G8 [redacted] 11/23

Let's go over the calculation which you didn't finish. Given $R, I, e, \bar{e}$ get $(eR, \bar{e}R, \alpha: eR/I \to \bar{e}R/I)$ add $((1-\bar{e})R, (1-\bar{e})R, 1-\bar{e})$ to get ~~complex~~

$$eR \oplus (1-\bar{e})R, \quad R, \quad \bar{e} \oplus (1-\bar{e}) \quad \text{mod } I$$

$$L \xleftarrow{q} M \xleftarrow{p} L$$

$$\begin{array}{c} \bar{e}R \\ \oplus \\ (1-\bar{e})R \end{array} \longleftarrow \begin{array}{c} eR \\ \oplus \\ (1-\bar{e})R \end{array} \longleftarrow \begin{array}{c} \bar{e}R \\ \oplus \\ (1-\bar{e})R \end{array}$$

What you want to do is to show that $M = eR \oplus (1-\bar{e})R$ together with $M/MI \simeq R/I$ yields an obj $M^{\#}$ of $P(\tilde{I})$. This means you need to produce an idempotent matrix over $\tilde{I}$. ~~Do this~~
Thus we want maps with composition = identity

$$M \xleftarrow{(p \;\; u)} \begin{array}{c} R \\ \oplus \\ R^{\oplus}? \end{array} \xleftarrow{\begin{pmatrix} q_2 \\ v \end{pmatrix}} M \qquad pq_2 + uv = 1$$

Important part of M:

$$\begin{tikzcd} eR & \begin{array}{c} \bar{e}R \\ \oplus \\ eR \end{array} \arrow[l, "e"'] \arrow[l, "e-e\bar{e}e", bend left] & eR \arrow[l, "2\bar{e}e-(\bar{e}e)^2"'] \arrow[l, "e-e\bar{e}e", bend left] \end{tikzcd}$$

$1 - pq_2 = y^2$

$y = e - e\bar{e}e$?

$$\begin{array}{c} eR \\ \oplus \\ (1-\bar{e})R \end{array} \xleftarrow{\begin{pmatrix} e\bar{e} & y & 0 \\ 0 & 1-\bar{e} & 0 & 0 \end{pmatrix}} \begin{array}{c} \bar{e}R \\ \oplus \\ (1-\bar{e})R \\ \oplus \\ eR \\ \oplus \\ (1-e)R \end{array} \xleftarrow{\begin{pmatrix} q_2 & 0 \\ 0 & 1-\bar{e} \\ y & 0 \\ 0 & 0 \end{pmatrix}} \begin{array}{c} eR \\ \oplus \\ (1-\bar{e})R \end{array}$$

$$\begin{pmatrix} g_2 & 0 \\ 0 & 1-\bar e \\ y & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix} e\bar e & 0 & y & 0 \\ 0 & 1-\bar e & 0 & 0\end{pmatrix} = \left(\begin{array}{cc|cc} g_2 p & 0 & g_2 y & 0 \\ 0 & 1-\bar e & 0 & 0 \\ \hline yp & 0 & y^2 & 0 \\ 0 & 0 & 0 & 0\end{array}\right)$$

$$\begin{pmatrix}\bar eR \\ (1-\bar e)R\end{pmatrix} \xleftarrow{\begin{pmatrix} g_2 & \\ & 1-\bar e\end{pmatrix}} \begin{matrix} eR \\ \oplus \\ (1-\bar e)R\end{matrix} \xleftarrow{\begin{pmatrix} e & \\ & 1-\bar e\end{pmatrix}} \begin{matrix} \bar eR \\ \oplus \\ (1-\bar e)R\end{matrix}$$

$$\begin{pmatrix} g_2 e\bar e & \\ & 1-\bar e\end{pmatrix} = \begin{pmatrix} g_2 p & \\ & 1-\bar e\end{pmatrix}$$

$$\bar e - x^2 = \bar e - (\bar e - \bar e e\bar e)^2 = 2\bar e e\bar e - \bar e e\bar e e\bar e$$

how to do it simply. You first look at

$$\begin{tikzcd} eR & \bar eR \arrow[l, "p = e\bar e"'] & eR \arrow[l, "g_2 = (2\bar e - \bar e e\bar e)e"'] \arrow[dl, "y = e - e\bar e e"] \\ & \oplus & \\ & eR \arrow[uul, "e - e\bar e e"] & \end{tikzcd}$$

$$pg_2 = 2e\bar e e - e\bar e e\bar e e$$
$$e - pg_2 = (\underbrace{e - e\bar e e}_{y})^2$$

This leads to idempotent

$$\begin{pmatrix} g_2 \\ y\end{pmatrix}\begin{pmatrix} p & y\end{pmatrix} = \begin{pmatrix} \overset{\bar e + I}{2\bar e e\bar e - \bar e e\bar e e\bar e} & \overset{\in I}{(2\bar e - \bar e e\bar e)(e - e\bar e e)} \\ \underset{\in I}{e\bar e - e\bar e e\bar e} & \underset{\in I^2}{(e - e\bar e e)^2}\end{pmatrix}$$

on $\begin{matrix}\bar eR \\ \oplus \\ eR\end{matrix}$ Combine with $\begin{pmatrix} 1-\bar e & 0 \\ 0 & 0\end{pmatrix}$ on $\begin{matrix}(1-\bar e)R \\ \oplus \\ (1-e)R\end{matrix}$

and you get an idempotent on $R^{\oplus 2}$.

$$1-(\bar e - \bar e e\bar e)^2$$

$$1 - \bar e + 2\bar e e\bar e - \bar e e\bar e e\bar e \qquad 2\bar e e - 3\bar e e\bar e e + \bar e e\bar e e\bar e e$$

$$e\bar e - e\bar e e\bar e \qquad (e - e\bar e e)^2$$

Try writing this out always working inside of R. First however try $(E, R, E/EI \simeq R/J)$ lift to

$$R \xleftarrow{q} E \xleftarrow{p} R$$

$x = 1 - qp$: $R \supset I \longleftarrow R$

$y = 1 - pq$: $E \supset EI \longleftarrow E$

$$R \xleftarrow{(q\ \sqrt{x})} \begin{matrix} E \\ \oplus \\ R \end{matrix} \xleftarrow{\begin{pmatrix} p \\ \sqrt{x} \end{pmatrix}} R \qquad \begin{pmatrix} p \\ \sqrt{x} \end{pmatrix}(q\ \sqrt{x}) = \begin{pmatrix} pq & p\sqrt{x} \\ \sqrt{x}q & x \end{pmatrix}$$

complementary

$$E \xleftarrow{(\sqrt{y}\ \ p)} \begin{matrix} E \\ \oplus \\ R \end{matrix} \xleftarrow{\begin{pmatrix} -\sqrt{y} \\ q \end{pmatrix}} E \qquad \begin{pmatrix} -\sqrt{y} \\ q \end{pmatrix}(-\sqrt{y}\ \ p) = \begin{pmatrix} y & -\sqrt{y}p \\ -q\sqrt{y} & qp \end{pmatrix}$$

what's needed is an embedding of E as a summand of $R^{\oplus N}$, so as to get the idempotent over $\tilde{I}$.

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trans. A right A-flat.

$P = A \quad Q = A$

$P' = \tilde{A} \quad Q = A$

$(A, \tilde{A}) \longrightarrow (\tilde{A}, \tilde{A})$ map of dual pairs.

$$\begin{array}{ccc} K_*(A \otimes_A \tilde{A}) & \longrightarrow & K_*(\tilde{A} \otimes_A \tilde{A}) \\ \downarrow \tau_A & & \downarrow \tau_{\tilde{A}} \\ K_*(A) & \hookrightarrow & K_*(\tilde{A}) \end{array}$$

Check: $e - e(2\bar{e}e - (\bar{e}e)^2)$

$= e - 2e\bar{e}e + e\bar{e}e\bar{e}e$

$(e - e\bar{e}e)^2 = e - 2e\bar{e}e + e\bar{e}e\bar{e}e$

So what comes next? $\mathbb{Z}_2$-graded version

Go on!!!! Let's try the following. Suppose you have a quasi-homom $A \rightrightarrows R \supset I$ and a Morita equiv. $\begin{pmatrix} I & V \\ U & B \end{pmatrix}$. You want to understand the map $K_0(A) \to K_0(I) \to K_0(B)$.

<u>You</u> have your triple E

Problem: Given $C \rightrightarrows R \supset B = P \otimes_A Q$

Better suppose $R = \mathrm{Mult}\{(P,Q)\}$ ~~and~~

~~we have~~

$$C \rightrightarrows R = \text{Mult ring of } (P,Q) \text{ over } A$$
$$= \left\{ \overset{(l,r)}{} {}_A\mathrm{Hom}_{A^{op}}(P,P) \times \mathrm{Hom}_A(Q,Q)^{op} \;\middle|\; \langle q, l(p) \rangle = \langle (q)r, p \rangle \right\}$$

Then R contains an ideal I = image of $P \otimes_A Q$

$$(p \otimes q)(p') = p \langle q, p' \rangle$$

$$r(p \otimes q)\, p' = (rp \otimes q) p' = rp \langle q, p' \rangle$$

$$\big(r(p\otimes q)\big)(p') \overset{?}{=} r\big((p\otimes q)p'\big) = r\big(p\langle q, p'\rangle\big)$$

$P \otimes_A Q \longrightarrow I$ is square 0 extension.

So you should get a map $K_0(C) \longrightarrow K_0(P \otimes_A Q) = K_0(A)$.
(need to assume $A = QP$ probably.)

 So how does this subproblem proceed?

Suppose you have $L, M \in \mathcal{P}(R^{op})$ and an isom modulo I: $L/LI \simeq M/MI$

Then this **step** seems tricky.

The real problem is ~~unclear~~ how to go from $\mathcal{P}(\tilde{I})$ to what you need. But here's a better idea: ~~Consider~~ ~~~~! Suppose you have this K_0-class given by (L, M, α) over R. How are you going to represent it over A?

~~In other words you will have nothing at all keep on trying ~~ First look at R, I and suppose you have ~~a~~ I-(nuclear) complex U. ~~~~

U complex of right modules $\ni$ identity of U deforms to a

$$U \underset{R}{\otimes} \mathrm{Hom}_{R^{op}}(U, R) \longrightarrow \mathrm{Hom}_R(U, U)$$

$$U \underset{R}{\otimes} I \underset{R}{\otimes} \mathrm{Hom}_{R^{op}}(U, R) \nearrow$$

So I have to review my K_0' paper.

Basically you want U, V, pairing $V \otimes_{\mathbb{Z}} U \to \mathbb{C}$

$\langle v, \mu u \rangle = \langle v\mu, u \rangle$

get $\underset{\text{ring}}{U \otimes_{\mathbb{C}} V} \longrightarrow \mathrm{Hom}_{\mathbb{C}^{op}}(U, U) \overset{\circ}{\times} \mathrm{Hom}_C(V, V)^{op}$

$$\downarrow \mathrm{Hom} \qquad \swarrow$$

try doing this with multiplier

$$\mathrm{Hom}_{C^{op}}(U \otimes_{\mathbb{Z}} V, C)$$

$$\begin{tikzcd} U \otimes_C V \arrow[r] \arrow[d] & \mathrm{Hom}_C(V,V)^{op} \arrow[d] \\ \mathrm{Hom}_{C^{op}}(U,U) \arrow[r] & \mathrm{Hom}_{C,C^{op}}(V \otimes U, C) \end{tikzcd}$$

$$\mathrm{Hom}_{C^{op}}(U,U) \longrightarrow \mathrm{Hom}_{C^{op}}(U, \mathrm{Hom}_C(V,C))$$

multiplier algebra is the fibre product. So then you deform ~~[illegible]~~ R.

Suppose you have then such a gadget over C. and a Morita eq. $\begin{pmatrix} C & Y \\ X & A \end{pmatrix}$

Then given $(U, V, \langle\ \rangle)$ get $U \otimes_C Y$, $X \otimes_C V$ with pairing

$$\begin{tikzcd} (X \otimes_C V) \otimes_{\mathbb{Z}} (U \otimes_C Y) \arrow[d] \\ X \otimes_C C \otimes_C Y \arrow[d] \\ A \end{tikzcd}$$

$$\begin{tikzcd} (U \otimes_C Y) \otimes_A (X \otimes_C V) \arrow[r] \arrow[d] & \mathrm{Hom}_A(X \otimes_C V, X \otimes_C V) \arrow[d] \\ \mathrm{Hom}_{A^{op}}(U \otimes_C Y, U \otimes_C Y) \arrow[r] & \mathrm{Hom}_{A,A^{op}}(X \otimes_C V \otimes_{\mathbb{Z}} U \otimes_C Y, A) \end{tikzcd}$$

74 Multiplier alg of ~~??~~ U, V maps to that of $U \otimes_C Y$, $X \otimes_C V$?

$$\begin{array}{ccc} \mathrm{Hom}_{C^{op}}(U,U) & & \mathrm{Hom}_C(V,V) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{A^{op}}(U\otimes_C Y, U\otimes_C Y) & & \mathrm{Hom}_A(X\otimes_C V, X\otimes_C V) \end{array}$$

$$\begin{array}{ccc} \langle (x\otimes v), (\mu u \otimes y)\rangle & & \langle (x\otimes v\mu), u\otimes y\rangle \\ \| & & \| \\ x\langle v, \mu u\rangle y & & x\langle v\mu, u\rangle y \end{array}$$

Also check that have hom.

$$\begin{array}{ccc} U\otimes_C V & \longrightarrow & U\otimes_C Y \otimes_A X \otimes_C V \\ & & \downarrow \text{nilp. extn.} \\ & & U \otimes_C C \otimes_C V \end{array}$$

good condition is the possibility of deforming the identity map in the mult. alg into $U \otimes_C C^{(\infty)} \otimes_C V$

So consider U, V, $V \underset{\mathbb{Z}}{\otimes} U \longrightarrow C$ dual pair ~~and~~

Assume $U \otimes_C V \longrightarrow \mathrm{Hom}_{C^{op}}(U,U) \times_{\mathrm{Hom}_{C,C^{op}}(V\otimes U, C)} \mathrm{Hom}_{C^{op}}(V,V)$

$\exists$ deformation of the identity in the mult. algebra.

~~Assume $\exists$ def. of the identity~~ Suppose working with complexes

I think you know that ~~??~~, $CU \longrightarrow U$ is a homotopy equivalence. You have to go over these arguments.

75 So consider

$C \quad V$
$U \quad U \otimes_C V$

$$B = U \otimes_C V$$

$$B \otimes_B B = (U \otimes_C V) \otimes_B (U \otimes_C V)$$

Let's understand. You need to begin with the class $\chi \in U \otimes_C V$ whose action on U, V is homotopic to the identity.

$$\chi \in U \otimes_C V \qquad \chi \in U \otimes_C V$$

$$u \mapsto \chi \otimes u$$

$$U \longrightarrow (U \otimes_C V) \otimes U \longrightarrow U \otimes_C C \longrightarrow U$$

$$U \otimes_C V \longrightarrow (U \otimes_C V) \otimes (U \otimes_C V) \longrightarrow U \otimes_C C \otimes_C V \longrightarrow U \otimes_C V$$

So you get an interesting structure

$$\Rrightarrow B \otimes B \otimes B \rightrightarrows B \otimes B \longrightarrow B$$

So it seems that the arguments are formal ~~homotopy equ~~ Homotopy equivalences, etc.

Maybe review why such a (U, V) is heq to a finite proj. complex. So suppose you have ~~two~~ (U, V) complexes of ~~left~~ right + left unitary R-modules such that $\exists$ $\chi \in U \otimes_R V$ whose actions on U, V are homotopic to 1.

I ~~should~~ can take $V = \mathrm{Hom}_{R^{op}}(U, R)$

$$U \otimes_R \mathrm{Hom}_{R^{op}}(U, R) \longrightarrow \mathrm{Hom}_{R^{op}}(U, U)$$

What does this mean $1 - f = [d, h]$ where f comes from $\chi \in U \otimes_R V$.

 Go over this point again

Suppose you have a cycle in $U \otimes_R V$.
What's the analogue of $x = \sum u_i \otimes v_i$
What do you hope for? dominated by a ~~p~~ perfect complex. Thus you want a perfect complex P and map $P \underset{i}{\overset{j}{\rightleftarrows}} U$ such that $ji \sim 1$. In this case you get $1 \in P \otimes_R \check{P} \xrightarrow{j \otimes {}^t i} U \otimes_R U^\vee$.
Better would be to have?

~~Look at~~ Replace U by a complex of free modules, then U is ~~b~~ a filtered colimit of ~~proj~~ strictly perfect complexes.

$$x \in U \otimes_R U^\vee = \varinjlim P_i \otimes_R U^\vee$$

~~So~~ have $P_i \xrightarrow{a} U$ and $x' \in P_i \otimes_R U^\vee$
i.e. $x' : U \xrightarrow{b} P_i$ such that

$$\begin{array}{ccc} x' \in & P_i \otimes_R U^\vee & \ni b \\ \downarrow & \downarrow a & \\ x \in & U \otimes_R \check{U} & \end{array} \qquad \text{i.e. } U \xrightarrow{b} P_i \xrightarrow{a} U_i \text{ is } \sim 1.$$

Should work with V. Namely $x \in U \otimes_R V = \varinjlim P_i \otimes_R V$ so you get $P_i \xrightarrow{a} U$ and $b \in P_i \otimes_R V \to \mathrm{Hom}_{R^{op}}(U, P_i)$

~~Therefore you find~~

Argument. Given $x \in U \otimes_R V$ write $x = \sum u_i \otimes v_i$ where u_i, v_i are homog. ~~Then~~ Let F be the free R^{op}-module with n-generators ~~and~~ graded appropriately, so that (u_i) defines $F \xrightarrow{v} U$ and (v_i) defines ~~v~~ $v: F^\vee \to V$ whence v
~~Res~~ $U \to V^\vee \to F^{\vee\vee} = F$. Then x_u is the comp. $U \xrightarrow{v} F \xrightarrow{u} U$

77. Now equip F with diff udv:

$$udv\, udv = ud\,X\, dv = uX\, d^2v = 0.$$

So ~~xxxx~~ if X is htpic to $1_{\mathcal{U}}$, $\mathcal{U}$ is dominated by F. ~~[illegible]~~

Program. Take the thing you get from a quasi-hom.

$$B \rightrightarrows R \supset I \leftarrow Y \otimes_A X, \text{ namely, } P, \bar{P} \in \mathcal{P}(R^{op})$$

and an isom $\alpha : P/PI \simeq \bar{P}/\bar{P}I$. Let $\mathcal{U} : P \underset{p}{\overset{q}{\rightleftarrows}} \bar{P}$

where p, q lift α and α^{-1}. Try thinking of $\mathcal{U}$ as perfect complex with homotopy. ~~The key idea is now to decide what to do.~~ I'm looking for a generalization of complex and homotopy. In any case we have this odd operator $\begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}$ and

$$X = 1 - \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}^2 = \begin{pmatrix} 1-qp & 0 \\ 0 & 1-pq \end{pmatrix} : \mathcal{U} \to \mathcal{U}I \quad \text{or}$$

$X \in \mathcal{U} \otimes_R I \otimes_R \check{\mathcal{U}}$. Now apply Morita invariance

$\mathcal{U}$ becomes $\mathcal{U} \otimes_R Y : P \otimes_R Y \rightleftarrows \bar{P} \otimes_R Y$

V becomes $X \otimes_R V : X \otimes_R \bar{P}^\vee \rightleftarrows X \otimes_R P^\vee$

The point is maybe that this "dual pair" $(\mathcal{U} \otimes_R Y, X \otimes_R V)$ over A ~~defines something over~~ with its d, h has a K_0 class. Make life easy take $P = eR$, $\bar{P} = \bar{e}R$

Then $\mathcal{U} \otimes_R Y : \underset{1}{eY} \underset{\bar{e}e}{\overset{e\bar{e}}{\rightleftarrows}} \underset{0}{\bar{e}Y}$

$X \otimes_R \check{\mathcal{U}} : \underset{0}{X\bar{e}} \rightleftarrows \underset{1}{Xe}$

$$\mathcal{U} \otimes_R Y \otimes_A X \otimes_R \check{\mathcal{U}} : \begin{pmatrix} eI'e & eI'\bar{e} \\ \bar{e}I'e & \bar{e}I'\bar{e} \end{pmatrix} \qquad I' = Y \otimes_A X$$

I get a "complex" over A^{op} $\quad eY \rightleftarrows \bar{e}Y \quad$ which because of the explicit dual defines an elt of $K_0(A)$.

I need the dual so as to "embed" in a perfect "complex".

$1 - f = [d, h]$

$$\begin{tikzcd} F \arrow[d, shift left, "j"] \\ U \arrow[u, shift left, "i"] \end{tikzcd}$$

$ji = f$

$(idj)i \qquad id \qquad idh$

$= id(1 - dh - hd) = idhd$

$idj \qquad (1 \mp hd)(1 \mp dh)$

$$\begin{tikzcd} F \arrow[loop above, "idj"] \arrow[d, shift left, "j"] \\ U \arrow[u, shift left, "i"] \end{tikzcd}$$

$idj\, i(1 \mp dh) \doteq i(1-dh)d$

$= ijid = ifd \qquad = ifd.$

similarly $(1-hd)j\, idj \qquad d(1-hd)j$

$= (1-hd)dfj \qquad = d(1-hd-dh)j$

$= dfj \qquad = dfj.$

$$\begin{tikzcd} F \arrow[loop above, "idj"] \arrow[d, shift left, "(1-hd)j"] \\ U \arrow[u, shift left, "i(1-dh)"] \end{tikzcd}$$

$(1-hd)_j\, i(1-dh)$

$= (1-hd)f(1-dh)$

$= f - hdf - fdh + \cancel{hdfdh}$

$= f -$

79.
$$\begin{array}{c} F \\ i\uparrow\ \downarrow j \\ U \end{array} \qquad ji = f \qquad 1-f = dh + hd$$

~~Try $i' = i(1-f)$~~

~~$id_j\, i(1-f) \overset{?}{=} i(1-f)d$~~

$$i' = i(1-dh) \qquad \underbrace{id_j\, i}(1-dh) \overset{?}{=} i(1-dh)d$$

$$\begin{array}{ccc} id_j\, i(1-dh) & & i(1-dh)d \\ \| & & \| \\ idf(1-dh) & & if\,d \\ \| & & \\ ifd(1-dh) & & \end{array}$$

$$j'i' = (1-hd)\overset{f}{\overline{ji}}(1-dh) = f - hdf - fdh + \cancel{hdfdh}$$

$$hdf = hd(1-hd)$$
$$fdh = (1-dh)dh$$
$$hdf + fdh = hd + dh - hdhd - dhdh$$
$$= [d,h] - [d, hdh] = [d, h']$$

$$h' = h - hdh$$

$$[d, h'] = (1-f) - (1-f)dh - hd(1-f)$$
$$= fdh + hdf$$

Is this a case of HPT?

$$\begin{array}{c} F \\ i\uparrow\ \downarrow j \\ E \end{array} \qquad \text{maps of complexes} \ni ji \simeq 1.$$

NOT obviously.

$$K_1(R/I) \longrightarrow K_0(R \times_{R/I} R) \longrightarrow K_0R \oplus K_0R \longrightarrow K_0(R/I)$$

80. $U : P \rightleftarrows \bar{P}$

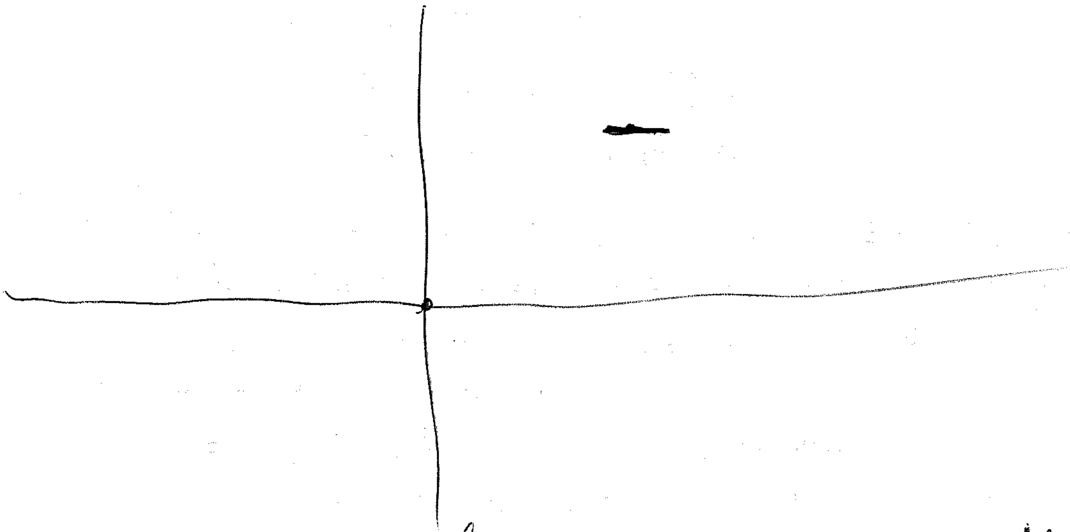
1 $\qquad\qquad\qquad \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$

$$0 \longrightarrow U \otimes_B I \otimes_R U^\vee \longrightarrow U \otimes_R U^\vee \longrightarrow U/UI \otimes_{R/I} U^\vee / IU^\vee \longrightarrow 0$$

You lift $\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$ to $\alpha = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}$: $\begin{matrix} P \\ \oplus \\ \bar{P} \end{matrix} \xleftarrow{\begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}} \begin{matrix} P \\ \oplus \\ \bar{P} \end{matrix}$

Then $1 - \alpha^2 = \begin{pmatrix} 1 - qp & 0 \\ 0 & 1 - pq \end{pmatrix}$. Question: Can you refine α so as to make $1 - \alpha^2 \equiv 0 \mod I^n$.

Look for an odd polynomial $p(\alpha)$ such that $p(1) = 1$ and $p(x) - 1$ divisible by $(x-1)^n$. Then $1 - p(\alpha)^2 = (1 - p(\alpha))(1 + p(\alpha)) \equiv 0$ modulo $(\alpha - 1)^n (\alpha + 1)^n = (\alpha^2 - 1)^n$. Graph of p



It seems that p has degree $\geq 2n+1$. Not true for $n = 1$.

If $p(x) - 1 \equiv 0 \mod (x-1)^n$

then $p'(x) \equiv 0 \mod (x-1)^{n-1}$

so $p'(x) \equiv 0 \mod (x^2 - 1)^{n-1}$

and the smallest degree possible is $p'(x) = c(x^2-1)^{n-1}$

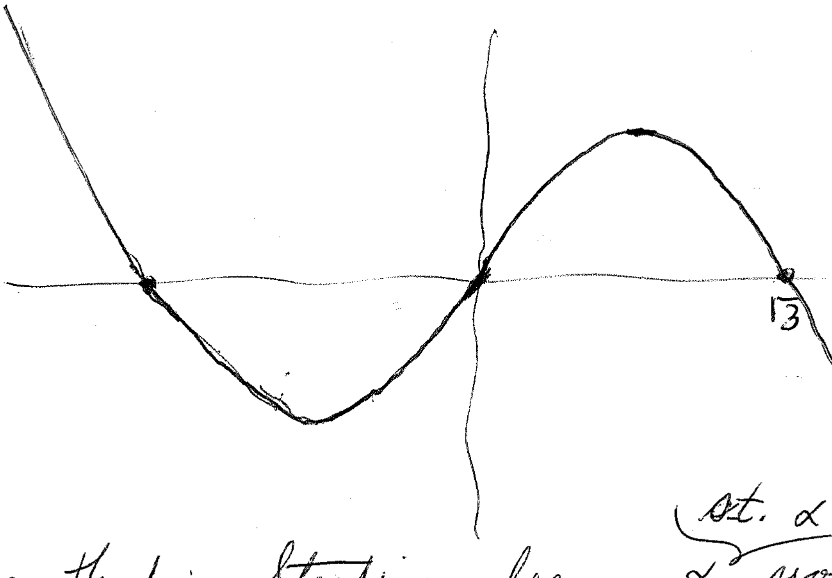
$$p(x) = c \int_0^x (x^2 - 1)^{n-1} dx \quad \text{where} \quad c = \frac{1}{\int_0^1 (x^2-1)^{n-1} dx}$$

81. Pf $n=1$.

$$c\int_0^x (1-x^2)\,dx = \frac{x-\frac{x^3}{3}}{1-\frac{1}{3}} = \frac{3x-x^3}{2}$$

$p(x) = \frac{1}{2}(3x-x^3)$ Check $p(1)=1$ ✓

$p'(x) = \frac{1}{2}(3-3x^2)$ $p'(1)=0$



Other method: Starting from $\underbrace{\alpha}_{\text{st. } \alpha^2 \text{ close to } 1}$ you want an involution Polar decomp. ~~$\alpha g(\alpha)$~~

$$\alpha(\alpha^2)^{-1/2} = \alpha\left(1-(1-\alpha^2)\right)^{-1/2}$$

$$= \alpha\sum_{n=0}^{\infty}\frac{(-1/2)(-3/2)\cdots(-\frac{2n-1}{2})}{n!}(-1+\alpha^2)^n$$

$$= \alpha\sum_{n=0}^{\infty}\frac{1\cdot3\cdots(2n-1)}{2^n n!}(-\alpha^2+1)^n$$

$$= \alpha\Big(1+\frac{1}{2}(1-\alpha^2)+\frac{1\cdot3}{2^2\cdot2!}(1-\alpha^2)^2+\underbrace{\frac{1\cdot3\cdot5}{2^3\,3!}}_{\frac{5}{16}}(1-\alpha^2)^3$$

$$2\cdot\frac{3}{8}+\frac{1}{4}=1$$

$$\frac{1\cdot3\cdot5\cdot7}{2^4\,4!}$$

$$2\cdot\frac{5}{16}+2\,\frac{1}{2}\cdot\frac{3}{8}=1$$

$\frac{1\cdot3\cdots 2n-1}{2^n\,n!}$ does this have power of 2 in den. YES.

$$\frac{(2n)!}{\cancel{2^n}\,n!\,\cancel{2^n}\,n!}\left(\frac{1-\alpha^2}{4}\right)^n$$

82. $$\frac{2n!}{n!\,n!} = \frac{2^n\, 1\cdot 3 \cdots 2n-1}{n!} \in \mathbb{Z}$$

$$\left[\frac{n}{2}\right] + \left[\frac{n}{4}\right] + \cdots + \leq n. \qquad n = 2^r$$

$n = 4 \qquad 4! = 2\cdot 4 \cdot 8$

$$2^{r-1} + 2^{r-2} + \cdots + 1 = 2^r - 1 = n-1$$

$$\frac{2n!}{n!\,n!} = \frac{(2n-1)!\, 2}{n!\,(n-1)!} = \frac{1\cdot 3 \cdots (2n-1)\, 2^{n-1}}{n!}\, 2$$

Approx. method. Given α, $\alpha^2 - 1 \in I$ I propose to change α to $\alpha\left(1 + \frac{1}{2}(1-\alpha^2)\right)$

$$\left[\alpha\left(1+\tfrac{1}{2}(1-\alpha^2)\right)\right]^2 = \alpha^2\left(1 + (1-\alpha^2) + \left(\frac{1-\alpha^2}{2}\right)^2\right)$$
$$\equiv 2\alpha^2 - \alpha^4 \mod I^2$$

$$1 - \left[\alpha\left(1 + \frac{1-\alpha^2}{2}\right)\right]^2 \equiv 1 - 2\alpha^2 + \alpha^4 \mod I^2$$
$$\equiv (1-\alpha^2)^2$$
$$\equiv 0$$

So your approximation method takes $\alpha = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}$

to $$\alpha\left(1 + \tfrac{1}{2}(1-\alpha^2)\right) = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}\begin{pmatrix} 1 + \frac{1-qp}{2} & 0 \\ 0 & 1 + \frac{1-pq}{2} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & q + \frac{q - qpq}{2} \\ p + \frac{p - pqp}{2} & 0 \end{pmatrix}$$

replace p by $\frac{3p - pqp}{2}$

83. 11/25/97

Recall last position: Given $(P, \tilde{P}, \theta)$ you let $U = P \oplus \tilde{P}$ $\quad V = U^\vee$

$$0 \to U \otimes_R I \otimes_R U^\vee \longrightarrow \underset{\substack{\| \\ \mathrm{Hom}_{R^{op}}(U,U)}}{U \otimes_R U^\vee} \longrightarrow \underset{\substack{\Psi}}{U/UI \otimes_{R/I} U^\vee / IU^\vee} \to 0$$

$$\begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & \theta^{-1} \\ \theta & 0 \end{pmatrix}$$

~~[illegible]~~ I know that I can view U as a complex in $\mathcal{L}^1(R, I)$ with p as diff'l and q as a homotopy operator. If I regard p as fixed, then this gives more information than just θ. I want somehow to exploit the symmetry between p and q ~~[illegible]~~ in order to obtain ~~[illegible]~~ a suitable notion of homotopy. Consider the determinant in the commutative case. ~~[illegible]~~

$E_1 \xrightarrow{p} E_0$ yields $\xi_p \in \Lambda^{max} E_0 \otimes (\Lambda^{max} E_1)^\vee$

~~[illegible]~~

Setup R comm. unital domain say, $E_i \in P(R)$ p lifts $\theta : E_1 / IE_1 \xrightarrow{\sim} E_0 / IE_0$. Then ~~[illegible]~~ E_0, E_1 have the same rank. ~~[illegible]~~

This may be misleading.

What sort of homotopy possibilities. Replace

Start with (U, V) a $\mathbb{Z}_2$-graded dual pair over $\mathbb{C}$

$$V \otimes U \longrightarrow C$$

Next idea $\quad U \otimes_C C \otimes_C V \longrightarrow U \otimes_C V$

What is the sort of stuff you need.

84. How do I decide whether there is a future here? ~~M~~ My problem is this. Let (U,V) be a super dual pair over C, then I need to find the notion of a diagonal in $U\otimes_C V$. It should be an even element which when pushed into the multiplier alg.

$$R = \mathrm{Hom}_{C^{op}}(U,U) \times \mathrm{Hom}_C(V,V) \qquad \text{odd}$$
$$\mathrm{Hom}_{C,C^{op}}(V\otimes U, C)$$

has the form $1-\alpha^2$ where α is odd

Assume $U\otimes_C V$ is unital $1 = \sum u_i \otimes v_i$

and $U \xrightarrow{(v_i)\cdot} C^n \xrightarrow{(u_i)\cdot} U$. Then what is R? R is $\mathrm{Hom}_{C^{op}}(U,U) = \mathrm{Hom}_C(V,V)^{op}$

What is an element μ of R? $(\mu\cdot, \cdot\mu)$ such that $\langle v\cdot\mu, u\rangle = \langle v, \mu\cdot u\rangle$. Thus $\cdot\mu$ and $\mu\cdot$ are transposes. If the pairing is non-degenerate on one side:

$$V \hookrightarrow \mathrm{Hom}_{C^{op}}(U,C)$$

then clearly $\cdot\mu$ is determined by $\mu\cdot$

If $V = \mathrm{Hom}_{C^{op}}(U,C)$ or $C\otimes_C \mathrm{Hom}_{C^{op}}(U,C)$ etc.

then $R = \mathrm{Hom}_{C^{op}}(U,U)$

 11/26/97

Recall $\alpha' = \alpha\left(1 + \frac{1-\alpha^2}{2}\right)$ satisfies

$$1-(\alpha')^2 = 1 - \alpha^2\left(1 + 1-\alpha^2 + \frac{(1-\alpha^2)^2}{4}\right)$$

$$= 1 - 2\alpha^2 + \alpha^4 + \frac{(1-\alpha^2)^2}{4}$$

$$= \left(1+\frac{1}{4}\right)(1-\alpha^2)^2$$

$$\alpha' = \alpha\left(\alpha^2\right)^{-1/2} = \alpha\left(1-(1-\alpha^2)\right)^{-1/2}$$

$$= \alpha \sum_{n=0}^{\infty} \frac{1.3\cdots 2n-1}{n!}\left(\frac{1-\alpha^2}{2}\right)^n$$

I have to carefully review my K_0' paper.

First I need to specify the kind of "complexes"

The initial example ~~[illegible]~~ is provided by Milnor's triples $\left(P, \bar{P}, P/PI \simeq \bar{P}/\bar{P}I\right)$

Question: Multiplier alg for A^∞ in the case $\bar{T}(V)$.

Given A we have a dual pair (A^∞, A^∞) over A whence a multiplier ring

$$\mathrm{Hom}_{A^{op}}(A^\infty, A^\infty) \times \mathrm{Hom}_A(A^\infty, A^\infty)^{op}$$

$$\mathrm{Hom}_{A,A^{op}}(A^\infty \otimes A^\infty, A^\infty)$$

If A is left or right flat then $A^{(n)} = A^n$ is firm flat. What happens? Can you describe the unital ring $\mathrm{Hom}_{A^{op}}(A^\infty, A^\infty)$. Form $\mathcal{M}(A)$ etc. ~~This seems to be hard.~~

$$\mathrm{Hom}_{A^{op}}(A^\infty, A^\infty) = \varprojlim \mathrm{Hom}_{A^{op}}(A^\infty, A^k)$$

$$= \varprojlim_k \varinjlim_j \mathrm{Hom}_{A^{op}}(A^j, A^k)$$

$A = \bar{T}(V) = V \otimes \tilde{A}$ $\quad\therefore\quad$ Hom

$A^n = V^{\otimes n} \otimes \tilde{A}$

$$\varinjlim_j \mathrm{Hom}_{A^{op}}\left(V^{\otimes j}\otimes \tilde{A}, A^k\right)$$

$$= \varinjlim_j A^k \otimes V^{*\otimes j} = A^k \otimes$$

First do $\varinjlim_j \mathrm{Hom}_{A^{op}}\left(\overbrace{A^j}^{V^{\otimes j}\otimes\tilde{A}}, \tilde{A}\right)$

$$= \varinjlim_j \tilde{A}\otimes V^{*\otimes j}$$

Note that ~~lim~~ $\mathrm{Hom}_{A^{op}}(A^j, X) = X \otimes V^{*\otimes j}$

Pretty clearly, this inductive system is what leads to the Toeplitz algebra, i.e. used the canonical $k \longrightarrow V\otimes V^*$:

$$X\otimes V^{*\otimes j} \longrightarrow \underbrace{X\otimes V}_{\downarrow\, X}\otimes\underbrace{V^*\otimes V^{*\otimes j}}_{\shortparallel\, V^{*\otimes j+1}}$$

Thus $\mathrm{Hom}_{A^{op}}(A^\infty, \tilde{A}) = \mathcal{O}_V$

~~Is it true that~~

$$\mathrm{Hom}_{A^{op}}(A^\infty, A^n) = A^n \otimes_A \mathcal{O}_V$$

$$0 \longrightarrow A^n \longrightarrow \tilde{A} \longrightarrow \underbrace{\tilde{A}/A^n}_{\text{nil}} \longrightarrow 0$$

So the left multiplier algebra is $\mathcal{O}_V$ and similarly the right one.

Let's analyze following: (U, V) super dual pair over B, whence canon

$$U \otimes_B V \longrightarrow \mathrm{Mult}(U,V) = \mathrm{Hom}_{B^{op}}(U,U) \times \mathrm{Hom}_B(V,V)^{op}$$

$\mathrm{Hom}_{B,B^{op}}(V \otimes U, B)$

Suppose given odd element α in the multiplier ring and an element $f \in U \otimes_B V$ mapping to $1-\alpha^2$. Multiplier rings are not functorial. But let's simplify things by assuming $V \xrightarrow{\sim} \mathrm{Hom}_{B^{op}}(U, B)$, in which case the multiplier ring is just the left multiplier ring $\mathrm{Hom}_{B^{op}}(U,U)$. So we ought to be able to picture everything using U, e.g. α is $\begin{pmatrix} 0 & \beta \\ \rho & 0 \end{pmatrix}$ on $U_1 \oplus U_0$, which I like to view as $\begin{pmatrix} 0 & h \\ d & 0 \end{pmatrix}$.

Choose

$$\begin{tikzcd} F = B^n \arrow[d, shift left, "j"] \\ U \arrow[u, shift left, "i"] \end{tikzcd}$$

$i = (v_i)$
$j = (u_i)$
such that $ji = f$

I know that U becomes homotopy equivalent to a finite proj complex. I should work out why. I recall the formulas were messy, but maybe manageable for length 1 complexes.

$$\begin{tikzcd} F_1 \arrow[d, shift left, "j_1"] & F_0 \arrow[d, shift left, "j_0"] \\ U_1 \arrow[u, shift left, "i_1"] \arrow[r, shift right, "d"'] & U_0 \arrow[u, shift left, "i_0"] \arrow[l, shift right, "h"'] \end{tikzcd}$$

$j_0 i_0 = f_0 = 1 - dh$
$j_1 i_1 = f_1 = 1 - hd$

What would you really like? What you do is to modify i_0 to $i_0(1-dh)$
j_1 to $(1-hd)j_1$

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An idea $\quad \iota = (v_i) \quad j = (u_i)$

$$\begin{pmatrix} B & V \\ U & U \otimes_B V \end{pmatrix}$$

replace B by $M_n(B)$
V by $V^{\oplus n}$, U by $U^{\oplus n}$

So you now work in this Morita context but entries are all super. Let's now try to understand the calculations to be done. Two stages in my paper, one involving my transition to a complex dominated by a f. free cx., the other being Ranicki's, which again was one-sided. Can I symmetrize somehow. At the moment we have this odd α in $\mathrm{Mult}(U,V)$ such that

$1-\alpha^2 = uv$

$$\begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix}^2 = \begin{pmatrix} vu & 0 \\ 0 & uv \end{pmatrix}$$

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d, shift left, "j_1"] & F_0 \arrow[d, shift left, "j_0"] \\ U_1 \arrow[u, shift left, "i_1"] \arrow[r, shift right, "d"'] \arrow[r, phantom, "\oplus", near start, yshift=20pt] & U_0 \arrow[u, shift left, "i_0"] \arrow[l, shift right, "h"'] \end{tikzcd}$$

$= j_0 i_0 (1-dh)$
$(1-dh)^2$

$j_1 i_1 = 1 - hd$

Anyway what

$$\begin{aligned} j_0 i_0 (1-dh) &= (1-dh)^2 \\ &= 1 - 2dh + dhdh \\ &= 1 - d(2h - hdh) \end{aligned}$$

$$\underbrace{(i_0 d j_1) i_1}_{1-hd} \overset{!}{=} i_0 (1-dh) d$$

$$d(1-hd) j_1 \overset{?}{=} \underbrace{j_0 i_0}_{1-dh} d j_1$$

$$\begin{tikzcd} F_1 \arrow[r, "i_0 d j_1"] \arrow[d, shift left, "(1-hd)j_1"] & F_0 \arrow[d, shift left, "j_0"] \\ U_1 \arrow[u, shift left, "i_1"] \arrow[r, shift right, "d"'] & U_0 \arrow[u, shift left, "i_0(1-dh)"] \arrow[l, shift right, "2h-hdh"'] \end{tikzcd}$$

Try a more straightforward idea. You work in the ~~multiplier~~ ring

$$\begin{pmatrix} B & V \\ U & Mult(U,V) \end{pmatrix}$$

What are trying to do? Life is difficult; You have α ~~even~~ odd $\in Mult(U,V)$ U, V even in U, V. What sort of thing do you want to achieve? I think you want to allow α to be perturbed modulo $U \otimes_B V$.

Look at that question B superalg, (U,V) super dual pair over B. ~~Then you~~ Work in Multiplier ring. Is this a kind of dilation problem?

Take

11/27 Work a bit on the multiplier ring of A^∞, where $A = \bar{T}(E)$. [redacted]

Let $\mathcal{O}_E$ be Cuntz alg of E

$$\mathcal{O}_E = \mathcal{T}_E / (1 - \sum \psi_i \psi_i^*)$$

~~$\tilde{A} = T(E)$~~

Review what I learned this morning about $\mathcal{T}_E = R$. We have the unital homom. $k \longrightarrow \mathcal{T}_E$. You have a dual pair over k $(T(E), T(E^*))$ direct sum of $(E^{\otimes n}, E^{*\otimes n})$ for $n \geq 0$ $E^{*\otimes n} \otimes E^{\otimes n} \to E^{*\otimes n-1}$ etc.

Then $T(E) \otimes_k T(E^*)$ should be ideal of finite rank operators in R.

$$R(1 - \sum s_i s_i^*) R$$
$$= T(E)(1 - \sum s_i s_i^*) T(E^*)$$

 [redacted] Map $K_*(R) \longrightarrow K_*(k)$ defined by a [redacted] Kasparov module given by

$$T(E)\otimes E \overset{?}{\rightleftarrows} T(E)$$

These are finitely generated R-modules$\overset{NO}{?}$. In fact

$$0 \longrightarrow \mathcal{J}_E \otimes E^* \longrightarrow \mathcal{J}_E \longrightarrow T(E) \longrightarrow 0$$

So $T(E) \in \mathcal{P}'(R)$. Use basis $s_\alpha s_\beta^*$ for $\mathcal{J}_E$ basis $s_\alpha 1$ for $T(E)$.

To compose $K_*(k) \to K_*(k) \longrightarrow K_*(R)$. Wait

The first ~~[illegible]~~ thing to understand is the map $K_0(R) \longrightarrow K_0(k)$ defined by

$$T(E)\otimes E \left(\overset{?}{\rightleftarrows} \right) T(E)$$

So you start with $P \in \mathcal{P}(R^{op})$ and tensor:

$$P \underset{R}{\otimes} T(E) \otimes E \overset{?}{\rightleftarrows} P \underset{R}{\otimes} T(E)$$

The construction of the operator [redacted] ? is the subtle part of Kasparov's theory.

So how do we proceed? One idea is ~~the~~ that you need a "connection" on P, and that such a thing comes from expressing P as a summand of a free module. So what happens?

We have $\mathcal{U} = \mathcal{U}_1 \oplus \mathcal{U}_0$ with the odd ?

In this example we have $\mathcal{U}_1 \oplus \mathcal{U}_0 = T(E)\otimes_k E \oplus T(E)$ as R-modules equipped by some odd ~~[illegible]~~ operator α ~~[illegible]~~ such that $1-\alpha^2$ is in $\mathcal{U} \otimes_R V$.

91 Be specific. Follow Pimsner [redacted]

Enlarge $T(E)\otimes E = T(E)_{>0}$ to $T(E)$ so that α can be chosen as $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$.

Now you want to tensor with $P \in \mathcal{P}(R^n)$.

Say $P = eR^{\oplus n}$. Then have operator e acting on $\mathcal{U}^{\oplus n}$, say $n=1$. What can you do?

What is the mechanism.

You have $e \in R$ acting on $\mathcal{U}_1 \oplus \mathcal{U}_0$ but not quite commuting with F. OKAY

$e\mathcal{U}_1 \oplus u\mathcal{U}_0 \qquad eFe = \alpha$

$$1-\alpha^2 = e - eFeFe = eF1Fe - eFeFe$$
$$= eF(1-e)Fe = eF(1-e)[F,e]$$
$$= e[F,1-e][F,e] = -e[F,e]^2$$

So you have $R = \mathcal{T}(E) \qquad \mathcal{U} = \mathcal{T}(E) \oplus T(E)$

$$1-F^2 = \begin{pmatrix}1-d^*d & \\ & 1-dd^*\end{pmatrix} \qquad F = \begin{pmatrix}0 & d^*\\ d & 0\end{pmatrix}$$

$$= \begin{pmatrix}0 & 0\\ 0 & p_0\end{pmatrix} \qquad p_0 \text{ projection onto } E^{\otimes 0}$$

$$p_0 = \sum s_i s_i^*$$

---

have difficulty analyzing what happens in this case, however there is another angle, namely to use a f.g proj resolution of $\mathcal{U}$. Actually you might first look at the process $P \rightsquigarrow P\otimes_R \mathcal{U}$: $P\otimes_R \mathcal{U}_1$ $P\otimes_R \mathcal{U}_0$ bring $I = \mathcal{U}\otimes_R V$ in. What is $\mathcal{U}/I\mathcal{U}$? What is $T(E)/IT(E)$?

$$0 \to R\otimes E^* \longrightarrow R \longrightarrow T(E) \longrightarrow 0$$

$$0 \to (R/I) \otimes E^* \to R/I \longrightarrow T(E)/IT(E) \to 0$$

I generated by $1-\sum_i s_i s_i^*$ 4:20 shortbread

$(1-\sum s_i s_i^*)\, T(E) = k$ so $T(E)/IT(E) = 0$.

Very precise example. This is very nice. Start

You have a very specific ~~situation~~, namely
$R = \mathcal{T}_E = T(E) \otimes T(E^*)$ in a funny way.
$R \supset I = T(E) \otimes T(E^*)$ defined by a dual pair
But in any case you have $R \supset I \underset{\text{equivalence}}{\overset{\text{Morita}}{\sim}} k$
But $R \subset \text{Mult alg}\,(T(E), T(E^*)) \subset \text{Hom}\,(T(E), T(E)) \times \text{Hom}\,(T(E^*), T(E^*))^{op}$

~~Suppose Lorthaug OK~~

Have R acting on $T(E)$ and $\underline{T(E) \otimes E = T(E)^{\oplus n}}$

You have two homoms. $R \overset{id}{\rightrightarrows} R$. the 2nd
is non unital rather 1 goes into $\sum s_i s_i^* = e^\perp$

$$T(E) \otimes E \subset T(E) \supset k$$

$R \longrightarrow$

$x \longmapsto x +$

$s_j \sum s_i s_i^* \sim s_j$

$e^\perp s_j(\underbrace{1-\sum s_i s_i^*}_{e^\perp})$

$= s_j e^\perp$

look at the generators.

$s_i \longmapsto e^\perp s_i e^\perp = s_i'$

$s_i^* \longmapsto e^\perp s_i^* e^\perp = s_i^{*\prime}$

$s_i^{*\prime} s_j' = e^\perp s_i^* \overbrace{e^\perp e^\perp s_j}^{s_j} e^\perp = e^\perp \delta_{ij}$

$s_i' s_i^{*\prime} = e^\perp s_i e^\perp s_i^* e^\perp$

there are the relation

$$s_i \longmapsto e^\perp s_i e^\perp \qquad s_i^* \longmapsto e^\perp s_i^* e^\perp$$

$$s_i' = s_i e^\perp \qquad s_i^{*\prime} = e^\perp s_i^*$$

$$s_i - s_i' = s_i e \in I \qquad s_i^* - s_i^{*\prime} = e s_i^* \in I$$

$$e^\perp s_i = \sum s_j s_j^* s_i = s_i$$

$$s_i^* e^\perp = \sum \underbrace{s_i^* s_j}_{\delta_{ij}} s_j^* = s_i^*$$

So we have a quasi-hom $R \underset{\alpha}{\overset{1}{\rightrightarrows}} R \supset I$.

So start with $P \in \mathcal{P}(R^{op})$. Then have $(P, \alpha_* P; \alpha)$. Now you want to ~~consider~~ convert this to

11/28/97 $R = \mathcal{T}_E$ $\quad I = \mathcal{T}_E(\underbrace{1 - \sum s_i s_i^*}_{e})\mathcal{T}_E \simeq T(E) \otimes T(E^*)$

$e^\perp =$ proj onto $T(E)_{>0}$

$$e^\perp s_i = s_i \qquad s_i^* = s_i^* e^\perp$$

$$s_i' = s_i e^\perp \qquad s_i'^* s_j' = e^\perp \underbrace{s_i^* s_j}_{\delta_{ij}} e^\perp = \delta_{ij} e^\perp$$

$$s_i'^* = e^\perp s_i^*$$

Means you get unital hom $R \xrightarrow{\tau} e^\perp R e^\perp$

$$s_i, s_i^* \qquad s_i e^\perp, e^\perp s_i^*$$

This means you get ~~a quasi~~ a homom. $R \underset{\sigma}{\overset{1}{\rightrightarrows}} R$

Let $P \in \mathcal{P}(R^{op})$. What is $\sigma_*(P) = P \otimes_R R = P \otimes_R e^\perp R$?

Look this way. Look at the repn on $T(E)$. Then σ is the rep. $T(E) \otimes E \oplus 0$ rep on k.

What do you need to get straight? ~~Not much~~

Let's go over the philosophy

 Philosophy. You have a ~~quasi~~-homom.
$B \rightrightarrows R \supset I$ and this induces $K_0(B) \to K_0(I)$
You also have a M og $\begin{pmatrix} I & Y \\ X & k \end{pmatrix}$ whence

an isom. $K_0(I) \xrightarrow{\sim} K_0(k)$. You need to get each of these in the best possible form.

Where to start? Universal case is $B = R \times_{R/I} R$
know $\mathcal{P}(B^{op})$ cat of $(P_1, P_0, P_1/P_1 I \simeq P_2/P_2 I)$. $K_0(B)$ is $\oplus$ Groth group of these triples, $K_0(I)$ is quotient of $K_0(B)$ by degenerate triples, really should write $K_0(R, I)$ and then prove excision $K_0(\tilde{I}, I) \xrightarrow{\sim} K_0(R, I)$.

Where to start? I think you want to take (P_1, P_0, α). Wait. Try first to relate the step $K_0(\tilde{I}, I) \xrightarrow{\sim} K_0(R, I)$ and the Morita invariance step $K_0(R, I) \xrightarrow{\sim} K_0(k)$. The problem is passing from (P_1, P_0, α) over R to similar data over k. You really have to get better control over your K_0' paper.

What might work? You start with $\mathcal{U} = (P_1, P_0, \bar{\alpha})$. and the dual triple $\mathcal{U} = (\check{P}_0, \check{P}_1, \bar{\alpha}^t)$. Then use M cat. $\begin{pmatrix} R & Y \\ X & k \end{pmatrix}$ to get $(\mathcal{U} \otimes_R Y, X \otimes_R \mathcal{U}^\vee)$ over k. This is a super dual pair over k.

$$0 \to \underbrace{\mathcal{U} \otimes_R I \otimes_R \mathcal{U}^\vee}_{Y \otimes_k X} \to \mathcal{U} \otimes_R \mathcal{U}^\vee \to \mathcal{U}/\mathcal{U}I \otimes_{R/I} \mathcal{U}^\vee / I\mathcal{U}^\vee \to 0$$

In this case you have $\bar{\alpha}$ in $\mathcal{U}/\mathcal{U}I \otimes_{R/I} \mathcal{U}^\vee / I\mathcal{U}^\vee$, can lift $\bar{\alpha}$

95 to an odd $\alpha \in U \otimes_R \check{U}$, then form $1-\alpha^2$

$1-\alpha^2 \in (U \otimes_R Y) \otimes_k (X \otimes_R U^\vee)$. The problem is now to convert the latter into an element of $K_0(k)$.

So here is the crucial problem. to treat the dual pair $(U \otimes_R Y, X \otimes_R U^\vee)$ over k, together with the element $1-\alpha^2$. This is probably not enough. You need α also. ~~α also~~ But you do have the multiplier $(\alpha \otimes 1, 1 \otimes \alpha^t)$ on $(U \otimes_R Y, X \otimes_R U^\vee)$. YES!!!

So exactly what is at hand over k. You get a dual pair $(L, M) = (U \otimes_R Y, X \otimes_R \check{U})$ over k together with an α, α^t in the mult. ring. ~~α, α^t in the mult. ring~~

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Idea $e, \bar{e} \in R$ $e - \bar{e} \in I$, then

$eU, \bar{e}U$ are commensurable

Start with (Y, X)

Somehow the basic construction amounts to the following. Let (Y, X) be non-deg. dual pair over k field, let $R =$ mult. ring, $I = Y \otimes_k X \subset R$. We want to describe $K_0(R \times_{R/I} R) \longrightarrow K_0(k)$. canonical map. An object of $\mathcal{P}(R \times_{R/I} R)_r$ is $(P, \bar{P}, \phi)$ apply $- \otimes_R Y$ get $P \otimes_R Y \underset{\longrightarrow}{\overset{\alpha \otimes 1}{\longleftarrow}} \bar{P} \otimes_R Y$ where α is any lifting of ϕ. $1 - \alpha^2 \equiv 0$ modulo I.

Let $f : P \to PI \subset P$. Then what about

$$f\otimes 1 : P\otimes_R Y \longrightarrow PI\otimes_R Y \subset P\otimes_R Y$$

What do I know about $P\otimes_R Y$? If you write P as a summand of $R^{\oplus n}$. ~~Typically you say that~~ then f is given by $M_n(I)$. You have

$$P\otimes_R Y \hookrightarrow Y^{\oplus n} \xrightarrow{M_n(I)} Y^{\oplus n} \to P\otimes_R Y$$

You know nothing about $P\otimes_R Y$ other than the fact it is a ~~subvector sp~~ complemented subspace of $Y^{\oplus n}$. But no other ~~structure~~ I can see. The point is that this $f\otimes 1$ is nuclear.

What picture $(P,\bar{P},\theta)$ might be obtained, arise.

For any object of $\mathcal{P}(R\times_{R/I}R)_r$ you choose lifting α of θ to get $P \overset{\alpha}{\rightleftarrows} \bar{P}$, then you get $P\otimes_R Y \overset{\alpha\otimes 1}{\rightleftarrows} \bar{P}\otimes_R Y$ such that $1-(\alpha\otimes 1)^2$ is nuclear.

Next take $R = \mathcal{T}_E$ $\quad \begin{pmatrix} R & Y \\ X & k \end{pmatrix}$ $\quad Y = T(E)$, $X = T(E^*)$, $I = Y\otimes_k X \subset R$.

We look at two actions of R on $T(E)$, the first is the obvious one, the second is the action on $T(E)\otimes E = T(E)_{>0}$ extended by 0.

$$\sigma(s_i) = s_i e^\perp$$
$$\sigma(s_j^*) = e^\perp s_j^*$$

$\sigma(1) = e^\perp$

This gadget should yield a map $K_*(R) \longrightarrow \mathbb{K}(k)$. What is it on $P \in \mathcal{P}(R^{op})$

 Somehow I should focus on the idea that $T(E)\otimes E$, $T(E)$ are almost isomorphic representations of R over k. From this viewpoint it is clear that $P\otimes_R T(E)\otimes E$, $P\otimes_R T(E)$ are almost isomorphic k-modules for any $P \in \mathcal{P}(R^{op})$.

Idea: Is there a way to dilate almost isomorphic reps?

Question: Graph of a unitary

What has to be done. We have two inf. diml reps. of $R = \mathcal{T}_E$ namely $T(E)\otimes E$ and $T(E)$ which are almost isomorphic. i.e. odd α on $T(E)\otimes E \oplus T(E)$ such that $[R,\alpha]$ and $1-\alpha^2$ finite rank. I think it's always possible to dilate α. ? $\alpha = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}$ $1-\alpha^2 = \begin{pmatrix} 1-qp & 0 \\ 0 & 1-pq \end{pmatrix}$ unitarily

Given a contraction $\alpha = \begin{pmatrix} 0 & c^* \\ c & 0 \end{pmatrix}$, then

$$\begin{pmatrix} \sqrt{1-c^*c} & c^* \\ c & -\sqrt{1-c^*c^*} \end{pmatrix} \underbrace{\begin{pmatrix} \sqrt{1-c^*c} & c^* \\ c & -\sqrt{1-cc^*} \end{pmatrix}}_{\alpha + \varepsilon\sqrt{1-\alpha^2}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$c\sqrt{1-c^*c} - \sqrt{1-cc^*}\,c$$

11/29/97 Problem $\begin{pmatrix} R = \mathcal{T}_E & Y = T(E) \\ X = T(E^*) & k \end{pmatrix}$ $I = Y\otimes X$

We have reps of R on $T(E)\otimes E$ and $T(E)$ which are almost isomorphic, specifically there are map $T(E)\otimes E \underset{\iota}{\overset{\iota^*}{\rightleftarrows}} T(E)$ such that $\alpha = \begin{pmatrix} & \iota^* \\ \iota & \end{pmatrix}$ satisfies $[R,\alpha] \subset I$ $1-\alpha^2 \in I$. Alternatively you can replace $T(E)\otimes E$ by ${}_\sigma T(E)$ which is $T(E)$ with R acting through the hom $\sigma: R \to R$ then $\alpha^2 = 1$. This gives a map $K_0(R) \longrightarrow K_0(k)$.

$P \in \mathcal{P}(R^{op}) \longmapsto P\otimes_{R^{\sigma}} T(E)$ $\quad P\otimes_R T(E)$. ~~and the result~~

Want: $\quad \overset{\shortparallel}{(P\otimes_R {}_\sigma R)\otimes_R T(E)}$

So you have $\underset{\substack{T(E) \text{ with} \\ \sigma \text{ action}}}{R\oplus T(E)\otimes E} \rightleftarrows \underset{\text{reg. action}}{T(E)}$

$$P\otimes_R T(E)\otimes E \qquad P\otimes_R T(E)$$

How do we produce the maps between these. One way is to write $P = \overset{e}{}R^{\oplus r}$, then have

$$(T(E)\otimes E)^{\oplus r} \rightleftarrows T(E)^{\oplus r}$$

Clearer $\quad P\otimes_R {}_\sigma T(E) = \overset{e}{}R^{\oplus r}\otimes_R {}_\sigma T(E)$

$$= \sigma(e)\left(T(E)^{\oplus r}_{>0}\right)$$

Simply you have $\quad T(E)\otimes E \rightleftarrows T(E)$

Take the ~~projection~~ idemp. matrix e over R. and act on both sides

$$e\left(T(E)^{\oplus r}\otimes E\right) \qquad eT(E)^{\oplus r}$$

My viewpoint. You have two R-modules $T(E)\otimes E$ and $T(E)$ so given $P\in \mathcal{P}(R^{op})$ get $P\otimes_R T(E)\otimes E$, $P\otimes_R T(E)$ two vector spaces. But the important part is that these ~~is an isomorph the~~ two R-modules are almost isomorphic. ~~Notice the~~ We know $T(E)\in\mathcal{P}(R)$ (this ~~may be~~ is probably irrelevant) because you do not use the map $\mathcal{P}(R)\longrightarrow\mathcal{P}(R/I)$ as $IT(E) = T(E)$

Keep on trying. The fundamental problem will be to clean up the idea that ~~[illegible]~~ elements of $K_0(A)$ are represented by super dual pairs (U,V) over A together with an odd $\alpha \in Mult(U,V)$ and even $\beta \in U \otimes_A V$ satisfying $f \longmapsto 1-\alpha^2$. I already know this is true because such data ~~[illegible]~~ yield a complex U of A^{op}-modules with homotopy operator:

$$\alpha = \begin{pmatrix} & h \\ d & \end{pmatrix}$$

and then $f = 1-\alpha^2 = \begin{pmatrix} 1-hd & \\ & 1-dh \end{pmatrix}$ is nuclear.

<u>But I</u> don't understand the equivalence relation.

Given (U,V) super dual pair over A

$\alpha \in Mult(U,V)$ $\quad \alpha$ odd

$f \in U \otimes_A V$ $\quad f$ even $\qquad \partial(f) = 1-\alpha^2$

$$0 \qquad U \otimes_A V \xrightarrow{\partial} Mult(U,V) \qquad \text{DGA.}$$

$\cancel{\partial(x\partial y) = \partial x \partial y}$

Suppose $\alpha' = \alpha + \partial g$. Then

$$1-\alpha'^2 = 1 - (\alpha^2 + \alpha\partial g + \partial g\,\alpha + \partial g \partial g)$$

$$= 1-\alpha^2 - \partial(\alpha g) - \partial(g\alpha) - \partial(g\partial g)$$

$$= \partial(f - \alpha g - g\alpha - g^2)$$ in the sense of the product in $U \otimes_A V$

Important is $\quad x\partial y = \partial x\, y \quad$ for $x,y \in U \otimes_A V$

$$\partial(\underbrace{xy}_{0}) = \partial x\, y - x \partial y.$$

Next thing is that given α, g $\quad \partial(g) = 1-\alpha^2$

one can improve g. Set

$$\alpha' = \alpha + \tfrac{1}{2}\underbrace{\alpha(1-\alpha^2)}_{\partial(\alpha \cancel{f})} = \alpha + \partial\left(\overbrace{\frac{\alpha f + f\alpha}{4}}^{g}\right)$$

$$f - \alpha\frac{\alpha f + f\alpha}{4} - \frac{\alpha f + f\alpha}{4}\alpha - \frac{\alpha f + f\alpha}{4}(1-\alpha^2)$$

Wait $\quad \alpha' = \alpha + \frac{1}{2}\alpha(1-\alpha^2)$

$$1-\alpha'^2 = \underbrace{1 - \alpha^2 - \alpha^2(1-\alpha^2)}_{\substack{1-2\alpha^2+\alpha^4 \\ \| \\ (1-\alpha^2)^2}} + \frac{1}{4}\alpha^2(1-\alpha^2)^2$$

$$f - \frac{1}{4}\left\{\alpha(\alpha f + f\alpha) + (\alpha f + f\alpha)\alpha + (\alpha f + f\alpha)\right.$$

---

first point is that $\quad \alpha f - f\alpha \xrightarrow{\partial} \alpha(1-\alpha^2) - (1-\alpha^2)\alpha = 0$

Try changing α to $\quad \alpha' = \alpha + \underbrace{\tfrac{1}{2}\alpha(1-\alpha^2)}_{\partial(\alpha f)}$

$g = \alpha f$

$$f - \alpha g - g\alpha - g\partial g$$

$$= f - \alpha\alpha f - \alpha f\alpha - \alpha f(1-\alpha^2)$$

~~$= f - \alpha^2 f - 2\alpha f\alpha + \alpha f\alpha^3$~~

$$\partial(f - \alpha g - g\alpha - g\partial g) = 1-\alpha^2 - \alpha\partial g - \partial g\,\alpha - \partial g\,\partial g$$

$$= 1 - (\alpha + \partial g)^2$$

101 $\quad \alpha' = \alpha + \frac{1}{2}\alpha(1-\alpha^2) = \alpha + \alpha\,\frac{1-\alpha^2}{2} = \alpha + \alpha\partial\left(\frac{f}{2}\right)$

$\cancel{1 - \alpha'^2 = 1 - (\alpha}$

Want $\partial g = \frac{1}{2}\alpha(1-\alpha^2)$

If so then

First ~~assume~~ replace $U \otimes_R V$ by image $UV \subset$ Mult.
Then $\partial : UV \longrightarrow$ Mult is injective. So

$$\partial(f\alpha) = (1-\alpha^2)\alpha = \alpha(1-\alpha^2) = \partial(\alpha f) \implies \alpha f = f\alpha.$$

~~Part 2 assume f commutes with α^2. I know that $\partial(\alpha f - f\alpha) = 0$ so that $\alpha f - f\alpha$ is killed by any ∂k $k \in U \otimes_A V$, in particular $1-\alpha^2$~~

$f \in I$

~~First change α~~

$$\partial(f^2) = -f\partial f + \partial f f$$

i.e. $\quad f(1-\alpha^2) = (1-\alpha^2)f \qquad f\alpha^2 = \alpha^2 f$

The first case to understand is where $I \subset$ Mult.

$$\alpha(\alpha f + f\alpha) - (\alpha f + f\alpha)\alpha = \alpha^2 f + \alpha f\alpha - \alpha f\alpha - f\alpha^2$$
$$= \alpha^2 f - f\alpha^2 = 0$$

$g = \frac{1}{4}(\alpha f + f\alpha) \qquad \partial(g) = \frac{\alpha(1-\alpha^2) + (1-\alpha^2)\alpha}{4}$

$$= \frac{1}{2}\alpha(1-\alpha^2).$$

and we know $\alpha g = g\alpha$

f gets changed to $f - \alpha g - g\alpha + g\frac{1}{2}\alpha(1-\alpha^2)$

$\alpha^2 f + 2\alpha f\alpha + f\alpha^2$

$$f - \frac{\alpha(\alpha f + \cancel{f\alpha}) + (\cancel{\alpha f} + f\alpha)\alpha}{4} + \underbrace{g\frac{1}{2}\alpha(1-\alpha^2)}_{\text{comes from } I \otimes I}$$

02

~~$f\frac{?}{?}\alpha g - \frac{1}{2}\alpha(1-\alpha^2)g$~~ Begin with $\partial(f) = 1-\alpha^2$

Then $f(1-\alpha^2) = f\partial f = (\partial f)f = (1-\alpha^2)f$

$\therefore$ f commutes with α^2. Replace f by

$$\frac{f+\alpha f\alpha}{2} \qquad\qquad y = x^2$$

$$k[x] \otimes_{k[y]} k[x] \longrightarrow k[x]$$

Form DGA gen. by x in deg 0 y in deg 1

$d(y) = 1-x^2$. truncated $k[x]y\,k[x]y\,k[x] \to k[x]y\,k[x]$

$$d(y f(x) y) = (1-x^2)f(x)y - y f(x)(1-x^2)$$

$$k[x]\otimes k[x] \big/$$

$$k[x] \otimes_{k[1-x^2]} k[x] \longrightarrow k[x] \longrightarrow k[x]/(1-x^2) \to 0$$

$$y \longmapsto 1-x^2$$

Form DGA. $M\otimes_B M \longrightarrow \overline{M} \longrightarrow \overline{R}$

$k[x]y\,k[x]y\,k[x] \longrightarrow k[x]y\,k[x] \longrightarrow k[x]$

where $dy = 1-x^2$. What is homology?

$$d(f_0 y f_1 y f_2) = f_0\underbrace{(1-x^2)f_1}\,y f_2 - f_0 y \underbrace{f_1(1-x^2)}f_2$$

so we obtain $k[x]\otimes k[x] \big/ f_0 g\otimes f_1 - f_0 \otimes g f_1$

for all $g \in (1-x^2)k[x]$.

$k[x] \otimes_{(1-x^2)k[x]} k[x]$ ~~This~~ What is this

~~[illegible]~~ $k[x] \otimes_{k[1-x^2]} k[x]$ is a rank 2 free right $k[x]$-module with basis $1\otimes 1, x\otimes 1$

$$0 \longrightarrow K \longrightarrow k[x] \otimes_{(1-x^2)k[x]} k[x] \longrightarrow (1-x^2)k[x] \longrightarrow 0$$

$f \otimes g \longmapsto f(1-x^2)g$

$(1-x^2)k[x] \cong k[x]$ as $k[x]$-bimod.

$$0 \longrightarrow K \longrightarrow k[x] \otimes_{(1-x^2)k[x]} k[x] \xrightarrow[\text{ring hom.}]{\mu} k[x] \longrightarrow 0$$

$K = \Omega^1_{k[x]/(1-x^2)k[x]}$

no non-comm

~~Clearly~~ K should be generated by $x\otimes 1 - 1\otimes x$ as $k[x]$ bimodule.

Look for $D: k[x] \to M$ a derivation with M a bimodule such that D kills $(1-x^2)k[x]$. D completely determined by Dx

$$(1-x^2)Dx + \underbrace{(D(1-x^2))}_{0}x = \underbrace{D((1-x^2)x)}_{0}$$

similarly $(Dx)(1-x^2) = 0$. So M is spanned by Dx, xDx, $(Dx)x$ ~~Also $D(1-x^2)=$~~

$0 = D(x^2-1) = (Dx)x + xDx$. So it seems that K is spanned by Dx, $xDx = -(Dx)x$. It's apparently related to $\Omega^1(k[F])$. Clear. ~~Relat~~

So where are we now?

Look at $k[x] \otimes_{(1-x^2)k[x]} k[x]$. The product is

$$(f_1 y g_1)(f_2 y g_2) = f_1 y g_1 f_2 (1-x^2) g_2$$

so this is associated the dual pair $(k[x], k[x])$ over $(1-x^2)k[x]$ where $\langle g, f\rangle = g(1-x^2)f$. What is the multiplier ring. $\mathrm{Hom}_{(1-x^2)k[x]}(k[x], k[x])$

Start again: To understand the algebra where you have ~~an odd oper~~ a supermodule equipped with an odd operator x such that $y = 1 - x^2$ is "compact". Better consider a dual pair (U, V) over A and form the DGA

$$U \otimes_A V \xrightarrow{d} Mult(U,V)$$

Study the algebra arising from elements $x \in Mult$, $y \in U \otimes_A V$ sat $d(y) = 1 - x^2$.

Universal case: Look at DGA gen. by x, y of degree $0, 1$ resp satisfying $d(y) = 1 - x^2$. This is

$$\longrightarrow M \otimes_R M \longrightarrow M \longrightarrow R$$

where ~~$M = R \otimes R$~~ $R = k[x]$, $M = R \otimes R = RyR$ and $dy = 1 - x^2$. We want to truncated this to

$$M / d(M \otimes_R M) \longrightarrow R$$

$$d(f_0 \underset{\otimes}{y} f_1 \underset{\otimes}{y} f_2) = f_0 (1-x^2) f_1 \underset{\otimes}{y} f_2 - f_0 \underset{\otimes}{y} f_1 (1-x^2) f_2$$

So you have $R \otimes R \big/ \{ f_0 g \otimes f_2 - f_0 \otimes g f_2$ where $g \in (1-x^2)k[x]$.

Thus our bimodule is $R \otimes_{(1-x^2)R} R$

Actually this arises from the dual pair $(k[x], k[x])$ over $(1-x^2)R$ where the pairing is $\langle g, f \rangle = g(1-x^2)f$

Check.

$$(f_0 y g_0)(f_1 y g_1) = f_0 y \underbrace{g_0 f_1 (1-x^2)}_{\langle g_0, f_1 \rangle} g_1$$

$$\mathrm{Hom}_{(1-x^2)k[x]}(k[x], k[x]) \longrightarrow k[x]$$
$$u \longmapsto u(1)$$
$$(g \mapsto gf) \longleftarrow f$$

This makes $k[x]$ a summand of the right mult. ring. So look at a u such that $u(1)=0$. Then u kills $(1-x^2)k[x]$ so

$$u \in \mathrm{Hom}_{(1-x^2)k[x]}\left(k[x]/(1-x^2)k[x],\ k[x]\right) = 0.$$

Next point calc. $R \otimes_{(1-x^2)R} R$

$$0 \to K \to R \otimes_{(1-x^2)R} R \longrightarrow R \longrightarrow 0$$

$D(f) = f \otimes 1 - 1 \otimes f = fy - yf.$

$D: R \longrightarrow K$ universal deriv. killing $(1-x^2)R$.

So only Dx can be $\neq 0$. We find

$$(x^2-1)Dx + \underset{\substack{\parallel \\ 0}}{D(x^2-1)}x = D((x^2-1)x) = 0$$

$$\therefore\ (x^2-1)Dx = 0 \qquad (Dx)(1-x^2) = 0.$$

At most 4 elts. $Dx \quad xDx \quad (Dx)x \quad xDxx$

$$0 = D(x^2-1) = Dx\,x + xDx \qquad Dx + xDx\,x = 0$$

And we do have such a D because

$$R \longrightarrow \underbrace{R/(1-x^2)R}_{k[F]} \xrightarrow{d} \Omega^1(k[F]).$$

Most element in R commute with y so $xy - yx$ and ~~$y-xyx$~~ are the two elements spanning the kernel.

106 Let's see if you can get anywhere

You have $x \in R$ $\quad d(y) = 1 - x^2$

You can refine x $\quad x' = x + \frac{1}{2}x(1-x^2)$

$$1-(x')^2 = 1 - x^2\left(1 + 1 - x^2 + \frac{1}{4}(1-x^2)^2\right)$$

$$= (1-x^2)^2 \cancel{\tfrac{3}{4}} - \frac{x^2}{4}(1-x^2)^2$$

$$= (1-x^2)^2\left(1 - \frac{x^2}{4}\right)$$

We know the kernel of d contains $xy - yx$ and ~~...~~ $x(xy-yx) = x^2y - xyx$. ~~Kernel~~ ~~of~~

$$\begin{array}{c} xy - yx \mapsto xyx - yx^2 \; \cancel{y - xyx} \\ \downarrow \\ x^2y - xyx = yx^2 - xyx \end{array}$$

So in the end you have this ~~...~~ bimodule of operators $k[x]y\,k[x] \simeq \dfrac{k[x] \otimes k[x]}{(x^2-1)k[x]}$ generated by y.

$$x(xy+yx) = x^2y + xyx$$
$$(xy+yx)x = xyx + yx^2$$

Look at the center of this bimodule all element ξ such that $x\xi = \xi x$.

$$f(x)y + yf(x)$$

example ~~...~~ $\quad f(x)(x^2-1)y$

because $x f(x)(x^2-1)y = y\, \underbrace{x f(x)(x^2-1)} = f(x)(x^2-1)y$.

So anything $\quad f(x)y + yf(x) \quad f(0) = 0$

$x^3y = xyx^2 \qquad x^3y = x(x^2-1)y + xy$

when we write elements down $f(x)y\,g(x)$ you first divide either f, g by x^2-1.
Anything $(x^2-1)f(x)y$ is in the center

$$x(xy-yx) = x^2y - xyx$$

$$\cancel{x(x^2y - xyx) = xyx^2 - yx^3}$$

$$\cancel{(x^2-1)\xi = d(-y)\xi = yd\xi}$$

$$(x^2-1)(xy-yx)$$
$$= y(x^2-1)x - y(x^2-1)x = 0.$$

$$y \longmapsto \overbrace{xy - yx}^{z}$$

$$xy-yx \longmapsto \begin{array}{l} x^2y - xyx \\ -xyx + yx^2 \end{array} = 2(x^2y - xyx)$$

$$z \longmapsto 2xz$$

$$\cancel{xz \longmapsto x^2(xy-yx) - (xy-yx)x^2 = 0.}$$

$$xz = x^2y - xyx$$
$$-zx = yx^2 - xyx$$

$$xz + zx = 0.$$

$$[x, z] = 2xz$$
$$[x, xz] = x[x,z] = x2xz = 2x^2z = 2z$$

$$[x, y] = z$$

$$[x, 2y - xz] = 2z - 2z = 0 \qquad \therefore\ 2y - x(xy-yx) \in \text{center}$$

center maps isom. to $(1-x^2)k[x]$.

basis ~~[illegible]~~

$(1-x^2)k[x]y$ $\quad z$

y, xy $\quad xz$

$$(1-x^2)x^ny \qquad n \geq 0$$
$$xy + yx$$
$$2y - x(xy-yx)$$

Thus it seems I can arrange ~~[illegible]~~ y to commute with x. Replace y by $y - \frac{1}{2}x[xy-yx]$

 10/30/97

$y - \frac{1}{2}x\underbrace{(xy - yx)}_{xz}$ should commute with x.

$$x(xz) = x(x^2y - xyx)$$
$$= \cancel{xyx}\ x^3y - x^2yx$$
$$= xyx^2 - yx^3$$
$$=$$

~~$[x, z] = xz - z$~~

~~$[x, xy - yx] = xz$~~

$$xz = x^2y - xyx = yx^2 - xyx = (yx - xy)x = -zx$$

$$[x, z] = xz - zx = 2xz$$

$$(1-x^2)z = dy\, z$$
$$= +y\, dz = 0.$$

$$[x, \tfrac{1}{2}xz] = \tfrac{1}{2}x[x, z] = \tfrac{1}{2}x\, 2xz = x^2z = z$$

$$[x, \underline{y - \tfrac{1}{2}xz}] = z - z = 0$$

Anyway what happens. Stop.

$x \quad dy = 1 - x^2$

$$x' = x + \tfrac{1}{2}x(1-x^2)$$
$$1 - x'^2 = 1 - x^2\left(1 + 1 - x^2 + \frac{(1-x^2)^2}{4}\right)$$
$$= 1 - 2x^2 + x^4 - \frac{x^2(1-x^2)^2}{4}$$
$$= (1-x^2)^2\left(1 - \frac{x^2}{4}\right)$$

What is y'. $\quad dy = 1 - x^2$ where $[x, y] = 0$

$$1 - x'^2 = dy\, dy\left(1 - \frac{x^2}{4}\right)$$
$$y' = y(1-x^2)\left(1 - \frac{x^2}{4}\right)$$

Basically we need equiv. relation

Kasparov module - generalization of an object of $\mathcal{P}(A^{op})$.

you want (U, V) super dual pair over A

together with $x \in \mathrm{Mult}(U, V)$ $\quad y \in U \otimes_A V$ & $dy = 1 - x^2$

109 Go back to $\mathcal{T}_E = R$. We have two dual pairs over R: (U_i, V_i) $i = 0, 1$.

For $i = 0$. $V_0 = T(E^*)$, $U_0 = T(E)$ direct sum of $E^{*\otimes n}$, $E^{\otimes n}$ for all $n \geq 0$. ~~So we have~~ $U_1 = T(E) \otimes E$, $V_1 = E^* \otimes T(E^*)$ ~~We have this basic~~ and we have

$$T(E) \otimes E \rightleftarrows T(E)$$

isos modulo compacts.

~~Basic idea~~ Over k have (U, V), $R \subset$ the multiplier ring. Get Kasparov bimodule R acting on a Kasparov module over k. Get $K_*(R) \to K_*(k)$. Big problem; understand higher K version.

Also we have homom. $k \to R$.

You want to compute the comp. $K_0(R) \to K_0(k) \to K_0(R)$

This composition is given by a Kasparov product. Basically $U_1 \otimes_k R$, $U_0 \otimes_k R$. This is a pair of R-bimodules. In principle this consists of $T(E) \otimes E \otimes R \oplus T(E) \otimes R$ and a subtle odd "operator" between them. What you want to do is to take $P \in \mathcal{P}(R^{op})$, then form

$$P \otimes_R T(E) \otimes E \oplus P \otimes_R T(E)$$

with some $\overset{?}{\rightleftarrows}$ Then tensor with R

$$P \otimes_R T(E) \otimes E \otimes R \overset{?}{\rightleftarrows} P \otimes_R T(E) \otimes R$$

Then someone you ~~hot~~ homotop the differential to get $(P \otimes_R T(E)) \otimes (E \otimes R \rightleftarrows R)$ ~~Not Problem~~

probably not true because it would give $(P \otimes_R T(E)) \otimes T(E^*)$ which is $P \otimes_R \overset{\text{finite rank}}{(T(E) \otimes T(E^*))}$

110 Is there a way to see what to do.
Somehow you will apply $P \otimes_R -$ to a bimodule $(T(E) \otimes E \rightleftarrows T(E)) \otimes_k R$. There should be some twisting)

$$P \otimes_R \left(T(E) \otimes E \rightleftarrows T(E) \right) \otimes_k R$$

Intuitively this is $P \otimes_R k \otimes_k R$ which should give P by the homotopy $u_t : R \to R$ joining the identity to the "augmentation".

If $P = R$ I must get R

What method? Can you construct a bimodule resolution

$$\to R \otimes E^* \otimes E \otimes R \longrightarrow R \otimes (E \oplus E^*) \otimes R \longrightarrow R \otimes R \longrightarrow R \longrightarrow 0$$

Look at $D : R \longrightarrow M$ derivations determined by $D(s_i)$ $D(s_i^*)$ $\quad D(s_j) s_i^* + s_j D s_i^* = 0$

$$R = T/J$$

$$0 \longrightarrow J/J^2 \longrightarrow R \otimes_T \Omega^1(T) \otimes_T R \longrightarrow \Omega^1(T/J) \longrightarrow 0$$

12/01/97 Let $R = T(E)$, have $R \to k \to R$ homos. and you want to understand $K_0(R) \to K_0(k) \to K_0(R)$. The idea will be to ~~compare with~~ use the bimodule resolution

$$0 \longrightarrow R \otimes E \otimes R \longrightarrow R \otimes R \longrightarrow R \longrightarrow 0$$

The problem is to link $P \otimes_R k \otimes_k R$ with P.

$$0 \longrightarrow P \otimes E \longrightarrow P \longrightarrow P \otimes_R k \longrightarrow 0$$

So

111 12/01/97 cont. $R = T(E)$, have homos $R \to k \to R$, to ~~[illegible]~~ understand $K_0(R) \to K_0(k) \to K_0(R)$, hopefully to see it's 1. Use somehow the bimodule resolution

$$0 \longrightarrow R \otimes E \otimes R \longrightarrow R \otimes R \longrightarrow R \longrightarrow 0$$

which gives a ~~[illegible]~~ functorial resolution

$$0 \longrightarrow P \otimes E \otimes R \longrightarrow P \otimes R \longrightarrow P \longrightarrow 0$$

Easier to look at

Waldhausen free product paper.

$$0 \longrightarrow R \otimes E \otimes M \longrightarrow R \otimes M \longrightarrow M \longrightarrow 0$$

What techniques do you know? filtered rings, projective line, Ranicki papers? Bass FT. ~~Suppose~~ ~~we consider~~ ~~Consider~~ ~~A[~~ ~~Kronecker~~ Kronecker quiver.

Let's revert to ~~old~~ old notation. A unital T variable, consider $R = A[T]$, filtered ring $F_p A[T] = A + AT + \dots + AT^p$ ~~assoc.~~ graded ring $\bigoplus_{p \geq 0} h^p F_p A[T] = A[T_0, T_1]$. ~~Suppose we~~ basis $h^p T^i$ $0 \leq i \leq p$ ~~A~~ $\bigoplus_{p \geq 0} h^p F_p R = A[h, hT]$. Use graded $A[T_0, T_1]$ modules, invert T_0 ~~to~~ to get

$$\bigcup_n \left(\begin{matrix} \text{graded} \\ A[T_0,T_1]/(T_0^n) \\ \text{mods} \end{matrix} \right) \qquad \begin{matrix} \text{graded} \\ A[T_0,T_1] \\ \text{modules} \end{matrix} \qquad \begin{matrix} \text{graded} \\ A[T_0,T_1][T_0^{-1}] \\ \text{modules} \\ \| \\ A[T] \text{ modules} \end{matrix}$$

Try again

I have to understand $K_0(A[T])$. Basic idea goes back to Grothendieck at least in the regular case. Basic idea — take $P \in \mathcal{P}(A[T])$ extend to something over $\mathbb{P}^1_A$ and then? Take $P \otimes_{A[T]} A[T,T^{-1}]$ and extend ~~over~~ to $A[T^{-1}]$. This can do somewhat ~~easily~~ by embedding P as a summand of $A^{\oplus n}$. $(P, P^\vee)$ over $A[T]$. ~~that~~

Given P you ~~form~~ consider the appropriate building of $A[T^{-1}]$ lattices L in $P[T^{-1}] = P \otimes_{A[T]} A[T,T^{-1}]$. P, L together define a "module" over $\mathbb{P}^1_A$.

Let's recall some of the ideas. Start with

$$GL_n(A[T,T^{-1}]) = \cancel{GL_n(M_n(A[T,T^{-1}]))}$$
$$= (M_n A)[T,T^{-1}]^\times$$

what's important is the order in T. There's a linearization procedure to reduce to an invertible $N\times N$ matrix of the form $aT + b$. This defines a vbundle over $\mathbb{P}^1_A$ of the form

$\mathcal{O} \otimes P_0 \oplus \mathcal{O}(-1) \otimes P_1$ roughly where $P_0 \oplus P_1 = A^{\oplus N}$. Why. Suppose $aT+b \in A[T,T^{-1}]$ is invertible

$$A[T^{-1}](aT+b), \quad A[T]$$
$$\left(T^{-1}A[T^{-1}], \quad (aT+b)^{-1}A[T]\right)$$
$$\cup$$
$$A[T]$$

So you find that

$$T^{-1}A[T^{-1}] + (aT+b)^{-1}A[T] = A[T,T^{-1}]$$

Look at $K = T^{-1}A[T^{-1}] \cap (aT+b)^{-1}A[T]$. It is projective over A^{op}.

$$0 \longrightarrow K \longrightarrow \begin{matrix} T^{-1}A[T^{-1}] \\ \oplus \\ (aT+b)^{-1}A[T] \end{matrix} \longrightarrow A[T,T^{-1}] \longrightarrow 0$$

$$K = (aT+b)^{-1}A[T] / A[T] \cong A[T]/(aT+b)A[T]$$

You can also look at

$$(aT+b)^{-1}A[T^{-1}], \quad A[T]$$
$$\Vert$$
$$(a+bT^{-1})^{-1}A[T^{-1}]T^{-1}, \quad A[T]$$
$$\cup$$
$$A[T^{-1}]T^{-1}$$

$$\cancel{T^{-1}A[T^{-1}], \quad (a+bT^{-1}) A[T]}$$
$$\cancel{\cup}$$
$$\cancel{A[T]}$$

$$(a+bT^{-1})^{-1}A[T^{-1}], \quad TA[T]$$
$$\cup$$
$$A[T^{-1}]$$

Basically you are computing $H^0(F)$

$$(a+bT^{-1})^{-1}A[T^{-1}] / A[T^{-1}] = A[T^{-1}] / (a+bT^{-1})A[T^{-1}].$$

14 I don't understand the geometry, but basically you are ~~[illegible]~~ examining a clutching function $aT+b$, i.e. a Kronecker module such that $az+b$ is invertible for $z \neq 0, \infty$, hence you have a torsion sheaf F over $\mathbb{P}^1$ with support at $0, \infty$. Then $A = H^0(F)$ and the splitting of A results from the splitting of the support.

12/02/97 So what next? Higman linearization.

<u>IDEA to be explored later</u>: ~~A nuclear maps~~
a map of R-modules $M \to N$ is nuclear when it is in the image of

$$\mathrm{Hom}_R(M,R) \otimes_R N \longrightarrow \mathrm{Hom}_R(M,N)$$

equivalently, if it factors $M \longrightarrow R^{\oplus n} \longrightarrow N$ for some. In representation theory one introduces a category of G-modules, triangulated category I think, in which maps factoring through a projective G-module are put equal to zero. Analogs of null homotopic maps.

Work out formulas again: Let $U = \{U_1 \to U_0\}$ be a complex of R-modules, such that 1_U is homotopic to an A-nuclear map. Find a heq with a f. proj. complex $T_1 \to T_0$ which is acyclic modulo A. We can choose $h: U_0 \to U_1$ such that $1-hd$ on U_0 and $1-dh$ on U_1 are A-nuclear. The main point is the case $A=R$.
let
$$1-dh = j_0 i_0 : U_0 \xrightarrow{i_0} T_0 \xrightarrow{j_0} U_0$$
$$1-hd = j_1 i_1 : U_1 \longrightarrow T_1 \longrightarrow U_1$$
where T_0, T_1 are finite free modules. Actually

115 you don't want to choose T_1 yet. Note that T_0 maps onto $H_0(U)$ so that if T_1 is the pull-back in

$$\begin{tikzcd} T_1 \arrow[r, "d"] \arrow[d, "j_1"] & T_0 \arrow[d, "j_0"] \\ U_1 \arrow[r, "d"] & U_0 \end{tikzcd}$$

then $(T_1 \to T_0) \xrightarrow{j} (U_1 \to U_0)$ is a quis. In fact j is a heq because its cone is contractible

$$0 \longrightarrow T_1 \underset{\begin{pmatrix} -j_1 \\ d \end{pmatrix}}{\overset{(-i_1 \;\; h^T)}{\rightleftarrows}} \begin{matrix} U_1 \\ \oplus \\ T_0 \end{matrix} \underset{(d \;\; j_0)}{\overset{\begin{pmatrix} h^U \\ i_0 \end{pmatrix}}{\rightleftarrows}} U_0 \longrightarrow 0$$

$$1 \text{ on } T_1 = i_1 j_1 + d h^T$$

$$\begin{pmatrix} -j_1 \\ d \end{pmatrix} (-i_1 \;\; h^T) + \begin{pmatrix} h^U \\ i_0 \end{pmatrix} (d \;\; j_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$j_1 i_1 = 1_{U_1} - h^U d \quad \text{nuclear on } U_1$$

argument

$$1_{T_1} = (-i_1 \;\; h^T)\begin{pmatrix} -j_1 \\ d \end{pmatrix} = (-i_1 \;\; h^T)\begin{pmatrix} 1 - h^U d & -h^U j_0 \\ -i_0 d & 1 - i_0 j_0 \end{pmatrix}\begin{pmatrix} -j_1 \\ d \end{pmatrix}$$

OKAY. ~~YES~~

116. What do I do next? $K_0(A[T]) = K_0(A) \oplus \mathrm{Nil}_0(A)$

Let $P \in \mathcal{P}(A)$ ν nil endo of P

$$A[T] \otimes_A P \xrightarrow{T\otimes 1 - 1\otimes\nu} A[T] \otimes_A P \quad ?$$

Let $P \in \mathcal{P}(A[T])$, let $\bar{P} = P/TP \in \mathcal{P}(A)$
You want to compare P with $A[T] \otimes_A \bar{P}$. Have a surj $P \twoheadrightarrow \bar{P}$ and since $\bar{P}$ proj over A can lift to $\bar{P} \xrightarrow{i} P$ and then you get $A[T] \otimes_A \bar{P} \longrightarrow P$. What to hope for. Over a field you have a vector bundle over the affine line

Need to understand $A[T]$.

I need to understand $A[T]$!

Let's begin with M a f.g. $A[T]$-module. Want to extend to a module over $\mathbb{P}^1_A$. Use $A[[T^{-1}]]$? Take v.b. over $\mathbb{A}^1$ look at building at ∞.

$$M \otimes_{A[T]} A[[T^{-1}]][T]$$

To get ~~started~~ we might use graded module approach. and see what this yields.

Begin with M ~~[illegible]~~ over $A[T]$, choose F_0M a f.g.

$$1 \in \mathrm{Hom}_{A[T]}(M, A[T]) \otimes_{A[T]} M$$

12/03/97. Let's make a systematic attempt to calculate $K_0(A[T])$. List ideas to pursue

- graded modules over $S = A[T_0, T_1]$
- modules over $X = \mathbb{P}^1_A$.
- diagonal class in $K^0(X \times X)$ ~ ~~[illegible]~~ review modules over $\mathbb{P}^1$ and K-modules Kronecker

Here's something worth pursuing. A bimodule resolution like the Koszul complex

$$\longrightarrow R\otimes\Lambda^2V\otimes R \longrightarrow R\otimes V\otimes R \longrightarrow R\otimes R \longrightarrow R \longrightarrow 0$$

in the case $R = S(V)$ yields a ~~functorial~~ functorial projective resolution for any module:

$$\longrightarrow R\otimes V\otimes M \longrightarrow R\otimes M \longrightarrow M \longrightarrow 0$$

Idea of diagonal approximation, Thom form for the tangent bundle, etc.

What is the analog for modules over $X = \mathbb{P}^1$

F coh. module over $\mathbb{P}^1$

$$H^1(F(-1)) = 0 \implies H^1(F(-1)) = H^1(F(0)) = H^1(F(1)) = \cdots 0$$

$$0 \to F(n-1) \to F(n)^{\oplus 2} \longrightarrow F(n+1) \longrightarrow 0$$

$$0 \longrightarrow H^0(F(-1)) \to H^0(F)^{\oplus 2} \longrightarrow H^0(F(1)) \longrightarrow 0$$

$$0 \longrightarrow H^0(F) \to H^0(F(1))^{\oplus 2} \longrightarrow H^0(F(2)) \to 0$$

So $0 \to F' \to \mathcal{O}\otimes H^0(F) \longrightarrow F \to 0$

$$H^0(F') = H^1(F') = 0.$$

$$\cancel{\mathcal{O}\otimes H^0(F'(-1)) \longrightarrow F'(-1)}$$

$$H^1(F') = 0 \implies \mathcal{O}\otimes H^0(F'(1)) \twoheadrightarrow F'(1)$$

So $H^1(F(-1)) = 0 \implies 0 \to \mathcal{O}(-1)\otimes H^0(F(-1)) \to \mathcal{O}\otimes H^0(F) \longrightarrow F \to 0$

$$F = \left\{ \mathcal{O}(-1)\otimes R\Gamma(\underbrace{F(-1)}) \longrightarrow \mathcal{O}\otimes R\Gamma(F) \right\}$$

vaguely really involves $\Lambda^2 V$.

 Tilting object $\mathcal{O} \oplus \mathcal{O}(-1) = T$

has $R^i\mathrm{Hom}(T,T) = \begin{cases} 0 & i \neq 0 \\ \begin{pmatrix} k & V \\ 0 & k \end{pmatrix} \end{cases}$ $\quad V = \Gamma(\mathcal{O}(1))$

$\mathcal{E}nd_{\mathcal{O}}(T) = \begin{pmatrix} \mathcal{O} & \mathcal{O}(1) \\ \mathcal{O}(-1) & \mathcal{O} \end{pmatrix}$

Get equivalence of Derived Cats.

Idea: Start with a vb over $A[T]$

~~extend~~ regard as quasi-coherent sheaf on $\mathbb{P}^1_A$ over $A[T^{-1}]$

It is flat so it is a filtered ind. limit of f.g. free modules.

Let's be naive. Suppose we have M an $A[T]$-mod

and

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[rr, bend right, "1"'] & A[T]^{\oplus n} \arrow[r, "f"] & M \end{tikzcd}$$

$$M^\vee \xleftarrow{\ t_i\ } A[T]^{\oplus n} \xleftarrow{\ t_f\ } M^\vee$$

M an $A[T]$-module $\quad R = A[T]$.

Let $F_0 M$ be a f.g. A-submodule of M generating M.

set $F_p M = F_0 + TF_0 + \cdots + T^p F_0$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & A[T]^n & \longrightarrow & M & \longrightarrow & 0 \\ & & \cup & & \cup & & \cup & & \\ 0 & \longrightarrow & N \cap F_p A[T]^n & \longrightarrow & F_p A[T]^n & \twoheadrightarrow & F_p M & \longrightarrow & 0 \end{array}$$

Want finitely presented

$$0 \longrightarrow M \longrightarrow A[T]^{n_1} \longrightarrow A[T]^{n_0} \longrightarrow M \longrightarrow 0$$

Filtered mess.

119 Problem: To understand $K_*(A[t])$.

I think you want to start with the FT of Bass about $K_1(A[t,t^{-1}])$, as this is closer to the geometry. The point is ~~...~~ to use modules over $\mathbb{P}^1_A$ where you have canonical resolutions & finiteness. An element $g \in GL_n(A[t,t^{-1}])$ is a clutching function for a v.b. E over $\mathbb{P}^1_A$. On the other hand twisting $E \rightsquigarrow E(n)$ yields a v.b. with canonical resolution

$$\to \mathcal{O}(-1) \otimes_A \underbrace{H^0(E(n-1))} \to \mathcal{O} \otimes_A \underbrace{H^0(E(n))} \to E(n) \to 0.$$

where $\in \mathcal{P}(A)$.

Now translate into Čech matrix calculations.

$$A[t] \xrightarrow{u} A[t,t^{-1}] \xleftarrow{v} A[t^{-1}]$$

$$A[t]^n \qquad\qquad A[t^{-1}]^n$$

$$\mathcal{E} = (E_+, E_-, u_*E_+ \simeq v_*E_-)$$

Work inside $A[t,t^{-1}]^n$ with $A[t]^n$, and $g^{-1}(A[t^{-1}]^n)$

$$\Gamma = \{ \underset{\substack{\deg\\ z\neq 0}}{f_+}, \underset{\substack{\deg\\ z\neq 0}}{f_-} \mid g(f_+) = f_- \}.$$

~~Is it possible to find~~

Let's go over Davydov again.

R unital $\quad e$ idempotent in R

$$\begin{tikzcd} 0 \arrow[r] & ReR \arrow[r] & R \arrow[r] & R/ReR \arrow[r] & 0 \\ & Re \otimes_{eRe} eR \arrow[u] & & e^\perp Re^\perp / e^\perp ReRe^\perp \arrow[u, phantom, "\cong"] & \end{tikzcd}$$

Assumptions equiv.
$$Re \otimes_{eRe} eR \xrightarrow{\sim} ReR$$
$$e^\perp Re \otimes_{eRe} eRe^\perp \xrightarrow{\sim} e^\perp Re Re^\perp$$

$$Re \in \mathcal{P}(eRe^{op})$$

Why is this true? [redacted]

But wait: Take $R = \mathcal{T}_E \qquad \dim(E) = 1$
$$= k\langle z, z^* \rangle / \; z^* z = 1$$

$e = 1 - zz^* \qquad R/ReR = k[z, z^{-1}]$.

Now ReR should ~~~~ satisfy excision

Thus you get

$$\begin{tikzcd} & K_1(k) \arrow[d] & \\ K_1(k) \arrow[r, "0?"] & K_1(R) \arrow[r] & K_1(k[z, z^{-1}]) \arrow[dll, bend left] \\ K_0(k) \arrow[r, "0?"] & K_0(R) \arrow[r] & K_0(k[z, z^{-1}]) \end{tikzcd}$$

Here k can be any unital ring.

Use that $K_1(k[z, z^{-1}]) = K_1(k) \oplus N_1(k) \oplus N_1(k) \oplus K_0(k)$

Thus $\quad K_*(R) = K_*(k[z]) \oplus_{K_*(k)} K_*(k[z^{-1}])$

What is $\quad e^\perp Re^\perp \simeq e^\perp T(E) \otimes T(E^*) e^\perp = \bar{T}(E) \otimes \bar{T}(E^*)$

$e^\perp = \sum z_i z_i^*$ kills 1. reproduces rest.

$e^\perp Re = \bar{T}(E)$

so look at basis $S_\alpha S_\beta^*$ for β.

$$e^\perp s_\alpha s_\beta^* e^\perp = s_\alpha s_\beta^*$$

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$$e^\perp s_\alpha = \begin{cases} 0 & \text{if } |\alpha| = 0 \\ s_\alpha & \text{if } |\alpha| > 0 \end{cases}$$

$\therefore \quad e^\perp s_\alpha s_\beta^* e^\perp = s_\alpha s_\beta^*$ if $|\alpha|, |\beta|$ both ≥ 1.

$e^\perp s_\alpha e^\perp = s_\alpha e^\perp \qquad |\alpha| \geq 1$

$\sum s_i s_i^* s_\alpha = s_\alpha$

$s_\alpha e^\perp = \sum s_\alpha s_i s_i^*$

$e^\perp$

$s_\alpha e^\perp \qquad e^\perp s_\beta^*$

$s_\alpha s_\beta^*$

$s_{i_1} e^\perp s_{i_2} e^\perp \cdots s_{i_n} e^\perp$

$= s_{i_1} \cdots s_{i_n} e^\perp$

Consider $s_i e^\perp \quad e^\perp s_j^*$ satisfy same relations

$e^\perp s_j^* s_i e^\perp = \delta_{ji} e^\perp$.

$s_i e^\perp s_j^* = s_i \sum_k s_k s_k^* s_j$

$e^\perp = ss^* \qquad e^\perp R e^\perp$

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ss^*

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$\shortparallel$

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