February 27, 1981

Return to the many-body problem.

$$Z = tr\left(e^{-\beta(H-\mu N)}\right) = \sum_n \left(e^{\beta\mu}\right)^n tr\left(e^{-\beta H_n}\right).$$

In the classical limit $\hbar \to 0$ one has

$$tr\left(e^{-\beta H_n}\right) = \frac{1}{n!}\int \prod_j \frac{d^3p_j\, d^3q_j}{(2\pi\hbar)^3}\, e^{-\beta\left(\sum \frac{p_j^2}{2m} + U_n(q_1,\dots,q_n)\right)}$$

$$= \frac{1}{n!}\left(\beta^{-\frac{3}{2}}\gamma\right)^n \int dq_1\cdots dq_n\, e^{-\beta U_n(q_1\cdots q_n)}$$

where $\gamma = \left(\frac{\sqrt{m}}{\sqrt{2\pi\hbar}}\right)^3$ is a constant.

and hence

$$Z = \sum_n \frac{z^n}{n!}\int dq_1\cdots dq_n\, e^{-\beta U_n(q_1,\dots,q_n)}$$

where $z = \beta^{-3/2} e^{\beta\mu}\gamma$.

In practice one is interested in averages as follows. Given a function $f(x)$ of position one extends it as a 1-particle function

$$\tilde{f} : \{x_j\} \longmapsto \sum_j f(x_j)$$

and then takes the average of $\tilde{f}$. Thus

$$\langle \tilde{f} \rangle = \sum \frac{z^n}{n!}\int dq_1\cdots dq_n \sum f(q_j)\, e^{-\beta U_n} / Z$$

$$= \int dx\, f(x)\, \underbrace{\sum \frac{z^n}{(n-1)!}\int dq_1\cdots dq_{n-1}\, e^{-\beta U_n(x,q_1,\dots,q_{n-1})}} / Z$$

$$= \int dx\, f(x)\, \rho(x)$$

Similarly if $f(x,y)$ is a symmetric fn, then it extends to the gas as

$$\tilde{f} : \{x_j\} \longmapsto \frac{1}{2}\sum_{i\neq j} f(x_i, x_j)$$

and then

$\langle \tilde{f} \rangle$ [redacted] $= \frac{1}{2}\int dx\,dy\, f(x,y)\, \underbrace{\sum_n \frac{z^n}{n!}\, n(n-1) \int dq_1 \cdots dq_{n-2}\, e^{-\beta U_n(x,y,q_1,\dots,q_{n-2})} \Big/ Z}_{G_2(x,y)}$ [redacted]

Thus

$$G_2(x,y) = z^2 \sum_n{}' \frac{z^n}{n!} \int dq_1 \cdots dq_2\, e^{-\beta U_{n+2}(x,y,q_1,\dots,q_n)} \Big/ Z.$$

More generally I could define partition function where z [redacted] is replaced by activities $z(x)$ depending on x. Thus if I put $z(x) = e^{J(x)}$, I have

$$Z = \sum \frac{1}{n!} \int dq_1 \cdots dq_n\, e^{\sum_i J(q_i) - \beta U_n(q_1 \cdots q_n)}$$

and

$$G_1(x) = \frac{\delta}{\delta J(x)} \log Z$$

$$G_2(x,y) = \frac{\delta^2 Z}{\delta J(x)\, \delta J(y)} \cdot \frac{1}{Z} \qquad \times$$

Possible project: Find the quantum [redacted] mechanical analogues of these Green's functions.

$\times$ The formulas aren't quite right. Recall that if we have a Taylor series

$$F(z) = \sum \frac{1}{n!} \int dx_1 \cdots dx_n\, f_n(x_1,\dots,x_n)\, z(x_1) \cdots z(x_n)$$

then

$$\frac{\delta F}{\delta z(x)} = \sum{}' \frac{1}{n!} \int dx_1 \cdots dx_n\, f_{n+1}(x, x_1, \dots, x_n)\, z(x_1) \cdots z(x_n).$$

so if $G_2(x,y)$ is defined so that it gives the 2-particle fn. averages as above, we have

$$G_2(x,y) = z(x)\,z(y)\,\frac{\delta^2 Z}{\delta z(x)\,\delta z(y)} \cdot Z^{-1}$$

where here $z(x) = e^{J(x)}$. Thus

$$\frac{1}{Z}\,\frac{\delta^2 Z}{\delta J(x)\,\delta J(y)} = \left(z(x)\frac{\delta}{\delta z(x)}\right)\left(z(y)\frac{\delta}{\delta z(y)}\right) Z \cdot \frac{1}{Z}$$

is not $G_2(x,y)$, because of the term on the diagonal.

$$= z(x)\,z(y)\,\frac{\delta^2 Z}{\delta z(x)\,\delta z(y)}\,\frac{1}{Z} + z(x)\,\delta(x-y)\,\frac{\delta Z}{\delta z(y)}\,\frac{1}{Z}$$

For example if all $U_n = 0$, then

$$Z = \sum \frac{1}{n!}\int dq_1 \cdots dq_n \; z(q_1)\cdots z(q_n) = e^{\int z(q)\,dq}$$

and so

$$\langle n(x)\rangle = z(x)\frac{\delta Z}{\delta z(x)}\,\frac{1}{Z} = z(x)\frac{\delta}{\delta z(x)}\left(\int z(q)\,dq\right) = z(x)\int \delta(x-q)\,dq = z(x)$$

$$\frac{\delta Z}{\delta z(x)} = e^{\int z(q)\,dq}\underbrace{\frac{\delta}{\delta z(x)}\int z(q)\,dq}_{1} = Z$$

Thus

$$G_2(x,y) = \frac{z(x)z(y)}{Z}\,\frac{\delta^2 Z}{\delta z(x)\,\delta z(y)} = z(x)\,z(y)$$

nice smooth function of x, y. However

$$\frac{1}{Z}\,\frac{\delta^2 Z}{\delta J(x)\,\delta J(y)} = z(x)\,z(y) + z(x)\,\delta(x-y)$$

or

$$\langle n(x)\,n(y)\rangle = \langle n(x)\rangle\langle n(y)\rangle + \langle n(x)\rangle\,\delta(x-y)$$

which is a sort of standard part of the Poisson distribution:

$$\langle n^2\rangle = \sum n^2\,\frac{e^{-\lambda}\lambda^n}{n!} = \sum\left[(n^2-n)+n\right]\frac{e^{-\lambda}\lambda^n}{n!}$$

$$= \lambda^2 + \lambda = \langle n\rangle^2 + \langle n\rangle$$

So at least conjecturally we feel more or less the singularities for the interacting gas should be the same, that is $G_2(x,y)$ is smooth.

February 25, 1981

Review the classical gas:

$$Z = \sum_n \frac{1}{n!} \int \prod_1^n z(q_j)\,dq_j \; e^{-\beta U_n(q_1 \cdots q_n)}$$

define

$$g_n(x_1, \ldots, x_n) = z(x_1) \cdots z(x_n) \frac{\delta^n Z}{\delta z(x_1) \cdots \delta z(x_n)} \cdot \frac{1}{Z}$$

Then we have seen that, e.g. for $n=2$ if we take a 2-particle function $\tilde{f} : \{x_j\} \longmapsto \frac{1}{2} \sum_{i \neq j}' f(x_i, x_j)$

on the gas, then

$$\langle \tilde{f} \rangle = \frac{1}{2} \int dx\,dy\; f(x,y)\, g_2(x,y)$$

If I put $z(q) = e^{J(q)}$, then

$$G_2(x,y) = \frac{1}{Z} \frac{\delta^2 Z}{\delta J(x)\, \delta J(y)} = \frac{1}{Z}\, z(x) \frac{\delta}{\delta z(x)}\, z(y) \frac{\delta}{\delta z(y)} Z$$

$$= g_2(x,y) + \delta(x-y)\, g_1(x)$$

is what you need to compute averages over pairs where you include diagonal pairs.

Question: Is there a smoothed-out version of this gas? One normally likes to have a continuum picture of a gas. I guess what I want is to [redacted] replace n_ξ by a smooth density [redacted].

So therefore what I want to do is to find a field theory:

$$Z = \int \mathcal{D}\phi \; e^{-S(\phi) + \int J\phi}$$

with the same Green's functions as the classical gas.

February 27, 1981

The problem is whether there is a way to smooth out a classical gas so that the configurations become densities instead of subsets. So I first wanted to look at the case where the particles dont interact.

Consider first the case of fixed N, where there are just n particles. A 1-particle function $\tilde{f} : \{x_j\} \longmapsto \sum_j f(x_j)$ has expectation

$$\langle \tilde{f} \rangle = \frac{\frac{1}{n!} \int dq_1 \cdots dq_n \sum_j f(q_j)\, e^{-\beta U_n(\vec{q})}}{\frac{1}{n!} \int dq_1 \cdots dq_n\, e^{-\beta U_n(\vec{q})}}$$

$$= n \int dx\, f(x)\, g_1(x)$$

where

$$g_1(x) = \int dq_1 \cdots dq_{n-1}\, e^{-\beta U_n(x,\vec{q})} \Big/ \int dq_1 \cdots dq_n\, e^{-\beta U_n(\vec{q})}$$

Similarly a 2-particle function $\tilde{f}_2 : \{x_j\} \longmapsto \frac{1}{2} \sum_{i \neq j} f(x_i, x_j)$ has

$$\langle \tilde{f}_2 \rangle = \frac{n(n-1)}{2} \int dx\, dy\, f_2(x,y)\, g_2(x,y)$$

where

$$g_2(x,y) = \int dq_1 \cdots dq_{n-2}\, e^{-\beta U_n(x,y,\vec{q})} \Big/ \int dq_1 \cdots dq_n\, e^{-\beta U_n(\vec{q})}$$

If the particles are independent, ~~[illegible]~~ then

$$U_n(\vec{q}) = \sum_{j=1}^{n} U(q_j) \quad , \text{ so that}$$

$$g_1(x) = e^{-\beta U(x)} \Big/ \int e^{-\beta U(x)} dx$$

$$g_2(x,y) = e^{-\beta U(x)} e^{-\beta U(y)} \Big/ \left(\int e^{-\beta U(x)} dx \right)^2$$

Thus $g_1(x)$ = density $\rho(x)$ predicted by Boltzmann's law and

$$g_2(x,y) = \rho(x)\,\rho(y)$$

which says the probability of finding a particle at x and another at y is the product of the separate probabilities.

If we use the grand ensemble, we find

$$\mathcal{Z} = \sum \frac{1}{n!} \int \prod_j dq_j \, z(q_j) \cdot e^{-\beta U_n(\vec{q})}$$

$$= \exp\left\{ \int dx \, z(x) e^{-\beta U(x)} \right\}$$

and so

$$g_1(x) = z(x) \frac{\delta}{\delta z(x)} \log \mathcal{Z} = z(x) e^{-\beta U(x)}$$

$$g_2(x,y) = g_1(x) g_1(y) + g_2^{(c)}(x,y)$$

$$g_2^{(c)}(x,y) = z(x) z(y) \frac{\delta^2}{\delta z(x) \delta z(y)} \log \mathcal{Z} = 0$$

$$= g_1(x) g_1(y)$$

which gives the same answer provided we set $z(x) = z$ for all x and choose $z = 1/\int e^{-\beta U(x)} dx$.

Now the viewpoint I want to adopt goes as follows: From the gas of discrete particles, one constructs Green's functions which embody all the useful information. In the independent particle case this information consists simply of the density function $g_1(x)$, because the remaining Green's fns. are products. So it might be possible to replace classical configurations by smooth densities.

March 2, 1981

Let us now consider a quantum gas. The underlying Hilbert space is a Fock space

$$\mathcal{F} = \Lambda \mathcal{H} \qquad \text{or} \qquad S(\mathcal{H})$$

made up of the different N-particle spaces $\mathcal{F}_N = \Lambda^N \mathcal{H}$ or $S^N(\mathcal{H})$ for each N. Now I have to consider the sort of quantities one takes averages of. When the gas is in equilibrium at inverse temperature β, the average of an operator A is

$$\langle A \rangle = \frac{tr\left(A e^{-\beta(H-\mu N)}\right)}{tr\left(e^{-\beta(H-\mu N)}\right)}$$

where μ is adjusted so that $\langle N \rangle$ is what you want.

Typically the operators A of interest are 1-particle operators like density at x

$$\rho(x) = \sum a_k^* \langle k|x\rangle\langle x|l\rangle a_l$$

or 2-particle operators like the potential energy. It might be useful to understand the algebra of these operators, that is operators generated by the basic operators $a_k^* a_l$

In the boson situation what we get is the algebra of differential operators generated by the basic operators

$$a_k^* a_l = z_k \frac{\partial}{\partial z_l} \qquad \text{basis for } gl_n$$

which is the algebra of all differential operators in $\mathbb{C}[z_i, \frac{\partial}{\partial z_j}]$ which are of total degree zero. This is not the universal enveloping alg. of gl_n.

Suppose we were to consider the classical situation: Here we have a set X of 1-particle states and instead of operators I [redacted] consider functions

$$f = \{f_n(x_1,\dots,x_n)\} \quad \text{on} \quad \coprod_{n\geq 0} X^n/\Sigma_n$$

There is a natural filtration here, and one can ask about the structures.

So what exactly is this algebra? $\coprod_{n\geq 0} X^n/\Sigma_n$ is the free abelian monoid generated by X; call it M. Hence our functions $f = \{f_n\}$ are elements of $\mathrm{Maps}(M,\mathbb{C}) = \mathrm{Hom}(\mathbb{C}[M],\mathbb{C})$ and $\mathbb{C}[M] = S(\mathbb{C}X)$. Thus $\mathrm{Map}(M,\mathbb{C})$ = dual of $S(\mathbb{C}X)$ which if X is finite is isomorphic to the divided power algebra on the vector space $\mathrm{Map}(X,\mathbb{C})$. The divided power algebra contains a very natural set of elements, namely, the exponential functions

$$\sum_n \underbrace{\gamma_n(\lambda)}_{\lambda^{\otimes n}} \qquad \lambda \in \mathrm{Map}(X,\mathbb{C}).$$

[redacted]

So to summarize, for each [redacted] λ [redacted] $: X \to \mathbb{C}$ we get a very nice family of functions on the different configurations of the gas, namely

$$\{x_j\}_{1\leq j\leq n} \longmapsto \prod_{j=1}^{n} \lambda(x_j)$$

Actually I can make this clearer. A homomorphism

$$\mathbb{C}(X^n/\Sigma_n) = S_n(\mathbb{C}X) \longrightarrow \mathbb{C}$$

is completely determined by what it does to the elements [redacted] $\lambda^{\otimes n}$. (Recall $S_n \xrightarrow{\sim} \Gamma_n$ and the $\lambda^{\otimes n} = \gamma_n(\lambda)$ span Γ_n)

Therefore a measure on the gas configurations is known once you say what it does to the functions [redacted] $\lambda^{\otimes n}$ for each n and $\lambda: X \to \mathbb{C}$. It is perhaps simplest to introduce a new parameter t and work with $\sum t^n$ [redacted] $\lambda^{\otimes n}$ which as a function on the gas is

$$\{x_j\} \longmapsto t^n \prod_1^n \lambda(x_j)$$

This is a function on the gas, and its integral is

$$\sum_n \frac{1}{n!} \int t^n \prod_1^n \lambda(x_j)\, dx_1 \dots dx_n\, e^{-\beta U_n(\vec{x})}$$

~~The functions $f = \{f_n(x_1, \dots, x_n)\}$ form an algebra in two ways. The point is that we know~~

$$\cancel{\bigoplus_n H_*(E\Sigma_n \times^{\Sigma_n} X^n)}$$

~~is a Hopf ring object. ?~~

Here's another approach. First recall that a measure $d\tilde{\mu}_n$ on X^n/Σ_n is of the form $p_*\left(\frac{1}{n!} d\mu_n\right)$, where $p : X^n \to X^n/\Sigma_n$ is the projection, and $d\mu_n$ is a symmetric measure on X^n. Thus

$$\int_{X^n/\Sigma_n} f\, d\tilde{\mu}_n = \frac{1}{n!} \int_{X^n} (p^* f)\, d\mu_n$$

Thus the integral of the function $\{x_j\}_1^n \longmapsto \lambda(x_1) \dots \lambda(x_n)$ on the gas is

$$\sum_n \frac{1}{n!} \int \lambda(x_1) \dots \lambda(x_n)\, d\mu_n$$

So therefore the measure on $\coprod_n X^n/\Sigma_n$ is determined by its effect on the functions

$$(*) \qquad \{x_j\}_1^n \longmapsto \lambda(x_1) \dots \lambda(x_n)$$

where $\lambda : X \to \mathbb{C}$.

This all has a nice interpretation via Bochner's thm.

We have a measure $d\mu = \{d\tilde{\mu}_n\}$ on the free abelian monoid

$$M = \coprod X^n/\Sigma_n$$

generated by the space X. This monoid sits inside the free abelian group

$$G = \coprod_X \mathbb{Z}$$

whose characters are the same as maps $z : X \longrightarrow S^1$. Thus the function $(*)$ is just the character z (or generalized character) of G restricted to M. Bochner's thm. says that measures on G are certain functions on the character gp. $\hat{G}$ of such z. Intuitively, supported in M for the measure means that the integral

$$Z(z) = \sum \frac{1}{n!} \int \prod_{j=1}^{n} z(x_j)\, d\mu_n$$

is analytic inside $|z| < 1$.

For example if $d\mu_n = \overset{n}{\otimes}\, d\mu_1$, then

$$Z(z) = e^{\int z(x)\, d\mu_1}$$

So the important thing to remember is that because $M = \coprod X^n/\Sigma_n$ is a monoid you have a special class of functions on it, namely characters, which span all functions by harmonic analysis. Thus a measure on M is known from its effect on characters.

Inverting a power series revisited

Start with $$Z(J) = \int e^{\lambda(Jx - f(x))} dx$$

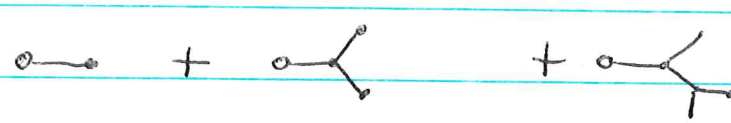
where $f(x) = \frac{a}{2}x^2 + \frac{b}{3!}x^3 + \cdots$. Then we have

$$Z(J) = e^{\lambda(Jx_c - f(x_c))} \frac{1}{\sqrt{2\pi\lambda f''(x_c)}}\left(1 + O\left(\tfrac{1}{\lambda}\right)\right)$$

and the [redacted] exponent $Jx_c - f(x_c)$ is computed using tree diagrams.

Better: Look at all tree diagrams contributing to $\bar{x}$. These give x_c as a function of J

x_c :



$$x_c = \frac{1}{a}J + \frac{(-b)}{a^3}\frac{J^2}{2!} +$$

If one applies Dyson to these tree diagrams, one gets

$$x_c = \frac{1}{a}J + \frac{1}{a}(-b)\frac{x_c^2}{2!}$$

or

$$a x_c + b\frac{x_c^2}{2!} = J \qquad \text{as it should be}$$

Thus you see that one can directly establish that x_c is given by tree diagrams using Dyson's decomposition.

Better: Define x_c using tree diagrams, then verify using [redacted] Dyson that $x_c(J)$ satisfies $f'(x_c) = J$.

March 4, 1981

In the case of a classical gas, we have seen that we get a measure on a space of distributions on X, and hence by pushing this measure forward we get a measure on all distributions ϕ. Then for any smooth function J on X we can form the function $\phi \longmapsto \int \phi J\, dx$ on distributions and form the integral

$$Z(J) = \int e^{i\int \phi J dx}\, \mathcal{D}\phi$$

where $\mathcal{D}\phi$ denotes the measure we are interested in. So the problem with this business is to start from a generating function and understand the "measure" it comes from, i.e. its support. By support I mean something fairly general, i.e. a space Y with a map to distributions on X.

Note that once you have $Y \to$ dist on X and a measure on Y, then you can map $C_0^\infty(Y) \to$ dist on X, which is reminiscent of the Schwarz kernel thm. Such a map is given by a distribution on $X \times Y$

$$f \longmapsto \int K(x,y) f(y)\, dy$$

Let's check this: Given $K \in \mathcal{D}(X \times Y)$ it should give rise to a generating function. Let $J \in C^\infty(X)$. ? There seems to be something missing. Let us start with $Y \to \mathcal{D}(X)$, $y \longmapsto \phi_y$ and the measure dy. Then

$$y \longmapsto \int \phi_y J$$

is a function of y, so we can form $y \mapsto e^{i\int \phi_y J}$ and then get the generating function

$$Z(J) = \int \mathcal{D}y\, e^{i\int \phi_y J}$$

If however instead of $y \mapsto \phi_y$ you were to give only $f \mapsto \int K(x,y) f(y)\, dy$, a map $C_0^\infty(Y) \to \mathcal{D}(X)$, then given J, you have only a distribution on Y:

$$f \longmapsto \int \left(\int dx \text{ [redacted] } J(x) K(x,y) \right) f(y)\, dy$$

and you have the problem of defining its exponential. This is roughly the same problem as defining the powers of a distribution.

Proceed formally

$$\int \mathcal{D}y \, e^{i\int \phi_y J} = \int \mathcal{D}y \sum_{n \geq 0} \frac{(i)^n}{n!} \left(\int \phi_y(x) J(x)\, dx \right)^n$$

$$= \int \mathcal{D}y \sum_{n \geq 0} \frac{(i)^n}{n!} \int dx_1 \dots dx_n \;\; \phi_y(x_1) \cdots \phi_y(x_n) \;\; J(x_1) \dots J(x_n)$$

and so one sees that the problem involves restricting [redacted] the product

$$\phi_{y_1}(x_1) \cdots \phi_{y_n}(x_n) \qquad \text{dists on } (Y \times X)^n$$

to the diagonal $Y \times (X)^n$ and then integrating.

Review [redacted] Wiener - Khinchin.

Let $x(t)$ be a random real-valued function of t. This means that there is some ensemble of such functions with a prob: measure on the ensemble. Let's suppose it is Gaussian with mean zero. This means that for any $f(t) \in C_0^\infty(\mathbb{R})$ we get a Gaussian r.v. (mean 0) by integrating:

$$\int f(t)\, x(t)\, dt$$

Such a r.v. is characterized by its variance

$$(*) \qquad \left\langle \left(\int f(t)\, x(t)\, dt \right)^2 \right\rangle = \int dt\, dt'\, f(t) f(t') \underbrace{\langle x(t)\, x(t') \rangle}_{K(t,t')}$$

where $K(t,t')$ is a distribution on $\mathbb{R} \times \mathbb{R}$ with the property that the integral on the right is ≥ 0 for any f. Conversely given such a K one gets a Gaussian process by Kolmogorov's theory.

Now assume stationarity: $K(t,t') = K(t-t')$. Then you can get a Hilbert space $\mathcal{H}$ by completing $C_0^\infty(\mathbb{R})$ under the norm $(*)$ and you have a 1-parameter unitary group given by translation. The spectral theorem should then give a measure $d\mu(\omega)$ on the line such that

$$\mathcal{H} \simeq L^2(\mathbb{R}, d\mu) \qquad \text{assume mult. 1}$$

$$f(t) \longleftrightarrow \hat{f}(\omega)$$

$$\left(f(t) \longmapsto f(t+a) \right) \longleftrightarrow \left(\hat{f}(\omega) \longmapsto \hat{f}(\omega) e^{-i\omega a} \right)$$

$$\int f(t)\, f(t')\, K(t-t')\, dt\, dt' = \int |\hat{f}(\omega)|^2\, d\mu(\omega)$$

So it seems that in fact the distribution $K(t)$ is the Fourier transform of the measure $d\mu$. [redacted] So it seems that for any [redacted] stationary Gaussian process, the [redacted] variance distribution $K(t-t')$ is of the form

$$K(t) = \int e^{-i\omega t}\, d\mu(\omega)$$

so that then if $\hat{f}(\omega) = \int e^{i\omega t} f(t)\,dt$, then

$$\|f\|^2 = \int dt\,dt'\, f(t) f(t') \int e^{-i\omega(t-t')}\, d\mu(\omega)$$

$$= \int \hat{f}(-\omega)\hat{f}(\omega)\, d\mu(\omega) = \int |\hat{f}(\omega)|^2\, d\mu(\omega)$$

Summary: Equivalence between stationary Gaussian processes on $\mathbb{R}$ and even measures [redacted] $d\mu(\omega)$ on $\mathbb{R}$, given by

$$\langle x(t)x(t')\rangle = \int e^{-i\omega(t-t')}\, d\mu(\omega)$$

The generating function for the process is:

$$\left\langle e^{i\int f(t)x(t)\,dt} \right\rangle = e^{-\frac{1}{2}\int f(t)f(t')K(t-t')\,dt}$$

$$= e^{-\frac{1}{2}\int |\hat{f}(\omega)|^2 d\mu(\omega)}$$

I think I may have forgotten the condition

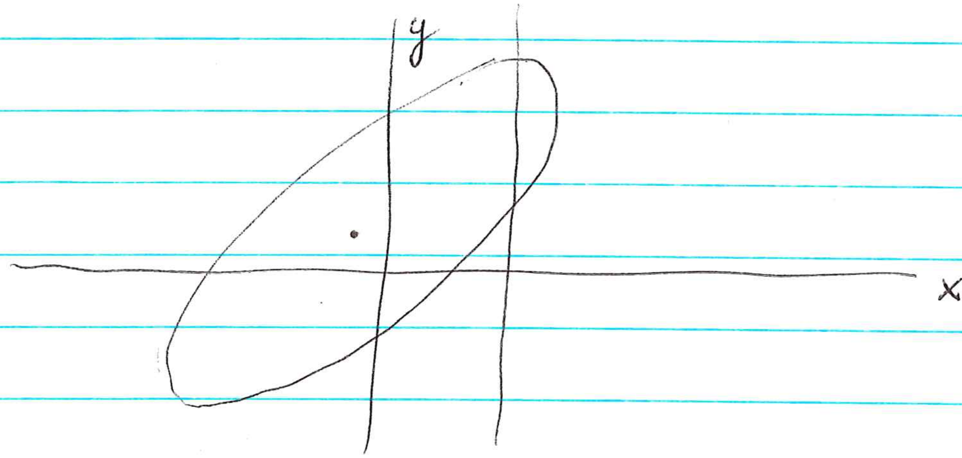
$$\int \frac{d\mu(\omega)}{1+\omega^2} < \infty \qquad ?$$

No, [redacted] this occurs when you want $\frac{1}{\lambda-\omega}$ to be in the Hilbert space when $\lambda \in \mathbb{C}-\mathbb{R}$.

We are looking at Gaussian processes on the line which are stationary and have found they are described by even measures on the line which are tempered distributions:

$$\langle x(t)x(t')\rangle = \int e^{-i\omega(t-t')}d\mu(\omega)$$

Let's now to describe which of these processes are Markov. Markov means that if one fixes $x(t_1)\cdots x(t_n)$ where $t_1<\cdots<t_n$ the probability distribution of $x(t)$ for $t>t_n$ depends on $x(t_n)$ but not the $x(t_j)$ for $j<n$. Because we have a Gaussian process, the distribution of $(x(t_1),\dots,x(t_n),x(t))$ in $\mathbb{R}^{n+1}$ is Gaussian, and also the distribution of $(\underbrace{x(t_1),\dots,x(t_{n-1})}_{x},\underbrace{x(t)}_{y})$ in $\mathbb{R}^n$ for a fixed $x(t_n)$ is Gaussian (but mean may be $\neq 0$). Put x, y and consider what it means for the distribution of y for a fixed x to be independent of the choice of x.

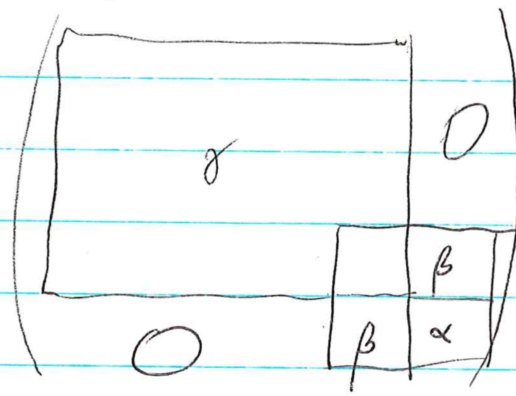


This can happen only if the ellipsoid isn't tilted. If the distribution of x,y is

$$p(x,y) = e^{-\frac{1}{2}(\alpha y^2+2\beta y+\gamma)}\Big/\text{const}$$

where α is const, $\beta(x)$ is of degree ≤ 1, $\gamma(x)$ is of deg ≤ 2, then we must have $\frac{\partial\beta}{\partial x}\equiv 0$.

Thus if we look at the covariance matrix for the distribution of $x(t_1), \ldots, x(t_n), x(t)$ it has the form



i.e.
$$p = e^{-\frac{1}{2}(\alpha y^2 + 2\beta y + \gamma)} / \text{const} \qquad \beta = \lambda x(t_n)$$

Now when you integrate over $y = x(t)$, so as to get the distribution of $x(t_1) \ldots x(t_n)$, you essentially evaluate at the critical point $y = -\beta/\alpha$ and get
$$p = e^{-\frac{1}{2}(-\beta^2/\alpha + \gamma)} / \text{const.}$$

Thus you get the quadratic form γ in $x(t_1) \cdots x(t_n)$ with the coeff of $x(t_n)^2$ modified. So by induction one sees that γ has to be a J-matrix. Therefore

Prop: A Gaussian process is Markov where for any $t_1 < \cdots < t_n$, the covariance matrix of $x(t_1), \cdots, x(t_n)$ is a J-matrix.

Actually I guess I have been a bit stupid, because the good way to think of a Markov process is in terms of propagators $K(xt, x't')$ satisfying
$$K(xt, x't') = \int dx_1\, K(xt, x_1 t_1) K(x_1 t_1, x' t')$$
for $t' < t_1 < t$. [redacted]

March 9, 1981

Poisson process on the line: Here's the derivation in Feynman's path integral book. Suppose we have atoms which decay independently with a certain rate λ atoms/sec. Take N atoms and a time interval $[-T/2, T/2]$ and suppose the j-th atom decays at time t_j. We get and ensemble of possible decay times: $\{t_j\} \in [-T/2, T/2]^N$. If we have an $f \in C_0^\infty(-T/2, T/2)$ (our measuring apparatus) the response is $\sum_j f(t_j)$ and the generating function is

$$\left\langle e^{i\sum_j f(t_j)} \right\rangle = \frac{1}{T^N} \int_{-T/2}^{T/2} dt_1 \cdots dt_N \, e^{i\sum f(t_j)}$$

$$= \left(\frac{1}{T} \int_{-T/2}^{T/2} e^{if(t)} dt \right)^N = \left(1 + \frac{1}{T} \int_{-T/2}^{T/2} (e^{if(t)} - 1) dt \right)^N$$

Now let $N, T \to \infty$ with $N/T = \lambda$, and you get

$$e^{\lambda \int_{-\infty}^{\infty} (e^{if(t)} - 1)\, dt}$$

which is the generating function for the Poisson process.

March 11, 1981

Levy-Khinchin: suppose we have an infinitely divisible probability measure on the line. This means that there is a 1-parameter family of prob. measures, which I will denote $K_t(x)dx$ such that

$$e^{tg(\xi)} = \int dx\, K_t(x) e^{-ix\xi} \qquad t \geq 0.$$

The typical example is a Poisson process

$$K_t(x) = \sum_{n\geq 0} \frac{(\lambda t)^n}{n!} e^{-\lambda t}\, \delta(x-na)$$

$$\int dx\, K_t(x) e^{-ix\xi} = \sum_{n\geq 0} \frac{e^{-\lambda t}}{n!} \left(\lambda t e^{-ia\xi}\right)^n = e^{t\lambda(e^{-ia\xi}-1)}$$

and an independent mixture of these:

$$g(\xi) = \int \underbrace{d\lambda(a)}_{\text{some measure}} \left(e^{-ia\xi}-1\right)$$

The Levy-Khinchin theorem says any inf. div. prob. measure is a mixture of Poisson processes and a Gaussian one. Let's try to understand this.

$$\frac{e^{tg(\xi)}-1}{t} = \int dx\, \frac{K_t(x)-\delta(x)}{t}\, e^{-ix\xi}$$

The left side is a continuous function of ξ which pointwise converges as $t \downarrow 0$ to $g(\xi)$; hence its Fourier transform converges to a distribution

$$\frac{K_t(x)-\delta(x)}{t} \longrightarrow \rho(x)$$

I guess the way to see this is to multiply by $\hat{f}(\xi)$ where

$f(x) \in C_0^\infty(\mathbb{R})$. Then

$$\int \frac{d\xi}{2\pi}\, \hat{f}(\xi)\, \frac{e^{tg(\xi)}-1}{t} = \int dx\, \frac{K_t(x)-\delta(x)}{t}\, f(x)$$

$$\Big\downarrow \text{conv. by Lebesg.}$$

so $\dfrac{K_t(x)-\delta(x)}{t} \to \rho(x)$ where $\hat{\rho} = g$

$$\int \frac{d\xi}{2\pi}\, \hat{f}(\xi)\, g(\xi)$$

Also we see that $\int dx\, \rho(x) f(x) \geq 0$ if $f \geq 0$ and $f(0)=0$. Thus we know $\rho(x)\,dx$ is a measure for $x \neq 0$, and that its behavior at $x=0$ depends on 2 constants.

To simplify let's suppose that $\int x^2 \rho(x)\,dx < \infty$, for example if ρ is compactly supported. Then

$$g(\xi) = \int dx\, \rho(x)\left(e^{-ix\xi} - 1 + ix\xi\right) + \underbrace{\int \rho(x)\,dx}_{0 \text{ as } g(0)=0} - i\xi \int dx\, \rho(x)\, x$$

$$= \int dx\; x^2 \rho(x) \left(\frac{e^{-ix\xi} - 1 + ix\xi}{x^2}\right) - i\xi \int dx\; x\rho(x)$$

March 14, 1981

So return to mean field theory:

Recall derivation of Weiss theory: One has spin configurations $\vec{s} = \{s_i\}$ with energy

$$E(\vec{s}) = -H\sum s_i - [redacted]\,\frac{1}{2}\sum_{i\neq j} J_{ij}s_i s_j .$$

One wants to compute $\langle s_i\rangle$[redacted]. [redacted] Think always in terms of the ergodic principle: One has a dynamical system with mysterious insides causing these spins to change in time. What s_i does depends upon the other spins $\{s_j\}_{j\neq i}$ thru the energy primarily. The other spins produce a local field

$$B_i = H + \sum_{j\neq i} J_{ij}s_j$$

at s_i. One assumes B_i can be approximated by

$$\langle B_i\rangle = H + \sum_{j\neq i} J_{ij}\langle s_j\rangle$$

i.e. that fluctuations are negligible. If so, then

$$\langle s_i\rangle = \frac{e^{\beta\langle B_i\rangle} - e^{-\beta\langle B_i\rangle}}{e^{\beta\langle B_i\rangle} + e^{-\beta\langle B_i\rangle}}$$

(treating s_i as acting thru the average field $\langle B_i\rangle$)

and one gets the Weiss theory.

The idea here involves these steps. One has [redacted] an energy $E(\phi)$ for the configuration ϕ. Given ϕ, the local field at x is

$$B_x = -\frac{\delta}{\delta\phi_x}E(\phi)$$

~~[illegible]~~ [redacted] Given a local field function $B = \{B_x\}$, [redacted] one can compute $\langle\phi_x\rangle$ assuming the field reacts with the others thru the mean field. Finally one has a self-consistency requirement.

For example take a single spin system $s = \pm 1$ with energy $E(s) = -Hs - \frac{J}{2}s^2$

Then $B = -\frac{\partial E}{\partial s} = H + Js$. Given B if s interacts with B one has

$\langle s\rangle$ = average of s [redacted] with Boltzmann weight $e^{-\beta(-Bs)}$

$$\boxed{\langle s\rangle = \frac{e^{\beta B} - e^{-\beta B}}{e^{\beta B} + e^{-\beta B}}}$$

This is then to be combined with the consistency requirement $\boxed{B = H + J\langle s\rangle}$.

(In this example there is no phase transition as predicted by mean field theory; the exact formula is

$$\langle s\rangle = \frac{e^{\beta H} - e^{-\beta H}}{e^{\beta H} + e^{-\beta H}}$$

In fact J doesn't matter.)

Next let's consider the case of a zero-diml field $x \in \mathbb{R}$ and instead of something like $x = \pm$ let's suppose there is given a potential energy $f(x)$. So we have [redacted] a [redacted] local field B, then

$$\langle x\rangle = \frac{\int x e^{Bx - f(x)}\,dx}{\int e^{Bx - f(x)}\,dx}$$

say $\beta = 1$. Let us now interpret the idea that

x should interact thru a mean field, better: the interaction is an independent particle interaction with a mean field, to mean that a Gaussian approx. is to be used: Thus

$$\langle x \rangle \doteq x_c \quad \text{where } \frac{d}{dx}(Bx - f(x)) = B - f'(x) = 0 \text{ at } x = x_c.$$

I am a bit confused as to what to do next, but it's clear that interpreting an independent particle model as ~~with the~~ a Gaussian approximation is a key idea.

March 18, 1981

Q: Is it possible to apply "dominant term" ideas to a classical gas? Suppose the partition function is

$$(1) \qquad Z = \sum \frac{z^n}{n!} \int dq_1 \cdots dq_n \, e^{-\beta U_n(\vec{q})}$$

In the case whe $U_n(\vec{q}) = \sum_{j=1}^{n} u(q_j)$, this partition function is an exponential

$$Z = e^{\int z e^{-\beta u(q)} dq}.$$

Intuitively space is broken into chunks $\Delta q_1, \cdots, \Delta q_n$ and then the partition function is an exponential series

$$e^{\sum_{i=1}^{n} z e^{-\beta u(q_i)} \Delta q_i}.$$

We should first understand dominant term for

$$e^x e^y = \sum_{m,n} \frac{x^m y^n}{m!\, n!}$$

$$\log\left(\frac{x^m y^n}{m!\, n!}\right) = \text{[redacted]}$$

$$= m \log x - \log m! + n \log y - \log n!$$

$$\frac{\partial}{\partial m}(\quad) = \log x - \log m = 0 \implies m = x$$

Similarly $n = y$, so the dominant term is $\frac{x^x}{x!} \frac{y^y}{y!} \sim e^x e^y$.

Return to $\quad Z = e^{\int z e^{-\beta u(q)} dq}.\quad$ Note

$$N = z \frac{\partial}{\partial z} \log Z = \int z e^{-\beta u(q)} dq.$$

Corresponding to splitting q-space into chunks Δq_i, we get

$$N = \sum n_i \qquad n_i = z e^{-\beta u(q_i)} \Delta q_i$$

and the exponential series becomes

$$e^{n_1+\cdots+n_r} = \sum_{m_1,\dots,m_r} \frac{(n_1)^{m_1}}{(m_1)!}\cdots\frac{(n_r)^{m_r}}{(m_r)!}$$

so the dominant term is evidently when $m_i = n_i$.

Here are some ideas which emerge from this computation. First of all one has to break up the space in pieces in order to get a dominant term. In other words, [redacted] one should not write the partition fn. in the infinite symmetric product form ①, but rather one should write it as an integral over divisors on q-space. Secondly the accuracy of the dominant term approximation depends on the n_i being fairly large, which means that even when N is large, you don't want to chop up $\mathcal{V}$-space too much.

The real interesting point is whether there is an interesting infinite limit. Two possibilities: $N \to \infty$, V fixed or $N \to \infty$, $V \to \infty$, N/V fixed.

Digression on the Γ-function: Suppose we try to get the dominant term approx for $\sum \frac{w^n}{n!}$ using a generating function:

$$\sum_{n\geq 0} (e^{iJ})^n \frac{w^n}{n!} = e^{we^{iJ}}$$

By Fourier inversion

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{we^{iJ}} e^{-iJx}\, dJ = \sum \frac{w^n}{n!}\delta(x-n)$$

or

$$\frac{w^n}{n!} = \frac{1}{2\pi}\int_0^{2\pi} e^{we^{iJ}} e^{-iJn}\, dJ = \frac{1}{2\pi i}\oint e^{wz} z^{-n}\frac{dz}{z}$$

A good generalization of the last formula is

$$\frac{w^{-n}}{\Gamma(n+1)} = \frac{1}{2\pi i}\int e^{wz}\, z^{-n}\,\frac{dz}{z}$$

which is valid for $w > 0$ and any n. We can suppose $w = 1$ by homogeneity. [redacted] If we apply saddle point to the integrand:

$$\frac{d}{dz}\left(z - n\log z\right) = 1 - \frac{n}{z} = 0 \implies z = n$$

$$z - n\log z = n - n\log n + \frac{1}{2}\frac{n}{n^2}(z-n)^2$$

$$\frac{1}{2\pi i}\int e^{\frac{1}{2n}(z-n)^2}\frac{dz}{z} \doteq \frac{1}{2\pi i}\int e^{-\frac{n}{2}y^2}\frac{d(1+iy)}{1+iy} \doteq \frac{1}{\sqrt{2\pi n}}$$

we get [redacted] $n\uparrow$ Stirling's formula:

$$\frac{1}{\Gamma(n+1)} \sim e^{n-n\log n}\frac{1}{\sqrt{2\pi n}}$$

March 20, 1981

The problem is to see if a dominant term distribution can be found for a classical gas of independent particles. The partition function is an exponential

$$\sum \frac{z^n}{n!}\int dq_1 \cdots dq_n \, e^{-\sum \beta U(q_j)} = e^{z\int e^{-\beta U(q)}dq}$$

and so I have some sort of idea as to what dominant term should mean. One point is that if I chop up q-space into pieces dq_i which are not too fine, then the dominant configuration should have $ze^{-\beta U(q_i)}dq_i$ points in this volume.

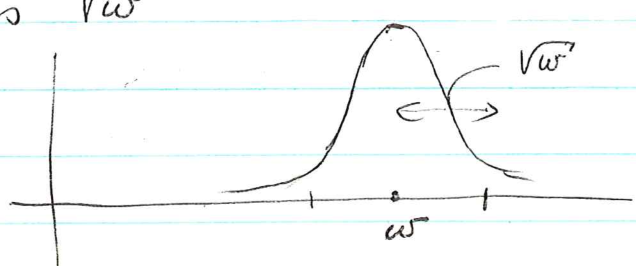
It seems to be relevant to discuss the dispersion as Einstein did. We have a general Poisson distribution so for any region Ω of q-space, the number of particles in Ω has a Poisson distribution

$$\langle n_\Omega \rangle = z\int_\Omega e^{-U(q)}dq$$

Now [redacted] for a Poisson distribution the dispersion is

$$\langle n_\Omega^2 \rangle - \langle n_\Omega \rangle^2 = \langle n_\Omega \rangle$$

So what this means is that for $\langle n_\Omega \rangle = \rho \operatorname{vol}(\Omega)$ large, the dispersion is comparatively small. For example if we plot the Poisson distribution $\frac{\omega^n}{n!}e^{-\omega}$, then the peak occurs at $n = \omega$, and the dispersion is ω so the spread is $\sqrt{\omega}$



Therefore it appears unreasonable to expect a good dominant term approximation unless one keeps things sufficiently coarse. My old idea was to allow fine subdivisions, and pull some sort of Γ-fn. trick to interpolate between small values of n. It's not clear this can be done because of the dispersion.

Is it possible to make [redacted] a Gaussian approximation? Just what this might be? We probably have enough to do this. Look at the generating function

$$e^{z\int(e^{iJ(q)}-1)\,e^{-\beta U(q)}\,dq} \sim e^{z\int_\Omega\left(iJ - \frac{J^2}{2}\right)e^{-\beta U(q)}\,dq}$$

so over any [redacted] set you get a Gaussian variable with mean and dispersion $= z\int_\Omega e^{-\beta U(q)}\,dq$

So what is the problem: I take a classical gas of independent particles and I would like to understand how to make a continuum limit out of it. I now understand the Gaussian approximation more or less and I am searching for something finer.

Possibility: The gas gives one a Poisson process on q-space. To each subset A the random variable n_A = no. of particles in A is a Poisson random variable with

$$\langle n_A \rangle = z\int_A e^{-\beta u(q)}\,dq$$

and hence $\langle n_A^2\rangle - \langle n_A\rangle^2 = \langle n_A\rangle$. Therefore the Gaussian approximation assigns to each subset A the Gaussian r.v. with this mean + dispersion.

Now Gaussian processes are described by a generating function of the form

$$e^{\,i\int J(q)\mu(q)\,dq \;-\; \frac{1}{2}\int dq\,dq'\; J(q)\,\nu(q,q')\,J(q')}$$

so it is clear that the Gaussian approximation to our gas has

$$\mu(q) = z e^{-\beta u(q)} \qquad \nu(q,q') = \mu(q)\,\delta(q-q')$$

A variance matrix ν supported on the diagonal more or less means that disjoint subsets of q-space are independent. There seems to be a Gaussian process where the variance matrix is a local operator, e.g. $-\Delta$.

The possibility is whether there is an analogue of a Poisson process where the variance matrix is local