1 Sunday Oct 28 0920

Consider stability. You must do something here.
F infinite field. Basic stability is achieved by letting G_n acts on a certain complex.

~~Ideas: Let us consider sytel~~

The first thing is the simplicial cx of ind subsets. vector + dual vector

Wait. V vector space. Consider

First ex. F char $\neq$ ~~0~~, ~~H_* coeffs $\mathbb{Q}$.~~ coeffs ~~prime to~~ ch. Q

X = s.complex of ind. subsets of V^n

Make G_n act on chains ~~of~~ on X.

~~Not much~~

$$X_p^n = \mathbb{Z}\left[G_n / \begin{pmatrix} \Sigma_p & * \\ 0 & G_{n-p} \end{pmatrix}\right] \otimes \mathbb{Z}^{sgn}$$

$$H_*(G_n, X_p^n) = H_*\left(\begin{pmatrix} \Sigma_p & * \\ 0 & G_{n-p} \end{pmatrix}, \mathbb{Z}^{sgn}\right)$$

$$= H_*\left(\begin{pmatrix} \Sigma_p & 0 \\ 0 & G_{n-p} \end{pmatrix}, \mathbb{Z}^{sgn}\right)$$

$$= \underbrace{H_*(\Sigma_p, \mathbb{Z}^{sgn})}_{0 \ \ p \geq 2} \otimes H_*(G_{n-p})$$

$$\begin{array}{cccccccc}
\cdot & \cdot & & & & & & \\
H_{n-1}(G_n) \leftarrow H_{n-1}(G_{n-1}) & & 0 & 0 & 0 & & & \\
\cdot \quad \cdot & \cdot & 0 & 0 & 0 & & & \\
H_1(G_n) & H_1(G_{n-1}) & 0 & 0 & 0 & H_1(G_n, I_{n+1}^n) & & \\
H_0(G_n) & H_0(G_{n-1}) & 0 & 0 & 0 & H_0(G_n, I_{n+1}^n) & 0 & \cdot \\
\hline
& 1 & & & & n+1 & & \\
X_0^n & X_1^n & & & & X_n^n & I_{n+1}^n &
\end{array}$$

What range does this give?

You find

$$H_n(G_{n-1}) \longrightarrow H_n(G_n) \to H_0(G_n, I^n_{n+1}) \longrightarrow H_{n-1}(G_{n-1}) \longrightarrow H_{n-1}(G_n) \longrightarrow 0$$

Now we have generators for $H_0(G_n, I^n_{n+1})$. I think Suslin computes this as K^M_* symbols.

$$H_2(G_1) \to H_2(G_2) \to H_0(G_2, I^2_3) \to H_1(G_1) \to H_1(G_2) \to 0$$

The basic conclusion is that ~~are~~ I can't do better than

$$H_n(G_{n-1}) \twoheadrightarrow H_{n-1}(G_n) \xrightarrow{\sim} H_{n-1}(G_{n+1}) \xrightarrow{\sim}$$

together with some generators for the kernel of. But maybe there's something secondary.

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other ideas: $\bigoplus_{n \geq 0} H_*(G_n) = H_*(\underbrace{\coprod BG_n}_{M})$

$$\cdots M \times M \times M \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; M \times M \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; M \rightrightarrows pt$$

get bar construction for $H_*(M) = R$. any alg.

M monoid, has class. space $BM =$ geom. real of above s. space. Spec. sequence

$$E^2_{pq} = \mathrm{Tor}^{H_*(M)}_p(\mathbb{Z}, \mathbb{Z})_q \Longrightarrow H_*(BM).$$

Tor_1 should be $\bar{R}/\bar{R}^2$ ind. $\bar{H}_*(M) / \bar{H}_*(M)\bar{H}_*(M)$.

 Something is happening here I don't understand.

$H_*(B^{st}_3)$ ~~So what can we do?~~

$H_*(BG_1)$ Somehow stability is not being discussed.

However you know that ~~[illegible]~~ the Steinberg homology ~~is~~

10/27 D545 Problem, to relate stability for the Q-category to the desired stability for GL. List possible approaches ~~[illegible]~~

Try to put things into words.

Idea: Recall you tried to make a cat ~~[illegible]~~ containing the groups $\mathrm{Aut}(P)$

~~The~~ objects P

1600 discuss various stability ideas.

maybe you want instead ~~[illegible]~~ a good model for ΩBQ.

go back to a field.

Q cat: vector spaces, maps subquotients.

$V \underset{Q}{\dashrightarrow} V'$ is an isom $V \simeq V_2/V_1$ where $0 \subset V_1 \subset V_2 \subset V'$.

This cat has ~~one~~ one object for each $n \geq 0$.

Filtration F_pQ $p \geq 0$

$$\begin{array}{ccc} EGL_p \times^{GL_p} X_p & \longrightarrow & BGL_p \\ \downarrow & & \downarrow \\ F_{p-1}Q & \subset & F_pQ \end{array}$$

X_p = poset of proper subquotients of V_p. = cone on building.

You can ask what this stability means.

Homology should ~~[illegible]~~

~~[illegible]~~ $H_i(F_p, F_{p-1}) =$

$$H_i(F_p, F_{p-1}) = H_i(\text{Thom space})$$

$$X_p \; = \Sigma \text{ Tits building on } V_p$$
$$= \vee S^{p-2}$$

$$X_p = \vee S^{p-1}$$

so it seems that F_p/F_{p-1} begins in degree p.

$$\boxed{H_i(F_p, F_{p-1}) = H_{i-p}(GL_p, J_p) \quad ?}$$

Check: $F_0 = pt$ $\quad F_1 = \Sigma BG_1$

$$H_i(F_2, F_1) = H_{i-2}(G_2, J_2)$$

what is J_2? involves $\mathbb{P}^1(F)$ all lines in V_2. ~~So what can I do ab~~

10/28/97 1210 Can you do anything at all about stability? Stability

I have all these ideas which don't seem very good. Let's return to Mor inv.

$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ everything firm.

P right A-flat $\Longleftrightarrow$

$\left(\begin{array}{l} P \text{ is } A^{op} \text{ flat} \Longleftrightarrow P \otimes_A Q = B \text{ is } B^{op} \text{ flat} \\ Q \text{ is } B^{op} \text{ flat} \Longleftrightarrow Q \otimes_B P = A \text{ is } A^{op} \text{ flat.} \end{array} \right.$

So entries are right flat. Then you get $K_*(A) \simeq K_*(B)$.

Now one thing I learned about is mag

5 But you probably missed something critical. Idea. Given B idemp. you can choose $P \twoheadrightarrow B$ B-module map P firm fl $/B$.

$$(B, P) \twoheadrightarrow (B, B) \qquad \begin{matrix} P & \\ \| & \\ A = B\otimes_B P & B \\ P & B \end{matrix}$$

$$A = B \otimes_B P$$

So what am I doing? Ans. You've forgotten. ~~But it's clear~~ The point is that any idemp. B is naturally a right flat one. Namely choose $Q \xrightarrow{f} B$ of B^{op}-mods. Q B^{op}-flat firm. Then $(Q, B) \xrightarrow{(f,1)} (B, B)$

so $A = Q \otimes_B B \Rightarrow Q$

$$\begin{matrix} A = Q & Q \\ B & B \end{matrix}$$

Q is B^{op} flat $\Leftarrow A = Q \otimes_B B = Q$ is A^{op} flat

~~So typically we end up with a coherent sheaf.~~

Thus if we choose simp. B^{op}-res.

$$\cdots \substack{\to\\\to\\\to} A_1 \rightrightarrows A_0 \to B$$

~~Then~~ where A_i is B^{op}-flat firm, then apply BGL

10/30 Let's consider this argument, go over it again. $B = B^2$. Construct simplicial B-module over B.

$$\substack{\to\\\to\\\to} \quad A_1 \rightrightarrows A_0$$

$$B \quad \substack{\to\\\to\\\to} \quad B \rightrightarrows B$$

We want each A_p to be B-flat. ~~Key idea~~ Idea here is that if E is a firm B-module equipped with $E \twoheadrightarrow B$, then get dual pair (B, E) with $B \otimes_B E = E$.

6. Can examine

$$0 \longrightarrow M \longrightarrow E \xrightarrow{f} B \longrightarrow 0$$

extension such that

product: $B \otimes_B E$ $\qquad (b_1 \otimes e_1)(b_2 \otimes e_2) = b_1 \otimes \underset{\overset{\|}{f(e_1)b_2}}{\langle e_1, b_2\rangle} e_2$

$$(b_1 \otimes e_1)(b_2 \otimes e_2) = b_1 \otimes f(e_1)\, b_2 e_2$$
$$= b_1 f(e_1) \otimes b_2 e_2 \qquad ?$$

Point: $e_1 \cdot e_2 = f(e_1) e_2$ defines a ring structure

~~What~~ Recall $M \xrightarrow{f} B$ B-bimod. map

get dialg. $\quad m_1\ m_2 \qquad \begin{matrix} f(m_1) m_2 \\ m_1 f(m_2) \end{matrix}$

~~Multiplier algebra!!~~

so it's clear.

~~What the hell happened~~

~~So it's clear.~~ NO.

$$(e_1 e_2) e_3 = f(f(e_1) e_2) e_3$$
$$e_1 (e_2 e_3) = f(e_1) f(e_2) e_3$$

Point is that $e_1 e_2 = f(e_1) e_2$ says that $\mathrm{Ker}(f) \cdot E = 0$, so the ~~ring hom.~~ extension

$$0 \longrightarrow M \longrightarrow E \longrightarrow B \longrightarrow 0$$

has $M^2 = 0$ and $MB = 0$.

~~The following is true~~.

Now look at ~~GL~~

$$1 \longrightarrow M_n(I) \longrightarrow GL_n(E) \longrightarrow GL_n(B) \longrightarrow 1$$

the action of $GL(B)$ on $M(I)$ is obvious ~~first left mult. The~~ conjugation action | right mult. is trivial. Try again you idiot.

10/31/97 1205

~~Let~~ I want to carefully go over Morita inv. for K_* to see if I can also derive Suslin's result about the obstruction to excision.

maybe also his obstruction to stability.

~~Ideas~~ try to find a good model for BGL^+

new idea is non unital framework

which suggests ways to glue GL and affine groups together.

~~Key~~ basic data. You seek a space X ~~receiving~~ together with maps $BGL_n(A) \longrightarrow X$ for all n, also with homotopies.

$$\begin{tikzcd}
B\left(\begin{array}{c|c} GL_n & * \\ \hline 0 & 1_m \end{array}\right) \arrow[r, hook] \arrow[d] & BGL_{n+m} \arrow[d] \\
BGL_n \arrow[r] \arrow[ur, phantom, "\Rightarrow"] & X
\end{tikzcd}$$

also for

$$\begin{tikzcd}
B\left(\begin{array}{c|c} GL_n & 0 \\ \hline * & 1_m \end{array}\right) \arrow[r] \arrow[d, hook] & BG_n \arrow[d] \\
BGL_{n+m} \arrow[r] \arrow[ur, phantom, "\Rightarrow"] & X
\end{tikzcd}$$

Specifically you consider [redacted] $P \xrightarrow{u} P'$

$$\begin{tikzcd} P\otimes_A P'^* \arrow[r, "u\otimes 1"] \arrow[d, "1\otimes u^t"'] & P'\otimes_A P'^* \\ P\otimes_A P^* & \end{tikzcd}$$

two cases are
$u: P \hookrightarrow P'$
$u: P \twoheadrightarrow P'$

inj. case $\tilde{A}^n \hookrightarrow \tilde{A}^{n+m} \twoheadrightarrow \tilde{A}^m$ ~~so that~~

surj. case $\tilde{A}^m \hookrightarrow \tilde{A}^{n+m} \twoheadrightarrow \tilde{A}^n$

~~The~~ One problem is how to assemble these suitably. ~~[illegible]~~ I don't see how to do this. ~~But nothing works~~.

$$Gr(V) = \coprod_p Gr_p(V)$$

I seem to recall the homology of this is an exterior algebra. ~~[illegible]~~ on $H_*(Gr_1(V)) = H_*(\mathbb{P}(V))$. What was the idea? Splitting principle. Look for a correspondence.

$$\begin{tikzcd} & D_{p-1,p} \arrow[dl] \arrow[dr] & \\ Gr_{p-1} & & Gr_p \end{tikzcd}$$

both are proj space bundles.

~~[illegible]~~ Take V large so that $Gr_p(V) = BU_p$, then the flag manifold $D_{1,2,\ldots,p}(V) = (BU_1)^p$

so H

9. Critical idea is to link Gr_{p-1}, Gr_p via the corresp $Gr_{p-1,p}$ on which lives a canonical line bundle. ~~Thus~~ This gives

$$H^*(BU_1) \longrightarrow H^*(Gr_{p-1,p})$$

yielding
$$H^*(BU_1) \to Hom(H^*(Gr_p), H^*(Gr_{p-1}))$$
$$\longrightarrow \qquad {}_{p-1} \qquad {}_{p}$$

Check the Clifford relations

$$H^*(BU_1) \otimes H^*(Gr_p) \longrightarrow H^*(Gr_{p,p-1}) \longrightarrow H^*(Gr_{p-1})$$

not so obvious.

Go back to analyzing ~~sq zero~~ a ring ~~an~~ extension

$$0 \longrightarrow M \longrightarrow A \longrightarrow B \longrightarrow 0$$

where $MA = 0$, i.e. $M^2 = 0$ and $MB = 0$. Then ~~A~~ is a B-module. You want to show that if A, B ~~is~~ are B-flat, then $H_*(BGL(A)) \xrightarrow{\sim} H_*(BGL(B))$.

This is a special case of your general Morita invariance for flat rings. Somehow you want to rearrange the arguments cleanly. See what's involved.

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~~http://www.246.ne.jp/~angel/mathind.jpg~~

$$\phi \to GL(M) \to GL(A) \to GL(B) \to 1$$

$$E^2_{pq} = H_p(GL(B), H_q(GL(M)))$$

already $H_0(GL(B), GL(M)) = 0$ is needed

In fact this should be $\cancel{M/[B,M]}$. M/BM

$$\cancel{(1+b)(1+m) - (1+m) = b+m-1+bm}$$

$$\underbrace{(1+b)(\cancel{1}+m)(1+b)^{-1}} - (\cancel{1}+m)$$

$$= (1+b)m - m = bm$$

So you have ~~some trouble~~ immediate problems unless $BM = M$.

Next:

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suppose $0 \to I \to P \to B \to 0$ s.e.s of B-modules

P firm: $B \otimes_B P \cong P$ ~~and then I want~~

(B, P) form dual pair over B $\quad A = B \otimes_B P = P$

~~Assume~~ $\begin{pmatrix} A & Q = B \\ P = A & B \end{pmatrix}$ $\quad$ B is B^{op}-fl $\Leftrightarrow$ $B \otimes_B P = A$ is A^{op}-fl

why? Suppose $B = \varinjlim Q_i$ $\quad$ Q_i filtered ind system $\tilde{B}^{n_i}$ transition maps of left mult by matrices in B.

Then $A = B \otimes_B P$
$= \varinjlim (Q_i \otimes_B P) \simeq \tilde{B}^{n_i} \otimes_B P = A^{n_i}$ left mult by matrices in A.

11. Take B right flat. Let P be a ~~right~~ left B module, $f : P \to B$ a left B-mod. map make P a ring by $pp' = f(p)p'$. Examine

$$GL(P) \longrightarrow GL(B)$$

Write $B = \varinjlim Q_i \qquad Q_i \simeq \tilde{B}^{n_i}$ right B-modules, transitions given by left mult by matrices over B. Each Q_i is a ring and $Q_i \to B$ is a homom.

$$\begin{tikzcd} GL(\overset{A}{\overset{\shortparallel}{P}}) \arrow[r] & GL(B) \\ GL(\underset{\tilde{A}^{n_i}}{\underset{\shortparallel}{Q_i \otimes_B P}}) \arrow[r] \arrow[u] & GL(\underset{\tilde{B}^{n_i}}{\underset{\shortparallel}{Q_i \otimes_B B}}) \arrow[u] \end{tikzcd}$$

See if there's hope here. You have the sq. zero extension $I \hookrightarrow A \overset{f}{\twoheadrightarrow} B$ where $IA = 0$ so that A is a B-module. You assume B is B^{op}-flat, then $B = \varinjlim Q_i \qquad Q_i = \tilde{B}^{n_i}$ and transitions are given by left mult by matrices over B? ~~Let $Q_i \simeq \tilde{A}^{n_i}$~~

$B = \varinjlim Q_i \qquad Q_i = \tilde{B}^{n_i}$ trans. are left mult. by mat.
right modules.

$Q_i \otimes_B A = A^{n_i}$ right mod. trans. given by mat. over B. Consider all possible lifts of these matrices to A.

$$\tilde{A}^n \longrightarrow \tilde{A}^p$$

$$\tilde{B}^n \qquad \tilde{B}^p$$

12.

$$\begin{tikzcd}
\tilde{A}^n \arrow[r] \arrow[dr, "0"'] & \tilde{A}^p \arrow[d] & \\
& A \arrow[r, equal] & A
\end{tikzcd}$$

If A is A^{op} flat then $A \otimes_{\tilde{A}} \tilde{B} = A \otimes_{\tilde{A}} (\tilde{A}/I)$
$= A/AI$
is B^{op}-flat

$$\begin{tikzcd}
\tilde{B}^n \arrow[r] \arrow[dr, "0"'] & \tilde{B}^p \arrow[r] \arrow[d] & \tilde{B}^q \arrow[d] \\
& B \arrow[r, equal] & B
\end{tikzcd}$$

$0 \to I \to \tilde{A} \to \tilde{B} \to 0$

$0 \to A \otimes_A I \to A \to A \otimes_A \tilde{B} \to 0$

$0 \to A \otimes_A I \to A \otimes_A A \to A \otimes_A B \to 0$

Start with B firm, choose $A \twoheadrightarrow B$ map of firm B-modules. Then (B, A) is a firm dual pair over B so what about I?

$$\begin{pmatrix} A = B \otimes_B A & B \\ A & B \end{pmatrix}$$

Say A is B-flat firm.

First case. Start with B idempotent, choose a B-mod surj. $A \xrightarrow{f} B$ with A flat firm B-mod. Let I = Kernel. Is I firm over B

$$0 \to I \to A \to B \to 0$$

$$\mathrm{Tor}_n^{\tilde{B}}(\mathbb{Z}, B) \xrightarrow{\sim} \mathrm{Tor}_{n-1}^{\tilde{B}}(\mathbb{Z}, I)$$

$$0 \to \underline{\mathrm{Tor}_1^{\tilde{B}}(\mathbb{Z}, B)} \to \mathbb{Z} \otimes_B I \to \mathbb{Z} \otimes_B A \to \mathbb{Z} \otimes_B B \to 0$$

$$\mathbb{Z} \otimes_B I = I/BI, \quad \mathbb{Z} \otimes_B A = A/BA, \quad \mathbb{Z} \otimes_B B = B/B^2$$

$$0 \to B \to \tilde{B} \to \mathbb{Z} \to 0$$

$$0 \to \mathrm{Tor}_1^{\tilde{B}}(\mathbb{Z}, B) \to B \otimes_B B \to B \to B/B^2 \to 0$$

So if B is B^{op}-flat, then I is h-unitary over B.

If B right flat

13 ~~[illegible]~~ Assume A right flat.

$$0 \to A \otimes_A I \to \underset{\substack{\| \\ A}}{A \otimes_A A} \to \underset{\substack{\| \\ A/AI}}{A \otimes_A B} \to 0$$

$$I \otimes_A I \to A \otimes_A I \to B \otimes_A I \to 0$$

Check it. If B right flat, then $B \otimes_B B \xrightarrow{\sim} B$ so $\operatorname{Tor}_1^{\tilde{B}}(\mathbb{Z}, B) = 0$. So

$$0 \to I \to A \to B \to 0$$

$$\underset{\substack{\| \\ 0}}{\operatorname{Tor}_1^{\tilde{B}}(\mathbb{Z}, B)} \to \underset{\substack{\| \\ I/BI}}{\mathbb{Z} \otimes_B I} \to \underset{\substack{A/BA \\ \| \\ 0}}{\mathbb{Z} \otimes_B A} \to \underset{\substack{\| \\ 0}}{\mathbb{Z} \otimes_B B} \to 0.$$

$\therefore \; I = BI = AI$ so

$A \otimes_{\tilde{A}} \tilde{B}$

Better. Start with $B = B^2$ choose $0 \to I \to A \to B \to 0$ ~~[illegible]~~ B-modules with A firm. Then $I = BI \Leftrightarrow B$ firm. Assume this. ~~[illegible]~~ Assume A rt flat. Then $A \otimes_A \tilde{B}$ is right B-flat. But

$$0 \to I \to \tilde{A} \to \tilde{B} \to 0$$

$$0 \to A \otimes_A I \to A \otimes_A \tilde{A} \to A \otimes_A \tilde{B} \to 0$$

$$\underset{\substack{\| \\ BI = I}}{AI} \to \overset{\substack{\| }}{A} \to \overset{\substack{\| }}{A \otimes_A \tilde{B}}$$

$\therefore \; B = A \otimes_A \tilde{B}$ is right flat.

14. Now go back to ~~[illegible]~~ the limit.

If A is flat, then $A = \varinjlim E_i$ $\qquad E_i := \tilde{A}^{n_i}$
~~[illegible]~~ transitions are *matrices* over A.

$$\begin{tikzcd} & A^q \arrow[r, phantom] & A^r \\ \tilde{A}^p \arrow[ur, "\alpha."] \arrow[r] & \tilde{A}^q \arrow[u, phantom, sloped, "\subset"] \arrow[ur, "\alpha."] \arrow[r] & \tilde{A}^r \arrow[u, phantom, sloped, "\subset"] \end{tikzcd}$$

$$I^p \xrightarrow{\alpha.} I^q \xrightarrow{\alpha'.} I^r$$

Somehow I need to relate $GL(A)$ $\qquad GL(B)$

$$GL(A) = \varinjlim GL(A^{n_i})$$

You have $GL(B)$ acting on $H_*(M(I))$
and you want the homology to be zero.

You have to get some picture of ~~$GL(B)$~~
$H_*(GL(B), H_*(M(I)))$ where I is a B-mod.
Use cyclic homology $H_*(gl(B))$,
You know the answer leads to $\mathbb{Z} \overset{L}{\otimes}_{\tilde{B}} I$.
When ~~[illegible]~~ this is zero? : I h-unitary

Basic result to prove in general, namely
that $H_*(GL(B), H_{>0}(M(I))) = 0 \iff \mathbb{Z} \overset{L}{\otimes}_{\tilde{B}} I = 0$.
So

15. 11/01/97 0545

Yesterday while pacing at the end I recalled the gap between what I know about Minv. and Suslin's excision thm. I work with K_* $K_*(A) = \mathrm{Ker}\{K_*(\tilde{A}) \to K_*(\mathbb{Z})\}$, or with the fibre of $BGL(\tilde{A})^+ \to BGL(\mathbb{Z})^+$, for which I know the add. result. Suslin somehow understands the map $BGL(A) \to$ this fibre. This became clear when thinking about the Lie analogue, namely, one uses the action of $\mathfrak{gl}(k)$ on $\mathfrak{gl}(\tilde{A})$. Taking $\mathfrak{gl}(k)$ invariants is like the + construction.

The cyclic complex arises from $C.(\mathfrak{gl}(\tilde{A}), \mathfrak{gl}(k)) = \Lambda(\mathfrak{gl}(\tilde{A})/\mathfrak{gl}(k))_{\mathfrak{gl}(k)}$. Question: When ~~does~~ this gives the same as $\Lambda \mathfrak{gl}(A)$. You need to have $\mathfrak{gl}(k)$ acting trivially on the homology of $\mathfrak{gl}(A)$. This is the excision question. How did I understand this? Wodzicki's filtration of $C.(\tilde{A})$ corresp. to $F_0\tilde{A} = k$, $F_1\tilde{A} = \tilde{A}$

0935 Do cyclic version. Basic theorem is

$$C.(\mathfrak{gl}(A))_{\mathfrak{gl}(k)} = S\{C.(A)\}$$

Because $\mathfrak{gl}(k)$ is reductive one knows (Koszul thesis) homology behavior like a filtration

$$* \qquad C.(\mathfrak{gl}(k)) \to C.(\mathfrak{gl}(\tilde{A})) \to C.(\mathfrak{gl}(A))_{\mathfrak{gl}(k)}.$$

for $\mathfrak{h}$ reductive $\subset \mathfrak{g}$ I think one has

$$\Lambda \mathfrak{h}^* \leftarrow \Lambda \mathfrak{g}^* \leftarrow \Lambda(\mathfrak{g}/\mathfrak{h})^{*\mathfrak{h}}$$

forms on H — form on G — basic forms = forms on G/H.

16. Pass to primitives to get Δ

$$C.(k) \longrightarrow C.(\tilde{A}) \longrightarrow C.(A)$$

~~So it seems that arrows go wrong.~~

Point is $C.(A) = \mathrm{Prim}\; C.(gl(A))_{gl(k)}$.

But you might be interested in ~~C(A)~~ $C.(gl(A))$ (Hanlon). Approach this by

$$\big(C.(gl(k)) \otimes C.(gl(A))\big)_{gl(k)} = C.(gl(\tilde{A}))_{gl(k)}$$

You want ~~give~~ to split $C.(gl(A))$ according to the irred repn of $gl(k)$. Roughly amounts to

$$\big(gl(k)^{\otimes p} \otimes C.(gl(A))\big)_{gl(k)}.$$

Hanlon's theory says ~~you form glch rep~~ you look at $C(\tilde{A})$

Important is $C.(A) = \mathrm{Prim}\{C.(gl(A))_{gl(k)}\}$

~~relative homology for $gl(k) \subset gl(\tilde{A})$~~

puzzle about $GL(\mathbb{Z}) \to GL(R)$ $\quad$ R unital. No.

So $C.(A)$ is the analogue of $K_*(A) = K_*(\tilde{A})/K_*(\mathbb{Z})$. Now what am I going to do? Think. Ask about M inv.

In particular $\quad 0 \to I \to A \to B \to 0 \quad IA = 0.$

$$\mathrm{gr}\, C.(A) = C.(\overset{\circ}{I} \oplus \overset{1}{B})$$ semi-direct product?

something funny is happening. Treat as an extension

~~http://personal.fidelity.com/fidbin/focus_today? 21/05/97~~

so that we resolve $C(B)$ by $C(A \oplus I[1])$

17. So consider $0 \to I \to A \to B \to 0$

Can take I adic filt of A: $\quad A \supset I \supset 0$

Then

$$\mathrm{gr}\, C_\cdot(A) = C_\cdot(B \oplus I)$$

$$= C_\cdot(B) \oplus \underset{\substack{\parallel \\ k \overset{L}{\underset{B}{\otimes}} I}}{I \overset{L}{\underset{B}{\otimes}}} \oplus \underset{\substack{\parallel \\ \left(k \overset{L}{\underset{B}{\otimes}} I\right)^{\otimes 2}_\lambda}}{\left(I \overset{L}{\underset{B}{\otimes}} I \overset{L}{\underset{B}{\otimes}}\right)_\lambda} \oplus \cdots$$

So it seems that if $k \overset{L}{\underset{B}{\otimes}} I \simeq 0$, then <u>$C_\cdot(A) \to C_\cdot(B)$</u> is a quis.

$C_\cdot(A)$ describe $H_*(\mathfrak{gl}(\tilde{A}), \mathfrak{gl}(k))$ similar to $K_*(\tilde{A})/K_*(\mathbb{Z})$ I think.

Consider $0 \to I \to A \to B \to 0 \qquad I^2 = 0.$

Use I-adic filt of $C_\cdot(A)$. $\quad \mathrm{gr}\, C_\cdot(A) = C_\cdot(B \oplus I)$

$$= C_\cdot(B) \oplus \left(I \overset{L}{\underset{B}{\otimes}}\right) \oplus \left(I \overset{L}{\underset{B}{\otimes}}\right)^{(2)}_\lambda \oplus \cdots$$

Assume $IB = 0$. Then $\left(I \overset{L}{\underset{B}{\otimes}}\right) = k \overset{L}{\underset{B}{\otimes}} I$ cyclic

$H_*\left(k \overset{L}{\underset{B}{\otimes}} I\right) = \mathrm{Tor}_*^{\tilde{B}}(k, I)$. This vanishes if $I = BI$ and I is B flat. $k \overset{L}{\underset{B}{\otimes}} I = 0$ means I is h-unitary, i.e. I has a resolution by firm flat B-modules. If this is the case then we seem to have $C_\cdot(A) \sim C_\cdot(B)$

18. The other thing to do is to use the extension i.e. $\underset{DG\,alg}{C_\cdot(I \to A)} \xrightarrow{quis} C_\cdot(B)$ which

gives $C(B) \sim C_\cdot(I \to A)$ where $gr\, C_\cdot(I \to A)$

$= C_\cdot(A \oplus I[1]) = C.(A) \oplus \Sigma [I \overset{\mathbb{L}}{\otimes}_A]^\cdot \oplus \Sigma^2 [I \overset{\mathbb{L}}{\otimes}_A]^{(2)}_\sigma \oplus \ldots$

In the case $IA = 0$. $I \overset{\mathbb{L}}{\otimes}_A = k \overset{\mathbb{L}}{\otimes}_A I$ has $H_* = \mathrm{Tor}_*^{\tilde{A}}(k, I)$. Vanishes $\iff$ I is h-unitary over A, i.e. $\exists$ flat res. So get

I h-unitary over $A \Rightarrow C.(A) \sim C.(B)$.

Does this check out? ~~So the basic point is clear.~~

HC_0?
$$0 \to I \to A \to B \to 0$$

$$HC_1(A) \to HC_1(B) \to I/[A,I] \to A/[A,A] \to B/[B,B] \to 0$$
$$\| $$
$$I/AI \quad \text{if} \quad IA = 0.$$

So it seems to work!! ~~Next point: Take yild.~~

So what next?

What's going on is that I have the analogue of $K_*(A) = K_*(\tilde{A})/K_*(\mathbb{Z})$. But I still haven't ~~found~~ understood why $A = A^2$ and A left flat $\Rightarrow H_*(gl(A)) = H_*(gl(\tilde{A}), gl(k))$.

Here are the ideas. In general $gl(k)$ acts on $C_\cdot(gl(A))$ and this complex splits. I think I can understand this by tensoring ~~with~~ $C_\cdot(gl(k)) \otimes C_\cdot(gl(A))$ Then taking coinvariants under $gl(k)$.

19. I don't know the details, but let's assume we get $C.(\mathfrak{g}(\tilde{A}))_{\mathfrak{gl}(k)}$ out of this. The good case is where $\mathfrak{gl}(k)$ acts trivially on $H_*(\mathfrak{gl}(A))$. This means that $C.(\mathfrak{gl}(A))$ splits into $C.(\mathfrak{gl}(A))_{\mathfrak{gl}(k)}$ the inv. part and the rest is acyclic.

You know $C_\cdot(\mathfrak{gl}(k), Z) = 0$ for any nontrivial irred. Z. So the process of tensoring with $C.(\mathfrak{gl}(k)$ and taking coinv. yield $C.(\mathfrak{gl}(k))_{\mathfrak{gl}(k)} \otimes C.(\mathfrak{gl}(A))_{\mathfrak{gl}(k)}$. Basically we get something like $C_\cdot(k \oplus A)$. I seem to have the wrong idea.

11/02 0815

Yesterday viewpoint. Look at cyc. homol.

$C.(\mathfrak{gl}(A))$ $\quad C_p = (\mathfrak{g} \otimes A)^{\otimes p} \otimes_{\Sigma_p} (\mathrm{sgn})$ $\quad \mathfrak{g} = \mathfrak{gl}(k)$

$\mathfrak{g}_n$ acts on $C.(\mathfrak{gl}_n(A))$, canonical splitting according to irreducible reps of $\mathfrak{g}_n$.

$\mathfrak{g} \otimes C.(\mathfrak{gl}(A))$.

I know that $C.(A) = \mathrm{Prim}\ C.(\mathfrak{gl}(A))_{\mathfrak{g}}$ is the analog of $K_*(A) = K_*(\tilde{A})/K_*(\mathbb{Z})$. Put another way $C.(\mathfrak{gl}(A))_{\mathfrak{g}} = S\{C.(A)\}$. The good case is when $C.(\mathfrak{gl}(A)) \longrightarrow C.(\mathfrak{gl}(A))_{\mathfrak{g}}$ is a quis, i.e when $\mathfrak{g}$ acts trivially on $H.(\mathfrak{gl}(A))$. ~~To find when this happens it's enough to tensor with $C.(\mathfrak{gl}(A))$ with an~~ $\mathfrak{g}$ acts trivially on $H.(\mathfrak{gl}(A))$

$\Longleftrightarrow$ $H.(\mathfrak{gl}(A)) \otimes$?

20. How do you know when $\mathfrak{g}$ acts trivially on $M = H_\cdot(\mathfrak{gl}(A))$. When for enough rep W we have

$(W \otimes M)_{\mathfrak{g}} = V_{\mathfrak{g}} \otimes M$. enough reps means $V^{\otimes p}$ $\forall p$.

V standard rep of $\mathfrak{g}$: $\mathfrak{g} = V \otimes V^*$. So you want $\left((V \otimes V^*)^{\otimes p} \otimes C_\cdot(\mathfrak{gl}(A)) \right)_{\mathfrak{g}}$ to be

quis $\left((V \otimes V^*)^{\otimes p} \right)_{\mathfrak{g}} \otimes C_\cdot(\mathfrak{gl}(A))_{\mathfrak{g}}$ for all p.

~~Diagrams~~ graphs in which you have p inc. and p outgoing external vertices. Somehow this leads to bar homology. Hanlon's thm.

Also consistent with

$$C_\cdot(\mathfrak{gl}(k) \oplus \mathfrak{gl}(A))_{\mathfrak{g}}$$
$$\| $$
$$S\{C_\cdot(\tilde{A})\}.$$

NO

so you want $C_\cdot(\tilde{A}) \sim C(A) \oplus C(k)$

But isn't this always true.

In general for R unital have Δ

$$C(k) \longrightarrow C(R) \longrightarrow \bar{C}(R)$$

Where are you? You have a fairly simple proof that A h-unital $\Longleftrightarrow$

$$H_*(\mathfrak{gl}(A)) \xrightarrow{\sim} H_*(\mathfrak{gl}(\tilde{A}), \mathfrak{gl}(k)) \quad \text{i.e.}$$

$$C_\cdot(\mathfrak{gl}(A)) \xrightarrow{\text{quis}} C_\cdot(\mathfrak{gl}(A))_{\mathfrak{gl}(k)}.$$

This is based on invariant theory.

21. You want the analogue for GL:

$$BGL(A) \longrightarrow \mathrm{Fib}\{BGL(\tilde{A})^+ \to BGL(\mathbb{Z})^+\}$$

is a homology isom. This should be equiv. to $GL(\mathbb{Z})$ acting trivially on $H_*(GL(A))$. This is Suslin's excision result.

How do you propose to analyze this situation? Suslin's result is via the affine group.

How to proceed? You feel that you should do the following. Given B h-unital, you can construct a simplicial resolution $Q.$ of B by firm flat B modules. Then get $(B, Q.)$ s. dual pair, $A. = B\otimes_B Q. = Q.$ is s. ring. And ?

$$\begin{array}{ccc} GL(A.) & & GL(B) \\ \downarrow & & \downarrow \\ GL(\tilde{A}.) & \longrightarrow & GL(\tilde{B}) \end{array} \qquad \begin{pmatrix} A & A=Q \\ P=B & B \end{pmatrix}$$

$A=Q$ is A-flat $\Leftrightarrow P\otimes_A Q$ is B-fl

$A.$ is left flat.

WRONG side

~~The nice way to say this is to assume P has a nuclear B-action i.e. gives $B \longrightarrow P\otimes_A P^*$ P' has something slightly stronger than a nuclear C-action namely $C \longrightarrow P'\otimes_B B \otimes_B \check{P}'$.~~

~~Maybe the simplest symmetrical thing is~~

~~$B \longrightarrow P\otimes_A A \otimes_A \check{P}$~~

~~OK what next????~~

22. B h-unital, $P_\bullet$ simp. f-flat B-mod res. of B

~~(P_n, B) ... $A_n = B \otimes_B B$~~ $(\underset{Q}{\overset{\shortparallel}{B}}, P_n)$ $A_n = B \otimes_B P_n = P_n$

$$\begin{pmatrix} A & Q = B \\ A = P & B \end{pmatrix} \qquad P \text{ is } B\text{-flat} \implies Q \otimes_B P = B \otimes_B P = \overset{\shortparallel A}{P} \text{ is } A\text{-flat}$$

Thus each A_n is left flat. Now by M-inv I know that $K_*(A_\bullet)$ is constant. ~~So let's see.~~ What do I know in this situation?

$$\overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} GL(A_1) \rightrightarrows GL(A_0) \to GL(B)$$

$$\overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} GL(\tilde{A}_1) \rightrightarrows GL(\tilde{A}_0) \longrightarrow GL(\tilde{B})$$

~~What~~ I know that $K_*(\tilde{A}_n)$ const in n.

So this construction reduces to understanding the case of flat modules. So you take B a left flat, idemp. ring.

Look carefully. ~~A unital~~ You want to understand the case where A is A-flat. Look first at $A \in \mathcal{P}(\tilde{A}^{op})$.

Review this case: $A \in \mathcal{P}(\tilde{A}^{op})$, Consider the dual pair $P = A$, $Q = \mathrm{Hom}_{A^{op}}(A, A) = \mathrm{Hom}_{A^{op}}(A, \tilde{A})$

$$\begin{pmatrix} \underset{P}{\overset{A}{\shortparallel}} & \underset{B}{\overset{Q}{\shortparallel}} \end{pmatrix} \qquad P \otimes_A Q = A \otimes_A Q \qquad ?$$

You should take $Q = A \otimes_A \mathrm{Hom}_{A^{op}}(A, A)$

confused.

23. $\underset{P}{\overset{A}{\|}} \in \mathcal{P}(\tilde{A}^{op})$ $\qquad A = A^2$

$$Q = \check{P} = \mathrm{Hom}_{A^{op}}(P, \tilde{A})$$

Well $\quad \mathrm{Hom}_{A^{op}}(P, W) = W \otimes_A \check{P}$

$$\therefore \quad \mathrm{Hom}_{A^{op}}(A, A) = \underbrace{A}_{\substack{\| \\ A \otimes_A A}} \otimes_A \mathrm{Hom}_{A^{op}}(A, \tilde{A})$$

$$= A \otimes_A \mathrm{Hom}_{A^{op}}(A, A). \quad \text{OK}$$

So B is unital etc. Examine the B-reciprocal dual pair $(\underset{B}{\overset{Q}{\|}}, P)$ over B with P a left unitary B-module equipped with $P \xrightarrow{f} \mathrm{Hom}_B(B, B) = B$ such that $P \otimes_{\mathbb{Z}} B \twoheadrightarrow B$. Think of P as a left ideal in unital $B \ni PB = B$.

So we have unital ring B a B-module map $f : A \to B \ni f(A)B = B$. The question now is whether $GL(A) \longrightarrow GL(B)$ is a homology isomorphism. You suppose A is a left ideal in B unital such that $AB = B$.

We have $f : A \to B$ homom. $\therefore GL(A) \to GL(B)$ Also have B acting on $A \in \mathcal{P}(\tilde{A}^{op})$ $\therefore$ some sort of map $GL(B) \longrightarrow GL(A)$. You want these to be inverse on homology.

24. Special case $A \twoheadrightarrow B$. Then we have $A = B \oplus L$ as B-modules. A is an affine ring $\begin{pmatrix} B & L \\ 0 & 0 \end{pmatrix}$ So we have an arb. unitary B-module L

You need to see that $H_*\left(GL\begin{pmatrix} B & L \\ 0 & 0 \end{pmatrix}\right) = H_*(GL(B))$

So ask why $H_*(GL(B), H_*^{so}(M(L))) = 0$

~~So consider~~ ~~So ask about GL(B)~~

Functor from $\mathrm{Mod}(B)$ $L \mapsto H_*\left(GL\begin{pmatrix} B & L \\ 0 & 0 \end{pmatrix}\right)$

$\boxed{\text{Yes}}$ What next??? You can replace L by a complex and get a semi-simplicial thing. Thus if you choose a free resolution

$AB = B \qquad yx = 1 \qquad y \in A,\ x \in B$

say $A = By$. $GL(xBy) \to GL(By) \longrightarrow GL(B)$

~~$A \ne B$ ... on ... A~~ hom. $A \to B$, $A = By$, $By \hookrightarrow B$

Also have B acting on $By = A$ and A^{op}-module summand of $\tilde{A}$ how: $A \to \tilde{A} \to A$

$A \xrightarrow{x\cdot} \tilde{A} \xrightarrow{y\cdot} A$

$\overset{a}{\widehat{by}} \mapsto x\overset{a}{\widehat{by}} \longmapsto yx\overset{a}{\widehat{by}} = \overset{a}{\widehat{by}}$

$B \to A \hookrightarrow B$

$b \mapsto xby \mapsto xby$

hom. $B \longrightarrow A$

$b \longmapsto xby$

$A \subset B \longrightarrow A$

$a \quad a \mapsto xay$

25. functors. $P(\tilde{A}^{op}) \qquad P(\tilde{B}^{op})$

$$W \otimes_B A \longleftarrow\!\shortmid\; W$$

$$V \longmapsto V \otimes_A B$$

$$\begin{matrix} A & B \\ A & B \end{matrix}$$

$$A \otimes_A B = B$$

$$B \otimes_B A = A$$

via the hom.

$$W \longmapsto W \otimes_B A \longmapsto W \otimes_B A \otimes_A B$$

$$V \longmapsto V \otimes_A B \longmapsto V \otimes_A B \otimes_B A = V \otimes_A A$$

Check this carefully. You have ~~Try unital ring~~ Take $A \twoheadrightarrow B$ i.e. $A = B \oplus L$

B is unital but A is not. In the end you must use

$$0 \to A \to \tilde{A} \to \mathbb{Z} \to 0$$

Shanuel's lemma.

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & V/AV \otimes_{\mathbb{Z}} A \arrow[d] & \\
 & & & V/AV \otimes_{\mathbb{Z}} \tilde{A} \arrow[d] & \\
0 \arrow[r] & V \otimes_A A \arrow[r] & V \arrow[r] & V \otimes_A \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

basic bimodule

$$\begin{tikzcd}
 & A \arrow[d] \\
J \arrow[r] \arrow[d] & {}_0\tilde{A}_1 \arrow[d] \\
\tilde{A}_{11} \arrow[r] & \mathbb{Z}_{00}
\end{tikzcd}$$

Note: In the above situation $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ $\quad B \in P(B^{op})$, $A \in P(A^{op})$

Here A is any B-module equipped with $f : A \to B$ B-map $\ni f(A)B = B$. ~~[illegible]~~ special case of B B^{op}-flat $\Rightarrow$ A is A^{op} flat.

26. To write A as a summand of a free $\tilde{A}^{op}$ module you need to choose $\sum f(a_i)b_i = 1$. So there's a difficulty ~~[illegible]~~ to go from expressing B as a filtered colimit of fg proj B^{op}-modules to a similar expression for A.

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$A \in P(\tilde{A}^{op})$ $\qquad B = \mathrm{Hom}_{A^{op}}(A,A) = A \otimes_A \mathrm{Hom}_{A^{op}}(A,\tilde{A})$

So what. Does A have to be idempotent. NO

$$\underset{A}{X\mathbb{Z}[X]} \subset \underset{\tilde{A}=B}{\mathbb{Z}[X]} \quad ? \quad \text{Anyway} \quad \cancel{B=A}$$

tensor alg + such things. So you must assume that $A^2=A$. Then $B = A \otimes_A B$

$$0 \longrightarrow \mathrm{Hom}_{A^{op}}(A,A) \longrightarrow \mathrm{Hom}_{A^{op}}(A,\tilde{A}) \longrightarrow \mathrm{Hom}_{A^{op}}(A,\mathbb{Z})$$

$$0 \to A \otimes_A \mathrm{Hom}_{A^{op}}(A,A) \longrightarrow \underset{\substack{\parallel \\ \mathrm{Hom}_{A^{op}}(A,A)}}{A \otimes_{\mathbb{Z}} \mathrm{Hom}_{A^{op}}(A,\tilde{A})} \longrightarrow A/A^2 \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(A/A^2,\mathbb{Z})$$

So apparently $0 \to A \otimes_A B \longrightarrow B \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A/A^2, A/A^2)$

This should be interesting ~~[illegible]~~ later.

Back to $A \in P(\tilde{A}^{op})$ $\quad A = A^2$. Then

$$B = \mathrm{Hom}_{A^{op}}(A,A) = A \otimes_A \mathrm{Hom}_{A^{op}}(A,\tilde{A}) = A \otimes_A B$$

and $A = B \otimes_B A$ so we have ~~[illegible]~~ a map $f : A \to B$ in $\mathrm{Mod}(\tilde{B})$ $\ni$ $B = f(A)B$

Now continue. You really want to try to understand the relation between $GL(A)$ and $GL(B)$.

Try some special cases.

Take $A \subset B$ left ideal + $AB = B$

assume $\exists y \in A$, $x \in B$ $yx = 1$. Then

$A \supset By$ $By \supset Byxy \supset Bxy \supset By$

Therefore $By = Be$ is the smallest A can be given $\exists y \in A$, $x \in B \ni yx = 1$.

~~Next~~ Assume $A = Be$. Then $B = \begin{pmatrix} eBe & eBe^{\perp} \\ e^{\perp}Be & e^{\perp}Be^{\perp} \end{pmatrix}$

$$A = \begin{pmatrix} eBe & 0 \\ e^{\perp}Be & 0 \end{pmatrix}$$

Say $A = Be$ $e = xy$ $yx = 1$.

So what to say? ~~You want that A works.~~

~~You need to refine the following thus~~

~~Say A ≠ Be. Possibly~~

Let's analyze ~~these~~ these ideas? Lie theory

$H.(\mathfrak{gl}(A))$ analyze as repn. of $\mathfrak{gl}(k)$. Consider

$\mathfrak{g}^{\otimes n} \otimes C.(\mathfrak{gl}(A))$. Apply invariant theory

and out comes the bar complex

~~Let us consider the coherent sheaves~~

Not much ~~to~~ understood. $A = A^2$

Can you handle this $A \in P(\tilde{A}^{op})$ situation?

Can you actually prove that $GL(A) \to GL(B)$ is a homology isom.

Let's try

28. Let's fix B unital and consider firm A's which are left M eq to B. This means we have firm dual pair (B, P) over B, i.e. $P \to \mathrm{Hom}_{B^{op}}(B, B) = B$ such that $P \otimes B \twoheadrightarrow B$. In other words we have a unitary B-module map $f: P \to B \ni f(P)B = B$ and then $A = \cancel{P \otimes_B} \; B \otimes_B P = P$. So the category we have consists of un B-mod ~~such~~ maps $f: A \to B \ni f(A)B = B$. ~~Any A ...~~ Such an $A \in \mathcal{P}(\tilde{A}^{op})$, so by Suslin we know ~~that~~ for any such $A \to B$ that $GL(A) \to GL(B)$ is a homology isomorphism. Can I prove this somehow? Special case: where $f: A \twoheadrightarrow B$ then pick $e \in A$ $f(e) = 1$. Then $ea = f(e)a = a$. So A has a left identity

When does A have a left identity? Wodicki says ~~...~~ True $\Leftrightarrow$ $\mathbb{Z} \in \mathcal{P}(\tilde{A}^{op})$.

$$0 \to A \to \tilde{A} \to \mathbb{Z} \to 0$$

$\mathbb{Z} \in \mathcal{P}(\tilde{A}^{op}) \Leftrightarrow$ this splits as A^{op}-modules, equiv. $\exists$ $1-e$, $e \in A$ such that $(1-e)A = 0$ i.e. $a = ea$ $\forall a \in A$.

The preceding is new to me. Does it help? So how does this help? ~~...~~ Look at

$$0 \to A \to \tilde{A} \to \mathbb{Z} \to 0$$

as an exact sequence ~~in~~ in $\mathcal{P}(\tilde{A}^{op})$, and note that the left action is via ~~the~~ an affine group.

29. This is very instructive. You reach a simple case. The ring theory is very simple namely $\tilde{A} = A \oplus \tilde{A}/A$ as right A-module and so ~~the~~ left mult by a is given by $\begin{pmatrix} aA & a \\ 0 & 0 \end{pmatrix}$ on $A \oplus \mathbb{Z}$.

Go back over stability.

Take

11/05 1423. Lets look carefully at $GL(B \oplus M)$

Techinques you have? ~~Start with~~

You want to use the s.gp resolution $GL(A.)$ ~~one~~ of $GL(B)$, associated to

$$\cdots A_2 \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} A_1 \rightrightarrows A_0 \to B \to 0$$

Start with B h-unital. Then you know such an A exists with A_i flat ferin over B. Get resolution $GL(A.)$ of $GL(B)$. You need to know that $H_j(GL(A.))$ is a constant simp. obj. This may have very little to do with B. One knows that $A_1 =_{s_0} A_0 \oplus \mathrm{Ker}\{d_0 : A_1 \to A_0\}$. Be more precise. ~~Enough to take~~ Consider $A_1 \underset{d_0}{\overset{s_0}{\rightleftarrows}} A_0$

So you need to know that $A_1 = A_0 \oplus M$ has same $H_*GL(?)$ for A_0, M flat flat

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You need to understand $H_*(GL(A))$

~~You should~~ Let me try some more. Main idea. Begin with B h-unital, then you can find firm flat B-mod. resolution

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \to B \to 0$$

Convert to a simplicial flat B-mod res.

$$A_2 \Rrightarrow A_1 \rightrightarrows A_0 \to B \to 0$$

where $A_n \simeq \bigoplus\limits_{p,\ [n]\twoheadrightarrow[p]} F_p$ $\qquad A_1 = F_1 \oplus s_0F_0$

$$A_2 = F_2 \oplus s_0 F_1 \oplus s_1 F_1 \oplus s_1 s_0 F_0$$

You ~~would like~~ want to show $H_j(GL(A_\bullet))$ is constant. You have s.s.

$$E^1_{pq} = H_p H_q(GL(A_p)) \Rightarrow H_q(GL(B))$$

Special case. Suppose $F_p = 0$ for $p \geq 2$.

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow B \longrightarrow 0$$
$$F_0 = A_0$$

Then $F_n = A_0 \times_B \cdots \times_B A_0$ $n+1$ times.

What happens in this case? You have a Cech cxd

$$\begin{array}{ccccccc}
B & & B & & B & & \\
\uparrow & & \uparrow & & \uparrow & & \\
A\times_B A\times_B A & \Rrightarrow & A\times_B A & \rightrightarrows & A & \to & B \\
 & & \uparrow & & \uparrow & & \\
I\oplus I\oplus I & & I\oplus I & & I & &
\end{array}$$

You wonder about

31. So what happens. Suppose you know that $A = B \oplus I$. Does this give you a contracting homotopy?

$I \times I \times I \qquad I \times I \qquad I$

The problem is that $H_*(GL(-))$ is ~~a bright~~ non linear.

You would like to have a simplicial ring homotopy between $A_\bullet$ and B, ~~[illegible]~~ so that you get a simp. group homotopy between $GL(A_\bullet)$ and $GL(B)$.

~~11/02/2000~~

11/08 1649 I had some ideas today.

1. M-inv. for ~~B~~ K_* of right (or left) flat firm rings $\Longrightarrow$ K_* defined for Roos cat. Can you find an intrinsic defn?

2. You apparently can prove K_0 is M-invariant for idempotent rings. Because if $B = B^2$ and $A \twoheadrightarrow B$ is a B-mod surj with A -firm flat, then $K_0 A \Rightarrow K_0 B$, since $\mathcal{P}(\tilde{A}) \simeq \mathcal{P}(\tilde{B})$, by nilpotent extension stuff. Can you find a direct proof?

Assume $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ $\ni$ $\begin{matrix} A^2 = A & PA = P & AQ = Q & QP = A \\ B^2 = B & BP = P & QB = Q & PQ = B. \end{matrix}$

then show that $K_0(A) \simeq K_0(B)$. Recall your paper showing this for K_0'. Look at

$\begin{pmatrix} \tilde{A} & Q \\ P & \tilde{B} \end{pmatrix} = R \qquad e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

32. $R = \begin{pmatrix} eRe & eRe^\perp \\ e^\perp Re & e^\perp Re^\perp \end{pmatrix} \supset \begin{pmatrix} eRe^\perp Re & eRe^\perp \\ e^\perp Re & e^\perp ReRe^\perp \end{pmatrix}$

You want $K_0(A) \xrightarrow{\sim} K_0\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$. So not yet clear. Another idea

$$eRe = \tilde{A} \subset \begin{pmatrix} \tilde{A} & Q \\ P & B \end{pmatrix} = ReR$$

Observe that $0 \to \begin{pmatrix} \tilde{A} & Q \\ P & B \end{pmatrix} \to \begin{pmatrix} \tilde{A} & Q \\ P & \tilde{B} \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{Z} \end{pmatrix} \to 0$

so in this case $K_*(ReR) = \mathrm{Ker}\{K_*(R) \to K_*(\mathbb{Z})\}$

so we are trying to show $K_0(eRe) = K_0(ReR)$.

It seems this reduces us to the case where A unital

~~Case~~ Thus suppose given $\tilde{A}$ unital and (P, Q) any dual pair over A. $P' = \tilde{A} \oplus P$, $Q' = \tilde{A} \oplus Q$

$P' \otimes_A Q' = \begin{pmatrix} \tilde{A} & Q \\ P & P \otimes_A Q \end{pmatrix}$. So what do you know?

$\begin{pmatrix} \tilde{A} & Q \\ P & P \otimes_A Q \end{pmatrix}$ What are you trying to say?

Basically you want $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ comp. iden. to yield $K_0 A \simeq K_0 B$.

~~$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$~~

Y

X

33. Try thing generated by $X \subset P$, $Y \subset Q$.

$$\begin{matrix} YX & Y & & X & Y \\ X & XY & & XY & YX \\ & & & XYX & YXY \\ & & & XYXY & YXYX \end{matrix} \qquad X^2 = 0, \quad Y^2 = 0$$

So what comes next? Try K_0. ~~Yes, No~~,

Idea: Go back to $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ $\quad QP = A$, $PQ = B$

$$K \hookrightarrow P \otimes_A Q \twoheadrightarrow B \qquad K(P \otimes_A Q) = 0$$

$$\left(\sum p_i \otimes q_i\right) pq = \underbrace{\sum p_i q_i p}_{0} \otimes q$$

~~Anyway what next?~~ $\therefore K^2 = 0$, so

$$\begin{pmatrix} A & 0 \\ 0 & K \end{pmatrix} \hookrightarrow \begin{pmatrix} A & Q \\ P & P \otimes_A Q \end{pmatrix} \twoheadrightarrow \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

Similarly can suppose $Q \otimes_B P \xrightarrow{\sim} A$

Next consider ~~$\begin{pmatrix} A & Q \\ P & P \otimes_A Q \end{pmatrix}$~~ $\qquad$ wait.

Basic idea: Start with $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ ~~s. idem.~~

~~Where~~ $\ni$ $PQ = B$, $QP = A$. Choose $P' \twoheadrightarrow P$

34. Take $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ $\qquad QP = A \qquad PQ = B.$

11/12/97 1303. To show in above situation that $K_0(A) \simeq K_0(B)$. Idea: let $C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ and show $A \subset C \supset B$ induce isos. on K_0. Actually, I think you can show that $PQ = B \Rightarrow K_0(A) \xrightarrow{\sim} K_0(C)$ and then by symmetry $QP = A \Rightarrow K_0(B) \xrightarrow{\sim} K_0(C)$.

~~First steps will~~ Fact. K_0 preserved for nilpotent extensions. Assuming $PQ = B$, we know $P \otimes_A Q \to B$ is a square zero extension. (If $k = \sum p_i \otimes q_i$ is in the kernel K, then $(pq)k = \sum pqp_i \otimes q_i = p \otimes q \sum \cancel{q} p_i q_i = 0$ so $BK = KB = 0$. ✓) Thus $\begin{pmatrix} A & Q \\ P & P\otimes_A Q \end{pmatrix} \twoheadrightarrow \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ induces an iso on K_0.

Fact: I ideal in R unital, have exact seq.

$$K_1(R) \to K_1(R/I) \to K_0(I) \to K_0(R) \to K_0(R/I)$$

You can then ~~drop R unital~~ replace R by a nonunital ring. $R = \tilde{A}$ where $I \subset A$ ideal

~~Thus~~. $K_i(\tilde{A}) = K_i(A) \oplus K_i(\mathbb{Z})$ and sim for $\tilde{A}/I$.

Consider

$$\begin{array}{ccccc} \begin{pmatrix} A & Q \\ P & B \end{pmatrix} & \underset{\text{ideal}}{\subset} & \begin{pmatrix} \tilde{A} & Q \\ P & B \end{pmatrix} & \longrightarrow & \mathbb{Z} \\ \cup & & \cup & & \| \\ A & \subset & \tilde{A} & \longrightarrow & \mathbb{Z} \end{array}$$

~~Rows~~ rows give 5 term exact. seq as above which split as $\mathbb{Z}$ lifts back. $\therefore$

$$\begin{array}{ccccccccc} 0 & \to & K_0(A) & \to & K_0(\tilde{A}) & \to & K_0(\mathbb{Z}) & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & K_0\begin{pmatrix} A & Q \\ P & B \end{pmatrix} & \to & K_0\begin{pmatrix} \tilde{A} & Q \\ P & B \end{pmatrix} & \to & K_0(\mathbb{Z}) & \to & 0 \end{array}$$

So we reduce to the case ~~of any dual pair P, Q over A~~ of $A \subset \begin{pmatrix} A & Q \\ P & P\otimes_A Q \end{pmatrix} = C$ where A is unital and P, Q are unitary A-mods.

$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad A = eCe$

and $eC = (A \ \ Q)$, $Ce = \begin{pmatrix} A \\ P \end{pmatrix}$

$$Ce \otimes_A eC = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \ \ Q) = C.$$

Note $\tilde{C} = \begin{pmatrix} A & Q \\ P & \tilde{B} \end{pmatrix}$.

Actually you should leave off $P\otimes_A Q \xrightarrow{\sim} B$ until later. You start with $C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix} \ni PQ = B$

Then enough to handle $\begin{pmatrix} \tilde{A} & Q \\ P & B \end{pmatrix}$, so can ass. (A, P, Q) unital with QP arbitrary. Then

$R = \tilde{C} = \begin{pmatrix} A & Q \\ P & \tilde{B} \end{pmatrix} \ni \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad C = \tilde{C} e \tilde{C}$ $\quad A = e\tilde{C}e$

Ultimately we reach the case of $\begin{pmatrix} eRe & eR \\ Re & ReR \end{pmatrix}$ and we need to show $K_0(eRe) \xrightarrow{\sim} K_0(ReR)$.

Look at dual pairs

$$(A, A) \subset (A\oplus P, A) \subset (A\oplus P, A\oplus Q)$$

First case $\begin{pmatrix} A & A \\ P & P \end{pmatrix}$

$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \subset \begin{pmatrix} A & Q \\ 0 & 0 \end{pmatrix}$ OKAY

$\begin{pmatrix} A & Q \\ 0 & 0 \end{pmatrix} \subset \begin{pmatrix} A & Q \\ P & PQ \end{pmatrix}$

$eRe \subset Re \subset ReR$

rt id $\quad$ left ideal

 Work it out assuming everything is firm

$A \subset \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ $\qquad$ $(A, A) \subset \left(\begin{pmatrix} A \\ P \end{pmatrix}, (A \;\; Q) \right)$

$\begin{pmatrix} A \\ 0 \end{pmatrix}, (A \;\; Q)$

Let $A' = \begin{pmatrix} A & Q \\ 0 & 0 \end{pmatrix}$ $\qquad$ $P' = \begin{pmatrix} 0 & 0 \\ P & B \end{pmatrix}$

You want to compute the dual pair over A' yielding C.

$$\begin{tikzcd} \mathcal{M}(A) \arrow[r, "\begin{pmatrix} A \\ P \end{pmatrix} \otimes_A -"] & \mathcal{M}(C) \\ \mathcal{M}(A') \arrow[u, "A \otimes_{A'} -"] \arrow[ur] & \end{tikzcd}$$

$A' = A \oplus Q$
s.t. $QA' = 0$
so $M \in \mathcal{M}(A')$
is killed by Q.

$\begin{pmatrix} A \\ P \end{pmatrix} \otimes_A A \otimes_{A'} - = \begin{pmatrix} A \\ P \end{pmatrix}$ $\quad$ obvious left C-action right A' action thru A.

$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \otimes \begin{pmatrix} A & Q \\ 0 & 0 \end{pmatrix}$ $\quad \cong \quad$ $\begin{pmatrix} A \\ P \end{pmatrix} \otimes_{A'} A'$

So my guess is that the dual pair over A' consists of $\begin{pmatrix} A \\ P \end{pmatrix}$ with right action of $(a \;\; q)$ given by $\cdot a$.

$\begin{pmatrix} a \\ p \end{pmatrix} (a_1, q) = \begin{pmatrix} a a_1 \\ p a_1 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} A \\ P \end{pmatrix} \otimes_{A'} A'$

and A' with ~~obvious action of~~ left mult action A' $= \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \;\; Q) = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

37. We have $A' = \begin{pmatrix} A & Q \end{pmatrix} \subset \begin{pmatrix} A & Q \\ P & B \end{pmatrix} = C$

$P' = \begin{pmatrix} A \\ P \end{pmatrix} \subset C$ and $Q' = A'$. Then

A' is a subring $\quad Q'P' = \begin{pmatrix} A & Q \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ P & 0 \end{pmatrix} = \begin{pmatrix} A^2+QP & 0 \\ 0 & 0 \end{pmatrix}$

$$P'Q' = \begin{pmatrix} A & 0 \\ P & 0 \end{pmatrix}\begin{pmatrix} A & Q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A^2 & AQ \\ PA & PQ \end{pmatrix} \subset C.$$

So to prove given $C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ with $PQ = B$

that $K_0 A \xrightarrow{\sim} K_0 C$. Assume true and try

to generalize to $\begin{matrix} A & \subset & Y \\ \cap & & \cap \\ X & \subset & C \end{matrix}$ $\ni$ $A^2 \subset A,\ AY \subset Y,\ XA \subset X,\ YX \subset A$

$C^2 \subset C,\ CX \subset X,\ YC \subset Y,\ XY = C$

$$YC = YXY \subset AY \subset Y$$
$$CX = XYX \subset XA \subset X$$

So given this consider $\begin{pmatrix} A & Y \\ X & C \end{pmatrix} \subset \begin{pmatrix} C & C \\ C & C \end{pmatrix}$

Assuming result $K_0(A) \xrightarrow{\sim} K_0\begin{pmatrix} A & Y \\ X & C \end{pmatrix} \xleftarrow{\sim} K_0(C)$

If you know also $YX = A$, then

$XY = C$
$\Rightarrow C \subset C^2$
$\therefore C = C^2$

$$\begin{tikzcd} K_0(A) \arrow[r] & K_0\begin{pmatrix} A & Y \\ X & C \end{pmatrix} \arrow[d] \\ K_0(C) \arrow[r, "\sim"] & K_0\begin{pmatrix} C & C \\ C & C \end{pmatrix} \end{tikzcd}$$

?

First reduction from $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ to $\begin{pmatrix} \tilde{A} & Q \\ P & B \end{pmatrix}$

38. Review

$$\begin{tikzcd}
0 \arrow[r] & K_0 A \arrow[r] \arrow[d] & K_0 \tilde{A} \arrow[r] \arrow[d] & K_0 \mathbb{Z} \arrow[r] & 0 \\
0 \arrow[r] & K_0 \begin{pmatrix} A & Q \\ P & B \end{pmatrix} \arrow[r] & K_0 \begin{pmatrix} \tilde{A} & Q \\ P & B \end{pmatrix} \arrow[r] & K_0 \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

So can assume A unital, P Q unitary mods over A

Can also assume $B = P \otimes_A Q$.

$$\begin{pmatrix} \tilde{A} \\ P \end{pmatrix} \otimes_A \begin{pmatrix} \tilde{A} & Q \end{pmatrix} = \begin{pmatrix} \tilde{A} & Q \\ P & B \end{pmatrix}.$$

So the point is simple, namely for any dual pair (P, Q) over A to prove that

$$K_0(\tilde{A}) \xrightarrow{\sim} K_0 \begin{pmatrix} \tilde{A} & Q \\ P & P \otimes_A Q \end{pmatrix}$$

Why should this be true? You should be able to reduce to $P = \tilde{A}^n$ by ~~direct~~ ind lims. $\longrightarrow$

So can take $P = \tilde{A}^n$. Can

What's important about the ring $ReR = \begin{pmatrix} \tilde{A} & Q \\ P & B \end{pmatrix}$ where $R = \begin{pmatrix} \tilde{A} & Q \\ P & \tilde{B} \end{pmatrix}$ $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$? Answer: ReR is M-equiv. to the unital ring $eRe = \tilde{A}$. M-context is $\begin{pmatrix} eRe & eR \\ Re & ReR \end{pmatrix}$ Thus

$$Re \in \mathcal{P}(ReR)$$
$$eR \in \mathcal{P}(ReR^{op})$$

are dual and $eR \otimes_R Re = eRe$. So what?

The better case is where $Re = \begin{pmatrix} \tilde{A} \\ P \end{pmatrix} \in \mathcal{P}(\tilde{A}^{op})$ because then

39. What was my understanding of the good case? $A \in \mathcal{P}(\tilde{A}^{op})$ and $A = A^2$.

$$\begin{pmatrix} A & \mathrm{Hom}_{A^{op}}(A, \tilde{A}) \\ A & \mathrm{Hom}_{A^{op}}(A, A) \end{pmatrix}$$

Not defined: the pairing $\check{P}P = \tilde{A}$ is possible

$$\begin{matrix} A & \mathrm{Hom}_{A^{op}}(P, \tilde{A}) \\ P & \underbrace{P \otimes_A \mathrm{Hom}_{A^{op}}(P, \tilde{A})}_{\mathrm{Hom}_{A^{op}}(P, P)} \end{matrix} \qquad \begin{matrix} A & \check{P} \\ P & P \otimes_A \check{P} \\ & \overset{\shortparallel}{\mathrm{End}(P)}. \end{matrix}$$

~~the pairing A~~ So what's going on is that if $A \in \mathcal{P}(\tilde{A}^{op})$, then ???

You don't understand. Take A such that $A \in \mathcal{P}(\tilde{A}^{op})$, then form $A^\vee = \mathrm{Hom}_{A^{op}}(A, \tilde{A})$ and $A \otimes_A A^\vee = \mathrm{Hom}_{A^{op}}(A, A)$. Wait.

$$0 \to \mathrm{Hom}_{A^{op}}(A, A) \longrightarrow \mathrm{Hom}_{A^{op}}(A, \tilde{A}) \longrightarrow \mathrm{Hom}_{A^{op}}(A, \mathbb{Z}) \to 0$$

$$\mathrm{Hom}_{A^{op}}(A, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(A/A^2, \mathbb{Z})$$

So what happens?

I am completely confused. ~~Let's take a coh. sh.~~

$P \in \mathcal{P}(\tilde{A}^{op})$ $\qquad \check{P} = \mathrm{Hom}_{A^{op}}(P, \tilde{A})$

$$\begin{pmatrix} \tilde{A} & \check{P} \\ P & P \otimes_A \check{P} \end{pmatrix} \qquad P \otimes_A \check{P} = \mathrm{End}_{A^{op}}(P, P)$$

Now assume $P = PA$

$\langle \check{P}, P \rangle$ ideal in $\tilde{A}$

~~If $P \neq PA$, then~~.

$$0 \longrightarrow \mathrm{Hom}_{A^{op}}(P, A) \longrightarrow \mathrm{Hom}_{A^{op}}(P, \tilde{A}) \longrightarrow \mathrm{Hom}_{A^{op}}(P, \mathbb{Z}) \longrightarrow 0$$

$$\mathrm{Hom}_{A^{op}}(P, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(P \otimes_A \mathbb{Z}, \mathbb{Z})$$

40. So the interesting point is that $P \neq PA$ means (note that $P \otimes_A \mathbb{Z} \in \mathcal{P}(\mathbb{Z}^{op})$) that $\exists$ $f \in \check{P}$ such that $f(P) \not\subset A$. So we have an interesting situation.

$$0 \to A \otimes_A \check{P} \longrightarrow \check{P} \longrightarrow \check{P}/A\check{P} \longrightarrow 0$$

$\check{P}/A\check{P}$ $=$ dual of P/PA over $\mathbb{Z}$.

Interesting case is tensor alg. $\bar{T}(V) = A = V \otimes_{\mathbb{Z}} \tilde{A}$

$T(V)$ $\quad$ $T(V) \otimes V^*$

$V \otimes T(V)$ $\quad$ $V \otimes T(V) \otimes V^*$

Recognize similarly with Pimsner.

End this digression.

Important: If you have $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ firm ~~ctxt~~ then B is unital $\Longleftrightarrow$ $P \in \mathcal{P}(A^{op})$ $Q \in \mathcal{P}(A)$ are dual.

~~So if $P \in \mathcal{P}(A^{op})$ $P = PA$~~

Let $P \in \mathcal{P}(A^{op})$ $\quad$ A, P firm

$Q = \check{P} = P^* \in \mathcal{P}(A)$, $\quad B = P \otimes_A P^*$ unital.

$$\begin{pmatrix} \overset{A}{\overset{\shortparallel}{Q \otimes_B P}} & Q \\ P & B \end{pmatrix}$$

Assume $QP = A$ generating condition

41. So suppose $A \in \mathcal{P}(\tilde{A}^{op})$ $A = A^2$

What do I mean by $\mathcal{P}(\tilde{A})$? images of idemp. matrices?

$$\begin{tikzcd} A^n \arrow[r, two heads] \arrow[rr, bend right, "e"'] & P \arrow[r, hook] & A^n \end{tikzcd}$$

$e \in M_n A$
$e^2 = e$.

Then $x \in eA^n$ $x_i = \sum e_{ij} a_j$

$$\sum_i e_{ki} x_i = \sum e_{ki} e_{ij} a_j = \sum e_{kj} a_j = x_k$$

M   ~~$x = ex$~~

$e = e^2$ in A. $eA \subset AA$

$eA = e^2A \subset eAA$

$\therefore (eA)A = eA$ so eA is a firm projective module.

So I'm reviewing $\mathcal{P}(\tilde{A}) \subset$ full subcat of ~~Mod($\tilde{A}$) consisting~~ $\mathcal{P}(\tilde{A}^{op})$ consisting of M such that $AM = M$. So $\tilde{A}^N = M \oplus M'$
$A^N = M \oplus AM'$

Assertion should be that $\mathcal{P}(A)$ ~~consists~~ consists of images of idempotent matrices over A. Why

Let $M = \mathrm{Im}\{e : \tilde{A}^N$ . .

42. Take $e : \tilde{A}^N \longrightarrow \tilde{A}^N$ $e^2 = e$
and suppose $P = e\tilde{A}^N \subset A^N$, i.e. $e \in M_N(A)$.
Then $P = eP$. Use right modules
$e = (e_{ij}) \in M_N(A)$. $P = \{ev \mid v \in \tilde{A}^N\}$.
$\tilde{A}^N = P \oplus P'$ $P \subset A^N$
$A^N = AP \oplus AP'$

$A^N = A^N \cap (P + P') = P + A^N \cap P'$ modular
$\therefore$ $AP = P$ and $AP' = A^N \cap P'$.

A nuclear. namely
$$\mathrm{Hom}_{A^{op}}(P, V) = V \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$$
for all right modules V. Equiv. conditions
$$1 = \sum p_i \otimes q_i \qquad \begin{tikzcd} P \arrow[r] \arrow[rrr, bend right, "1"'] & A^N \subset \tilde{A}^N \arrow[rr] & & P \end{tikzcd}$$
$$p = \sum p_i (q_i p)$$
Then things are clear I think.

So what was I doing yesterday?
$A \in \mathcal{P}(\tilde{A}^{op})$ ~~with~~ $A = A^2$.

When is ~~a~~ an idempotent ring A
mor to a unital ring. Answer when $\mathcal{M}(A)$
has a small proj. generator, call it Y. You
can arrange Y to be A by moving A.
Choose $X \in \mathcal{M}(A^{op})$ with $Y \otimes_{\mathbb{Z}} X \twoheadrightarrow A$, then
if you make $\begin{pmatrix} A & Y \\ X & X \otimes_A Y = A' \end{pmatrix}$ you know $Y \mapsto X \otimes_A Y = A'$

43. So ~~we~~ can assume $A \in \mathcal{P}(A^{op})$
Then put $B = \mathrm{Hom}_{A^{op}}(A, A)$
$= A \otimes_{A^{op}} \mathrm{Hom}_{A^{op}}(A, A)$
this is unital etc. So what have you learned??

Yesterday I started trying to handle Morita invariance of K_0 for idempotent rings. And my idea was to use unital

---

Symmetries of the Cuntz alg $\mathcal{O}_E$ = quotient of $T(E \oplus E^*)$ by relations $\iota_\lambda e_\xi = \langle \lambda, \xi \rangle$ and $\sum \xi_i \lambda_i = 1$

$\psi(\lambda)\psi^*(\xi) = \langle \lambda, \xi \rangle$
$\sum \psi_i^* \psi_i = 1$.

Ask about deformations
relations $\psi_i \psi_j^* = \delta_{ij}$ $\qquad \sum \psi_i^* \psi_i = 1$.

$\dot{\psi}_i \psi_j^* + \psi_i \dot{\psi}_j^* = 0 \qquad \sum \dot{\psi}_i^* \psi_i + \psi_i \dot{\psi}_i^* = 0$

~~Anyway~~

Idea: The Toeplitz alg $\mathcal{T}$ has the following modules: ~~vector~~ Hilb space H together with ~~iso~~ embeddings $x_i : H \longrightarrow H$ $\quad i = 1, .., n$ such that $x_i H \perp x_j H$ $\quad i \neq j$. The Cuntz alg $\mathcal{O}$ is the quotient ~~of~~ alg ~~has~~ whose modules are such $(H, x_1, .., x_n)$ with $H = \bigoplus x_i H$

44. Use non unital $\mathcal{T}$, call it $\bar{\mathcal{T}}$ its quot of $\bar{T}(E\oplus E^*)$. This may not be important. What is important is ~~that~~ the fact that $e=\sum x_i y_i$ is an idempotent

$$\mathcal{T} = k\langle x_i, y_i\rangle / (y_i x_j = \delta_{ij}) \simeq k\langle x_i\rangle \otimes k\langle y_i\rangle$$

normal ordering

~~So let us consider~~ $\mathcal{T}$ has ~~so~~ ~~have~~ idemp. e and you want $\mathcal{T}$-mods M such that $eM = M$, i.e. $e^\perp M = 0$. Leads to $\mathcal{O} = \mathcal{T}/\mathcal{T}e^\perp\mathcal{T}$. But also you

$$\mathrm{Mod}(R/Re^\perp R) \to \mathrm{Mod}(R) \to$$

What about ReR

All this should be familiar from the Davydov analysis.

$$\mathrm{Mod}(R)$$

R unital ring with an idempotent e

$Re \in \mathcal{P}(R)$ $\qquad \mathrm{Hom}_R(Re, Re) = eRe$

$$\begin{tikzcd} \mathrm{Mod}(eRe) \arrow[r, hook, shift left] \arrow[d, phantom, "\wr"] & \mathrm{Mod}(R) \arrow[l, shift left] \\ \mathcal{M}(ReR) & \end{tikzcd}$$

$$\begin{pmatrix} eRe & eRe^\perp \\ e^\perp Re & e^\perp ReRe^\perp \end{pmatrix}$$

$$\text{``} \; ReR$$

$$\begin{tikzcd} \mathrm{Mod}(R/ReR) \arrow[r, hook] & \mathrm{Mod}(R) \arrow[r, two heads] & \mathcal{M}(ReR) \arrow[d, phantom, "\wr"] \\ & & \mathrm{Mod}(eRe) \end{tikzcd}$$

But what I really want is

$$K_*(eRe) \longrightarrow K_*(R) \longrightarrow K_*(R/ReR) \quad ?$$

45. 11/16/97 1100

Remark made by Hannabuss at Jacek's lecture yesterday that there exists an interesting action of something like $SO(d^{+\varepsilon}, 1)$ on the Cuntz algebra $\mathcal{O}_d$. Work of Carey-Evans.

Recall $\mathcal{O}_d$ has generators ψ_i $i = 1, \dots, d$ satisfying $\psi_i \psi_j^* = \delta_{ij}$ and $\sum_{i=1}^d \psi_i^* \psi_i = 1$. A *rep of $\mathcal{O}_d$ on a Hilbert H is the same as an isomorphism

$$\begin{array}{ccc} \mathbb{C}^d \otimes H & \xrightarrow{\sim} & H. \\ \| & \nearrow_{(\psi_i^*)} & \\ \bigoplus_{i=1}^d H & & \end{array}$$

There should be an obvious action of $U(d)$ on $\mathcal{O}_d$, but I don't see anything further.

Recall Pimsner article: $T(E) = \bigoplus_{n \geq 0} E^{\otimes n}$, operators ψ_ξ^* = left mult by $\xi \in E$

ψ_λ = int. mult of left by $\lambda \in E^*$

Then $\psi_\lambda \psi_\xi^* = \langle \lambda, \xi \rangle$, $\sum_{i=1}^d \psi_i^* \psi_i$ = ~~the~~ projn onto $T(E)$ with kernel $k = E^0$

$\psi_i^* = \psi_{\xi_i}^*$, $\psi_i = \psi_{\lambda_i}$ where ξ_i basis, λ_i dual basis

$\mathcal{T}_E$ = Toeplitz alg of E is the alg. generated by these ψ_ξ^*, ψ_λ. Known $\mathcal{T}_E \xleftarrow{\sim} T(E) \otimes T(E^*)$ normal ordered picture

$$\mathcal{T}_E \xleftarrow{\sim} T(E \oplus E^*) / \{\psi_\lambda \psi_\xi^* = \langle \lambda, \xi \rangle\}$$

Then dividing by the last relation $\sum \psi_i^* \psi_i = 1$ yields $\mathcal{O}_E$.

45. At one point I used to understand this well. If $R = \mathcal{T}_E$, then $\sum \varphi_i^* \varphi_i = e$ is an idempotent in R:

$$\sum_{i,j} \varphi_i^* \underbrace{\varphi_i \varphi_j^*}_{\delta_{ij}} \varphi_j = \sum \varphi_i^* \varphi_i .$$

and $\mathcal{O}_E = R/Re^{\perp}R$. Picture of $\mathcal{T}_E$:

$$\begin{tikzcd}[column sep=tiny]
 & & & k \arrow[dd] & & & \\
 & & E \arrow[dd] & & E^{*} \arrow[dd] & & \\
 & E^{\otimes 2} \arrow[dd] & & E\otimes E^{*} \arrow[dd] & & E^{*\otimes 2} \arrow[dd] & \\
E^{\otimes 3} & & E^{\otimes 2}\otimes E^{*} & & E\otimes E^{*\otimes 2} & & E^{*\otimes 3} \\
 & {} & & {} & & {} &
\end{tikzcd}$$

$\mathcal{O}_E$ is the limit of the vertical arrows. $\mathcal{O}_E$ is $\mathbb{Z}$-graded

$$\mathcal{O}_E^{m} = \varinjlim_{p} E^{\otimes p+m} \otimes E^{*\otimes p}$$

where maps are
$$E^{\otimes p} \otimes E^{*\otimes q} \longrightarrow E^{\otimes p} \otimes E \otimes E^{*} \otimes E^{*\otimes q}$$
$$\alpha \otimes \beta \qquad\qquad \sum_i \alpha \otimes \varphi_i^* \otimes \varphi_i \otimes \beta$$

Now set $R = \mathcal{T}_E$ and try to relate $K_*(R)$ with $K_*(R/Re^{\perp}R)$

$$K_*(R/Re^{\perp}R) \text{ with } R/Re^{\perp}R = \mathcal{O}_E$$

47. So [illegible] I have to review Davydov. and especially the key hypothesis. Recall that one idea is that R/ReR has proj. dim 1. so you can define $K_*(R/ReR) \longrightarrow K_*(R)$ by resolution thm.

$$0 \xrightarrow{\ ReR \text{ firm}\ } Re \otimes_{eRe} eR \longrightarrow R \longrightarrow R/ReR \longrightarrow 0$$

assume that $\boxed{eR \in \mathcal{P}(eRe)}$, whence $ReR \in \mathcal{P}(R)$. So how does it go?

~~I think that $\mathcal{P}(R)$~~ You somehow define a functor ~~$\mathcal{P}(R)$~~ from $\mathcal{P}(R)$ to $\mathcal{P}(R)$.

~~So you ha~~ ~~Take Pe~~ Basic maps

$$\begin{tikzcd} K_*(eRe) \arrow[r, shift left, "Re\otimes_R -"] & K_*(R) \arrow[l, shift left, "eR\otimes_R -"] \arrow[r, shift left, "\text{hom}"] & K_*(R/ReR) \arrow[l, shift left, "\text{res.}"] \end{tikzcd}$$

$$Y \longmapsto Re\otimes_{eRe} Y_{\ P} \longmapsto P/ReP$$

$$\begin{tikzcd} \mathcal{P}(eRe) \arrow[r, shift left] & \mathcal{P}(R) \arrow[l, shift left] \arrow[r, shift left] & \mathcal{P}(R/ReR) \arrow[l, shift left, dotted, "\text{resolution}"] \end{tikzcd}$$

$$eR\otimes_R P \longleftarrow\!\shortmid\ P \qquad\qquad X$$

$$0 \longrightarrow Re\otimes_{eRe} eP \longrightarrow P \longrightarrow P/ReP \longrightarrow 0$$

This is the argument I recall

$X \in \mathcal{P}(R/ReR)$,

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

$$\mathrm{Tor}_2^R(R/ReR, R/ReR)$$

48. $0 \longrightarrow ReR \longrightarrow R \longrightarrow R/ReR \longrightarrow 0$

$$\mathrm{Tor}_1^R(R/A, R/A) \longrightarrow R/A \otimes A \longrightarrow R/A \xrightarrow{\sim} R/A \otimes_R R/A \longrightarrow 0$$

$$R/A \otimes A = A/A^2$$

So where does the ~~world~~ K.

Idea: a Hilbert space rep of $\mathcal{O}_2$ is a Hilbert space H + isom $\mathbb{C}^2 \otimes H \xrightarrow{\sim} H$. Still I don't see a connection. Map of $H \longrightarrow$ continuous function on $\{\pm 1\}^{\mathbb{N}}$ with values in H of some sort. ..

What do I know about H? In a natural way it's a module over the ring of cont. functions on $\{\pm 1\}^{\mathbb{N}}$. All you do is keep track of the finer decomposition of H.

Go to the algebra. Look at $\mathcal{T}_E$, $e = 1 - \sum \psi_i^* \psi_i$.

$$T(E) \supset \bar{T}(E) = E \otimes \bar{T}(E)$$

...

M free over $\bar{T}(E) \iff E \otimes M \xrightarrow{\sim} M$

i.e. M is an $\mathcal{O}_E$-module.

$$R = \mathcal{T}_E \ni e = 1 - \sum \psi_i^* \psi_i \quad \Big| \quad \mathcal{O}_E = R/ReR$$

Check Davydov conditions.

$$K_*(R) = K_*(eRe) \oplus K_*(\overbrace{R/ReR}^{\mathcal{O}_E}) \qquad \textbf{YES}$$

So what next?

$\mathcal{T}_E$ acts on $T(E)$ as shift operators.

49. What does e do $e = 1-\sum \varphi_i^*\varphi_i$

e is the projection onto the constants.

$$eT(E)\otimes T(E^*)e = k.$$

I think that $eR = eT(E)\otimes T(E^*) = T(E^*)$

$Re = T(E^*)$

$$e = 1-\sum_i \varphi_i^*\varphi_i$$

$$R = T(E)\otimes T(E^*)$$
$$\parallel$$
$$\mathbb{C}1 \oplus \cancel{\bar{T}(E)} \oplus \bar{T}(E^*) \oplus \cancel{\bar{T}(E)\otimes \bar{T}(E^*)}$$

$eR = eT(E^*)$ $Re = T(E^*)e$

11/19 0643 I have been studying the Toeplitz algebra $\mathcal{T}_E$ hoping to show $k \to \mathcal{T}_E$ induces an isomorphism on K_* by some variant of the proof in Pimsner's paper. One can define a map $K_*(\mathcal{T}_E) \to K_*(k)$ using excision as follows.

Given an extension $I \to L \to L/I$ where I satisfies excision in K_*, one should have a fibration

$$\mathbb{K}(I) \longrightarrow \mathbb{K}(L) \longrightarrow \mathbb{K}(L/I)$$

Given two homos. $R \rightrightarrows L$ which are congruent modulo I, there "difference" $\mathbb{K}(R) \to \mathbb{K}(L)$ is zero in $\mathbb{K}(L/I)$, whence whence we get $\mathbb{K}(R) \to \mathbb{K}(I)$

$$\begin{tikzcd}
\mathbb{K}(I) \arrow[r] \arrow[d] & FP \arrow[r, shift left] \arrow[d, dash] & \mathbb{K}(L) \arrow[l, shift left] \arrow[d, dash] \\
\mathbb{K}(I) \arrow[r] & \mathbb{K}(L) \arrow[r] & \mathbb{K}(L/I)
\end{tikzcd}$$

50. SAVE FOR IDEAS

Try for some proofs

Is $\mathcal{T}$ K-same as k

The idea would be some version of the Eilenberg trick. Put $R = k\langle x,y\rangle/(yx=1)$

$$0 \longrightarrow R \xrightarrow{\cdot y} R \longrightarrow R/Ry \longrightarrow 0$$

Let R act on $k[x]$ with $x =$ mult. by x and $y = \iota_y$. So what happens?

There should be a pr

Question. What about $\mathcal{T}/I \simeq \mathcal{J}$?

What can you say about I? $I = \mathcal{T}(1-\sum x_i y_i)\mathcal{T}$

$I = \mathcal{T}e\mathcal{T}$. Use R in place of $\mathcal{T}$.

$\mathcal{T}$ basis $x^\alpha y^\beta$ $\alpha,\beta \in \mathbb{N}$

$e = \sum 1 - x_i y_i$ kills $x^i y^j$ for $i > 0$.

$e\mathcal{T} \xleftarrow{\sim} ek[y]$ $\mathcal{T}e = k[x]e$

$e\mathcal{T}e$ spanned by $ex^i y^j e = 0$ unless $i=j=0$.

$\therefore$ $e\mathcal{T}e = k$ and $\mathcal{T}e \otimes_k e\mathcal{T} \longrightarrow \mathcal{T}e\mathcal{T}$

You probably can see that this is an isom. just by simplicity of matrices. $\mathcal{T}e \otimes_k e\mathcal{T}$ should be fin. supp. matrices.

51 11/20 1608 Aim: to understand Pimsner's calculation of K_* for the Toeplitz alg $\mathcal{T}_E$. Simplest example is $E = k$, then $\mathcal{T}_E = k\langle z, z^* \rangle / (z^*z - 1)$. The approach: Define a map $K_*(\mathcal{T}) \to K_*(k)$ by an "even" "Kasparov" module, namely, you have two repns. of $\mathcal{T}$ on $T(E)$ and $T(E) \otimes E$ and a map between them:

$$T(E) \otimes E \hookrightarrow E$$

which almosts commutes with the $\mathcal{T}$ action and almost is an isomorphism. Here almost means modulo finite rank operators. Pimsner actually handles this by constructing a graded F. Thus extends the $\mathcal{T}$ action on $T(E) \otimes E$ by zero to get a different $\mathcal{T}$ action on $T(E)$, whence we have two homos. $\mathcal{T} \rightrightarrows \mathcal{L}(T(E))$ which are congruent modulo fin rank ops. $\mathcal{K}$ Use excision then for $\mathcal{K} \subset \mathcal{L} \to \mathcal{Q}$ to get a difference map $\widetilde{BGL}(\mathcal{T}) \to \widetilde{BGL}(\mathcal{K})$.

This construction goes beyond Grothendieck's methods - use of ~~cx~~ perfect complexes, ~~so life goes on. You must be careful~~ it seems. You might study this a bit. There are various angles.

First: What sort of $\mathcal{T}_E$-module is $T(E)$? It a cyclic module $T(E)/\mathfrak{a}$ $\mathfrak{a}$ annihilator of 1 left ideal generated by ψ_i^*. $\mathfrak{a}$ has basis $\psi_\alpha \psi_\beta^*$ with $|\beta| \geq 1$. Is $T(E)$ a projective module. zz^* is an idempotent killing $1 \in T(E)$ and reproducing $T(E)E$. Seems that $\mathfrak{a} = \mathcal{T}_E zz^*$ in general

$\sum z_i z_i^*$ is idempotent $\sum z_i \underbrace{z_i^* z_j}_{\delta_{ij}} z_j^* = \sum z_i z_i^*$

reproduces $T(E)E$ and kills 1,

$$z_j^* \sum z_i z_i^* = \sum \delta_{ji} z_i^* = z_j^* .$$

So this is clear. Thus $T(E) \in \mathcal{P}(\mathcal{T}_E)$. Look carefully $0 \to \mathfrak{a} \to \mathcal{T}_E \to T(E) \to 0$. $\mathfrak{a}$ should have a right identity element. $\sum z_i z_i^*$

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{a} \arrow[r] & \mathcal{T}_E \arrow[r] \arrow[d, equal] & T(E) \arrow[r] & 0 \\
& T(E)\otimes T(E^*)E^* & T(E)\otimes T(E^*) & &
\end{tikzcd}$$

Is it true that $\mathfrak{a} \simeq \mathcal{T}_E^{\oplus n}$ as left modules.

~~This~~ Clear: $\mathcal{T}$ has basis $z^\alpha z^{*\beta}$

$\mathfrak{a}$ has basis $z^\alpha z^{*\beta}$ with $|\beta| \geq 1$. YES.

$\therefore 0 \to \mathcal{T}_E^{\oplus n} \to \mathcal{T}_E \to T(E) \to 0$ is exact

so in $K_0(\mathcal{T})$ we have $[T(E)] = (1-n)[\mathcal{T}_E]$. Hope in the end that

$$\begin{tikzcd}
K_*(\text{compact}) \arrow[rr] & & K_*(\mathcal{T}) \arrow[dl] \\
& K_*(\mathcal{O}) \arrow[ul] &
\end{tikzcd}$$

hope that $\begin{cases} K_0(\mathcal{O}) = \mathbb{Z}/(n-1) \\ K_1(\mathcal{O}) = 0 . \end{cases}$

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & K_0\mathcal{O} \arrow[d] \\
K_1\mathcal{O} \arrow[u] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Anyways compute $K_*(k) \to K_*(\mathcal{T}_E) \to K_*(k)$.
You have to understand how the exact sequence
$0 \to T(E)\otimes E \to T(E) \to k \to 0$

manages to give the correct result – what's this mean? V over $k \longmapsto \mathcal{T}_E \otimes_k V$. To compare

$$0 \longrightarrow \mathcal{T}_E \otimes E^* \longrightarrow \mathcal{T}_E \longrightarrow T(E) \longrightarrow 0$$

$$0 \longrightarrow T(E)\otimes E \longrightarrow T(E) \longrightarrow k \longrightarrow 0$$

somehow. So what do I need?

You have $\mathcal{T}_E$ acting

You have two reps of $\mathcal{T}_E$ namely $T(E)$ and $T(E)\otimes E$. You are confronted with two representations which are almost isomorphic.

Maybe the idea would be to resolve $T(E)$ and $T(E)\otimes E$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{T}_E \otimes E^* \otimes E & \longrightarrow & \mathcal{T}_E \otimes E & \longrightarrow & T(E)\otimes E & \longrightarrow & 0 \\
\\
0 & \longrightarrow & \mathcal{T}_E \otimes E^* & \longrightarrow & \mathcal{T}_E & \longrightarrow & T(E) & \longrightarrow & 0
\end{array}$$

almost isom.

So there might be a way to assemble the four ~~gadgets~~ f.g. free $\mathcal{T}_E$ modules into a complex with homology k.

11/21 0800 The problem is roughly this. You are given two homos. $A \underset{v}{\overset{u}{\rightrightarrows}} R$ congruent mod $I \subset R$

$A \longrightarrow R \times_{R/I} R$ universal case

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & R \times_{R/I} R & \longrightarrow & R & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & R/I & \longrightarrow & 0
\end{array}$$

54. Lesson: If R is a semi-direct product $R = R/I \ltimes I$, I satisfies exc. then $K_*(R) = K_*(R/I) \oplus K_*(I)$.
Somehow this is an extension of additivity

Understand the Kasparov method to handle this. Uses dilation.

Maybe I should go back to the example of $\mathcal{T}_E$. You have $\mathcal{T}_E$ acting on $T(E)$ and $T(E) \otimes E$ and a map $T(E) \otimes E \to T(E)$ which ~~does not prese~~ respects the action modulo compact operators.

Try to understand what happens on the K_0 level. What is the situation? You have zilch.

Just on the K_0 level.

Two homos. $k \rightrightarrows R$ congruent modulo I i.e. two idempotents e $\bar{e}$ congruent modulo I. Leads to what? An element of $K_0(I)$. How?

Let's try to relate the quasi-hom. $A \rightrightarrows R \to R/I$ especially what it does to idempotents in A to complexes $\mathcal{L}^1(R, I)$. You want to free yourself from complexes in some fashion.

So how do you go from a pair $e, \bar{e}$ of idempotents in R which are cong. mod I to an element of $K_0(I)$? There's a standard formula which you should remember. ~~Can you with zilch?~~

Geometry ~~...~~ $e, \bar{e}$ determine $P, \bar{P} \in \mathcal{P}(R)$ which are congruent modulo I so you have a triple $(P, \bar{P}, \alpha : P/I \xrightarrow{\sim} \bar{P}/I)$, whence $X = \mathcal{P}(R \times_{R/I} R)$. Add the complement of $\bar{P}$ i.e. $(Re, R\bar{e}, \alpha) \oplus (R\bar{e}, R\bar{e}^{\perp}, 1)$.

making $\bar{e}$ standard. So end up with $\left(R(e\oplus\bar{e}),\ R,\ \text{obvious } \alpha\right)$. What to do?

Try this: You begin with $P = Re$, $\bar{P} = R\bar{e}$ and the isom $P/IP = (R/I)e = (R/I)\bar{e} = \bar{P}/I\bar{P}$. Then you can lift to a complex $P \longrightarrow \bar{P}$. So actually what happens? You want to apply some sort of Morita equivalence later. R will be a ring of multipliers on some dual pair - maybe.

So what to do? ~~You begin with~~

Try to understand.

Let's go back to K_0'. U complex of R modules:

$$\overbrace{\mathrm{Hom}_R(U,A)}^{V} \otimes_R U \longrightarrow \mathrm{Hom}_R(U,U)$$

$f = 1 - [d,h]$ $\qquad h \in \mathrm{Hom}_R(U,U)$,

and we want f to be in the image of the above map.

We need (V,U) dual pair over A.

i.e. $U \otimes_{\mathbb{Z}} V \longrightarrow A$ $\qquad \left|\begin{array}{l} V \text{ right module cx.} \\ U \text{ left} \text{ — cx.} \end{array}\right.$

$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ $\qquad (V,U)$ over A to $\qquad (P\otimes_A U)\otimes_{\mathbb{Z}}(V\otimes_A Q)$

$\qquad (V\otimes_A Q,\ P\otimes_A U)$ over B. $\qquad \downarrow$

$\qquad P\otimes_A A \otimes_A Q \to B$

If $f \in V\otimes_R U \longrightarrow \mathrm{Hom}$

 So it seems that we might try to construct K out of complexes.

$P \otimes_A Q \twoheadrightarrow PQ = I$ Question: You have $e, \bar{e}$ over R such that $e - \bar{e} \in I$, so you will get ~~an e~~ a class in $K_0(I)$ which can be lifted to $K_0(P \otimes_A Q)$, because of nilpotent extns. But can you do it directly?

The first problem to solve I think should be going from $e, \bar{e}$ to an elt of $K_0'(I)$.

We have $\underset{1}{Re} \underset{h = \cdot e}{\overset{d = \cdot \bar{e}}{\rightleftarrows}} \underset{0}{R\bar{e}}$. Call this U

Then $\mathrm{Hom}_R(U, R) = \underset{0}{\bar{e}R} \longrightarrow \underset{-1}{eR}$. Then

$$\begin{tikzcd}
Re \arrow[r, "d = \cdot \bar{e}"] \arrow[d, "hd = \cdot \bar{e}e"'] & R\bar{e} \arrow[dl, "h = \cdot e"'] \arrow[d, "dh = \cdot e\bar{e}"] \\
Re \arrow[r, "\cdot \bar{e} = d"'] & R\bar{e}
\end{tikzcd}$$

$\bar{e} - e\bar{e}$

$\bar{e} - e\bar{e} = \; = (1-e)\bar{e}$

$1 - hd = \cdot (e - \bar{e}e) = \cdot (1-\bar{e})e$

$$f = \begin{cases} \cdot (1-\bar{e})e & \text{on } Re \\ \cdot (1-e)\bar{e} & \text{on } R\bar{e} \end{cases}$$

~~And you can do this by means of zilch~~

So what gives? You have in the (R, I) situation

57. So ~~A~~ let's begin again.

Idea: Given ~~$B = P\otimes_A Q$~~ $B = P\otimes_A Q$,
$R = \mathrm{Mult}(P, Q, \langle\ \rangle)$, $I = \mathrm{Im}\{B \to R\}$, $e, \bar e$ idemp. in R such that $e - \bar e \in I$. Then you should have a class in $K_0(I) = K_0(B) = K_0(A)$. The idea is to understand the construction of this class, — you want a symmetric construction if possible.

First step: $Re \underset{\cdot \bar e e}{\overset{\cdot e\bar e}{\rightleftarrows}} R\bar e$

$$\begin{tikzcd}
Re \arrow[r, "\cdot\bar e"] \arrow[d, "\underbrace{\cdot e - \bar e e}_{(1-\bar e)e}"'] & R\bar e \arrow[d, "\underbrace{\cdot \bar e - e\bar e}_{(1-e)\bar e}"] \arrow[dl, "\cdot e"'] \\
Re \arrow[r, "\cdot\bar e"] & R\bar e
\end{tikzcd}$$

What sort of ~~s~~ structures. Answer: You have a perfect complex $Re \xrightarrow{\cdot\bar e} R\bar e$ and a deformation of the identity map to something I-nuclear

$$\begin{array}{ccc}
\mathrm{Hom}_R(X, I)\otimes_R X & \longrightarrow & \mathrm{Hom}_R(X, X) \\
\| & & \| \\
X^\vee \otimes_R I \otimes_R X & \longrightarrow & X^\vee \otimes_R X \\
f & & 1
\end{array}$$

What other structure? X is graded — leave this aside, and go through the M equivalence. Fix step is to replace I by $B = P\otimes_A Q$. ~~Then~~

$$\check X \otimes_R P \otimes_A Q \otimes_R X$$

What is happening? You start with $e, \bar e \in R$, $e - \bar e \in I$ and ultimately construct an elt of $K_0(\bar{\bar I}) \subset K_0(\tilde I)$. On one level you have $F = 2\bar e - 1$, $\varepsilon = 2e - 1$. $F - \varepsilon \in I$. So the ring of interest is $\mathbb{Z}[\mathbb{Z}_2 * \mathbb{Z}_2]$

58 You take free product [redacted] $\mathbb{Z}e * \mathbb{Z}\bar{e}$

has basis $e, \bar{e}, e\bar{e}, \bar{e}e$, etc. ~~Anyway what~~

~~free product~~ $\mathbb{Z}e$ $\mathbb{Z}\bar{e}$ Rather arrows

Ideal generated by $e-\bar{e}$ ~~consisting~~ of basis

~~$e-\bar{e}$, $e\bar{e}-\bar{e}e$, $e\bar{e}e-\bar{e}ee$, ...~~

better basis $\cancel{1}$ de de^2

e ede ede^2

Q So what am I going to do?

$$Q\tilde{A} = \tilde{A} * \tilde{A} \supset \underset{q\tilde{A}}{\underset{\parallel}{J}} \supset J^2 \supset \cdots$$

So if you forget 1 You have the ideal J.

Note that $Q\tilde{A} \supset J$ There is a canon element elt in $K_0(J) = \mathrm{Ker}\{K_0(Q\tilde{A}) \to K_0(\tilde{A})\}$

Look at the universal case

$$R = \mathbb{Z}[e] * \mathbb{Z}[\bar{e}] = Q(\tilde{\mathbb{Z}e}) = \tilde{\mathbb{Z}e} \ltimes q(\tilde{\mathbb{Z}e})$$

Saturday 11/22/97. You continue with the Toeplitz algebra $\mathcal{T}_E$. You are led to study the map on K-theory associated to a quasi-hom. $A \rightrightarrows R \supset I$, where I satisfies excision. ~~The first~~ first case: $K_0(A) \longrightarrow K_0(I)$. Take $A = \mathbb{Z}e$. Given E in $\mathcal{P}(\tilde{A})$ get two objects in $\mathcal{P}(R)$ together with isom modulo I: $P_0, P_1, \alpha : P_0/IP_0 \to P_1/IP_1$ by Milnor get $M(P_0, P_1, \alpha) \in \mathcal{P}(R \times_{R/I} R)$, have split extn. $I \longrightarrow R \times_{R/I} R \xrightarrow{pr_2} R$ so $[M(P_0,P_1,\alpha)] - [M(P_1,P_1,1)] \in K_0(I)$. Note that $M(P_1,P_1,1) = \Delta_*(P_1) = (R \times_{R/I} R) \otimes_R P_1$. So ~~you first add~~ this difference can be found by first adding $M(P_1, Q_1, 1)$ to $M(P_0, P_1, \alpha)$ to make P_1 free and then ~~getting~~ the difference class is ~~rep by~~ $[M(P_0 \oplus Q_1, \underbrace{P_1 \oplus Q_1}, \alpha \oplus 1)] - [R^k, R^k, 1]$. Somehow you will actually a $P \in \mathcal{P}(R)^{R^k}$ equipped with an isom $P/IP \xrightarrow{\sim} (R/I)^k$, and

The problem: Given $e, \bar{e}$ write down an idempotent over $\tilde{J}$, probably a 2×2 matrix, describing the difference class.

$$Re \xrightarrow{\cdot\bar{e}} R\bar{e}$$

$$Re \oplus R(1-\bar{e}) \xrightarrow{\begin{pmatrix}\cdot\bar{e} & 0\\ 0 & 1\end{pmatrix}} R$$

Again $Re \xrightarrow{\cdot\bar{e}} R\bar{e}$ two objs of $\mathcal{P}(R)$ + isom. mod I.

$$Re \oplus R(1-\bar{e}) \longrightarrow R\bar{e} \oplus R(1-\bar{e}) = R$$

Recall argument. Given $(P_0, P_1, \alpha : P_0/IP_0 \xrightarrow{\sim} P_1/IP_1)$

Basic idea was to form the fibre products $M(P_0, P_1, \alpha)$. To see f.g. proj

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you will be able to show that the fibre product situation
$$\begin{array}{ccc} \tilde{I} & \longrightarrow & R \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & R/I \end{array}$$
yields an object of $\mathcal{P}(\tilde{I})$.

~~So suppose $e, \bar{e}$ idempotents in R, $e - \bar{e} \in I$~~

$$\begin{array}{ccc} F & & \longrightarrow \mathbb{Z}^k \\ \downarrow & & \downarrow \\ P & \longrightarrow & P/IP \simeq R/I^k \end{array}$$

So now what? Given $P \in \mathcal{P}(R)$

to show $F \in \mathcal{P}(\tilde{I})$. Need maps between F and $\tilde{I}$

Start with

$$\begin{array}{ccccccccc} & & & & & & R^k & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & IP & \longrightarrow & P & \longrightarrow & (R/I)^k & \longrightarrow & 0 \end{array}$$

$$0 \longrightarrow P^\vee I \longrightarrow P^\vee \longrightarrow (R/I)^k$$

get $p_i \in P$ $\quad q_i \in P^\vee \quad i = 1, .., k$

such that $\langle p_i, q_j \rangle \equiv \delta_{ij}$ mod I

$$1 - \sum q_i p_i : P \longrightarrow IP$$

60. Take $k=1$. $p \in P$ $q \in \check{P}$

such that $pq \in 1 + I$ $1 - qp$?

Try again

$$\begin{array}{ccccc} P/IP & \xrightarrow{\sim} & (R/I)^k & \xrightarrow{\sim} & P/IP \\ \uparrow & & \uparrow & & \uparrow \\ P & \xrightarrow{q} & R^k & \xrightarrow{p} & P \end{array}$$

$$pq \equiv 1_P \qquad qp \equiv 1_{R^k} \qquad \text{mod } I^2$$

$1_P - pq = uv : P \xrightarrow{v} IP \subset P \xrightarrow{u} IP \subset V$

$1_{R^{\oplus k}} - qp = xy : R^{\oplus k} \xrightarrow{y} I^{\oplus k} \subset R^{\oplus k} \xrightarrow{x} I^{\oplus k} \subset R^{\oplus k}$

what does $1 = pq + uv$ mean?

$$P \xrightarrow{\begin{pmatrix} q \\ v \end{pmatrix}} \begin{pmatrix} R^{\oplus k} \\ P \end{pmatrix} \xrightarrow{(p \;\; u)} P$$

Geometrically you have E over X trivialized over Y take the triv. $E_Y \xrightarrow{\sim} \mathbb{R}^k_Y$ and extend to maps $E \rightleftarrows \mathbb{R}^k_X$ which are inverse isos. on Y.

So what is going on? You have P, and

$P/IP \xrightarrow{\sim} (R/I)^{\oplus k}$

$P \in \mathcal{P}(R)$ $P/IP \simeq (R/I)^{\oplus n}$. Lift

$$\begin{array}{ccccc} P & \xrightarrow{q} & R^{\oplus n} & \xrightarrow{p} & P \\ \downarrow & & \downarrow & & \downarrow \\ P/IP & \xrightarrow{\bar{q}} & (R/I)^{\oplus n} & \xrightarrow{\bar{p}} & P/IP \end{array}$$

Then

$1 - pq : P \longrightarrow IP$

$1 - qp : R^{\oplus n} \longrightarrow IR^{\oplus n}$

61. Assume you can factor

$$\begin{array}{ccc} P & \xrightarrow{1-pq} & P \\ & {\scriptstyle v}\searrow & \cup \\ & & IP \\ & & \uparrow {\scriptstyle u} \\ & & IR^{\oplus m} \subset R^{\oplus m} \end{array} \qquad R^{\oplus n} \xrightarrow{1-qp} R^{\oplus n}$$

It's a little confusing but you want

$$P \xrightarrow{\begin{pmatrix} q \\ v \end{pmatrix}} \begin{matrix} R^{\oplus n} \\ \oplus \\ R^{\oplus m} \end{matrix} \xrightarrow{(p \quad u)} P \qquad pq + uv = 1_P$$

So you have an idempotent $\begin{pmatrix} q \\ v \end{pmatrix}(p \quad u) = \begin{pmatrix} qp & qu \\ vp & vu \end{pmatrix}$ on $R^{\oplus(n+m)}$. If you reduce modulo I, you would like $\begin{pmatrix} q \\ v \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad (p \quad u) \rightsquigarrow (1 \quad 0)$. You thus want u, v to be cong 0 modulo I. [redacted]

Do this for $e, \bar{e}$. But first given p, q

$$P \xrightarrow{q} R^{\oplus n} \xrightarrow{p} P \qquad \overbrace{1-pq}^{y} : P \to IP$$

$$\overbrace{1-qp}^{x} = x : R^{\oplus n} \to IR^{\oplus n}$$

Replace q by $q_n = q(1-pq)$.

$$q_n = q(1+y+\cdots+y^{n-1}) = (1+x+\cdots+x^{n-1})q$$

Then $1 - pq_n = 1 - \overbrace{pq}^{1-y}(1+y+\cdots+y^{n-1}) = 1-(1-y^n) = y^n$

Sim $1 - q_n p = x^n$. $\quad x = 1-qp$

So $1 - q_2 p = x^2$.

$$\begin{pmatrix} q_2 \\ y \end{pmatrix}(p \quad y) = \begin{pmatrix} q_2 p & q_2 y \\ yp & y^2 \end{pmatrix}$$

$$(p \quad y)\begin{pmatrix} q_2 \\ y \end{pmatrix} = pq_2 + y^2 = pq_2 + 1 - pq_2 = 1$$

62. So continue to $e, \bar e \in R$ congruent mod I.

$$P = Re \oplus R\bar e^{\perp} \xrightarrow{g} R\bar e \oplus R\bar e^{\perp} = R$$

$$g = \begin{pmatrix} \bar e e & \\ & 1-\bar e \end{pmatrix} \qquad p = \begin{pmatrix} e\bar e & 0 \\ 0 & 1-\bar e \end{pmatrix}$$

$$1 - gp = -\begin{pmatrix} \bar e e \bar e & \\ & 1-\bar e \end{pmatrix} + \begin{pmatrix} \bar e & \\ & 1-\bar e \end{pmatrix} = \begin{pmatrix} e - \bar e e \bar e & \\ & 0 \end{pmatrix}$$

$$1 - pg = -\begin{pmatrix} e\bar e e & 0 \\ 0 & 1-\bar e \end{pmatrix} + \begin{pmatrix} \bar e & 0 \\ 0 & 1-\bar e \end{pmatrix} = \begin{pmatrix} \bar e - e\bar e e & 0 \\ 0 & 0 \end{pmatrix}$$

~~$x = e - \bar e e \bar e p$~~ $\quad P \longleftarrow \quad R \xleftarrow{g} P$

$$eR \oplus (1-\bar e)R \xleftarrow{\begin{pmatrix} e\bar e & 0 \\ 0 & 1-\bar e \end{pmatrix}} \bar e R \oplus (1-\bar e)R \xleftarrow{\begin{pmatrix} \bar e e & 0 \\ 0 & 1-\bar e \end{pmatrix}} eR \oplus (1-\bar e)R$$

$$p = \begin{pmatrix} e\bar e & \\ & 1-\bar e \end{pmatrix} \qquad g = \begin{pmatrix} \bar e e & \\ & 1-\bar e \end{pmatrix}$$

$$gp = \begin{pmatrix} \bar e e \bar e & \\ & 1-\bar e \end{pmatrix} \qquad \cancel{x} = 1 - gp = \begin{pmatrix} \bar e - \bar e e \bar e & 0 \\ 0 & 0 \end{pmatrix}$$

$$pg = \begin{pmatrix} e\bar e e & \\ & 1-\bar e \end{pmatrix} \qquad y = 1 - pg = \begin{pmatrix} e - e\bar e e & 0 \\ 0 & 0 \end{pmatrix}$$

So the natural question is what is

$$e - e\bar e e = e(e - \bar e)e = -ede$$

63. So in this situation you have to perform ~~lots of things~~ one iteration

$$1-qp = x \qquad 1-pq = y$$ invertible modulo an ideal

and you want an idempotent

Odd case $\quad g, g^{-1}$ in R/I lift to p, q

$$x = 1-qp \qquad y = 1-pq. \qquad q_n = q(1+y+\cdots+y^{n-1})$$
$$= (1+x+\cdots+x^{n-1})q$$

$$1-q_n p = 1-(1+x+\cdots+x^{n-1})(1-x) = x^n$$
$$1-pq_n = y^n$$

Idempotent: $\quad 1 = q_2 p + x^2 = \begin{pmatrix} q_2 & x \end{pmatrix}\begin{pmatrix} p \\ x \end{pmatrix}$

$$e = \begin{pmatrix} p \\ x \end{pmatrix}\begin{pmatrix} q_2 & x \end{pmatrix} = \begin{pmatrix} pq_2 & px \\ xq_2 & x^2 \end{pmatrix}$$

Or $\quad 1 = pq_2 + y^2 = \begin{pmatrix} p & y \end{pmatrix}\begin{pmatrix} q_2 \\ y \end{pmatrix}$

$$e = \begin{pmatrix} q_2 \\ y \end{pmatrix}\begin{pmatrix} p & y \end{pmatrix} = \begin{pmatrix} q_2 p & q_2 y \\ yp & y^2 \end{pmatrix}.$$

$e, \bar{e} \in R \qquad e-\bar{e} \in I$

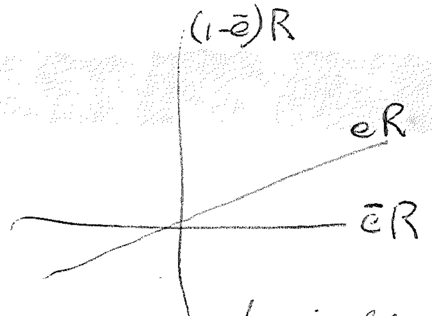
$$\begin{matrix} eR \\ \oplus \\ (1-\bar{e})R \end{matrix} \xrightarrow[q]{\begin{pmatrix} \bar{e} & 1-\bar{e} \end{pmatrix}} \begin{matrix} \bar{e}R \\ \oplus \\ (1-\bar{e})R \end{matrix} \xrightarrow[p]{\begin{pmatrix} e & 0 \\ 0 & 1-\bar{e} \end{pmatrix}} \begin{matrix} eR \\ \oplus \\ (1-\bar{e})R \end{matrix}$$

$$1-qp = \begin{pmatrix} \bar{e}-\bar{e}e & 0 \\ 0 & 0 \end{pmatrix} \qquad 1-pq = \begin{pmatrix} e-e\bar{e} & 0 \\ 0 & 0 \end{pmatrix} \text{ on } \begin{matrix} eR \\ \oplus \\ (1-\bar{e})R \end{matrix}$$

on $\bar{e}R \oplus (1-\bar{e})R = R$

Is there a subspace way to view this. Namely You have a splitting $R = \bar{e}R \oplus (1-\bar{e})R$ and a subspace

 eR close to $\bar{e}R$.



~~So what~~

Can you describe what's happening geometrically.

Start again. You have $e, \bar{e} \ni e-\bar{e} \in I$

Interpretation $(eR, \bar{e}R, \alpha : eR/I \simeq \bar{e}R/I)$

Add $((1-e)R, (1-\bar{e})R, 1)$ to get

$(P, R, P/PI \simeq R/I)$. Then lift to

$P \underset{p}{\overset{q}{\rightleftarrows}} R$ such that p, q are inverses modulo I

Missing the near final step. From (P, R, α) you get a fg projective $\tilde{I}$ module: $(P_{\mathbb{Z}}, \tilde{I}, \alpha)$

The point is to prove that $P_{\mathbb{Z}} = \{\xi \in P \mid \alpha(\xi + PI) \in \mathbb{Z}\}$ is a summand of $\tilde{I}^{\oplus k}$ - maybe $k=2$. P is a summand of R^2. Want $P_{\mathbb{Z}}$ to be summand of $\tilde{I}^{\oplus k}$. Recall the idea. You lift α and α^{-1} to p, q and then p, q take care of things mod I, so

$$\begin{tikzcd} P & R \arrow[l, "p"'] & P \arrow[l, "q"'] \arrow[ll, bend left, "1-pq"] \end{tikzcd}$$

You need to see that $P_{\mathbb{Z}}$ is a summand $\tilde{I}^{\oplus k}$.

You want P to be a summand of $R^{\oplus k}$ in such a way that $P_{\mathbb{Z}}$ is a summand of $\tilde{I}^{\oplus k}$.

$1-pq : P \longrightarrow IP$ I can factor this into $P \leftarrow R^{\oplus 4} \hookleftarrow I^{\oplus 4} \leftarrow P$

 So you learn that u, v must go somewhere else than P.

First idea: Take $(L,\ M,\ \alpha : L/LI \xrightarrow{\sim} M/MI)$

lift α, α^{-1} to $L \underset{q}{\overset{p}{\rightleftarrows}} M$, then form

$$1 - qp = x \qquad \begin{tikzcd} L & M \arrow[l, "q"'] & L \arrow[l, "p"'] \arrow[ll, bend left, "x"] \end{tikzcd}$$

$$1 - pq = y \qquad \begin{tikzcd} M & L \arrow[l, "p"'] & M \arrow[l, "q"'] \arrow[ll, bend left, "y"] \end{tikzcd}$$

get
$$\begin{pmatrix} q & \sqrt{x} \end{pmatrix} \begin{pmatrix} p \\ \sqrt{x} \end{pmatrix} = qp + x = 1_L$$

$$L \xleftarrow{\begin{pmatrix} q & \sqrt{x} \end{pmatrix}} \begin{matrix} M \\ \oplus \\ L \end{matrix} \xleftarrow{\begin{pmatrix} p \\ \sqrt{x} \end{pmatrix}} L$$

or
$$\begin{pmatrix} p & \sqrt{y} \end{pmatrix} \begin{pmatrix} q \\ \sqrt{y} \end{pmatrix} = pq + y = 1_M$$

$$M \xleftarrow{\begin{pmatrix} p & \sqrt{y} \end{pmatrix}} \begin{matrix} L \\ \oplus \\ M \end{matrix} \xleftarrow{\begin{pmatrix} q \\ \sqrt{y} \end{pmatrix}} M$$

So what happens here is that we obtain two idempotents on $L \oplus M$.

$$\begin{pmatrix} p \\ \sqrt{x} \end{pmatrix} \begin{pmatrix} q & \sqrt{x} \end{pmatrix} = \begin{pmatrix} pq & p\sqrt{x} \\ \sqrt{x}q & x \end{pmatrix} \quad \text{on } \begin{matrix} M \\ \oplus \\ L \end{matrix}$$

$$\cancel{\begin{pmatrix} q \\ \sqrt{y} \end{pmatrix} \begin{pmatrix} p & \sqrt{y} \end{pmatrix} = \begin{pmatrix} qp & q\sqrt{y} \\ \sqrt{y}p & y \end{pmatrix} \quad \text{on } \begin{matrix} L \\ \oplus \\ M \end{matrix}}$$

change sign
$$\begin{pmatrix} -\sqrt{y} \\ q \end{pmatrix} \begin{pmatrix} -\sqrt{y} & p \end{pmatrix} = \begin{pmatrix} y & -\sqrt{y}p \\ -q\sqrt{y} & qp \end{pmatrix} \quad \text{on } \begin{matrix} M \\ \oplus \\ L \end{matrix}$$

 ~~Then~~ Since $p\sqrt{x} = p\sqrt{1-qp} = \sqrt{1-pq}\,p = \sqrt{y}\,p$
these two idempotents are complementary.

These formulas above are not right for my problem.

Go back to $(eR, \bar{e}R, \alpha : eR/I \simeq \bar{e}R/I)$
add $((1-\bar{e})R, (1-\bar{e})R, 1)$ to get $(eR \oplus (1-\bar{e})R,\ R,\ \alpha \oplus 1_{(1-\bar{e})R})$

$$\begin{array}{ccccc} L & \xleftarrow{\ q\ } & M & \xleftarrow{\ p\ } & L \\ \bar{e}R & \begin{pmatrix} \bar{e} & 0 \\ 0 & 1-\bar{e} \end{pmatrix} & eR & \begin{pmatrix} e & 0 \\ 0 & 1-\bar{e} \end{pmatrix} & \bar{e}R \\ \oplus & \longleftarrow & \oplus & \longleftarrow & \oplus \\ (1-\bar{e})R & & (1-\bar{e})R & & (1-\bar{e})R \end{array}$$

$$x = 1 - qp = \begin{pmatrix} \bar{e} & 0 \\ 0 & 1-\bar{e} \end{pmatrix} - \begin{pmatrix} \bar{e}e & 0 \\ 0 & 1-\bar{e} \end{pmatrix} = \begin{pmatrix} \bar{e}-\bar{e}e & 0 \\ 0 & 0 \end{pmatrix} \text{ on } \begin{array}{c} \bar{e}R \\ \oplus \\ (1-\bar{e})R \end{array}$$

$\bar{e}(1-e)\bar{e}$ in $\bar{e}R\bar{e}$ left acting on $\bar{e}R$

$\bar{e}$ $\qquad \bar{e}(\bar{e}-e)\bar{e} \in \bar{e}I\bar{e}$

seems to work, but we need to improve the parametrix.

So ~~take~~ replace q by $(1+x)q = q + xq$

$$\begin{pmatrix} \bar{e} & 0 \\ 0 & 1-\bar{e} \end{pmatrix} + \begin{pmatrix} \bar{e}(1-e) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \bar{e} & 0 \\ 0 & 1-\bar{e} \end{pmatrix}$$

$$q_2 = \begin{pmatrix} \overbrace{\bar{e} + \bar{e}(1-e)\bar{e}}^{2\bar{e}-\bar{e}e\bar{e}} & \\ & 1-\bar{e} \end{pmatrix}$$

$1-q_2p$ has UL corner $\bar{e} - (\bar{e} + \bar{e}(1-e)\bar{e})e$
$= \bar{e} - (\bar{e} + \bar{e} - \bar{e}e\bar{e})e$
$= \bar{e} - 2\bar{e}e + \bar{e}e\bar{e}e$

$$\bar{e}(1-e)\bar{e}(1-e) = \bar{e}\big((1-e)(\bar{e}-\bar{e}e)\big) = \bar{e}(\bar{e}-\bar{e}e-e\bar{e}+e\bar{e}e)$$

$$g = \begin{pmatrix} \bar e e & 0 \\ 0 & 1-\bar e \end{pmatrix} \qquad p = \begin{pmatrix} e\bar e & 0 \\ 0 & 1-\bar e \end{pmatrix}$$

$$x = 1 - gp = \begin{pmatrix} \bar e - \bar e e \bar e & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \bar e(\bar e - e)\bar e & 0 \\ 0 & 0 \end{pmatrix}$$

$$g_2 = (1+x)g \qquad 1 - g_2 p = 1 - (1+x)\overset{1-x}{gp} = 1-(1-x^2) = x^2$$

$$= \left(\bar e + \bar e(\bar e - e)\bar e\right)\bar e e$$

$$g_2 p = \left(\bar e + \bar e(\bar e - e)\bar e\right)e\bar e = \left(2\bar e - \bar e e \bar e\right)e\bar e$$

$$=$$

$$1 - g_2 p = \bar e - 2\bar e e \bar e + \bar e e \bar e e \bar e$$

$$x^2 = \left(\bar e - \bar e e \bar e\right)\left(\bar e - \bar e e \bar e\right)$$

$$= \bar e - 2\bar e e \bar e + \bar e e \bar e e \bar e$$

OK it seems that things are simple, namely, you have

$$\bar e R \xleftarrow{\bar e e = g} eR \xleftarrow{e\bar e = p} \bar e R$$

$$x = 1 - gp = \bar e - \bar e e \bar e$$

$$g_2 = (1+x)g = g + xg = \bar e e + (\bar e - \bar e e \bar e)\bar e e$$

$$= 2\bar e e - \bar e e \bar e e$$

$$1 - g_2 p = \bar e - \left(2\bar e e - \bar e e \bar e e\right)e\bar e$$

$$= \bar e - 2\bar e e \bar e + \bar e e \bar e e \bar e$$

$$= \left(\bar e - \bar e e \bar e\right)^2 \qquad 1 = g_2 p + x^2$$

Thus $\bar e R \xleftarrow{(g_2\ x)} \begin{matrix} eR \\ \oplus \\ \bar e R \end{matrix} \xleftarrow{\binom{p}{x}} \bar e R$