•a)

invertible on W_λ for $\lambda \neq 0$. ~~So the key~~ It remains to analyze the case $\lambda = 0$. Assume $V = V_0$ etc. Filter $F_n V = \{ v \mid (fg)^n v = 0 \}$ sim for W.

$n=1$. Then we have a supercomplex and there should be four indecomposables

$$\begin{tikzcd} k \arrow[r, shift right] & 0 \arrow[l, shift right] \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r, shift right] & k \arrow[l, shift right] \end{tikzcd}$$

$$\begin{tikzcd} k \arrow[r, shift right, "0"'] & k \arrow[l, shift right, "1"'] \end{tikzcd}$$

$$\begin{tikzcd} k \arrow[r, shift right, "1"'] & k \arrow[l, shift right, "0"'] \end{tikzcd}$$

Why?

$$\begin{array}{cc} X_0 & X_1 \\ \cup & \cup \\ Z_0 & Z_1 \\ \cup & \cup \\ B_0 & B_1 \end{array}$$

you split these filtrations: $X_i = \underbrace{B_i \oplus \tilde{H}_i}_{Z_i} \oplus C_i$, then ~~this~~ X splits into the supercomplexes

$$\begin{tikzcd} \tilde{H}_0 \arrow[r, shift right] & 0 \arrow[l, shift right] \end{tikzcd} \qquad \begin{tikzcd} C_0 \arrow[r, shift right, "\sim"'] & B_1 \arrow[l, shift right, "0"'] \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r, shift right] & \tilde{H}_1 \arrow[l, shift right] \end{tikzcd} \qquad \begin{tikzcd} B_0 \arrow[r, shift right] & C_1 \arrow[l, shift right, "\sim"'] \end{tikzcd}$$

Now you want to understand the general case. One possibility is Jordan form for $X = \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix}$ on $\begin{matrix} V \\ \oplus \\ W \end{matrix}$, keeping track of the $\mathbb{Z}_2$-action.

Actually you work with $k[\varepsilon, X]$ $\quad \varepsilon^2 = 1$

$\varepsilon X = -X\varepsilon$.

Equivariant torsion sheaves on A^1 for $\mathbb{Z}_2$-action $\lambda \mapsto -\lambda$. torsion at $\lambda = 0$. Let M be such a module. Look at the filtration $M \supset XM \supset X^2 M \supset \dots$ You ~~can split~~ choose a complement: $M = L \oplus XM$ and average to get equivariant splitting of

$$0 \longrightarrow XM \longrightarrow M \longrightarrow M/XM \longrightarrow 0$$

b)

You must be more careful. What do you do in the $k[X]$ case? You look at the n such that $X^n M \neq 0$ and $X^{n+1}M = 0$.

$$M/XM \xrightarrow{X^n} {}_X M$$

$$M/XM \twoheadrightarrow M/X^nM \xrightarrow{\sim} X^nM \subseteq {}_XM$$

Idea is that to take a proj hull $(k[X]/(X^{n+1}))^\alpha \twoheadrightarrow M/X^nM$ and an inj. hull

$$X^nM \hookrightarrow (k[X]/(X^{n+1}))^\alpha.$$

Actually since M/X^nM is killed by X, these hulls are just given by tensor product. Any choose lifting

$$\begin{tikzcd} k[X]/(X^{n+1}) \otimes_k M/X^nM \arrow[r] \arrow[rr, bend left, "\text{reduction mod } X"] & M \arrow[r] & M/X^nM \end{tikzcd}$$

and an extension

$$\begin{tikzcd} M \arrow[r] & \text{inj hull of } X^nM \arrow[d, phantom, "\parallel"] \\ & k[X]/(X^{n+1}) \otimes_k X^nM \end{tikzcd}$$

Make this lifting + extension equivariant by ~~lifting~~ averaging. Then map from proj hull to inj hull should be an isomorphism, which means ~~we split~~ this module splits off M. This should mean that the four cases arising for $n=0$ extend to each $n \geq 0$. ~~So we have four~~ So for $\lambda = 0$

we have four cases for each $n \geq 0$. ~~This~~ c)
~~corresponds to~~

Cases: one for ~~each~~ all $\lambda \neq 0, \infty$ and $n \geq 0$
four for $\lambda = 0$ and all $n \geq 0$

K cases: One for all $\lambda \neq 0, \infty$ and $n \geq 0$

---

so what do we find? It seems that

$\bullet \rightleftarrows \bullet$ cases: for $\lambda \neq 0, \infty$ one for all $n \geq 0$.
for $\lambda = 0$ four cases and all $n \geq 0$.

~~K~~ $\bullet \rightrightarrows \bullet$ cases: for $\lambda \in \mathbb{P}^1$ one case for all $n \geq 0$
+ two cases for all $n \geq 0$.

So there is the possibility of a correspondence between ~~[illegible]~~ indecomposables. The four cases at $\lambda = 0$ correspond to the torsion ones at $\lambda = 0, \infty$ and the special ones $\xi^+(\mathcal{O}(n))$, $\xi^-(\mathcal{O}(-n-1))$. So what sort of thing do I look at next. These two examples look pretty different from the Serre subcategory viewpoint

What about the derived categories? Also tilting? Tilting object T has proj dim ≤ 1. ~~The~~ other condition is $\mathrm{Ext}^j(T,T) = 0$ for $j \neq 0$. and it generates the D.C. ~~The next question is whether~~ Wait: did you decide ~~that~~ if the ~~situation~~ ideal before is h-unital? ~~So~~ Recall, we have $V = \bigoplus_x P^x$ $= P^a \oplus \bigoplus_{x \neq a} P^x = Pe^\perp \oplus Pe \simeq Re^\perp \oplus Re$. The question is whether $Re \overset{L}{\otimes}_{eRe} eR \xrightarrow{\sim} ReR$. I think this amounts to $e^\perp Re \overset{L}{\otimes}_{eRe} eRe^\perp \xrightarrow{\sim} e^\perp ReRe^\perp$

To decide if $e^\perp Re \overset{L}{\otimes}_{eRe} eRe^\perp \to eRe^\perp Re$ d)

$e = \bigoplus_{x \neq a} e^x \qquad e^\perp = e^a \qquad R = \bigoplus_{x,y} e^x R e^y$

$e^x R e^y = \mathrm{Hom}_R(Re^x, Re^y) = \mathrm{Hom}_a(P^x, P^y) = P^y_x$

spanned by paths from y to x.

Now I think I learned that eRe is meq the path algebra of the quiver with a deleted and one new arrow from x to y for each pair of arrows $x \to a$, $a \to y$. Why? $eRe = \mathrm{Hom}_R(Re, Re)$ $= \bigoplus_{\substack{x \neq a \\ y \neq a}} P^y_x$ path alg of paths in original quiver Q between two points $x, y \neq a$. Now such a path of length n is an admissible word in arrows of Q and if a occurs in p places then we get

$$\begin{tikzcd}
x_0 \arrow[r] & x_1 \arrow[r] & x_2 \arrow[r] \arrow[d, phantom, "\parallel"] & x_3 \arrow[r] & x_4 \arrow[r] & x_5 \arrow[r] \arrow[d, phantom, "\parallel"] & x_6 \arrow[r] & x_7 \\
 & & a & & & a & &
\end{tikzcd}$$

which gives a word

$$x_0 \to x_1 \longrightarrow x_3 \to x_4 \longrightarrow x_6 \to x_7$$

in the deleted quiver. Note this assumes no arrows $a \circlearrowleft$. What happens if there is such a loop? Actually I know that this Morita equivalence should hold, since I have calculated $\mathcal{A}/\mathcal{S} = {}^\perp\mathcal{S} = \mathcal{S}^\perp$. So now I am interested in the actual details. ~~What happens~~ Not clear that my calculation depends on the assumption that there are no loops at a.

$$\begin{tikzcd} x \arrow[r] & a \arrow[r] \arrow[loop above] & y \end{tikzcd} \quad \Big| \quad M_x \oplus M_a \longrightarrow M_a \longrightarrow M_y \oplus M_a$$

$$M_x \to M_a \to M_y$$

so ~~words involve a~~ lump all a stuff together

So what ~~do~~ I learn is that a quiver is the c)
analogue of a free monoid. Actually a quiver seems to be the same as an oriented graph, equivalently a diagram of sets $X_1 \overset{s}{\underset{t}{\rightrightarrows}} X_0$

X_0 set of vertices s = source
X_1 set of arrows t = target

Then [illegible] $X_1 \underset{(s,t)}{\times} \overset{n}{\cdots} \underset{(s,t)}{\times} X_1$ paths of lengths n.

2.

~~So~~ Now what to do? ~~But have something left~~

Analogue of the ~~[illegible]~~ tensor algebra of a bimodule $T_S(M)$ where $S = k[X_0]$, $M = k[X_1]$.

Go back to the vertex a. I am after $R = \overbrace{Re^{\perp}}^{P^a} \oplus \overset{\bigoplus_{x \neq a} P^x}{Re}$

Idempotent ideal ReR should be meq the deleted quiver. You want to understand this directly.

Context $\begin{pmatrix} ReR & Re \\ eR & eRe \end{pmatrix}$ No! eRe is ~~[illegible]~~ not the path algebra of the deleted quiver. So we need a meq to R'. An idea is ~~that~~ to look at paths, the monoids and bireps. ~~So~~ eRe has basis paths $x \rightsquigarrow y$ where $x, y \neq a$. The idea is pretty clear ~~[illegible]~~ namely any path can be written as a composition of paths outside a and those at a.

$\xrightarrow{x_1} \xrightarrow{x_2} a \xrightarrow{u_3} a \xrightarrow{u_4} a \xrightarrow{x_5}$ This is eRe.

Then we want to form inside eRe a left and a right ideal. ~~The left ide~~ The right ideal should be stable under R' on the left. The idea is to take a path ~~xp~~

f)

So review. $R = Re^{\perp} \oplus Re = P^a \oplus \bigoplus_{x \neq a} P^x$

$eRe = \bigoplus_{\substack{x,y \\ \neq a}} \underbrace{e^x R e^y}$ $\quad Hom_R(Re^x, Re^y) = P^y_x$ paths $x \longleftarrow y$

I want this to be meq to R' the path alg of the reduced quiver.

The problem: Assume no $\overset{\curvearrowright}{a}$ loops to simplify. We want a Meq between eRe and R', the path alg of the changed quiver Q'. Now what should the basic idea be? Suppose given a basis element for eRe, that is a path $y \longleftarrow x$ in Q between vertices $x, y \neq a$. Then you can assign to this path a basis element of R'. Why? each time an a occurs it does so

$$y \longleftarrow a \longleftarrow x$$

Thus it seems that in this case $R' = eRe$

Back to mathematics. Suppose no loops at a. Then $l_a \Rightarrow Mod(k)$ and conversely. The reduced quiver Q' ~~has vertices $Q_0 - a$ is the quiver~~ is the quiver obtained from Q by deleting the vertex a and adding extra arrows, one $\beta\alpha$ for each pair of arrows $x \xrightarrow{\alpha} a$, $a \xrightarrow{\beta} y$. Here $\alpha(\beta)$ is any arrow with target (source) a. So what happens next? Look at $eRe = \bigoplus_{\substack{x,y \\ \neq a}} e^x R e^y$ paths from y to x in R.

introduce symbols for the arrows. Four types of arrows $x \longrightarrow x'$ $\quad x \longrightarrow a$ $\quad a \longrightarrow x$ $\quad (a \longrightarrow a$ eventually$)$

$\longleftarrow y$

Anyway when there are no loops at a g)
there are three types of arrows

$\alpha : x \longrightarrow a$

$\beta : a \longrightarrow y$ here x, y are $\neq a$

$\gamma : x \longrightarrow y$

Any arrow $x \rightsquigarrow y$ is ~~a~~ an ad

MESA commuter between Phoenix + Tucson.

Review: Start with a quiver Q, let R be its path algebra, $R = \bigoplus_{x,y \text{ vertices}} e^x R e^y$, $e^x R e^y$ has basis the paths from y to x, $\mathcal{A} = \mathcal{M}od(R)$. a vertex of Q, $\mathcal{S}_a$ = Serre subcat of modules supported at a. In general $\mathcal{S}_a = \mathcal{M}od(T(\substack{\text{arrows} \\ a \to a}))$, but I first consider the case where no such arrows. $R = Re^\perp \oplus Re$ $\quad e^\perp = e^a \quad e = \bigoplus_{x \neq a} e^x$. Reduced quiver $Q' = Q/a$ has for vertices $x \neq a$ in Q and new arrows for each pair of arrows $x \xrightarrow{\alpha} a$, $a \xrightarrow{\beta} y$ in Q with x, y in Q'. R' = path alg of Q'. Claim that $R' = eRe$. Why because eRe has basis described by words involving arrows $\alpha : x \to a$, $\beta : a \to y$, $\gamma : x \to y$ where the ~~source~~ target ~~must mo~~ of each map matches the source of the following one.

$$\cdots \beta_2 \alpha_2 \gamma_q \cdots \gamma_{p+1} \beta_1 \alpha_1 \gamma_p \cdots \gamma_1$$

Whenever an α appears it must be followed by a β. So these words can be described equivalently as words in the reduced quiver.

Consider now the general case where b) there are loops at a. In this case we have these loops to consider in eRe but not R'.

I have to review the situation. Anyway. I think that the reduced quiver still describes the quotient cat. If M is a reduced module:

$$\begin{tikzcd} M_{x_i} \arrow[r, "\alpha_i"] & M_a \arrow[loop above] \arrow[r, "\beta_j"] & M_{y_j} \end{tikzcd}$$

What is a torsion submodule? This must be a T_a-submodule of M_a killed by all β_j. A torsion quotient mod. is a T_a-quot mod of M_a killing all α_i. No torsion q. mod of M_a $\iff$ the images $\mathrm{Im}\{\alpha_i : M_{x_i} \to M_a\}$ generate the T_a-mod M_a. No tors q mod $\iff$ $\{\beta_j\}$ cogenerate to T_a-mod M_a.

$$\bigoplus_{\alpha_i} T_a \otimes_k M_{x_i} \twoheadrightarrow M_a \hookrightarrow \bigoplus_{\beta_j} \mathrm{Hom}(T_a, M_{y_j})$$

I think firm + closed are clear. i.e first map $\simeq$ resp 2nd map.

I think this all means that $\mathcal{a}/\mathcal{s} = \mathrm{Mod}(R')$, but I already know $\mathcal{a}/\mathcal{s} \simeq \mathcal{M}(R, ReR) \simeq \mathrm{Mod}(eRe)$. $\therefore$ For general reasons R', eRe should be m.q. Can you check this directly? Anyway what happens?

eRe has basis $(\beta\tau^{n_1}\alpha)(\beta\tau^{n_2}\alpha)\cdots(\beta\tau^{n_r}\alpha)$

example: Q:
$$\begin{tikzcd} x \arrow[r, bend right, "\alpha"'] & a \arrow[l, bend right, "\beta"'] \arrow[loop right, "\tau"] \end{tikzcd}$$

eRe has basis

$$\beta\tau^{n_p}\alpha \cdots\cdots \beta\tau^{n_2}\alpha\,\beta\tau^{n_1}\alpha$$

for all p and $n_1, n_2, \dots, n_p \geq 0$. ~~It~~ looks as if I have a tensor algebra. with generators $\beta\tau^n\alpha$ for $n \geq 0$. The reduced quiver is

Q′: $\begin{tikzcd} x \arrow[loop right, "\beta\alpha"] \end{tikzcd}$ so $R' = k[\beta\alpha]$. Is it possible for these two to be meq?

Example:
$$\begin{tikzcd} x \arrow[r, bend right] & a \arrow[l, bend right] \end{tikzcd}$$
~~The path algebra~~

Is there some relation to the Toeplitz algebra?

You have probably made a mistake. I don't see how eRe embeds in a finite matrix ring over $R' = k[\beta\alpha]$.

$$\begin{tikzcd} \mathcal{S} \arrow[r] & \mathcal{A} \arrow[r, "j^*"] \arrow[d, equal] & \mathcal{A}/\mathcal{S} \\ & \mathrm{Mod}(R) & \end{tikzcd}$$

~~$\mathcal{S} = \{M \mid$~~ $\overbrace{\mathrm{Hom}_{\mathcal{A}}(V^a, M) = 0\}}^{M_a}$

$$R = \mathrm{End}\Big(\underbrace{\bigoplus_x V^x}_{\text{small proj gens for } \mathcal{A}.}\Big)^{op}$$

$$V = V^a \oplus \bigoplus_{x \neq a} V^x = V^a \oplus V' = Ve^\perp \oplus Ve$$

$$\mathcal{S} = \{M \mid \mathrm{Hom}_{\mathcal{A}}(V', M) = 0\} \subset \mathcal{A}.$$

Then V' is firm and projective, so j^*V' should be ~~give~~ a projective in $\mathcal{A}/\mathcal{S}$. A small proj. gen.

$V' = Re$ $\qquad M$ in $\mathcal{S} \iff \mathrm{Hom}_R(Re, M) = eM = 0$.

$\therefore\ \mathcal{S} = \mathrm{Mod}(R/ReR)$. $\qquad \mathcal{A}/\mathcal{S} = \mathrm{Mod}(R, ReR) = \mathcal{M}(ReR) = \mathrm{Mod}(eRe)$.

j)

$$\begin{tikzcd} x \arrow[r, bend right, "\alpha"'] & a \arrow[l, bend right, "\beta"'] \arrow[loop right, "\tau"] \end{tikzcd}$$

$$\begin{tikzcd} M_x \arrow[r, bend right, "\alpha"'] \arrow[d] & \overset{N}{M_a} \arrow[l, bend right, "\beta"'] \arrow[loop right, "\tau"] \arrow[d] \\ 0 \arrow[r, bend right, "0"'] & M_a/N \arrow[l, bend right, "0"'] \end{tikzcd}$$

?

[redacted]

$$\begin{tikzcd} M_x \arrow[r, bend right, "\alpha"'] \arrow[d, equal] & N_a \arrow[l, bend right, "\beta"'] \arrow[d, hook] \\ M_x \arrow[r, bend right, "\alpha"'] & M_a \arrow[l, bend right, "\beta"'] \arrow[loop right, "\tau"] \arrow[d] \\ 0 & M_a/N_a \end{tikzcd}$$

$\tau N_a \subset N_a$

$\alpha M_x = \alpha N_x \subset N_a$

So N_α ~~a~~ $\supset$ the τ-submodule of M_a generated by αM_x

$\therefore$ $\alpha : M_x \longrightarrow M_a$ generates the T_a-module M_a.

$$\begin{tikzcd} 0 \arrow[r, bend right] \arrow[d, hook] & N_a \arrow[l, bend right, "\beta"'] \arrow[d, hook] \\ M_x \arrow[r, bend right, "\alpha"'] & M_a \arrow[l, bend right, "\beta"'] \end{tikzcd}$$

$\tau N_a \subset N_a$

$\beta N_a = 0.$

Thus N_a is contained in the largest T_a-submodule X of M_a contained in the kernel of β.

$$\begin{tikzcd} M_a \arrow[r] \arrow[d, "\beta"'] \arrow[dr] & M_a/X \arrow[d] \\ M_x & \mathrm{Hom}_k(T_a, M_x) \arrow[l] \end{tikzcd}$$

So I have this problem.

~~That I'm discussing~~

I'm examining an example $\begin{tikzcd} x \arrow[r, bend right, "\alpha"'] & a \arrow[l, bend right, "\beta"'] \arrow[loop right, "\tau"] \end{tikzcd}$

Let's review. I ran into a problem yesterday and decided to study the example k)

$$\begin{tikzcd} Q: & x \arrow[r, bend right, "\alpha"'] & a \arrow[l, bend right, "\beta"'] \arrow[loop right, "\tau"] \end{tikzcd} \qquad \begin{tikzcd} Q': & x \arrow[loop right, "\beta\alpha"] \end{tikzcd}$$

Here eRe has basis

$$(\beta\tau^{n_k}\alpha)\cdots(\beta\tau^{n_2}\alpha)(\beta\tau^{n_1}\alpha)$$

so it is a tensor algebra with generators $\beta\tau^n\alpha$, $n \geq 0$. $R' = k[\beta\alpha]$ one generator. So ~~there~~ there's must be mistake in your reasoning. It's clear what you did wrong. A reduced Q-module M has

$$\begin{tikzcd} T_a \otimes M_x \arrow[r, two heads] \arrow[rr, bend right] & M_a \arrow[r, hook] & \mathrm{Hom}(T_a, M_x) \end{tikzcd}$$

so M_a can be recovered from M_x and the map of T_a-modules, which is equivalent to a ~~map~~ k-module map $T_a \otimes M_x \to M_x$. So your Q' is the wrong quiver. In general the same thing can be expected

---

So I now recognize the mistake and I know that ~~deleting~~ trying to remove a vertex a with a loop will ~~lead~~ lead to an infinite quiver provided there are arrows $x \to \overset{\circlearrowleft}{a} \to y$ (with possibly $x=y$). ~~The other conjecture~~ What happens if only ~~inside~~ incoming arrows to a, ~~To~~ i.e. no $a \to y$ $y \neq a$ ~~reduces~~

$$\bigoplus_i M_{x_i} \longrightarrow M_a$$

Ringel form. dim vector of a Q-module $(\dim M_x)_x$ vertex

Suppose this ~~specific~~ given (n_x) then you want the dimension of the corresp. variety of modules. You take linear space $\prod_{x,y} \mathrm{Hom}(M_x, M_y)$ and act on it via $G = \prod_x \mathrm{Aut}(M_x)$. Count dimensions. $\sum_{\alpha : x \to y} n_x n_y$, $\sum_x n_x^2$. ~~Question~~

Assume the action of G is non-singular. Any module M is of finite length, so the possible dim. vectors should be determined by the simple modules. The auto group of a simple module should be scalar. Thus it would seem that ~~the action~~ M simple $\Rightarrow$ $\sum_x n_x^2 - 1 \leq \sum_{\alpha : x \to y} n_x n_y$

examples: ① $a \rightrightarrows b$ $\quad n_a^2 + n_b^2$, $2n_a n_b$

Always $n_a^2 + n_b^2 \geq 2n_a n_b$

So $n_a^2 + n_b^2 \leq 1 + 2n_a n_b \Rightarrow (n_a - n_b)^2 \leq 1$

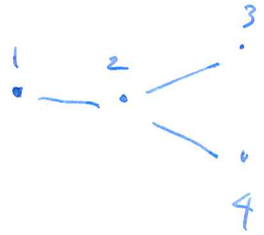
$\Rightarrow n_a - n_b = \begin{cases} 1 \\ 0 \\ -1 \end{cases}$ agrees with what happens for indecomposables.

② $\begin{tikzcd} a \arrow[loop right] \end{tikzcd}$ $\quad n_a^2 \leq 1 + n_a^2$ always. ?

③ $a \rightleftarrows b$ $\quad n_a^2 + n_b^2 \leq 1 + 2n_a n_b$

④ $\begin{tikzcd} a \arrow[loop right] \arrow[loop, out=30, in=-30, distance=3em] \end{tikzcd}$ $\quad n_a^2 \leq 1 + 2n_a^2$

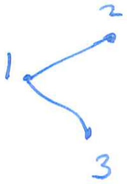
Tree.



$n_1^2 + n_2^2 + n_3^2 + n_4^2$

$n_1 n_2 + n_2 n_3 + n_2 n_4$

m)



$$n_1^2 + n_2^2 + n_3^2 - n_1 n_2 - n_1 n_3$$

$$= \left(n_1 - \tfrac{1}{2} n_2 - \tfrac{1}{2} n_3\right)^2 + \tfrac{3}{4} n_2^2 + \tfrac{3}{4} n_3^2 - \tfrac{1}{4} n_2 n_3$$

$$= \left(n_1 - \tfrac{1}{2} n_2 - \tfrac{1}{2} n_3\right)^2 + \tfrac{1}{4} \underbrace{\left(3 n_2^2 + 3 n_3^2 - n_2 n_3\right)}$$

$$3\left(n_2 - \tfrac{1}{6} n_3\right)^2 + \left(3 - \tfrac{3}{36}\right) n_3^2$$

$$n_1^2 + n_2^2 + n_3^2 + n_4^2 - n_1(n_2 + n_3 + n_4)$$

$$\left(n_1 - \tfrac{1}{2}(n_2 + n_3 + n_4)\right)^2 + \underbrace{n_2^2 + n_3^2 + n_4^2 - \frac{(n_2+n_3+n_4)^2}{4}}$$

$$\frac{1}{4}\left(3 n_2^2 + 3 n_3^2 + 3 n_4^2 - 2 n_2 n_3 - 2 n_2 n_4 - 2 n_3 n_4\right)$$

$$\cancel{3} n_2^2 + \cancel{3} n_3^2 + \cancel{3} n_4^2 - \frac{2}{3} n_2 n_3 - \frac{2}{3} n_2 n_4 - \frac{2}{3} n_3 n_4$$

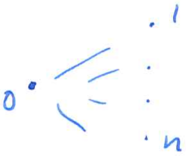
$$\left(n_2 - \tfrac{1}{3} n_3 - \tfrac{1}{3} n_4\right)^2 + \underbrace{\left(1 - \tfrac{1}{9}\right) n_3^2 + \left(1 - \tfrac{1}{9}\right) n_4^2 + \left(-\tfrac{6}{9} - \tfrac{2}{9}\right) n_3 n_4}$$

$$\frac{8}{9}\left(\underbrace{n_3^2 + n_4^2 - n_3 n_4}_{\geq 0.}\right)$$

$$x_0^2 + \overbrace{\sum_{j=1}^{n} x_j^2}^{c} - x_0 \overbrace{\sum_{j=1}^{n} x_j}^{b}$$ look at as a q.f. in x_0

$$b^2 - 4c = \cancel{\left(\sum x_j^2\right)^2 - 4\left(\sum x_j\right)} \quad \boxed{\left(\sum_{j=1}^{n} x_j\right)^2 - 4 \sum_{j=1}^{n} x_j^2}$$

a)



$$x_0^2 + \sum_{j=1}^n x_j^2 - x_0 \sum_{j=1}^n x_j$$

quadratic in x_0 minimum •

$$\sum_1^n x_j^2 - \frac{1}{4}\left(\sum_{j=1}^n x_j\right)^2$$

$x^2 + bx + c$

$x = -\frac{b}{2}$

min is $\frac{b^2}{4} - \frac{b^2}{2} + c$

$-\frac{b^2}{4} + c$

Suppose all $x_j = x$. Then

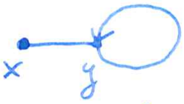
$$nx^2 - \frac14 n^2 x^2 = \left(n - \frac14 n^2\right)x^2 = n\left(1 - \frac{n}{4}\right)x^2 \quad \begin{cases} >0 & n<4 \\ =0 & n=4 \\ <0 & n>4 \end{cases}$$

So this checks. So there's this Ringel form which is $\sum_x n_x^2 - \sum_{x \to y} n_x n_y$, and presumably all connected graphs with this form [illegible] >0, ≥ 0 are known.

$n_1^2 + n_2^2 - 2n_1n_2$

$n^2 - n^2 = 0.$

$n^2 - 2n^2 = -n^2 < 0.$



$n_x^2 + n_y^2 - n_x n_y - n_y^2 = n_x(n_x - n_y)$ hyp.

So no simple loops allowed.

[redacted]

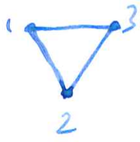
$$\sum_{j=1}^n x_j^2 - x_1x_2 - x_2x_3 - \dots - x_{n-1}x_n.$$

$n_1^2 + n_2^2 - 2n_1n_2$

$x_1^2 + x_2^2 - x_1x_2$

$$\begin{array}{ccc} 1 & \frac12 & \\ \frac12 & 1 & \frac12 \\ & \frac12 & 1 \\ & & \ddots \end{array}$$

$x_1^2 + x_2^2 + x_3^2$
$-x_1x_2 - x_2x_3 - x_3x_1$



$(n_1 - n_2)^2 + n_3^2 - n_2n_3$

$(n_1 - n_2)^2 + \underbrace{n_3(n_3 - n_2)}_{\text{hyp.}}$

0)

$$\cancel{\left(\frac{1}{2}+\frac{1}{2}\right)}x_1^2 + \cancel{x_2}^2 + \cancel{x_3}^2 - x_1x_2 - x_2x_3 - x_3x_1$$

$$= \frac{1}{2}(x_1-x_2)^2 + \frac{1}{2}(x_2-x_3)^2 + \frac{1}{2}(x_3-x_1)^2 \geqslant 0.$$

This ~~to~~ should be enough to show chains  $\geqslant 0$.

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3$$

$$= \frac{1}{2}x_1^2 + \frac{1}{2}(x_1-x_2)^2 + \frac{1}{2}(x_2-x_3)^2 + \frac{1}{2}x_3^2$$

$$\frac{1}{2}x_1^2 + \frac{1}{2}(x_1-x_2)^2 + \cdots + \frac{1}{2}(x_{n-1}-x_n)^2 + \frac{1}{2}x_n^2 - x_{n-1}x_n \;!$$

$$x_0^2 - 2x_0x_1 + x_1^2 - x_1x_2 + x_2^2 - \cdots + x_{n-1}^2 - x_{n-1}x_n + x_n^2$$

$$= \underbrace{-x_0x_1 + \frac{1}{2}x_0^2 + \frac{1}{2}(x_0-x_1)^2}_{} + \cdots + \frac{1}{2}(x_{n-1}-x_n)^2 + \frac{1}{2}x_n^2$$

$$\underbrace{x_0^2 - 2x_0x_1 + \frac{1}{2}x_1^2}_{(x_0-x_1)^2 - \frac{1}{2}x_1^2}$$

$$x_1^2 - x_1x_2 + x_2^2 - 2x_2x_3 + x_3^2$$
$$= \left(x_1 - \frac{1}{2}x_2\right)^2 - \frac{1}{4}x_2^2 + (x_2-x_3)^2$$

 indefinite

$$x_1^2 - 2x_1x_2 + x_2^2 - x_2x_3 + x_3^2 = (x_1-x_2)^2 + \left(-\frac{1}{2}x_2 + x_3\right)^2 - \frac{1}{4}x_2^2$$

 • $\geqslant 0$ semi. def.

 def.

 indef.

$$x_0^2 - x_0\sum_1^n x_j + \sum_1^n x_j^2 \quad \text{has min.} \quad \sum_1^n x_j^2 - \frac{1}{4}\left(\sum x_j\right)^2$$

$$= \frac{3}{4}\sum x_j^2 - \frac{1}{2}\sum_{i<j} x_ix_j$$

$ax^2+bx+c \qquad 2ax+b=0$ p)

$$a\left(-\frac{b}{2a}\right)^2+b\left(-\frac{b}{2a}\right)+c = c-\frac{b^2}{4a}$$

$f = x_1^2 - x_1x_2 + x_2^2 - \cdots + x_n^2$ minimize with x_1 fixed

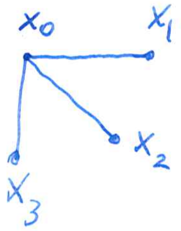
$$[illegible] = \frac{1}{2}x_1^2+\frac{1}{2}(x_1-x_2)^2+\cdots+\frac{1}{2}(x_{n-1}-x_n)^2+\frac{1}{2}x_n^2$$

Use calc. $\frac{\partial}{\partial x_2}f = -x_1+x_2$ $\qquad \frac{\partial}{\partial x_n}f = -x_{n-1}+x_n$

Thus only stationary value of f is $x_1 = x_2 = \cdots = x_n$ where f is x_1^2. NO.

$\frac{\partial f}{\partial x_2} = -x_1+2x_2-x_3$

$\frac{\partial f}{\partial x_n} = -x_{n-1} + 2x_n \qquad \therefore\ 2x_n = x_{n-1} = x_{n-2} = \cdots = x_1$ ✗ $\qquad x_1 = \frac{x_0}{2}$



$$\begin{aligned} f = x_0^2 - x_0x_1 + x_1^2 &\qquad -x_0+2x_1=0 \\ - x_0x_2 + x_2^2 &\qquad -x_0+2x_2=0 \\ - x_0x_3 + x_3^2 &\qquad -x_0+2x_3=0 \end{aligned}$$

$f = -x_0x_1 + x_1^2 - x_1x_2 + x_2^2$

$\frac{\partial f}{\partial x_1} = -x_0+2x_1-x_2$, $\frac{\partial f}{\partial x_2} = -x_1+2x_2$

Better:

$2x_1-x_2=0$

$-x_1+2x_2-x_3=0$

$-x_2+2x_3-x_4=0$

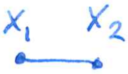
$x_2 = 2x_1$

$x_3 = -x_1+4x_1 = 3x_1$

$x_4 = -x_2+2x_3$

$= (-2+6)x_1$

$= 4x_1$



$f = x_1^2 - x_1x_2 + x_2^2$

$\frac{\partial f}{\partial x_2} = -x_1+2x_2 = 0$

$\therefore\ x_2 = +\frac{1}{2}x_1 \qquad \therefore f = x_1^2\left(1-\frac{1}{2}+\frac{1}{4}\right) = \frac{3}{4}x_1^2$

$f = x_1^2 - x_1x_2 + x_2^2 - x_2x_3 + x_3^2$

$\frac{\partial f}{\partial x_2} = -x_1+2x_2-x_3 = 0 \qquad \frac{\partial f}{\partial x_3} = -x_2+2x_3 \qquad \therefore\ x_3 = \frac{1}{2}\underset{\frac{2}{3}x_1}{x_2}$ OK ~~$= \frac{1}{3}x_1$~~

$x_1 = 2x_2 - x_3 = \left(2-\frac{1}{2}\right)x_2 = \frac{3}{2}x_2 \qquad x_2 = \frac{2}{3}x_1 \qquad x_3 = \frac{1}{3}x_1$

$$f = x_1^2\left(1-\frac{2}{3}+\underbrace{\left(\frac{2}{3}\right)^2-\frac{2}{3}\frac{1}{3}+\left(\frac{1}{3}\right)^2}_{4-2+1=3}\right) = x_1^2\left(\frac{1}{3}+\frac{3}{9}\right) = x_1^2\frac{2}{3}$$