT_0 \subset T_0 \xrightarrow{j_0} U_0$ with $T_0 = \tilde{A}^{\oplus N}$. Then define T_1 by means of

$$0 \leftarrow T_1 \underset{\binom{+j_1}{-d}}{\overset{(\cancel{\ \ }\ i_1 - h)}{\rightleftarrows}} \begin{matrix} \cancel{U}U_1 \\ \oplus \\ T_0 \end{matrix} \underset{(d\ j_0)}{\overset{\binom{h}{i_0}}{\rightleftarrows}} U_0 \leftarrow 0$$

~~This is to keep~~ What I am trying to do is ~~to take the complex $U_1 \xrightarrow{d} U_0$~~ to study the process of passing from a cx $U_1 \xrightarrow{d} U_0$ whose id can be deformed to an A-nuclear map, to a homotopy equivalent f. projective complex.

Alternative. Choose h then either factor: $1-hd = f_0 c_0$ or factor $1-dh = f_1 c_1$

A

$$\begin{tikzcd}
U_1 \times_{U_0} T_0 \arrow[r] \arrow[d] & T_0 \arrow[d] \\
U_1 \arrow[r, "d"] \arrow[d] & U_0 \arrow[d] \\
T_1' \arrow[r] & T_1 \amalg_{U_1} U_0
\end{tikzcd}$$

~~$P/BP = P/PA$~~

$(P/BP)A = 0$

since $PA = PQP \subset BP$

?

~~Illegible~~ I am studying ~~the~~ a process going from U with ~~a~~ parametrix mod A to a htpy eq. f.p. cx.

Idea that seems to work. Start with ~~[illegible]~~ $\xi \in K_0(A)$, represent by ...... Basic idea: You have Morita equiv. for K_0' so you have to check ~~that~~ it descends to K_0. Start with $(U_1, U_0, \alpha: U_1/AU_1 \simeq U_0/AU_0)$ $U_i \in \mathcal{P}(\tilde{A})$, lift α to d, lift α^{-1} to h. Know $P \otimes_A (U,d)$ is homotopy equiv. to a f.pg complex $T_1 \to T_0$ ~~over~~ B

$[U_1 \xrightarrow{d} U_0]$ or $[U_0 \xrightarrow{h} U_1]$

$$K_0'(A) \;\xrightarrow{\sim}\; K^0(B)$$

$[P \otimes_A U_1 \xrightarrow{1 \otimes d} P \otimes_A U_0]$ or $[P \otimes_A U_0 \xrightarrow{1 \otimes h} P \otimes_A U_1]$

$\| \qquad\qquad\qquad\qquad \|$

$[T_1 \xrightarrow{d} T_0] \qquad [T_0 \xrightarrow{h} T_1]$

$$\begin{tikzcd}
K_0'(A) \arrow[d] & K^0(B) \arrow[d] \\
\left[(U_1, U_0, \alpha)\right] \in K_0(A) & K^0(B)
\end{tikzcd}$$

$[(U_1, U_0, \alpha)] \in K_0(A)$ $\quad$ $K^0(B)$ $\quad [(T_1, T_0, \bar{d})] = [(T_0, T_1, \bar{h})]$

$\|$

$[(U_0, U_1, \alpha^{-1})]$

~~[illegible]~~ Putting $U_i' = P \otimes_A U_i$, then you have as before

$$\begin{tikzcd}
T_1 \arrow[r, shift right, "\begin{pmatrix} f_1 \\ -d \end{pmatrix}"'] & U_1' \oplus T_0 \arrow[l, shift right, "(c_1\ -h)"'] \arrow[r, shift right, "(d'\ f_0)"'] & U_0' \arrow[l, shift right, "\begin{pmatrix} h \\ i_0 \end{pmatrix}"']
\end{tikzcd}$$

14a

$$\begin{array}{ccc} A & \tilde{A} & Q \\ A & A & AQ \\ P & P & B \end{array}$$

$$\parallel$$

$$D$$

$$K_0 A \simeq K_0 D$$

$$\begin{pmatrix} A & AQ \\ P & B \end{pmatrix} \longrightarrow \begin{pmatrix} A & Q \\ P & B \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & Q/AQ \\ 0 & 0 \end{pmatrix}$$

$$A(AQ) + \underset{QPQ}{\underset{\shortparallel}{QB}} \subset AQ$$

$$\begin{pmatrix} \tilde{A} & Q \\ P & \tilde{B} \end{pmatrix} \twoheadrightarrow \begin{pmatrix} \mathbb{Z} & Q/AQ \\ & \mathbb{Z} \end{pmatrix}$$

---

to prove $C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ $\quad PQ = B \Rightarrow \quad K_0(A) \simeq K_0(C)$

Your argument. $\quad D = \begin{pmatrix} A & AQ \\ P & B \end{pmatrix} \overset{B=PQ}{\subset} C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

$$C^2 = \begin{pmatrix} A^2+QP & AQ+QB \\ PA+BP & B^2+PQ \end{pmatrix} = \begin{pmatrix} A^2+QP & AQ \\ PA & B \end{pmatrix}$$

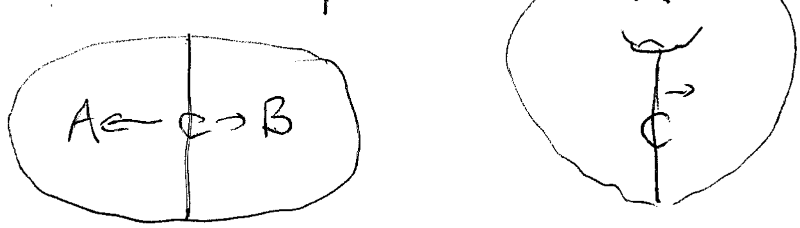
so $\quad C^2 \subset D \subset C$ $\qquad \begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix}$

$$K_0(C^2) \twoheadrightarrow K_0(D) \twoheadrightarrow K_0(C)$$

---

back to Waldhausen & Bass FT.

two Waldhausen pictures



$C \rightrightarrows A.$

~~Sketch~~ a part case of latter is [redacted] where $C = A$ and both arrows are the identity $X = Y \times S^1$

Waldhausen philosophy: The category of modules appears as a category of diagrams modulo nil objects.

What should the ~~deg~~ diagrams be? $\quad M_C \rightrightarrows M_A$

~~$M_C \rightrightarrows M_A$~~ Look at the group theory. $\quad \pi_1(C) \rightrightarrows \pi_1(A)$

~~problem.~~ so you have two subgps H_1, H_2 of ~~G~~ G and you want to find ~~an ex~~ a group containing G tog. with an element conjugating H_1 to H_2.

picture  $BH \rightrightarrows BG \longrightarrow BG \cup_{BH\times\dot{I}} BH\times I$

Covering of this space should consist of a G-set V and H-set E and maps $E \rightrightarrows V$ compatible with the two homos $H \overset{\alpha}{\underset{\beta}{\rightrightarrows}} G$. YES. So $(V, E \longrightarrow V\times V)$ is a graph.

$E \rightrightarrows V$. Try to describe modules - get a

G-module M_0 and an H-module M_1 together with maps $M_1 \rightrightarrows M_0$ compatible with $H \overset{\alpha}{\underset{\beta}{\rightrightarrows}} G$.

$\mathbb{Z}[H] \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \mathbb{Z}[G]$. You want a local system - need isos.

So a covering consists of a G-set V_0 and an H-set V_1 and bijections $V_1 \overset{\sim}{\underset{\sim}{\rightrightarrows}} V_0$ compatible with $H \overset{\alpha}{\underset{\beta}{\rightrightarrows}} G$.

~~You want~~ $A *_C B = R$, Attempt to understand $\mathrm{Mod}(R)$ in simpler terms. Diagram consists of

$$\begin{tikzcd} & M_A \\ M_C \arrow[ur] \arrow[dr] & \\ & M_B \end{tikzcd} \qquad \text{e.g.} \qquad \begin{tikzcd} & A \\ C \arrow[ur] \arrow[dr] & \\ & B \end{tikzcd}$$

$$\begin{tikzcd} & R\otimes_A M_A \arrow[dr] & \\ R\otimes_C M_C \arrow[ur] \arrow[dr] & & M \\ & R\otimes_B M_B \arrow[ur] & \end{tikzcd}$$

what will

So what do I need? Prolongation: I would like to find a filtration of R by diagrams.

143 bimodule resolution for $R = A *_C B$

$\Omega^1(R)$

$$\begin{tikzcd} R\otimes_C \Omega^1(C)\otimes_C R \arrow[r] \arrow[d] & R\otimes_B \Omega^1(B)\otimes_B R \arrow[d] \\ R\otimes_A \Omega^1(A)\otimes_A R \arrow[r] & \Omega^1(R) \end{tikzcd}$$

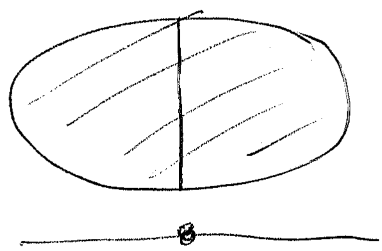
?

$$0 \to R\otimes_C \Omega^1(C)\otimes_C R \to \begin{matrix} R\otimes_B \Omega^1(B)\otimes_B R \\ \oplus \\ R\otimes_A \Omega^1(A)\otimes_A R \end{matrix} \to R\otimes R \to R \to 0$$

You have a functor from systems $\left\{ M_C \rightrightarrows \begin{matrix} M_A \\ M_B \end{matrix} \right\}$ to R-modules; extend to R then let M be the fibre product. Question: What systems yield zero?

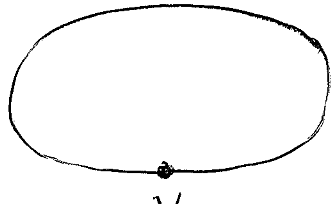
Dec 7. The problem is how to ~~attack~~ approach Waldhausen's theory. I would like a simple example to study. Perhaps $A[t,t^{-1}]$ or even $A[t]$. These two seem to have the same diagrams, but perhaps different notions of nil modules.

I think $A[t,t^{-1}]$ is a special case of what can be done for $C \rightrightarrows A$ two homos. The geometry in the group theory situation $h\mathrm{Coker}\{BH \rightrightarrows BG\}$. You should imagine a space, how? Codim 1 subman. X closed man Y codim 1 subm $\ni$ $X-Y$ conn. What is the geom. picture.



$X = U \cup V$

$$\begin{tikzcd} U\cap V \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cocart}"] & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$



Y

here you have a $\mathbb{Z}$-covering. what happens around Y? Open covering $(X-Y) \cup$ nbd of Y

$$\begin{tikzcd}
T_1 \arrow[r] \arrow[d] & T_0 \arrow[d] \\
U'_1 \arrow[r, dash] & U'_0
\end{tikzcd}$$

$$\begin{tikzcd}
 & & U'_1 \arrow[d] \arrow[dr, "d"] & & \\
0 \arrow[r] & T_1 \arrow[r] \arrow[dr, "-d"'] & U'_1 \oplus T_0 \arrow[r] \arrow[d] & U'_0 \arrow[r] & 0 \\
 & & T_0 & &
\end{tikzcd}$$

$$\begin{pmatrix} d & j_0 \\ i_1 & -h \end{pmatrix}\begin{pmatrix} h & j_1 \\ i_0 & -d \end{pmatrix} = \begin{pmatrix} \overset{1}{dh+i_0j_0} & \overset{0}{dj_1 - j_0 d} \\ \overset{0}{i_1h - hi_0} & \overset{1}{i_1j_1 + hd} \end{pmatrix}$$

$$\begin{pmatrix} h & j_1 \\ i_0 & -d \end{pmatrix}\begin{pmatrix} d & j_0 \\ i_1 & -h \end{pmatrix} =$$

Assume $d = h = 1$ on U.

$$\begin{pmatrix} 1 & j_0 \\ i_1 & -h \end{pmatrix}\begin{pmatrix} 1 & j_1 \\ i_0 & -d \end{pmatrix}$$

It seems that if two matrices

$$\begin{pmatrix} 1 & y \\ x & \end{pmatrix}$$

145 There's a lot to check here. First suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ are ~~then~~ inverses.

show that d and d' are invers.

$$cb' + dd' = 1 \,, \quad c'b + d'd = 1$$

$$aa' + bc' = 1 \qquad a'a + b'c = 1$$

~~so~~ Assume $a' = a^{-1}$, then $\boxed{bc' = 0 \quad | \quad b'c = 0}$

$$(c'b)^2 = 0 \quad \text{and} \quad (cb')^2 = 0.$$

$$\underline{ab' + bd' = 0}$$

Assume $a = a' = 1$.

$$\begin{pmatrix} 1 & b \\ c & d \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & d-cb \end{pmatrix} \begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d-ca^{-1}b \end{pmatrix} \begin{pmatrix} 1 & -a^{-1}b \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 \\ 0 & d-ca^{-1}b \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d-ca^{-1}b \end{pmatrix} \begin{pmatrix} 1 & +a^{-1}b \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a^{-1}b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & (d-ca^{-1}b)^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -ca^{-1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a^{-1} & -a^{-1}b(d-ca^{-1}b)^{-1} \\ 0 & (d-ca^{-1}b)^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -ca^{-1} & 1 \end{pmatrix}$$

$$= a^{-1} + a^{-1}b(d-ca^{-1}b)^{-1}ca^{-1}$$

Assume $a = 1$. $\boxed{\cancel{b(d-cb)^{-1}c = 0}}\ \boxed{\cancel{(d-cb)^{-1} = d^{-1}}}$

so $cb = 0$ and $bd^{-1}c = 0$

$$\begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c & d-cb \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & d-cb \end{pmatrix}$$

$$\begin{pmatrix} 1 & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (d-cb) \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -b \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (d-cb)^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -b \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ (d-cb)^{-1}c & (d-cb)^{-1} \end{pmatrix} = \begin{pmatrix} 1-b(d-cb)^{-1}c & -b(d-cb)^{-1} \\ (d-cb)^{-1}c & (d-cb)^{-1} \end{pmatrix}$$

assume $b(d-cb)^{-1}c = 0$ does this imply $cb = 0$?

d

$$\begin{tikzcd} T_1 \arrow[r, shift right, "\begin{pmatrix} j_1 \\ -d \end{pmatrix}"'] & U_1 \oplus T_0 \arrow[l, shift right, "(x_1, -h)"'] \arrow[r, shift right, "(d\ j_0)"'] & U_0 \arrow[l, shift right, "\begin{pmatrix} h \\ i_0 \end{pmatrix}"'] \end{tikzcd}$$

same as 2 projections on the same space

there's also an interesting spectral theory away from 2.

Ask about modules for $\mathbb{Z}e_0 * \mathbb{Z}e_1$

Look again

$\underline{g = \varepsilon F}$ $\qquad \varepsilon g \varepsilon^{-1} = \varepsilon\varepsilon F \varepsilon = F\varepsilon = g^{-1}$.

<u>Problem.</u> $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ assume a invertible and $a' = a^{-1}$ ~~and~~ show d invertible with inverse $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$

Start with Assume $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ invertible, show

a invertible $\iff$ d' invertible.

can assume $a = 1$.

$$\begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c & d-cb \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & d-cb \end{pmatrix}$$

$$\begin{pmatrix} 1 & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (d-cb)^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -(d-cb)^{-1}c & (d-cb)^{-1} \end{pmatrix} = \begin{pmatrix} 1+b(d-cb)^{-1}c & -b(d-cb)^{-1} \\ -(d-cb)^{-1}c & (d-cb)^{-1} \end{pmatrix}$$

$(d-cb)^{-1} =$

Look at from the viewpoint of e_1, e_0

$$
\begin{tikzcd}
 & U_1 \arrow[dl, shift left, "i_1"] \arrow[dr, shift left, "d"] & \\
T_1 \arrow[ur, shift left, "j_1"] \arrow[dr, shift left, "d"] & \bigcirc & U_0 \arrow[ul, shift left, "h"] \arrow[dl, shift left, "i_0"] \\
 & T_0 \arrow[ul, shift left, "h"] \arrow[ur, shift left, "j_0"] &
\end{tikzcd}
$$

Suppose e_1, e_0 given on M, let

~~$U_i = e_i M$ $T_i = e_i^\perp M$~~ $U_i = e_i M$

$$
\begin{tikzcd}
 & e_1 M = U_1 & \\
T_1 = e_0^\perp M \arrow[r, "\begin{pmatrix} e_1 e_0^\perp \\ e_1^\perp e_0^\perp \end{pmatrix}"'] & \oplus \arrow[r, shift right, "(e_0e_1 \;\; e_0e_1^\perp)"'] & e_0 M = U_0 \arrow[l, shift right] \\
 & e_1^\perp M = T_0 &
\end{tikzcd}
$$

$$e_0 e_1^\perp e_1^\perp e_0^\perp = e_0 e_1^\perp e_0^\perp = -e_0 e_1 e_0^\perp$$

$$
\begin{tikzcd}
e_0^\perp M \arrow[r, shift right, "-e_1^\perp e_0^\perp"'] \arrow[d, shift right, "e_1 e_0^\perp"'] & e_1^\perp M \arrow[l, shift right, "-e_0^\perp e_1^\perp"'] \arrow[d, shift right, "e_0 e_1^\perp"'] \\
e_1 M \arrow[u, shift right] \arrow[r, shift right, "e_0 e_1"'] & e_0 M \arrow[l, shift right, "e_1 e_0"'] \arrow[u, shift right, "e_1^\perp e_0"']
\end{tikzcd}
$$

Check: $e_0 - dh = e_0 - e_0 e_1 e_0 = e_0 e_1^\perp e_0$

$j_0 i_0 = e_0 e_1^\perp e_0$

Check again:

$$
\begin{tikzcd}
T_1 \arrow[r, shift right, "d"'] \arrow[d, shift left, "j_1"] & T_0 \arrow[l, shift right, "h"'] \arrow[d, shift left, "j_0"] \\
U_1 \arrow[u, shift left, "i_1"] \arrow[r, shift right, "d"'] & U_0 \arrow[l, shift right, "h"'] \arrow[u, shift left, "i_0"]
\end{tikzcd}
\qquad
\begin{tikzcd}
T_0 \arrow[r, "h"] \arrow[d, "j_0"] & T_1 \arrow[r, "d"] \arrow[d, "j_1"] & T_0 \arrow[d, "j_0"] \\
U_0 \arrow[r, "h"] & U_1 \arrow[r, "d"] & U_0
\end{tikzcd}
$$

$$\begin{tikzcd} T_1 \arrow[r, "d"] \arrow[d, "j_1"'] & T_0 \arrow[d, shift left, "j_0"] \\ U_1 \arrow[r, shift right, "d"'] & U_0 \arrow[l, shift right, "h"'] \arrow[u, shift left, "i_0"] \end{tikzcd}
\qquad
\begin{tikzcd} T_1 \arrow[r, shift right, "\begin{pmatrix} j_1 \\ -d \end{pmatrix}"'] & U_1 \oplus T_0 \arrow[l, shift right, "(i_1\ -h)"'] \arrow[r, shift right, "(d\ j_0)"'] & U_0 \arrow[l, shift right, "\begin{pmatrix} h \\ i_0 \end{pmatrix}"'] \end{tikzcd}$$

$$\begin{pmatrix} j_1 \\ -d \end{pmatrix}(i_1\ \ -h) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} h \\ i_0 \end{pmatrix}(d\ \ j_0)$$

$$\begin{pmatrix} 1-j_1 i_1 & +j_1 h \\ +d i_1 & 1-dh \end{pmatrix} = \begin{pmatrix} hd & hj_0 \\ i_0 d & i_0 j_0 \end{pmatrix}$$

2×2 invertible matrix = two splitting of same thing.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \& \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Think of the matrix as an isom
$$\begin{tikzcd} P \oplus Q \arrow[r, shift right, "\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}"'] & X \oplus Y \arrow[l, shift right, "\begin{pmatrix} a & b \\ c & d \end{pmatrix}"'] \end{tikzcd}$$

$$\begin{tikzcd} Q \arrow[r, shift right, "-\beta"'] \arrow[d, shift left, "\delta"] & \cancel{X} \arrow[l, shift right, "-c"'] \arrow[d, shift left, "a"] \\ \cancel{Y} \arrow[u, shift left, "d"] \arrow[r, shift right, "b"'] & P \arrow[l, shift right, "\gamma"'] \arrow[u, shift left, "\alpha"] \end{tikzcd}$$

Question: Given e, e' on M. get

$$\begin{array}{c} e'M \\ \oplus \\ e'^{\perp}M \end{array} \leftarrow \begin{pmatrix} e'e & e'e^{\perp} \\ e'^{\perp}e & e'^{\perp}e^{\perp} \end{pmatrix} \longleftarrow \begin{array}{c} eM \\ \oplus \\ e^{\perp}M \end{array}$$

$$\begin{tikzcd} X \arrow[r, "-c"] \arrow[d, "a"] & Q \arrow[r, "-\beta"] \arrow[d, "\delta"] & X \arrow[d, "a"] \\ P \arrow[r, "+\gamma"] \arrow[rr, bend right, "1"'] & Y \arrow[r, "+b"] & P \end{tikzcd}$$

computationally $b\gamma = 1 \Leftrightarrow \boxed{a\alpha = 0}$

Then $1 = \beta c + \alpha a$

$\beta c = 1 - \alpha a$ invertible

inverse $1 + \alpha a$

$(1+\alpha a)\beta c =$

$\alpha a \beta c = \alpha(-b\delta)c = +\alpha b \gamma a = +\alpha a$

$\alpha a(1 - \alpha a)$

$c\alpha a = -d\gamma a$

to show $\alpha a = 0$

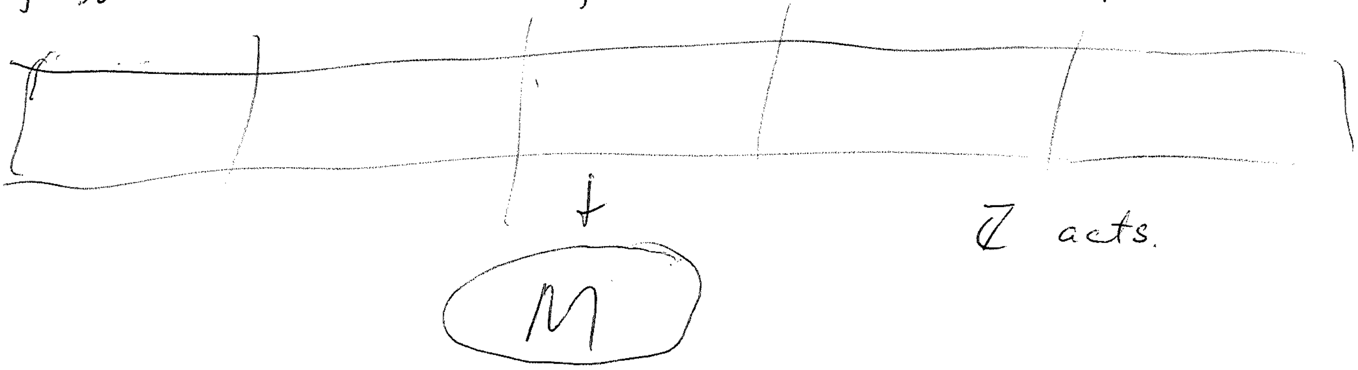
$\alpha a \beta c = \alpha a$

$a \alpha a = 0$ OK.

Return to geometry. Consider the problem of fibring over a circle. Consider a smooth ~~compact~~ closed manifold M with a homotopy class of maps $\cancel{X} \xrightarrow{f} S^1$. Same as element of $H^1(\cancel{X}, \mathbb{Z})$. By Thom can ~~smooth~~ f make transversal to zero, get submanifold $Y \subset X$ codim 1, conversely. Let N be pull back

$$\begin{tikzcd} N \arrow[r] \arrow[d] & \mathbb{R} \arrow[d] \\ M \arrow[r, "f"] & S^1 \end{tikzcd}$$

N open manifold with $\mathbb{Z}$ acting. You want to deform f to a submersion — df onto. What happens is

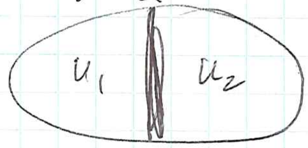


N is an open manifold with 2 ends ~~I think the problem then~~ The idea I think is the following. You want find a codim 1 submanifold of N which is homotopy equivalent to N. The inverse image of a generic point of $\mathbb{R}$ gives a submanifold and you want to do ~~surg~~ surgery. Another version is to put a boundary on one of the ends.

Roughly what happens: You first need to assume that N is homotopy equivalent to a finite complex. Presumably you can get a nasty group. N is an inf covering. Too much unfamiliar stuff.

What to do? How about trying to find a start on Waldhausens business, or the Bass F.T.

151 Picture to aim for: suppose we consider $K_*(A[t,t^{-1}])$. ~~[illegible]~~ Here one has B&FT. But ~~also~~ it would seem this is an example of Waldhausen's theory. a special case of $C \rightrightarrows A$. In the case $R = A *_C B$ he ~~[illegible]~~ uses systems M_A M_C M_B ~~and~~ with appropriate maps. What is the motivation? group rings, so that R is the group ring of $G_1 *_H G_2 = G$. Maybe a manifold M with fundamental group G. Look at universal covering $\tilde{M}$. Think of the man. as



~~One get a covering of. These open sets~~ ~~[illegible]~~ Take inverse image of these open sets, split into connected components. The resulting nerve should be the tree Serre constructed associated to the free product. ~~What [illegible] saying is some sort of [illegible]~~ I guess we look at chains on $\tilde{M}$, a finite ~~[illegible]~~ free complex of $\mathbb{Z}[G]$-modules. Maybe another idea is how you prove Mayer-Vietoris ~~for simplicial complexes~~ using singular chains. Step is to replace all chains by those subordinate to the covering.

Dec 8 - Graeme's split de Rham complex result. - homotopy equivalence rather than quis. Use Weil proof. $C^\circ(\mathcal{U}, \Omega^\circ)$. You need $\Omega^\circ(U_\sigma) \sim \mathbb{C}$ for each U_σ and also $C^\circ(\mathcal{U}, \Omega^p) \approx \Omega^p(X)$ for each p. This looks suspiciously like a partition of 1, ~~but~~

$$\Omega^\circ(X) \to \prod \Omega^\circ(U_i) \to \prod \Omega^\circ(U_i \cap U_j) \to \cdots$$

$$0 \to \Omega^\circ(X) \to \Omega^\circ(U) \times \Omega^\circ(V) \to \Omega^\circ(U \cap V) \to 0$$

2 surprising number of ~~the~~ candidates did not realize that the consistency conditions for solving ~~can~~ describe ~~[illegible]~~ image of lin. transf.

 Reviewed in passing

1) G proof of F coh over X comp. anal space $\Rightarrow H^*(X,F)$ fin dim

2) G(V?) ~~proof~~ arg that $H^k(X,F) = \text{Im}\{\check{H}^i(\mathcal{U},F) \to \check{H}^k(\mathcal{V},F)\}$ via $H^*(X, \prod_I F) = \prod_I H^*(X,F)$ for all sets I provided $\forall U \exists V \subset U \ni H^i(U,F) \to H^i(V,F)$ is zero all $i>0$

I want to ~~[illegible]~~ recall Verdier definition of the Wall obstruction. It might be possible to clarify the arguments. Yes.

3) Wall obstruction X CW cx. say conn. When is X htpy equiv. fin. complex? Ans. $\pi_1(X)$ must be fin. pres. and $C_\cdot(\tilde{X})$ must be ~~a perfect complex of~~ $\mathbb{Z}[\pi_1 X]$-modules, ~~i.e. $C_\cdot(\tilde{X})$ must be~~ $\sim$ finite cx of fin free. Wall assumes X dominated by a finite cx to get $\pi_1(X)$ f. pres and $C_\cdot(\tilde{X})$ perfect, then Wall obst in $K_0[\mathbb{Z}(\pi_1(X))]$ is defined. Constructs finite cx by attaching cells.

You want a Verdier version proving $\pi_1(X)$ fin pres & $C_\cdot(\tilde{X})$ perfect assuming X compact with nice local properties. You maybe want ~~[illegible]~~ to bring in $X \times X$ so as to work with kernels. e.g. assume X a compact top. manifold. How do you ~~[illegible]~~ verify $\pi_1(X)$ is discrete, finitely presented, etc?

Next return to Waldhausen - suppose $X = U_0 \cup U_1$, ~~[illegible]~~ U_0, U_1, U_{01} conn. ~~[illegible]~~ Maybe think X a CW cx U_0, U_1 subcomplexes, but infinite. What happens then? You assume $\underbrace{\pi_1(U_{01})}_{f.p.} \hookrightarrow \underbrace{\pi_1(U_i)}_{f.g.}$. Then

$$C_\cdot(\tilde{X}) = C(p^{-1}U_0) \underset{C(p^{-1}U_{01})}{\oplus} C(p^{-1}U_1)$$

using cellular chains. Now the tree business describes $\pi_0(p^{-1}U_i)$. What happens is that ~~[illegible]~~ the conn. components ~~[illegible]~~ of $p^{-1}U_0, p^{-1}U_1$ are subcomplexes of $\tilde{X}$ whose union is $\tilde{X}$ ~~[illegible]~~ and the nerve of this covering is the tree. Distinguish the components containing the basepoint of $\tilde{X}$, call then Z_i

 then

$$C_\cdot(\tilde{X}) = \mathbb{Z}[\pi_1 X] \otimes_{\pi_1 \mathcal{U}_0} C_\cdot(Z_0) \oplus \mathbb{Z}[\pi_1 X] \otimes_{\pi_1 \mathcal{U}_1} C(Z_1)$$

$$? \; ? \qquad \mathbb{Z}[\pi_1 X] \otimes_{\pi_1 \mathcal{U}_{01}} C(Z_{01})$$

Question: When is $\mathbb{Z}$ a perfect $\mathbb{Z}[G]$-module? $\Leftrightarrow$ G is of type FP. Does $\exists$ ~~exist~~ example of a G of type FP not of type FL? This would yield a $K(G,1)$ dominated by a finite cx but not essentially a fin. cx.

When is an ind object isom. to an essentially constant ind. obj?

Answer. When $\exists$ (X, e) $e^2 = e$, a map $f : \{L_i\} \to \mathrm{Im}(e)$ i.e. $f_i : L_i \to X$ $\ni$ $ef_i = f_i$

and a map $\mathrm{Im}(e) \xrightarrow{g} \{L_i\}$ i.e. $g : X \to L_{i_0}$ some i_0 such that $ge = g$.

want f, g to be inverse:

$\mathrm{Im}(e) \xrightarrow{g} \{L_i\} \xrightarrow{f} \mathrm{Im}(e)$ is 1 means

$X \xrightarrow{g} L_{i_0} \xrightarrow{f_{i_0}} X$ is e

$\{L_i\} \xrightarrow{f} \mathrm{Im}(e) \xrightarrow{g} \{L_i\}$ means, restricting to the cofinal cat of i under i_0, that for all i

$L_i \xrightarrow{f_i} X \xrightarrow{g} L_{i_0} \longrightarrow L_i$ and 1_{L_i}

~~can be~~ are coequalized by $L_i \to L_{i_1}$ for some $i \to i_1$

15 Back to W. From the geometry $X_A \underset{X_C}{\cup} X_B = X$ we get

$$R \otimes_A C_\cdot(\tilde{X}_A) \underset{R \otimes_C C_\cdot(\tilde{X}_C)}{\oplus} R \otimes_B C_\cdot(\tilde{X}_B) = C_\cdot(\tilde{X})$$

although if we replace X by the double mapping cylinder $X_A \underset{X_C \times 0}{\cup} (X_C \times I) \underset{X_C \times 1}{\cup} X_B$, then

$$\mathrm{Cone}\left\{ R \otimes_C C_\cdot(\tilde{X}_C) \longrightarrow \begin{matrix} R \otimes_A C_\cdot(\tilde{X}_A) \\ \oplus \\ R \otimes_B C_\cdot(\tilde{X}_B) \end{matrix} \right\} = C_\cdot(\tilde{X}).$$

~~[illegible]~~ You want to consider ~~systems~~ quivers
$$\begin{matrix} M_C & \to & M_B \\ \downarrow & & \\ M_A & & \end{matrix}$$
of modules (complexes?) where the arrows are compatible with the homos. $\begin{matrix} C & \to & B \\ \downarrow & & \\ A & & \end{matrix}$.
Such ~~systems~~ (quivers) are the same as modules over

$$\Gamma = \begin{pmatrix} C & 0 & 0 \\ A & A & 0 \\ B & 0 & B \end{pmatrix}$$

(recall C, A, B are unital so that a module M over this ring splits into $M_C \oplus M_A \oplus M_B$ etc.)

$K_*(\Gamma) = K_*(C) \oplus K_*(A) \oplus K_*(B)$. To a Γ module (M_A etc) we associate the length one complex

$$R \otimes_C M_C \longrightarrow \begin{matrix} R \otimes_A M_A \\ \oplus \\ R \otimes_B M_B \end{matrix}$$

Look at the H_0 - get functor $\mathrm{Mod}(\Gamma) \to \mathrm{Mod}(R)$. This must be $P \otimes_\Gamma -$ for P some R, Γ - bimodule, something like $(R \quad R \quad R)$ with some obvious right Γ action like

$$(r \quad r' \quad r'') \begin{pmatrix} c & & \\ a_1 & a & \\ b_1 & & b \end{pmatrix} = (rc + r'a_1 + r''b_1 \quad r'a \quad r''b)$$

~~[illegible]~~

This P is the bimodule giving the complex, so is $P_1 \oplus P_0$ with $d: P_1 \longrightarrow P_0$

155 So what? ~~[illegible]~~ Nil quivers - these become acyclic under this functor? examples are $M_A = A \otimes_C M_C$, $M_B = 0$ and the other way. <u>Question</u> Do nil modules split into these types?

Look at Kronecker quiver: $V_1 \overset{a}{\underset{b}{\rightrightarrows}} V_0$

suppose $V_1[t,t^{-1}] \xrightarrow[\sim]{a+bt} V_0[t,t^{-1}]$

So ~~suppose~~ you have $R\otimes_A (A\otimes_C M_C)$

$$R\otimes_C M_C \xrightarrow{\sim} \begin{array}{c} R\otimes_A M_A \\ \oplus \\ R\otimes_B M_B \end{array}$$

You would like to show that $M_C = (M_C \cap M_A) \oplus (M_C \cap M_B)$ and that $A\otimes_C (M_C \cap M_A) \xrightarrow{\sim} M_A$, $B\otimes_C (M_C\cap M_B) \xrightarrow{\sim} M_B$

$R = A *_C B$. Look at Kronecker case first. The proof somehow uses ~~$V_1[t,t^{-1}] \xrightarrow[\sim]{a+bt} V_0[t,t^{-1}]$~~.

~~use~~ replace V_1, V_0 by W, V. Actually $a+b$ is an isom. So $W = V$.

$$V[t] \oplus V[t^{-1}]t^{-1} = V[t,t^{-1}]$$

$$\cap$$

$$(at+b)^{-1}V[t]$$

$$V[t] + (at+b)t^{-1}V[t^{-1}] = V[t,t^{-1}]$$

Examples. ~~[illegible]~~ of filtrations

$$\begin{pmatrix} 0 & B \\ 0 & \end{pmatrix} \rightsquigarrow R$$

$$\cap$$

$$\begin{pmatrix} C & B \\ A & \end{pmatrix} \rightsquigarrow R$$

$$\cup$$

$$\begin{pmatrix} C & 0 \\ A & B \end{pmatrix} \text{ nil}$$

$$\begin{pmatrix} C & B \\ A & \end{pmatrix} \subset \begin{pmatrix} B & B \\ AB & \end{pmatrix} \longrightarrow \begin{pmatrix} B/C & 0 \\ AB/A & \end{pmatrix}$$

$$AB/A = A\otimes_C B / A\otimes_C C = A\otimes_C (B/C)$$

$$\cap \qquad\qquad \cap$$

$$\begin{pmatrix} A & BA \\ A & \end{pmatrix} \subset \begin{pmatrix} A+B & BA \\ AB & \end{pmatrix} \longrightarrow \begin{pmatrix} B/C & 0 \\ A\otimes_C B/C & \end{pmatrix}$$

$$\cap \qquad\qquad \cap$$

$$\begin{pmatrix} AB & BAB \\ AB & \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{AB}{A+B} & \frac{BAB}{\cancel{B}AB} \\ 0 & \end{pmatrix}$$

$$A/C\otimes_C B/C = \frac{AB}{A+B} \qquad B\otimes_C \bar A \otimes_C \bar B = \frac{BAB}{\cancel{B}AB} \qquad \frac{B\otimes_C A\otimes_C B}{A\otimes_C B}$$

$\bar A$ A $\bar B$ AB $\bar B \bar A$

C $\qquad A+B \qquad AB+BA$

$\bar B$ B $\bar A$ BA $\bar A\bar B$ BAB

$$R = C \quad \begin{matrix} \bar A & \bar A\otimes \bar B \\ \bar B & \bar B\otimes \bar A \end{matrix}$$

$$R\otimes_C M_C = M_C \oplus \begin{matrix} \bar A\otimes_C M_C \\ \bar B\otimes_C M_C \end{matrix} \oplus \begin{matrix} \bar A\otimes_C \bar B\otimes_C M_C \\ \bar B\otimes_C \bar A\otimes_C M_C \end{matrix} \oplus \cdots$$

$$\simeq M_A \oplus \bar B\otimes M$$

Dec 9, 1997 Philosophy is that Γ-modules have K-theory $K_*(C)\oplus K_*(A)\oplus K_*(B)$, while nil Γ-modules yield $K_*(C)\oplus K_*(C)$, so from $K_*(\mathrm{nil}\,\Gamma)\to K_*(\Gamma)\longrightarrow K_*(R)\longrightarrow$ you get MV: $K_*(C)\to K_*(A)\oplus K_*(B)\to K_*(R)\to\cdots$

Let's now work out the analogue for $C\rightrightarrows A$.

$$X = \mathrm{hcolim}\,\{X_C \rightrightarrows X_A\} = (X_C\times I)\underset{X_C\times \dot I}{\cup} X_A$$

You can't use sub CW cxs. – the closure of $X_C\times I^{\mathrm{int}}$ has the wrong htpy type. Instead subdivide I: Two subcomplexes

157 $X =$  $\cup$  and $\cap$ is $X_C \times \dot{I}$

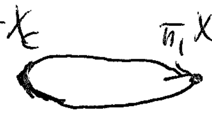
$\sim X_A$ $\qquad$ $\sim X_C$

This shouldn't be necessary - you should be able to work with the maps

$$\begin{tikzcd} X_C \times \dot{I} \arrow[r] \arrow[d, phantom, "\cap"] & X_A \arrow[d, phantom, "\cap"] \\ X_C \times I \arrow[r] & X \end{tikzcd}$$

~~can also collapse~~

Look at $\tilde{X}$. We assume $X_C \rightrightarrows X_A$ inj on π_1. ~~[illegible]~~

What am I after? Graph $\pi_1 X_C$  $\pi_1 X_A$

$G = \pi_1 X$ acts on an oriented tree transitively on vertices and on edges.

$E \rightrightarrows V$ $\qquad$ G

Pick fdl. domain ~~[illegible]~~ $\xi \in E$ $\quad v_0 \xrightarrow{\xi} v_1$

$G_\xi \subset G_{v_0}, G_{v_1}$ $\quad$ what's more $\exists\, g(v_0) = v_1$

~~Look at [illegible]~~) Check arbitrariness of t

Think of X_A as the complement of the codim 1 subman. X_C (tubular nbd of ?), so there are two maps $X_C \rightrightarrows X_A$. ~~Then~~ basepoint of X_C gives two points in X_A, join to basept of A, get a loop in X, which ~~can be fixed~~ is unique up to action of $\pi_1(X_A)$.

$$R = A *_C B = \begin{tikzcd}[column sep=small, row sep=small] & A \arrow[dr, dash] & & AB \arrow[dr, dash] & \\ C \arrow[ur, dash] \arrow[dr, dash] & & A+B \arrow[ur, dash] \arrow[dr, dash] & & AB+BA \\ & B \arrow[ur, dash] & & BA \arrow[ur, dash] & \end{tikzcd}$$

What's the link to the tree?

$G = G_A *_{G_C} G_B$ $\quad$ the tree is the graph $G_A \backslash G \leftarrow G_C \backslash G \rightarrow G_B \backslash G$

$G_A \backslash G$ is the G_A orbits on G $\qquad {}_A R \qquad {}_C R \qquad {}_B R$

if we choose section S, then $G = G_A \times S$

then $\mathbb{Z}[G] = \mathbb{Z}[G_A] \otimes \mathbb{Z}[S]$, so perhaps the tree appears as basis elements.

Let's try again. Look at chains on the universal covering $\tilde{X}$ – this means we need a cell structure on X. In the free product situation there are two models for X $\quad X_A \underset{X_C}{\cup} X_B$ and $X_A \underset{X_C}{\cup} X_C \times I \underset{X_C}{\cup} X_B$

What do you have in the $C \rightrightarrows A$ case? Two models probably

$$\begin{tikzcd} X_C \amalg X_C \arrow[r] \arrow[d] & X_A \arrow[d, dashed] \\ X_C \arrow[r, dashed] & X \end{tikzcd}$$

OK is $X_C \amalg X_C$ is a subcomplex of X_A

but in general

$$\begin{tikzcd} X_C \amalg X_C \arrow[r] \arrow[d, hook] & X_A \arrow[d, dashed] \\ X_C \times I \arrow[r, dashed] & X \end{tikzcd}$$

works.

Use 2nd model. Cells in X are either cells in A ~~of~~ (cells in X_C) $\times I$. Different picture – before X_C appears as codim 1 in X with complement X_A now X_A appears as codim 1 in X with comp. X_C). misleading as the latter embedding has no nbd. product structure, cells are another matter.

$$0 \longrightarrow C_\bullet(\tilde{X}_A) \otimes_{G_A} G \longrightarrow C_\bullet(\tilde{X}) \longrightarrow C_\bullet(\tilde{X}_C) \otimes_{G_C} G^{[1]} \longrightarrow 0$$

So the basic relation is

$$\mathrm{Cone}\left\{ R \otimes_C M_C \longrightarrow R \otimes_A M_A \right\} \simeq M$$

here you have two maps $M_C \rightrightarrows M_A$ compat with $C \rightrightarrows A$ so two maps $\quad A \otimes_C M_C \to M_A$, $\quad A \otimes_C M_C \to M_A$ and after $R \otimes_A$ you can identify the R modules $R \otimes_C M_C$ and $R \otimes_C M_C$

159 Examine Pimsner to see if there's a link.

Pimsner uses tensor alg of a bimodule

$$A \oplus M \oplus M \otimes_A M \oplus \cdots$$

for his ring. A objects, M morphisms of a quiver

$$\begin{tikzcd} C \arrow[r] \arrow[d] & B \\ A & \end{tikzcd}$$

top. Markov chains

Markov chain has matrix p_{xy}

If $\sum \mu_y = 1$, then $\sum_x \sum_y p_{xy}\mu_y$ should also be one. $\therefore \sum_x p_{xy} = 1$ for all y. Stochastic matrix. In a top. Markov chain the entries are 0 or 1. Set of objects ~~and arrows~~ subset of the product.

$\sum e_i = 1$

Cuntz-Krieger alg. $A = \bigoplus_{i=1}^{n} \mathbb{C}e_i$, an A-bimodule

$M = \bigoplus_{i,j} e_i M e_j$ YES. Still tensor alg of a bimodule.

Other example is

Notice It's not the tensor algebra $T_A(M)$ that is interesting, but rather the Cuntz alg $\mathcal{O}_M$

Go back to $C \overset{\rho}{\underset{\sigma}{\rightrightarrows}} A$ Form the A bimodule

$A_\rho \otimes_C {}_\sigma A$ the A-bimod with one gen. t subject to

$\rho(c)t = t\rho(c)$

~~OKAY~~ Then $T_A(AtA) = A \oplus AtA \oplus AtAtA \oplus \cdots$

Toeplitz alg? $T_A(E) = A \oplus E \oplus E \otimes_A E \oplus \cdots$

interior product operators are $\mathrm{Hom}_{A^{op}}(E, A)$

$\mathrm{Hom}_{A^{op}}(A_\rho \otimes_C {}_\sigma A, A) = \mathrm{Hom}_{C^{op}}(A_\rho, A_\sigma)$ is a left A-module

$$\phi : A {}_{\rho}\otimes_{C\,\sigma} A \longrightarrow A$$

$$A t A$$

$$\phi(a_1 t a_2) = \lambda(a_1) a_2$$

$$\lambda(a_1) = \phi(a_1 t)$$

$$\lambda(a_1 \rho(c)) = \phi(a_1 \rho(c) t) = \phi(a_1 t \sigma(c)) = \lambda(a_1)\sigma(c)$$

sats $\phi(a_1 \rho(c) t a_2) = \phi(a_1 t \sigma(c) a_2)$

$\therefore$ $\lambda(a_1 \rho(c)) a_2 \qquad \lambda(a_1)\sigma(c) a_2$

i.e $\boxed{\lambda(a_1 \rho(c)) = \lambda(a_1)\sigma(c)}$ $\therefore \lambda \in \mathrm{Hom}_{C^{op}}(A_\rho, A_\sigma)$

So now what? You have $\xi \in E = A {}_\rho\otimes_{C\,\sigma} A$

$\lambda \in \mathrm{Hom}_{C^{op}}(A_\rho, A_\sigma)$

and there's a pairing $\langle \lambda, \xi \rangle$ = Image of ξ under

$$A_\rho \otimes_{C\,\sigma} A \xrightarrow{\lambda\otimes 1} A_\sigma \otimes_{C\,\rho} A \xrightarrow{\text{mult}} A.$$

Have dual pair over A: $\big(E, \mathrm{Hom}_{A^{op}}(E, A)\big)$

whence

$$E \otimes_A \mathrm{Hom}_{A^{op}}(E, A) \longrightarrow \text{Mult} \xrightarrow{\sim} \mathrm{Hom}_{A^{op}}(E, E)$$

In the good case $E \in \mathcal{P}(A^{op})$, get $\sum \xi_i \otimes \lambda_i \longmapsto 1$

This implies $\sum \varphi(\xi_i)\,\varphi^*(\lambda_i) = 1$ on $T_A^{>0}(E)$

when $E = A_\rho \otimes_{C\,\sigma} A$ ~~this~~ means I guess that $A_\rho \in \mathcal{P}(C^{op})$. Assume this holds — is there some link between $\mathcal{O}_E$ and inverting t?

Take $\quad k \quad V \quad S^2(V) \qquad x, y$

t

$\otimes \quad \otimes$

$L^{-1} \quad B \quad L \quad L$

161 See what can be done about $E = A_\rho \otimes_C A$
assuming $A_\rho \in \mathcal{P}(C^{op})$ so that $E \in \mathcal{P}(A^{op})$
But if $A_\rho \in \mathcal{P}(C^{op})$, then ${}_\sigma A_\rho$ is a C-bimodule

Actually why not first look at

$T_C(A) = C \oplus A \oplus A \otimes_C A \oplus \cdots$ confusing.

~~[illegible]~~ What might be important is the ring of correspondences. Feature.

Go back to $A \oplus A t A \oplus \cdots$
and let's use our insight from the ~~case~~ group ring situation. This gives a model for inverting t.

~~[illegible]~~ differential operators: $\mathbb{Z}$-graded ring generated by $t, \frac{d}{dt}$

Then $-t \cdot \frac{d}{dt} + \frac{d}{dt} \cdot t = 1$. ~~[illegible]~~

$R = \bigoplus_{n \in \mathbb{Z}} R_n$. Have pairing $R_{-1} \otimes R_1 \longrightarrow R_0$

First look at R_1 as a R_0^{op} module. ~~typical~~

basis ~~[illegible]~~ R_0 has basis $(t\partial)^n$

$$(t\partial)^2 = t\partial t\partial = t^2\partial^2 + t\partial$$

$$\cancel{(t\partial)^3 = t\partial(t^2\partial^2 + t\partial)} \qquad \partial^2(t\partial) = \partial(t\partial^2 + \partial)$$

$$\cancel{= t^3\partial^3 + 3t^2\partial^2 +}$$

$$(t\partial)^3 = (t^2\partial^2 + t\partial)(t\partial)$$

$R_1 = tR_0$ $\qquad R_{-1} = R_0\partial$. Then $R_1 R_{-1} \subset tR_0\partial$

~~[illegible]~~ $R_0\partial$?

Go back to $A \oplus \underbrace{AtA}_{} \oplus$

$$A_\rho \otimes_C A_\sigma = E$$

assume $A_\rho \in \mathcal{P}(C^{op})$. Then
$$E^\vee = \mathrm{Hom}_{A^{op}}(E, A)$$
$$= \mathrm{Hom}_{C^{op}}(A_\rho, A_\sigma)$$
$$= A_\sigma \otimes_C \mathrm{Hom}_{C^{op}}(A_\rho, C)$$

It seems like you're taking the dual pair $(A_\rho, \mathrm{Hom}_C(A_\rho, C))$ over C and extending via the homom. $C \longrightarrow A_\sigma$

$$E \qquad \begin{array}{c} A \\ \downarrow \\ E \otimes_A E^\vee \end{array} \qquad E^\vee$$

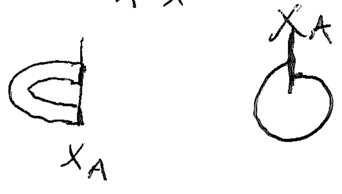
$$\begin{array}{c} A \\ \downarrow \\ A_\rho \otimes_C {}_\sigma A \otimes_A A_\sigma \otimes_{C} \mathrm{Hom}_{C^{op}}(A, C) \end{array}$$

leave momentarily.

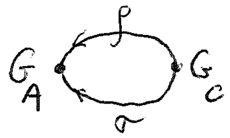
Go back to $X = \text{hocoker}\{X_C \rightrightarrows X_A\}$, X is a CW cx whose cells are those of X_A and suspensions of cells of X_C. $C_\cdot(\tilde{X})$ is a complex of free $\mathbb{Z}[G]$-modules, $G = \pi_1(X)$. G is the gp. generated by adding an elt t, quotient of $G * \mathbb{Z}$ by requiring t to conjugate ρ into σ, $\rho, \sigma : G_C \rightrightarrows G_A$. X is a union of X_A and $X_C \times (0,1)$, so $\tilde{X}$ is union of $X_A \times_X \tilde{X}$ and $(X_C \overset{\sigma}{\times}_X \tilde{X}) \times (0,1)$

Local geometry of X inherited upstairs. $X_A \times_X \tilde{X}$ disconnected

$\tilde{X}_A \times^{G_A} G = p^{-1}(X_A)$



163 So what happens? You have



~~quotient~~ graph

lifts to a oriented tree $G/G_A \leftleftarrows G/G_c$

vertices edges

go back to $A \underset{(C,\rho,\sigma)}{*} \mathbb{Z}[t,t^{-1}]$. We can always take $\underset{\shortparallel}{A} * \mathbb{Z}[t,t^{-1}]$ to get start, i.e. $G = G_A * \mathbb{Z}$

Question: Is $A * \mathbb{Z}[t,t^{-1}]$ a $\mathbb{Z}$-graded ring? Obvious. So what happens? So you, Conjugate. I guess the point is that $\underbrace{A * \mathbb{Z}[t,t^{-1}]} = B_\alpha[t,t^{-1}]$ where B_α is the degree zero part of. What Start over A with a dual pair (X,Y) and an action $A \xrightarrow{\varphi} \mathrm{Mult}(X,Y)$

Then X is a bimodule over A, Y also a bimodule can form Toeplitz algebra

$$A$$

$$X \qquad Y$$

$$X \otimes_A X \qquad X \otimes_A Y \qquad Y \otimes_A Y$$

Now suppose the dual pair gives an interesting [redacted] M. equiv. The ~~situations~~ you examined involved a perfect duality. What is your aim? ~~I am hoping to connect~~ [redacted] ~~up with something simpler.~~ I am hoping to link the Toeplitz algebra to the free product situation $A * \mathbb{Z}[t,t^{-1}]$.

$$A * \mathbb{Z}[t] = A \oplus A \otimes A \oplus \cdots = T_A(A \otimes A) \ ?$$

Infinite-dimensional Consider $J = \overline{T}(E)$

Multiplier alg for J^∞

 What is $\mathrm{Hom}_{T^{op}}(J^\infty, J^\infty)$

$$0 \to \mathrm{Hom}_{T^{op}}(J^\infty, J^n) \to \underbrace{\mathrm{Hom}_{T^{op}}(J^\infty, T)} \to \mathrm{Hom}_{T^{op}}(J^\infty, T/J^n)$$

$$\mathrm{Hom}_{T^{op}}(J^\infty, T/J^n) = 0$$

But $J^n = E^{\otimes n} \otimes T$

so
$$\mathrm{Hom}_{T^{op}}(J^n, T) = \mathrm{Hom}_{T^{op}}(E^{\otimes n} \otimes T, T)$$
$$= \mathrm{Hom}(E^{\otimes n}, T)$$
$$= T \otimes E^{*\otimes n} \quad \times \text{ not if } E \text{ is infinite diml.}$$

$$\varinjlim_n T \otimes E^{*\otimes n}$$

~~Anyway what happens.~~ Leave Toeplitz now

Go back to $A *_{C,\rho,\sigma} \mathbb{Z}[t, t^{-1}]$

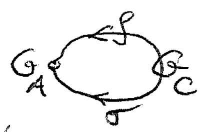
What do I know

$G = G_A *_{G_C,\rho,\sigma} \mathbb{Z}$ on a tree

having quotient

. Actually we have a quiver of groups $G_A \overset{\rho}{\underset{\sigma}{\leftleftarrows}} G_C$. What is a repn of this quiver?

A representation of



Recall reps of quivers, namely:

Dec. 10 I want to see how to go from the combinatorial structure, nerve, of $\tilde{X}$ to the appropriate quiver. In the case $X = X_A \cup_{X_C} X_B$, we split $\tilde{X}|_{X_A}$ etc. into conn components, and get a 1-diml cx which is the tree. Philosophy – You have a decomp. of X, which leads to a decomp. of chains on X. Refinement: think X as a variable space over $BG = BG_A \cup_{BG_C} BG_B$.

165 Other Ideas: Duality leads to considering a finite space together with its Spanier Whitehead dual. You want both cohomology and cohomology with compact supports. There should be some link with nuclear maps

Cuntz alg and invertible bimodules. Look at a $\mathbb{Z}$-graded ring $R = \bigoplus_{n\in\mathbb{Z}} R_n$ such that $R_1R_{-1} = R_{-1}R_1 = R_0$. Assuming $R_0R_n = R_nR_0 = R_n$, then $R_n = R_nR_0 = R_nR_1R_{-1} \subset R_{n+1}R_{-1} \subset R_n$ $\therefore$ $R_{n+1}R_{-1} = R_n$ and sim. $R_nR_1 = R_{n+1}$ and on the other side. Thus $\begin{pmatrix} R_0 & R_{-1} \\ R_1 & R_0 \end{pmatrix}$ gives a self M. equiv. of R_0. The idea is how universal is $\mathcal{O}_n$? Is $\mathcal{O}_n \simeq M_n(\mathcal{O}_n)$? So what

---

Consider $BG_A \cup_{BG_C} BG_B = BG$. How do you get?

~~Consider ...~~

I need a picture, a principle, an example to organize my thinking.

---

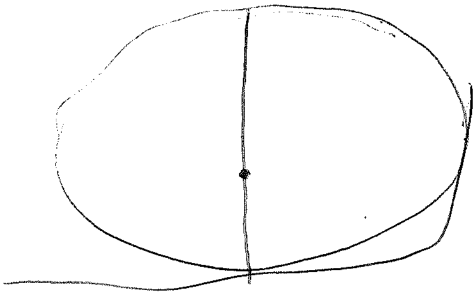
starting picture – conn. CW cx X conn. subcomplex X_C having nbd $X_C \times I$. What's maybe important is that X_C is 2 sided in X, I will be able to cut X along X_C getting ~~something~~ a CW complex $\overline{X-X_C}$ such that

$$\begin{tikzcd} X_C \amalg X_C \arrow[r, hook] \arrow[d, "\text{fold}"] & \overline{X-X_C} \arrow[d] \\ X_C \arrow[r] & X \end{tikzcd}$$

two cases are where $\overline{X-X_C}$ is connected or has two components. latter $\overline{X-X_C} = X_A \amalg X_B$ former $\overline{X-X_C} = X_A$

$$\begin{tikzcd} X_C \arrow[r, hook] \arrow[d] & X_B \arrow[d] \\ X_A \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} X_C \amalg X_C \arrow[r, hook] \arrow[d, "\text{fold}"] & X_A \arrow[d] \\ X_C \arrow[r] & X \end{tikzcd}$$

166 Example $C_\cdot(\tilde{X})$ $\qquad G=\pi_1(X)$ $\qquad$ need basepoint



Let review nil modules for $A[t,t^{-1}]$.

Take $at+b \in GL_1(A[t,t^{-1}])$.

$$\begin{tikzcd}
 & A[t]\oplus t^{-1}A[t^{-1}] \arrow[r, "\sim"] \arrow[d, phantom, "\cap"] & A[t,t^{-1}] \arrow[d, equal] & \\
0 \arrow[r] & H^0 \arrow[r] & (at+b)^{-1}A[t]\oplus t^{-1}A[t^{-1}] \arrow[r] & A[t,t^{-1}] \arrow[r] & 0
\end{tikzcd}$$

$$(at+b)^{-1}A[t]\oplus t^{-1}A[t^{-1}] \cong A[t]\oplus \underbrace{(a+bt^{-1})A[t^{-1}]}_{\subset A[t^{-1}]}$$

$$\begin{tikzcd}
0 \arrow[r] & H^0 \arrow[r] & A[t]\oplus(a+bt^{-1})A[t^{-1}] \arrow[r] \arrow[d] & A[t,t^{-1}] \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & A \arrow[r] & A[t]\oplus A[t^{-1}] \arrow[r] & A[t,t^{-1}] \arrow[r] & 0
\end{tikzcd}$$

Similarly have

$$0 \to H^0 \to \underbrace{(at+b)A[t]}\oplus A[t^{-1}] \to A[t,t^{-1}] \to 0$$

$$\cong$$

$$A[t]\oplus \underbrace{(at+b)^{-1}A[t^{-1}]}_{(a+bt^{-1})^{-1}t^{-1}A[t^{-1}]} \twoheadrightarrow A[t,t^{-1}]$$

$$\cup$$

$$A[t]\oplus t^{-1}A[t^{-1}] \xrightarrow{\sim} A[t,t^{-1}]$$

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$$0 \to A \to A[t] \oplus A[t^{-1}] \to A[t,t^{-1}] \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & H^0 \arrow[r] & (at+b)A[t] \oplus A[t^{-1}] \arrow[r] & A[t,t^{-1}] \arrow[r] & 0 \\
0 \arrow[r] & H^0 \arrow[r] & A[t] \oplus (a+bt^{-1})^{-1}t^{-1}A[t^{-1}] \arrow[r] \arrow[u, "\wr\ at+b"'] & A[t,t^{-1}] \arrow[r] \arrow[u, "at+b"'] & 0 \\
 & & A[t] \oplus t^{-1}A[t^{-1}] \arrow[r, "\sim"] \arrow[u, phantom, "\cup"] & A[t,t^{-1}] \arrow[u, equal] &
\end{tikzcd}$$

We have to abstract this argument.

You are working in $A[t,t^{-1}]$ autom. $at+b$.

What is the basic idea? torsion Sheaf on $\mathbb{P}^1$ supported at $0, \infty$. The sheaf splits

Maybe this is the idea. You have the following chain of lattices

$$\cdots \subset tA[t] \subset A[t] \subset t^{-1}A[t] \subset$$

~~So A you~~ $A[t] \oplus A[t^{-1}] \to A[t,t^{-1}]$

$$\begin{tikzcd}
 & & (at+b)A[t] \oplus A[t^{-1}] \arrow[r, two heads] \arrow[u] & A[t,t^{-1}] & \\
0 \arrow[r] & H^0 \arrow[r] & A[t] \oplus (at+b)^{-1}A[t^{-1}] \arrow[r, two heads] \arrow[u, "\wr"] & A[t,t^{-1}] \arrow[r] \arrow[u, "\wr"] & 0 \\
 & & A[t] \oplus t^{-1}A[t^{-1}] \arrow[u, phantom, "\cup"] \arrow[ur, "\sim"] & &
\end{tikzcd}$$

$$H^0 \xrightarrow{\sim} \frac{(at+b)^{-1}A[t^{-1}]}{t^{-1}A[t^{-1}]} = \frac{A[t^{-1}]}{(at+b)t^{-1}A[t^{-1}]} = A[t^{-1}]/(a+bt^{-1})A[t^{-1}]$$

What happens is that you get

~~$0 \to H^0 \to$~~

$$0 \to H^0 \to A \to A[t]/(at+b)A[t] \to 0$$

$$H^0 \cong A[t^{-1}]/(a+bt^{-1})A[t^{-1}]$$

rest should follow by symmetry.

168 To define a splitting

$$A = A[t]/(at+b)A[t] \oplus A[t^{-1}]/\underbrace{(a+bt^{-1})A[t^{-1}]}_{(at+b)t^{-1}A[t^{-1}]}$$

OKAY. Are you comparing the splittings

$$A[t,t^{-1}] = A[t] \oplus t^{-1}A[t^{-1}]$$

$$\cup$$

$$A[t,t^{-1}] = (at+b)A[t] \oplus (at+b)t^{-1}A[t^{-1}]$$

~~OKAY~~.

My idea.

$$A[t] \oplus A[t^{-1}]$$

$$\uparrow$$

$$(at+b)A[t] \oplus A[t^{-1}]$$

~~$A[t] \oplus (at+b)^{-1} A[t^{-1}]$~~

$$\uparrow$$

$$A[t] \oplus t^{-1}A[t^{-1}]$$

$$(at+b)t^{-1}A[t^{-1}] = (a+bt^{-1})A[t^{-1}] \subset A[t^{-1}]$$

$$\begin{array}{c} A[t] \oplus A[t^{-1}] \\ \cup \\ (at+b)A[t] \oplus A[t^{-1}] \\ \uparrow{\scriptstyle at+b} \\ A[t] \oplus t^{-1}A[t^{-1}] \end{array} \xrightarrow{at+b} A[t] \oplus (a+bt^{-1})A[t^{-1}]$$

$$\begin{array}{ccc} A[t] \oplus t^{-1}A[t^{-1}] & \xrightarrow{at+b} & A[t] \oplus (a+bt^{-1})A[t^{-1}] \\ \downarrow{\scriptstyle at+b} & & \downarrow \\ (at+b)A[t] \oplus A[t^{-1}] & \longrightarrow & A[t] \oplus A[t^{-1}] \end{array}$$

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$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & A[t] \oplus A[t^{-1}] \arrow[r] & A[t,t^{-1}] \\
0 \arrow[r] & H^0 \arrow[r] & (at+b)A[t] \oplus A[t^{-1}] \arrow[r] \arrow[u, phantom, "\cup"] & A[t,t^{-1}] \arrow[u, equal] \\
 & & A[t] \oplus t^{-1}A[t^{-1}] \arrow[r, "\cong"] \arrow[u, hook, "at+b"] & A[t,t^{-1}] \arrow[u, "at+b"', "\cong"]
\end{tikzcd}$$

In fact

$A[t]/(at+b)A[t] \oplus$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & A[t] \oplus A[t^{-1}] \arrow[r] & A[t,t^{-1}] \arrow[r] & 0 \\
 & & A[t] \oplus t^{-1}A[t^{-1}] \arrow[r, phantom, "\simeq"] \arrow[u, "{(at+b,\, at+b)}"] & A[t,t^{-1}] \arrow[u, "at+b"', "\simeq"] &
\end{tikzcd}$$

So next to generalize to Waldhausen's situation

Splitting goes as follows. Given $f(t)$, $g(t^{-1})$.

You take $\dfrac{f(t)+g(t^{-1})}{at+b}$

So for example take $a_0 + 0$ $\qquad a_0$

$(at+b)^{-1} a_0$

$(at+b)^{-1} = \sum c_i t^i$ $\qquad$ $\begin{cases} ac_{i-1} + bc_i = 0 & i \neq 0 \\ \phantom{ac_{i-1} + bc_i} = 1 & i = 0. \end{cases}$

also the other way

then take

$(at+b)^{-1} a = \sum c_i a_0 t^i$ split into

$\sum_{i<0} c_i a_0 t^i$ and $\sum_{i \geq 0} c_i a_0 t^i$

Then apply $at+b$

$(at+b) \sum_{i<0} c_i t^i = ac_{-1}$

$(at+b) \sum_{i \geq 0} c_i t^i = bc_0$

Anyway let us continue.

Consider
$$\begin{array}{ccc} M_C & \to & M_B \\ \downarrow & & \\ M_A & & \end{array}$$

such that $R\otimes_C M_C \xrightarrow{\sim} R\otimes_A M_A \oplus R\otimes_B M_B$

suppose $C=\mathbb{Z}$.

Go back to $R\otimes_C M_C \overset{at}{\underset{b}{\rightrightarrows}} R\otimes_A M_A$

Recall R $= A*\mathbb{Z}[t,t^{-1}]$ involves words AtA

Can ask whether $R = A*\mathbb{Z}[t,t^{-1}]$ has a structure like the Cuntz algebra $\mathcal{O}_n$, say generated by A-bimodules $\underbrace{AtA}_{E}$ $\quad \underset{E^*}{At^{-1}A}$

Then have dual pair and a ring $E\otimes_A E^* =$
$(AtA)\otimes_A(AtA) = A\otimes A\otimes A$

$A \simeq M_n(A) \qquad A \xrightarrow{\sim} \mathrm{Hom}_{A^{op}}(A^{\oplus n}, A^{\oplus n}) \qquad$ NO

You want to give $A \underset{s_j^*}{\overset{(s_i)}{\rightleftarrows}} A^{\oplus n}$ as Right A-modules

i.e. to give s_i s_j^*. If this true then for every right A-mod. M we have
$$M \underset{\cdot s_j^*}{\overset{\cdot s_i}{\rightleftarrows}} M^{\oplus n}$$

$$\underbrace{\sum_i c_i t^i\,(at+b)\sum_{i<0} c_i t^i}_{1} = \sum_i c_i t^i a c_{-1}$$

$\therefore$ $c_i a c_{-1} = 0$ for $i \geq 0$
$= c_i$ for $i<0$.

$\therefore$ $c_{-1} a c_{-1} = c_{-1}$ $\qquad ac_{-1}$.

correspondence interpretation of Kronecker quiver.

R in the $C \rightrightarrows A$ case is automatically a twisted Laurent poly ring. ~~R~~ R is $\mathbb{Z}$ graded

$$R = \bigoplus_{n \in \mathbb{Z}} R_0 t^n \qquad K_0$$

Dec. 11 Continue Let $\sum c_n t^n$ be inverse to $at+b$

Then $$(at+b)\sum_{n\geq 0} c_n t^n = \sum_{n\geq 0} ac_n t^{n+1} + bc_n t^n = bc_0$$

$$(at+b)\sum_{n<0} c_n t^n = \sum_{n<0} ac_n t^{n+1} + bc_n t^n = ac_{-1}$$

and $$\sum_n c_n t^n \,(at+b) \sum_{n\geq 0} c_n t^n$$

$$\overset{\shortparallel}{\sum_{n\geq 0} c_n t^n} = \sum_{n\geq 0} c_n bc_0 t^n$$

$\therefore$ $c_n bc_0 = 0$ for $n<0$ and $= c_n$ for $n \geq 0$

Let $e = bc_0$. Then $c_n e = 0$ for $n<0$
$c_n e = c_n$ for $n \geq 0$

By symmetry

$$\sum_{n\geq 0} c_n t^n (at+b) = \sum_{n\geq 0} c_n a t^{n+1} + c_n b t^n = c_0 b$$

$$c_0 b c_n = c_n \quad n \geq 0$$
$$= 0 \quad n<0.$$

$$c_0 b \sum_n c_n t^n \, bc_0 = \sum_{n\geq 0} c_n t^n$$

$$ac_{-1} \sum_n c_n t^n \, ac_{-1} = \sum_{n<0} c_n t^n$$

same true for $at+b$.

You want the philosophy here. You have the Kronecker quiver - Kronecker modules - linked to $\mathbb{P}^1$ $k[t]$ $k[t,t^{-1}]$. different kinds of nil modules none for $\mathbb{P}^1$, for $k[t]$ $\underset{\text{iso}}{a}$ $\underset{\text{nilp}}{ba^{-1}}$, for $k[t,t^{-1}]$ splitting of types.

There is a ~~quotient situation~~ which is not ~~the~~ abelian cat. ~~quotient~~.
nil modules do not form Serre subcategory.
Anyway what else. Next – structure of nil modules in more general situations.

$$\begin{tikzcd} & A \\ k \arrow[ur] \arrow[dr] & \\ & B \end{tikzcd} \qquad A * B \quad \text{Idemp}$$

consider case of 2 idempotents.

nil modules.

$$R \otimes M_k \xrightarrow{\sim} R \otimes_A M_A \oplus R \otimes_B M_B$$

$$R = k + \bar{A} + \bar{B}$$

Basically R has the same structure as the case $A = \tilde{ke}$, $B = \tilde{ke}$.

So $R \otimes_A M_A$ basis $b \quad bab \quad babab$

$R \otimes_B M_B$ — $a \quad aba \quad ababa$

use a, b for the idempotents in R.

$$R \otimes_A M_A = \bigoplus_{n \geq 0} b(ab)^n M_A + \bigoplus_{n \geq 0} (ab)^n M_A$$

$$R \otimes_B M_B = \bigoplus_{n \geq 0} a(ba)^n M_B + \bigoplus (ba)^n M_A$$

$$R \otimes_k M_C = M_C \oplus \sum a(ba)^n M_A \oplus \sum (ab)^n M_B \oplus \sum b(ab)^n M_B \oplus \sum (ba)^n M_B$$

and given ~~$M_C \to$~~

$$M_A \oplus bM_A \oplus abM_A \oplus babM_A \oplus \cdots$$

$$M_C \qquad bM_C \oplus aM_C \qquad abM_C \qquad baM_C$$

$$M_B \oplus aM_B \oplus baM_B \oplus abaM_B \oplus \cdots$$

$$\left\{\begin{matrix} M_A \oplus bM_A \oplus abM_A \oplus \\ M_B \oplus aM_B \oplus baM_B \oplus \end{matrix}\right\} \simeq \left\{ M_C \begin{matrix} bM_C \oplus abM_C \oplus \cdots \\ \oplus \quad\quad \oplus \\ aM_C \oplus baM_C \oplus \cdots \end{matrix}\right\}$$

example. $\mathbb{Z}/2 * \mathbb{Z}/2$. You want to consider then

Idea ~~chains of group result~~ a system $M_C \rightrightarrows \begin{matrix} M_A \\ M_B \end{matrix}$ should be a kind of coefficient system on the tree. Then $R\otimes_C M_C \longrightarrow R\otimes_A M_A \oplus R\otimes_B M_B$ is the complex of chains with these coefficients, and acyclic means the boundary operator d is invertible, so there's a Green's ~~[illegible]~~ operator analogous to $(at+b)^{-1}$. Then if you look at the fundamental edge you will find a splitting of M_C into submodules decaying on either side. This picture makes everything clear. Now see if you can find the formulas. Let's take the group ring case. Yes! Specifically $\mathbb{Z}/2 * \mathbb{Z}/2$. Well, first describe chains in general. The chain complex is

$$R\otimes_C M_C \longrightarrow R\otimes_A M_A \oplus R\otimes_B M_B$$

and you need to visualize this ~~[illegible]~~ as chains over the tree. What is a local system? ~~First keep the business straight~~

$$\underbrace{\mathbb{Z}[G] \otimes_{\mathbb{Z}[G_C]} M_C} \longrightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_A]} M_A \oplus \cdots$$

~~local coeff~~ You have the tree $E \begin{matrix} \nearrow V_A \\ \searrow V_B \end{matrix}$ $\qquad G/G_C \begin{matrix} \nearrow G/G_A \\ \searrow G/G_B \end{matrix}$

so the G_C module M_C yields

$\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_C]} M_C$ which is ~~the~~ M_C ~~[illegible]~~ transported to each edge. A better way to say things is that you look G equivariant maps $G/G_c \to \mathcal{A}b$.

You look at a G module over the set G/G_c.

a functor from the cat given by G acting on G/G_c.

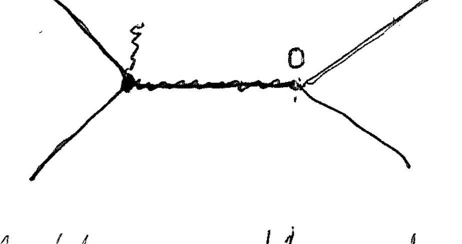
~~A system~~ We look at an equivariant system assigning to each simplex σ an abelian group, M_σ ~~[illegible]~~ equivariantly, means you ~~[illegible]~~ have G action on $\bigoplus_{\{\sigma\}} M_\sigma$ consistent with G ~~action~~ on $\{\sigma\}$. etc.

No problem about the meanings or the direction of the arrows since you talk about chains. So you have $M_A \leftarrow M_C \rightarrow M_B$. What about the Green's operator d^{-1}? Take an elt of M_C.

Problem: $R \otimes_C M_C \xrightarrow{d} R \otimes_A M_A \oplus R \otimes_B M_B$

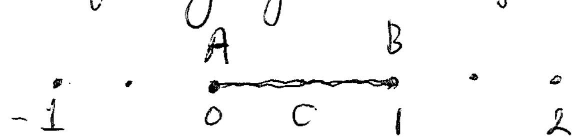
You can take an element ξ of M_A and look at $d^{-1}\xi$ But there is no way to split $d^{-1}\xi$. ??

Yes ~~[illegible]~~ when you pick your ~~base~~ fund. domain you divide the edges ~~[illegible]~~ having the A = vertex into the fund. domain and the others. So maybe you find ~~[illegible]~~ a Green's function which is zero on all edges ~~[illegible]~~

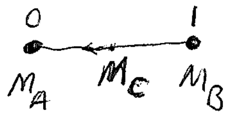


You have a Green's function at each vertex this is the 1-chain whose boundary is an element at the vertex.

Let's carefully go through the example of $\mathbb{Z}/2 * \mathbb{Z}/2$.



175 Example $k \to k[\mathbb{Z}/2]$, $k \to k[\mathbb{Z}/2]$ tree ~~is~~ is line with ~~vertices~~ $\mathbb{Z}$ vertices and reflections ~~at~~ each ~~vertex~~ integer. System



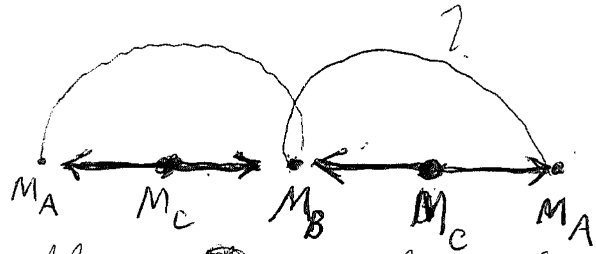
M_A, M_B are $\mathbb{Z}_2$-graded
M_C is ungraded.

So what's the complex? What the complex?

$R = k[\varepsilon, F]$. I am trying to find when this complex is contractible.

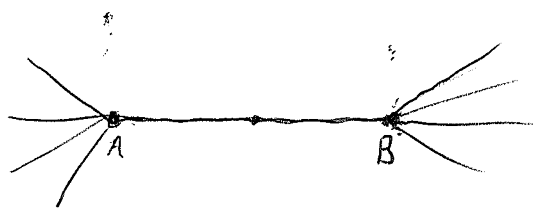
$$R \otimes M_C \xrightarrow{\sim} R \otimes_{k[\varepsilon]} M_A \oplus R \otimes_{k[F]} M_B$$

Look at the picture



I think it's clear that ~~[illegible]~~ this complex is $\mathbb{Z}$ periodic ~~so it will~~ It should amount to the same thing as a twisted Laurent poly.

$M_A^+ \oplus M_B^-$



Z'_l $\xrightarrow{\sim}$ M_C

$\oplus$ $M_A = M_A^+ \oplus M_A^-$

M_C

$\oplus$ $M_B = M_B^+ \oplus M_B^-$

$Z'_r \xrightarrow{+} M_B^+$

does appear that ~~[illegible]~~

$M_C \xrightarrow{\sim} M_A^+ \times M_B^-$

What happens when $M_A^- = 0$

$$Z'_l \oplus M_C \oplus Z'_r$$

$$\| \qquad\qquad \|$$

$$M_A^- \qquad M_A^+ \times M_B^- \qquad M_B^+$$

 Dec 12.

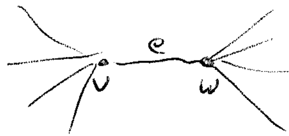
Go back to system on the tree X with trivial homology. Look at a vertex v. For each $m \in M_v$ $\exists!$ 1-chain whose d is m. $d^{-1}(m)$ splits according to the components of $X - v$. Thus M_v splits according to the edges ~~bour~~ containing v.

Each
$$M_v = \bigoplus_{\substack{(v,e) \\ v \in e}} M_{v,e}$$

$$M_{v,e} \xleftarrow{\sim} \text{1-cycles on the } e \text{ component of } X - v$$

$$\begin{array}{ccc} M_v & \xleftarrow[d]{\sim} & Z_1(X-v) \\ \| & & \| \\ \bigoplus M_{v,e} & & \bigoplus Z_1\big((X-v)_e\big) \end{array}$$

Consider an edge e with vertices v, w. $X - \mathring{e}$ has two components



$M_v \oplus M_w \xleftarrow[\sim]{d}$ 1-chains with boundary supported in $\{v, w\}$.

$$= \bigoplus_{e' \neq e} Z_1\big((X-v)_{e'}\big) \oplus M_e \oplus \bigoplus_{e'' \neq e} Z_1\big((X-w)_{e''}\big)$$

$$\therefore \quad M_{v,e} \oplus M_{w,e} \xleftarrow{\sim} M_e$$

Question: Can we split the system M using the fact that M_e has this canonical splitting for each edge e. If the graph is oriented, take

177 $M_v \oplus M_w \xleftarrow[d]{\sim} \bigoplus_{e' \neq e} Z_1((X-v)_{e'}) \oplus M_e \oplus \bigoplus_{e'' \neq e} Z_1((X-w)_{e''})$

induces $M_{v,e} \oplus M_{w,e} \xleftarrow{\sim} M_e$

therefore splitting M_e canonically. Given $\xi \in M_{v,e}$ you can lift it to $Z_1((X-v)_e) \subset M_e \oplus \bigoplus_{e'' \neq e} Z_1((X-w)_{e'})$

~~[illegible]~~ Notation $M_e = M_{e,v} \oplus M_{e,w}$ and for $\eta \in M_e$ you have $\eta \in M_{e,v}$ when η can be continued to an element of $Z_1((X-v)_e)$, i.e. the w boundary ~~compon~~ of η lies in $\bigoplus_{e'' \neq e} M_{w,e''}$

Decomp. M into $M^{v,e} \simeq (M_{v,e}, d^{-1}(M_{v,e}))$

~~What about nilpotence.~~

$$C_0 = \bigoplus_{\substack{v,e \\ v \in e}} M_{v,e}$$

$$C_1 = \bigoplus_{v \in e} d^{-1}(M_{v,e}) = \bigoplus_{e}$$

$$C_0 = \bigoplus_{v,w} M_{v,w}$$ oriented edges.

$$C_1 = \bigoplus_{v,w} d^{-1}(M_{v,w}) = \bigoplus_{v,w} M^1_{v,w}$$

$$\begin{tikzcd} 0 \arrow[r, no head] \arrow[d] & 0 \\ A & \end{tikzcd} \quad \subset \quad \begin{tikzcd} C \arrow[r] \arrow[d] & B \\ A & \end{tikzcd} \quad \subset \quad \begin{tikzcd} A \arrow[r] \arrow[d] & BA \\ A & \end{tikzcd}$$

cokernel $BA/B = B\otimes_C A / B\otimes C = B \otimes_C \bar{A}$

$$\begin{tikzcd} A+B \arrow[r] \arrow[d] & BA \\ AB & \end{tikzcd} \quad \subset \quad \begin{tikzcd} AB \arrow[r, Rightarrow] \arrow[d] & BAB \\ AB & \end{tikzcd}$$

$$\subset \quad \begin{tikzcd} AB+BA \arrow[r] \arrow[d] & BAB \\ ABA & \end{tikzcd}$$

---

look at

$$\begin{tikzcd} A+B \arrow[r] \arrow[d] & BA \\ A\otimes\bar{B} & \end{tikzcd} \qquad \begin{tikzcd} C\oplus\bar{A}\oplus\bar{B} \arrow[r] \arrow[d] & C\oplus\bar{A}\oplus\bar{B}\oplus\bar{B}\otimes\bar{A} \\ \bar{B}\oplus\bar{A}\otimes\bar{B} & \end{tikzcd}$$

---

construct in $C \rightrightarrows A$ You have tree with same analysis. ~~...~~ M_v M_w. So you ~~...~~ fix the fundamental edge, **get** $M_{v,e}$
For each vertex + edge **get something**

Look at the simple case of $A \rightrightarrows A$. The tree is just the line ~~...~~ + integers. You have ~~W, V~~ ~~...~~ quiver $W \overset{a}{\underset{b}{\rightrightarrows}} V$. Then system is

$$\begin{tikzcd}[column sep=small] V & & V & & V & & V \\ & W \arrow[ul, "a"] \arrow[ur, "b"'] & & W \arrow[ul, "a"] \arrow[ur, "b"'] & & W \arrow[ul, "a"] \arrow[ur, "b"'] & \end{tikzcd}$$

~~So the chain space is split.~~

So you have V at each vertex. Splitting of V and W, ~~similarly~~. What's the argument?

$$V = V^- \oplus V^+$$

where $v = d(d^{-1}v) = d\left[(d^{-1}v)_{>0} + (d^{-1}v)_{<0}\right)$

Need notation: You have ~~for each edge~~
for each vertex a splitting according to the incident edges.
for each edge a splitting corresp to the two incident vertices.
Important part maybe is that everything cancels nicely, namely ~~stalks~~ in degree zero have

$\bigoplus_{v<e} M_{v,e}$ and in degree 1 have $\bigoplus_{v<e} (M_{e,v})$

$$E \subset \frac{V \times V - \Delta}{\mathbb{Z}/2}$$

---

work out example of $\mathbb{Z}/2 * \mathbb{Z}/2$ in great detail
very similar to $\mathbb{Z}/2 \ltimes \mathbb{Z}$. These two groups are isomorphic, but will be described differently.

reflection at each vertex.

$A = B = \mathbb{Z}[\mathbb{Z}/2]$ $\quad C = \mathbb{Z}$

end up with $M_C \longrightarrow \underbrace{M_A \times M_B}_{\text{have } \mathbb{Z}/2 \text{ actions}}$

The actual chain complex should be independent of these actions. So you reach the case of

$$\begin{tikzcd}[column sep=small]
 & M_A & & M_B & & M_A & \\
\cdots & & M_C \arrow[ul] \arrow[ur] & & M_C \arrow[ul] \arrow[ur] & & \cdots
\end{tikzcd}$$

This is $\mathbb{Z}$ periodic - observe two maps $M_C \rightrightarrows M_A$ and two maps $M_C \rightrightarrows M_B$ needed before periodicity is applied; presumably these come from the original maps $M_C \to M_A, M_B$ and the $\mathbb{Z}/2$ actions.

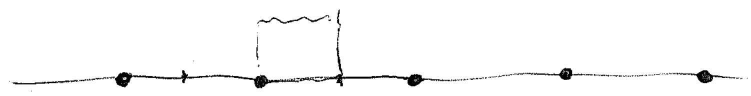
next $\mathbb{Z}_2 \ltimes \mathbb{Z}$ described by

$C = \mathbb{Z}[\mathbb{Z}/2]$ $\quad A = C \quad$ and two arrows $C \overset{1}{\underset{F}{\rightrightarrows}} A$. NO.

system M_C $\qquad C = \mathbb{Z} \oplus \mathbb{Z}\varepsilon$

$t = \varepsilon F \qquad F = \varepsilon t \qquad \underbrace{\varepsilon t}_{t^{-1}} \varepsilon t = 1$

Seems to be faulty:



dihedral acts - allow reflections at $\frac{1}{2}$ integers.

What about $\mathbb{Z} \rightrightarrows \mathbb{Z}[\varepsilon]$. Then you ask for ?.

Go back to $\mathbb{Z}/2 * \mathbb{Z}/2$.

First discuss things generally. You basically find.

X a tree = conn 1-diml s. complex.

M cosheaf on X zero homology. i.e.

$$\begin{array}{ccc} C_1(X,M) & \xrightarrow[\sim]{d} & C_0(X,M) \\ \| & & \| \\ \bigoplus_e M_e & & \bigoplus_v M_v \end{array}$$

Prop. $\quad M_v = \bigoplus_{v<e} M_{v,e} \quad , M_e = M_{e,v} \oplus M_{e,w} \quad$ if $e = \{v,w\}$

canon. isom. $\quad M_{v,e} = M_{e,v} \quad \forall e,v$. Thus

$C_1(X,M) \quad C_0(X,M) \quad$ both $\simeq \bigoplus_{v,e} M_{v,e}$

need to know isom. + inverse.

The iso is easy because $\quad M_{e,v} \xrightarrow{\sim} M_{v,e} \quad$ induced by d.

191 So you have this decomp. on both 0-chains and 1-chains, then d will be ~~the~~ isom. + something θ which is essentially nilpotent.

take $M_{v,e} \xrightarrow[\sim]{d^{-1}}$ 1-cycles finite supp on $(X-v)_e$

X a tree = contr. 1 dim'l s.cx.

M cosheaf on X $\qquad M_{\{x_0,x_1\}} \to M_{x_0}$, $\; M_{\{x_0,x_1\}} \to M_{x_1}$

Chain complex

$$\bigoplus_{e=\{x_0,x_1\}} M_e \xrightarrow{d} \bigoplus_{x} M_x$$

assume acyclic.

$$M_v = \bigoplus_{e \ni v} M_{v,e} \qquad M_e = M_{e,v_0} \oplus M_{e,v_1}$$

$$d: M_{e,v_0} \xrightarrow{\sim} M_{v_0,e}$$

decompose

So you have an index set I ordered 1-simplices

for each i module M_i. Each i has successors

Things I don't know the answer to. What happens with $A *_C B$. What happens with $A *_C B$?

Question: Splitting — Does M split in two ?. Look $M_e = M_e^+ \oplus M_e^-$

Does

192 Dec 13 Work out details. X tree, M cosheaf on X such that $C_\cdot(X,M)$ is acyclic. i.e.

$$d: \begin{array}{ccc} C_1(X,M) & \xrightarrow{\sim} & C_0(X,M) \\ \| & & \| \\ \bigoplus_\sigma M_\sigma & & \bigoplus_x M_x \end{array}$$

σ runs over 1-simp. $\{x_0,x_1\}$; d requires changing the sign of one of the arrows $M_{\{x_0,x_1\}} \rightrightarrows \begin{matrix} M_{x_0} \\ M_{x_1} \end{matrix}$

Fix x d sets up an isom. $Z_1(X,\{x\};M) \xrightarrow{\sim} M_x$

$$\begin{array}{ccc} Z_1(X,\{x\};M) & \xrightarrow{\sim} & M_x \\ \| & & \| \\ \bigoplus_{\substack{\sigma \\ x\in\sigma}} Z_1(Y_{x,\sigma},\{x\};M) & & \bigoplus M_{x,\sigma} \end{array}$$

components of $X-x$ correspond to edges σ w. vertex x.

~~$[M_{x,\sigma} \neq Z(X) \neq Z X_{\sigma}]$~~

$Y_{x,\sigma}$ subcomplex cons of x and all ~~vertices~~ simplices ~~which are~~ whose shortest path to x ends in σ.

$M_{x,\sigma}$ = boundaries of 1-chains supported in $Y_{x,\sigma}$ which are cycles mod x.

~~$M_{\sigma,x} = \{m \in M_\sigma\}$~~

Logic $\quad Z_1(Y_{x,\sigma}, x; M) \underset{\sim}{\xrightarrow{\partial}} M_{x,\sigma}$

$\quad Z_1(Y_{x',\sigma}, x'; M) \xrightarrow[\sim]{} M_{x',\sigma}$

as usual stuck on notation. If $\sigma = x,y$ let $X_{x,\sigma}$ = full subcomplex of x'

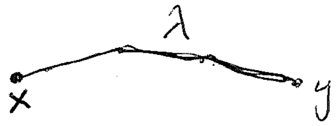
tree X, cosheaf M on X, $C_\cdot(X;M)$ exact.

Study fixed vertex v, derive splitting $M_v = \bigoplus_{\substack{\sigma \\ \text{cont.} \\ v}} M_{v,\sigma}$.

Study fixed vertex v, derive $M_v = \bigoplus_{\substack{\sigma \\ v\in\sigma}} M_{v,\sigma}$

$$\overset{M_v}{\underset{\Psi}{}}\ \xi_v \longmapsto d^{-1}\xi_v = \sum_{\sigma \ni v} (d^{-1}\xi_v)_\sigma \longmapsto \sum d(d^{-1}\xi_v)_\sigma$$

Next look at barycentric subdivision & appropriate cosheaf with $M_{\underbrace{\{v,\sigma\}}_{\text{new edge}}} = M_\sigma$ identity

Next look at two vertices 

1-chains on all

$$\mathbb{Z}_1(\text{left of } x) \oplus \mathbb{Z}_1(\lambda) \oplus \mathbb{Z}_1(\text{rt of } y)$$
$$\downarrow \wr$$
$$M_x \oplus M_y$$

so you get an isomorphism

$$\mathbb{Z}_1(\lambda) \xrightarrow{\sim} M_{x,\lambda} \oplus M_{y,\lambda}$$

Suppose $\lambda = \{x,y\}$, get $M_{\{x,y\}} \xrightarrow{\sim} M_{x,\lambda} \oplus M_{y,\lambda}$

Assertions are: Decomp of M_x all vertices x and decomp of $M_{\{x,y\}}$ all edges. Thus splittings

$C_0 = \bigoplus_{\substack{\{x,y\} \\ \text{or simp}}} M_{x,\{x,y\}}$ $\qquad C_1 = \bigoplus_{\text{simp.}} M_{\{x,y\},x}$

Now the isom d is given by a matrix. Each index (x,y) $|x-y| = 1$ has a shadow

I want to describe the situation taking place. 2 Splittings indexed by oriented edges

$$C_1 = \bigoplus_{\sigma=(x,y)} M^1_\sigma \xrightarrow[d_0 - d_1]{d} C_0 = \bigoplus_{\sigma} M^0_\sigma$$

~~Not what~~

$$\begin{tikzcd}[column sep=small]
M_x \arrow[d, phantom, "\shortparallel"] & \oplus & M_y \arrow[d, phantom, "\shortparallel"] \\
\left(\bigoplus_{\sigma'\neq\sigma} M_{x,\sigma'}\right) \oplus M_{x,\sigma} & & M_{y,\sigma} \oplus \left(\bigoplus_{\sigma''} M_{y,\sigma''}\right) \\
& M_\sigma \arrow[uul] \arrow[uur] \arrow[d, phantom, "\shortparallel"] & \\
& M_{\sigma,x} \oplus M_{\sigma,y} &
\end{tikzcd}$$

Analyze this situation.

$$\begin{array}{ccccc}
M_\sigma & \xrightarrow{\sim} & M_x \oplus M_y & = & M_{x,\sigma} \oplus M_{y,\sigma} \\
\shortparallel & & & & \oplus \\
M_{\sigma,x} \oplus M_{\sigma,y} & & & & \text{outside}
\end{array}$$

First ~~At the moment~~ ignore the outside, then it's simple namely the decomp of M_σ is rigged so [redacted] as to be direct sum of isos. $M_{\sigma,x} \to M_{x,\sigma}$ sim for y. How??
~~Answer not too clear.~~ Reminded of scattering theory. You should review the scattering matrix. [redacted]

$e^{i(kx-\omega t)}$

$$e^{ikx} \rightsquigarrow Ae^{ikx} + Be^{-ikx}$$
$$e^{-ikx} \rightsquigarrow Ce^{ikx} + De^{-ikx}$$

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$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{1}{\underbrace{AD-BC}_{\Delta}} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$$

$$\frac{1}{\Delta} e^{ikx} - \frac{C}{\Delta} e^{-ikx} \rightsquigarrow e^{ikx}$$

$$-\frac{C}{\Delta} e^{ikx} + \frac{A}{\Delta} e^{-ikx} \rightsquigarrow e^{-ikx}$$

Scattering matrix expresses outgoing in terms of incoming ~~$e^{ikx} \rightsquigarrow Ae^{ik}$~~

2 diml space of solution V. left and right pictures or bases incoming and outgoing.

$$e^{ikx} \rightsquigarrow Ae^{ikx} + Be^{-ikx}$$

~~$\frac{1}{A} e^{ikx} \rightsquigarrow e^{ikx} + \frac{B}{A} e^{-ikx}$~~

$$e^{-ikx} \rightsquigarrow Ce^{ikx} + De^{-ikx}$$

$$\underbrace{\frac{1}{D}}_{T} \underbrace{e^{-ikx}}_{\text{out}} \qquad \underbrace{\frac{C}{D}}_{R} \underbrace{e^{ikx}}_{\text{out}} + \underbrace{e^{-ikx}}_{\text{in}}$$

$$e^{ikx} - \frac{B}{D} e^{-ikx} \rightsquigarrow Ae^{ikx} + \cancel{Be^{-ikx}} - \frac{B}{D} Ce^{ikx} - \cancel{\frac{B}{D} De^{-ikx}} \qquad \left(A - \frac{BC}{D}\right) e^{ikx}$$

So if
$$e^{ikx} \rightsquigarrow Ae^{ikx} + Be^{-ikx}$$
$$e^{-ikx} \rightsquigarrow Ce^{ikx} + De^{-ikx} \quad \text{then}$$

$$\boxed{\begin{aligned} \frac{1}{D} e^{-ikx} &\longleftrightarrow \frac{C}{D} e^{ikx} + \underbrace{e^{-ikx}}_{\text{in}} \\ \underbrace{e^{ikx}}_{\text{in}} - \frac{B}{D} e^{-ikx} &\longleftrightarrow \frac{\Delta}{D} e^{ikx} \end{aligned}}$$

196 What I need to ~~[scribble]~~ review is the whole transmission game. — It might generalize to trees. First of all there is the linear algebra — 2 splittings. Then I have the example arising from an edge in a tree. Then you maybe have the $U(n,1)$ action on O_n

Let's try to ~~[scribble]~~ clean up the 1-simplex.

Vertices

$$\begin{tikzcd}
M_A^- \oplus M_A^+ & & M_B = M_B^- \oplus M_B^+ \\
& M_C \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Assumption is that $M_C \longrightarrow M_A/M_A^- \times M_B/M_B^+$ is an isomorphism. Thus M_C splits canonically into $M_C^+ \oplus M_C^-$. $M_C^+ = \mathrm{Ker}\ M_C \longrightarrow M_B/M_B^+$

$$\begin{tikzcd}
M_C^+ \arrow[r] \arrow[d] & M_C \arrow[d] \\
M_B \arrow[r] & M_B/M_B^+
\end{tikzcd}$$

first try to understand without splitting You give

$$\begin{tikzcd}
0 \arrow[r] & M_A^- \arrow[r] & M_A \arrow[r] & M_A/M_A^- \arrow[r] & 0 \\
& & & M_C \arrow[u] \arrow[d] & \\
0 \arrow[r] & M_B^+ \arrow[r] & M_B \arrow[r] & M_B/M_B^+ \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
& \circ \arrow[dl] \arrow[dr] & & \circ \arrow[dl] \arrow[dr] & \\
M_A \arrow[dr] & & M_C \arrow[dl] \arrow[dr] & & M_B \arrow[dl] \\
& M_A/M_A^- \arrow[dr] & & M_B/M_B^+ \arrow[dl] & \\
& & 0 & &
\end{tikzcd}$$

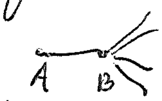
197 ~~[illegible]~~ Basic picture of an oriented simplex. Wait do the initial version, namely

$$M_C \xrightarrow{\sim} M_A/M_A^- \oplus M_B/M_B^+$$

Then $M_C = M_C^+ \oplus M_C^-$ where $M_C^+ =$ elements of M_C going into M_B^+. Then we have

$$\begin{tikzcd} & & M_C^+ \arrow[d] \arrow[dr, "\sim"] & & \\ 0 \arrow[r] & M_A^- \arrow[r] & M_A \arrow[r] & M_A/M_A^- \arrow[r] & 0 \\ & & & M_A^+ \arrow[u, "\wr"'] \arrow[ul] & \end{tikzcd}$$

so that $M_C^+ \longrightarrow M_A = M_A^+ \oplus M_A^-$ has two components the isomorphism $M_C^+ \xrightarrow{\sim} M_A^+$ above and some map $M_C^+ \xrightarrow{R} M_A^-$. ~~[illegible]~~ We also have a map $M_C^+ \xrightarrow{T} M_B^+$. I want to think of R and T as reflection and transmission "coefficients"

In our tree situation ~~[illegible]~~ what happens? M_A^+ is the subspace of M_A ~~[illegible]~~ consisting of boundaries of 1-cycles supported in the subcomplex $X_{A,B}$: 

Note that $M_B^- = M_{B,A}$, $M_B^+ \simeq \bigoplus_{X \neq A} M_{B,X}$. Given $\xi \in M_A^+$ look at corr. cycle z, let η be the ~~element~~ comp of $z \in M_C$. Write $z = \eta \oplus z'$, $\eta \in M_C$, z' ~~[illegible]~~ 1-chain support right side of B.

$dz' \in M_B^+$ clearly. $(dz)_A = \xi \Rightarrow (d\eta)_A = \xi$ and $(d\eta)_B = -dz'$

$\therefore$ $(d\eta)_A = \xi \in M_A^+$ and $(d\eta)_B \in M_B^+$

not just $(d\eta)_A \equiv \xi \bmod M_A^-$. So we find $\underline{R = 0}$.

Try to review reflection + transmission coefficients at some point. Where to start.

Consider $\quad \partial_t^2 \hat{u} = \left(-\partial_x^2 + V(x)\right)\hat{u} \qquad \hat{u} = e^{-i\omega t} u(x)$

$$\omega^2 u = -u'' + Vu$$

198 $\quad \frac{d}{dx}(u_1 u_2' - u_1' u_2) = u_1 u_2'' - u_1'' u_2$

$$= u_1(-\omega^2+V)u_1 - (-\omega^2+V)u_2 u_2 = 0$$

2dim'l space of solutions two splittings basis $e^{i\omega x}, e^{-i\omega x}$ at the far left and similarly on the far right.

$$e^{i\omega x} \longleftrightarrow Ae^{i\omega x} + Be^{-i\omega x}$$

$$e^{-i\omega x} \longleftrightarrow Ce^{i\omega x} + De^{-i\omega x}$$

Wronskian constant $\Rightarrow$ $AD - BC = 1$.

Reality Condition $C = \bar{B}$, $D = \bar{A}$ $\quad \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ $\quad |a|^2 - |b|^2 = 1.$ $SU(1,1)$

But now use the other (natural?) bases

inc.

$$\underset{\substack{\| \\ T}}{\overset{\text{out}}{\frac{1}{D}}} e^{-i\omega x} \longleftrightarrow \underset{\substack{\| \\ R}}{\overset{\text{out}}{\frac{C}{D}}} e^{i\omega x} + \overset{\text{incoming}}{e^{-i\omega x}}$$

$$A - \frac{BC}{D}$$

$$\overset{\text{in}}{e^{i\omega x}} + \overset{\text{out}}{\frac{B}{D} e^{-i\omega x}} \longleftrightarrow \overset{\text{out}}{\frac{1}{D} e^{i\omega x}}$$

matrix)
$$\begin{pmatrix} \frac{1}{D} & \frac{B}{D} \\ -\frac{C}{D} & \frac{1}{D} \end{pmatrix}$$

$$\frac{1}{|a|^2} + \frac{|b|^2}{|a|^2} = \frac{|a|^2}{|a|^2} = 1$$

$$\frac{-1\cdot\bar{c} + b}{|a|^2} = 0.$$

scattering matrix roughly $\begin{pmatrix} \frac{1}{a} & -\frac{\bar{b}}{\bar{a}} \\ \frac{b}{a} & \frac{1}{\bar{a}} \end{pmatrix} \in SU(2).$

So $SU(1,1) \hookrightarrow SU(2)$. matrices with diag $\neq 0$.