249-328 Dec. 4 – Jan 12, 1982

ρ operators on Fock space of $L^2(S^1)$ 258–, 276, 278

December 4, 1981

V vector space, $\tilde{V} = V \oplus V^*$ with quadratic fn. $(v,\lambda) \longmapsto \langle \lambda | v \rangle$. Let $\theta \in O(\tilde{V})$ say

$$\theta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \begin{array}{l} A \in \mathrm{Hom}(V,V) \\ B \in \mathrm{Hom}(V^*,V) \quad \text{etc.} \end{array}$$

Then $\theta \begin{pmatrix} v \\ \lambda \end{pmatrix} = \begin{pmatrix} Av + B\lambda \\ Cv + D\lambda \end{pmatrix}$ so θ orthogonal means that

$$\langle Cv + D\lambda | Av + B\lambda \rangle = \langle Cv | Av \rangle + \langle Cv | B\lambda \rangle + \langle D\lambda | Av \rangle + \langle D\lambda | B\lambda \rangle = \langle \lambda | v \rangle$$

for all (v,λ), which yields

$$C^tA + A^tC = 0, \quad D^tB + B^tD = 0, \quad B^tC + D^tA = I.$$

or

$$\begin{pmatrix} D^t & B^t \\ C^t & A^t \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Similarly from

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} D^t & B^t \\ C^t & A^t \end{pmatrix} = id$$

we get the equations

$$AD^t + BC^t = I, \quad AB^t + BA^t = 0, \quad CD^t + DC^t = 0.$$

Examples of orth. transf are

$$\begin{pmatrix} 1 & \alpha \\ & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & \\ \gamma & 1 \end{pmatrix} \quad \begin{pmatrix} (\beta^t)^{-1} & \\ & \beta \end{pmatrix}$$

where α, γ are skew-symmetric. From the normal product business we want to write Θ in the form

$$\theta = \begin{pmatrix} 1 & \alpha \\ & 1 \end{pmatrix} \begin{pmatrix} (\beta^t)^{-1} & \\ & \beta \end{pmatrix} \begin{pmatrix} 1 & \\ \gamma & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ & 1 \end{pmatrix} \begin{pmatrix} (\beta^t)^{-1} & 0 \\ \beta\gamma & \beta \end{pmatrix} = \begin{pmatrix} (\beta^t)^{-1} + \alpha\beta\gamma & \alpha\beta \\ \beta\gamma & \beta \end{pmatrix}$$

which gives us the equations (Change β to $(\beta^t)^{-1}$ in this)

$$\boxed{\beta = (D^t)^{-1} \qquad \alpha = BD^{-1} \qquad \gamma = D^{-1}C}$$

~~Maybe a better notation is to use $(\beta^t)^{-1} = D$, so that $\langle 0 | \tilde{\theta} | 0 \rangle =$~~

Let's review what $\tilde{\Theta}$ is supposed to do. We have that $\tilde{\theta}$ is an operator on ΛV of the form

$$\tilde{\theta} = e^{\frac{1}{2}a^* \alpha a^*} \Lambda(\beta)\, e^{\frac{1}{2}a\gamma a}$$

up to a scalar, determined by

$$\tilde{\theta}\, \rho\binom{v}{\lambda}\, \tilde{\theta}^{-1} = \rho\left(\theta\binom{v}{\lambda}\right) \qquad \rho\binom{v}{\lambda} = e(v) + i(\lambda)$$

So the thing to remember perhaps is that if I have an orthogonal transformation $\theta = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then when I realize it as $\tilde{\theta}$ in the Clifford algebra, the β is $(D^t)^{-1}$.

Now the case of interest to me is when I have a vector space W with ~~orthogonal transformation~~ an automorphism f, I put $\tilde{V} = W \oplus W^*$ and let $\theta = \begin{pmatrix} f & 0 \\ 0 & (f^t)^{-1} \end{pmatrix}$ on $\tilde{V}$. Suppose that $W = W_+ \oplus W_-$ so that $\tilde{V} = W_+ \oplus W_- \oplus W_+^* \oplus W_-^*$ and let $V = W_+ \oplus W_-^*$

$$\tilde{V} = V \oplus V^*$$

If $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ relative to $W = W_+ \oplus W_-$ and

$$(f^t)^{-1} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

then

$$\theta = \begin{pmatrix} f & \\ & (f^t)^{-1} \end{pmatrix} = \begin{pmatrix} a & b & & \\ c & d & & \\ & & a_1^t & c_1^t \\ & & b_1^t & d_1^t \end{pmatrix} \quad \text{rel. to } \tilde{V} = W_+ \oplus W_- \oplus W_+^* \oplus W_-^*$$

and

$$\theta = \left(\begin{array}{cc|cc} a & 0 & & b \\ 0 & d_1^t & b_1^t & \\ \hline & c_1^t & a_1^t & \\ c & & & d \end{array}\right) \quad \text{relative to } \tilde{V} = V \oplus V^* = W_+ \oplus W_-^* \oplus W_+^* \oplus W_-$$

Suppose V is a complex Hilbert space and $\tilde{V} = V \oplus V^* = V \oplus \bar{V}$ where the inner product on V is used to identify $\bar{V}$ and V^*. Then $\tilde{V}$ has a real structure and an orthogonal transformation θ of $\tilde{V}$ preserving the real structure is what ought to give a unitary $\tilde{\theta}$. θ is real when

$$\bar{\theta} = \begin{pmatrix} \bar{D} & \bar{C} \\ \bar{B} & \bar{A} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \theta$$

i.e. when θ is of the form

$$\theta = \begin{pmatrix} \bar{D} & \bar{C} \\ C & D \end{pmatrix}$$

Now $\tilde{\theta}$ is of the form

$$e^{\frac{1}{2}a^*\alpha a^*} \; \Lambda(\beta) \; e^{\frac{1}{2}a\gamma a}$$

up to a multiplication scalar. Applied to $|0\rangle$ we get

$$e^{\frac{1}{2}a^*\alpha a^*}|0\rangle \qquad\qquad \tfrac{1}{2}a^*\alpha a^* = \sum_{i<j} a_i^*\alpha_{ij}a_j^*$$

which one can see has the norm square

$$1 + \left\| \tfrac{1}{2}a^*\alpha a^*|0\rangle \right\|^2 + \cdots\cdots$$

Now by analogy with the symplectic case I should be able to show a canonical form for a skew-symmetric complex matrix under unitary transf. is

$$\begin{pmatrix} 0 & \lambda_1 \\ -\bar{\lambda}_1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \lambda_2 \\ -\bar{\lambda}_2 & 0 \end{pmatrix}$$

and since

$$\left\| e^{\lambda a_1^* a_2^*}|0\rangle \right\|^2 = \left\| |0\rangle + \lambda\, a_1^* a_2^* |0\rangle \right\|^2 = 1 + |\lambda|^2$$

it should be the case that

$$\left\| e^{\frac{1}{2}a^*\alpha a^*}|0\rangle \right\|^2 = \det(1 + \alpha^*\alpha)$$

Can check this as follows: $\alpha = BD^{-1}$ so that

$$1 + \alpha^*\alpha = 1 + (D^{-1})^* B^* B D^{-1} = (D^{-1})^* \underbrace{(D^* D + B^* B)}_{A^t D + C^t B = I} D^{-1} = (D^{-1})^* D^{-1}$$

and hence the constant needed to make $\tilde{\Theta}$ unitary is $\dfrac{1}{\sqrt{\det(1+\alpha^*\alpha)}} = |\det D|$

<u>December 5, 1981</u>

Let's consider a vector space V with a direct sum decomposition:

$$V = V_+ \oplus V_-$$

and let f be an automorphism of V with matrix

$$f = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \begin{array}{l} A \in \operatorname{Hom}(V_+, V_+) \\ B \in \operatorname{Hom}(V_-, V_+) \end{array} \quad \text{etc.}$$

We want to understand the action of f on

$$\begin{aligned} \Lambda(V) &= \Lambda(V_+) \otimes \Lambda(V_-) \\ &= \Lambda(V_+) \otimes \Lambda(V_-^*) \otimes \lambda(V_-) \\ &= \Lambda(V_+ \oplus V_-^*) \otimes \lambda(V_-) \\ &\cong \Lambda(V_+ \oplus V_-^*) \end{aligned}$$

The idea is that the line $\lambda(V_-)$ in $\Lambda(V)$ corresponds to the maximal isotropic subspace of $V \oplus V^*$ will kills this line, i.e. $V_- \oplus V_+^*$ (Think of $V \oplus V^*$ as $e(V) + i(V^*)$). Now from our previous calculations in the Clifford algebra we expect $\tilde{f} = \Lambda(f)$ to be given in a normal product form

$$\text{const.} \quad e^{?\, a^* a^*} \quad a\, a^* \quad e^{?\, aa}$$

corresponding to the following factorization of f

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ +D^{-1}C & 1 \end{pmatrix}$$

The last factor acts as identity on V_-. Notice that one must also D is invertible in order to make this factorization.

We can now see what $\tilde{f}$ does to the line belonging to V_-.

$$fV_- = \left\{ \begin{pmatrix} Bw \\ Dw \end{pmatrix} \middle| \, w \in V_- \right\}$$

$$= \left\{ \begin{pmatrix} BD^{-1}w \\ w \end{pmatrix} \middle| \, w \in V_- \right\}$$

So the assumption that D is invertible means the same as requiring $fV_- \subset V_+ \oplus V_-$ to be complementary to V_+, and hence the graph of a map $T : V_- \longrightarrow V_+$. In this case $T = BD^{-1}$.

~~Now suppose V is a Hilbert space and $V = V_+ \oplus V_-$ is an orthogonal decomposition. We can choose orth. bases for V_+, V_- so that T is diagonal. $T(e_i) = \lambda_i e_i'$. Then~~

~~$$\lambda(fV_-) \ni \underbrace{(e_1 + \lambda_1 e_1') \wedge (e_2 + \lambda_2 e_2') \wedge \cdots}$$~~

~~image of [redacted] $e_1 \wedge e_2 \wedge \cdots$~~

~~under [redacted] $e^{?a^*a^*} \; (\quad) \; e^{?aa} = \tilde{\theta}$~~

~~Consequently if I want~~

So now let's pass to the infinite-diml. case. Then we know how to form $\Lambda(V; V_-)$ and for $f \in Aut(V)$ preserving V_- up to commensurability we can attach a autom. $\tilde{f}$ of $\Lambda(V; V_-)$ provided we choose a generator for $\lambda(V_-; fV_-)$. Precisely one has

$$\Lambda(V; V_-) \xrightarrow{f_*} \Lambda(V; fV_-) = \Lambda(V; V_-) \otimes \lambda(V_-; fV_-)$$

[redacted] For some reason [redacted] when one is given a decomposition [redacted] $V = V_+ \oplus V_-$ and one knows fV_- is complementary to V_+, there is a canonical trivialization of $\lambda(V_-; fV_-)$ provided by the projection $P : fV_- \longrightarrow V_-$.

[redacted] In effect, one is assuming that $fV_- \cap V_-$

is a finite codimension in V_- and fV_- and that $P : fV_- \to V_-$ is an isomorphism.

So where are we? On the fat cell of complements to V_+ commensurable with V_- the line bundle $L \mapsto \lambda(V_-; L)$ is trivialized by the isomorphism $P : L \xrightarrow{\sim} V_-$. This means that provided we restrict ~~attention~~ to f such that fV_- is in this fat cell, we will actually get a cocycle $c(f', f)$.

December 6, 1981

Consider $\Lambda(V;L_0)$ and autos. f of V which preserve L_0 up to commensurability. Recall that to lift f to an automorphism $\tilde{f}$ of $\Lambda(V;L_0)$ we need to choose a generator for $\lambda(fL_0;L_0) = \lambda(L_0;fL_0)^\vee$

$$\begin{tikzcd} \Lambda(V;L_0) \arrow[r, "f_*"] \arrow[drr, "\tilde{f}"'] & \Lambda(V;fL_0) = \Lambda(V;L_0)\otimes\lambda(L_0;fL_0) \arrow[d, "s"] \\ & & \Lambda(V;L_0) \end{tikzcd}$$

Now I want to take the case where I am given a complement to L_0: $V = W \oplus L_0$ and where I restrict attention to f such that fL_0 remains complementary to W. Then if $P: V \longrightarrow L_0$ is the projection with kernel W, I know that $P: fL_0 \to L_0$ is an isomorphism $= 1$ on the common subspace $fL_0 \cap L_0$ of finite codim. Hence the isomorphism gives me a generator of $\lambda(fL_0;L_0)$.

We have $\qquad \Lambda(V;L_0) = \Lambda(W)\otimes\Lambda(L_0^\vee)$

and $\Lambda(P) =$ augmentation on $\Lambda(W)\otimes$ identity on $\Lambda(L_0^\vee)$. Inside $\Lambda(V;L_0)$ I have the line belonging to $fL_0 = L$ which is canonically isomorphic to $\lambda(L;L_0)$. It seems clear that I give $\lambda(L;L_0)$ the generator which under $\Lambda(P)$ goes into the generator 1 of $\Lambda(V;L_0)$. In other words if L is the graph of $T: L_0 \to W$, so that $L = \{x+Tx \mid x \in L_0\}$ then taking $x_1,\dots,x_q$ to be a basis mod $L\cap L_0$ and the rest $x_{q+1},\dots$ a basis for $L\cap L_0$, we have

$$|0\rangle = x_1 \wedge x_2 \wedge \cdots\cdots$$

$$\text{gen. for line belonging to } L = (x_1+Tx_1)\wedge(x_2+Tx_2)\wedge\cdots\cdots$$

So what I've done is to assign to f such that fL_0 is complementary to W and, of course, [redacted] commensurable with L_0 an operator $\tilde{f}$ on $\Lambda(V; L_0)$ [redacted] normalized such that $\langle 0|\tilde{f}|0\rangle = 1$. Here $|0\rangle$ denotes the canonical element 1 in $\Lambda(V; L_0)$, and $\langle 0|$ denotes the linear functional on $\Lambda^{\bullet}(W \oplus L_0^{\vee})$ vanishing on Λ^+ and [redacted] such that $\langle 0|0\rangle = 1$.

Now one is in a position to compute the cocycle of the central extension using the lifting $f \mapsto \tilde{f}$, at least for these f. Let's do the calculation in finite dimensions where we have the lifting $f \mapsto \Lambda(f)$ on $\Lambda(V)$, and so $\tilde{f}$ will differ from f by a scalar. Let

$$f = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad D : L_0 \to L_0 \qquad B : L_0 \to W \quad \text{etc.}$$

Our assumption that fL_0 is complementary to W means D is invertible. Since I am using matrices I have a bases $x_1, \dots, x_n$ chosen for [redacted] L_0 and $|0\rangle = x_1 \wedge \dots \wedge x_n$.

$$(\Lambda f)|0\rangle = fx_1 \wedge \dots \wedge fx_n$$

$$\begin{aligned}
\langle 0 | \Lambda f | 0 \rangle &= \langle x_1^* \wedge \dots \wedge x_n^* \,|\, fx_1 \wedge \dots \wedge fx_n \rangle \\
&= \det \langle x_i^* | f x_j \rangle \\
&= \det \langle x_i^* | Bx_j + Dx_j \rangle \\
&= \det \langle x_i^* | Dx_j \rangle = \det(D).
\end{aligned}$$

So therefore one sees that

$$\Lambda f = \det(D)\, \tilde{f}$$

Now let

$$f' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \qquad f = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad f'f = \begin{pmatrix} & \\ & C'B + D'D \end{pmatrix}$$

and we find that the cocycle is

$$c(f',f)^{-1} = \det(D')^{-1}\det(C'B+D'D)\det(D)^{-1}$$
$$= \det\left(I+(D')^{-1}C'BD^{-1}\right)$$

This formula should work in infinite dimensions, because the assumption that fL_0 is commensurable with L_0 means that $T = BD^{-1}$ is of finite rank.

Next suppose f is unitary. Then one wants a unitary lifting. So

$$f^{-1} = f^* = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \times \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

which gives $\qquad B^*B + D^*D = I$

Now

$$\tilde{f}|0\rangle = (x_1+Tx_1)\wedge(x_2+Tx_2)\wedge\cdots$$

and up to unitary transf. in W, L_0 one can make T diagonal, say $Tx_i = t_i x_i'$

$$\|\tilde{f}|0\rangle\|^2 = \prod(1+|t_i|^2) = \det(1+T^*T)$$
$$= \det\left(1+(D^{-1})^*B^*BD^{-1}\right)$$
$$= c(f^{-1},f)^{-1}$$

In finite dimensions one can write this as

$$\det\left(1+(D^{-1})^*B^*BD^{-1}\right) = \det(D^*)^{-1}\det(D)^{-1} = |\det D|^{-2}$$

and so unitary lifting of f is

$$|\det D|\,\tilde{f}.$$

In some sense it seems that whereas the determinant of $D = Pf$ is meaningless in general, this operator is a contraction operator with all but a finite number of eigenvalues on S^1. Hence $|\det D|$ makes sense.

Better:

$$1/\det(1+(D^*)^{-1}B^*BD^{-1}) = 1/\det((D^*)^{-1}D^{-1}) = \det(DD^*)$$

the point being that $DD^* = I +$ finite rank operator.

This is very interesting in the case where $V = L^2(S^1)$ and $L_0 = H_+$, $f =$ map $S^1 \to S^1$ rational function. Then $D = Pf$, so we would like to compute

$$\det(Pf\,P\bar{f})$$

or more generally compute $\det(Pf\,Pf^{-1})$ where f is a rational function.

Let do this more carefully. Recall

$$c(f',f) = \det(1+(D')^{-1}C'BD^{-1})^{-1}$$

where f', f are as on bottom p. 256. Now suppose $f' = f^{-1}$ so that $C'B + D'D = I$. Then

$$c(f^{-1},f) = \det(DD') = \det(Pf \cdot Pf^{-1})$$

Remark: When f, g commute, then

$$c(f,g)\,c(g,f)^{-1} = \text{commutator pairing } (\tilde{f},\tilde{g})$$

is independent of the choice of trivializations defining the cocycle. Hence $\boxed{c(f^{-1},f) = c(f,f^{-1})}$

Lie algebra viewpoint: Look at the case $V = L^2(S^1)$, $L_0 = H_+$. An element of the Lie algebra is a function $f: S^1 \to i\mathbb{R}$, say given by a Laurent polynomial. f acts on V just by multiplication (since $(1+\varepsilon f)v = v + \varepsilon(fv)$). Let z^n be usual basis of V and let a_n, a_n^* be corresponding annihilation and creation operators. Then the extension of f to ΛV is given by

$$\rho(f) = \sum a_n^* \underbrace{\langle z^n | f | z^m \rangle}_{\langle z^{n-m}|f\rangle} a_m = \sum_p{}' \langle z^p | f \rangle \sum_{m\in\mathbb{Z}}{}' a_{p+m}^* a_m$$

When we try to do the same thing on $\Lambda(V; L_0)$ we run into some problems with the infinite sum $\sum a^*_{p+m} a_m$. This operator is a derivation which replaces z^m by z^{p+m}. If $p \neq 0$, then when applied to $|0\rangle = 1 \wedge z^1 \wedge z^2 \wedge z^3 \wedge \cdots$, it gives a finite sum since the terms $a^*_{p+m} a_m$ kill $|0\rangle$ for $m < 0$ and $p+m \geq 0$. However if $p = 0$, then

$$
\begin{aligned}
a_p^* a_p |0\rangle &= |0\rangle & p \geq 0 \\
&= 0 & p < 0
\end{aligned}
$$

and so we get an infinite result.

The reason is that because of the central extension one expects only to be able to lift f to an operator on $\Lambda(V; L_0)$ up to an additive constant. For example

$$\sum_{n<q} a_n^* a_n - \sum_{n \geq q} a_n a_n^*$$

is a lifting of the operator $f = 1$. A different q changes this operator by a scalar since $a_n^* a_n + a_n a_n^* = 1$.

So we've seen that $\sum_m a^*_{p+m} a_m$ makes sense on $\Lambda(V; L_0)$ for $p \neq 0$. Let's compute the commutator carefully

$$
\begin{aligned}
\Big[\sum_{|n| \leq M} a^*_{-p+n} a_n, \sum_{|m| \leq M} a^*_{p+m} a_m\Big] &= \sum_{n,m} \big(a^*_{-p+n} \delta_{n,p+m} a_m \\
&\qquad - a^*_{p+m} \delta_{-p+n,m} a_n \big) \\
&= \sum_{-M \leq m \leq M-p} a_m^* a_m - \sum_{-M \leq m \leq M-p} a^*_{p+m} a_{p+m} \\
&= \underbrace{\sum_{-M \leq m < -M+p} a_m^* a_m}_{0 \text{ on any elt.}} - \underbrace{\sum_{M-p < m \leq M} a_m^* a_m}_{-p \text{ on any element for } M \text{ large}}
\end{aligned}
$$

So you get the formula

$$[\rho(z^{-p}), \rho(z^p)] = -p.$$

So far I have described the operators giving the central extension of the Lie algebra. Now what I ulti[redacted]mately want to see is the boson creation + annih. operators, but I guess this is now more or less clear. You have to fix the exterior degree, which normally is the eigenvalue of $\sum' a_n^* a_n$. This means that for degree 0 in $\Lambda(V; L_0)$ you are looking at the zero eigenspace of

$$\sum_{n<0}' a_n^* a_n - \sum_{n\geq 0}' a_n a_n^*$$

which is how you want to define $\rho(1)$. So now the boson creation operators are the

$$\frac{1}{\sqrt{p}} \rho(z^{-p}) = \frac{1}{\sqrt{p}} \sum_{m\in\mathbb{Z}} a_{-p+m}^* a_m \qquad p = 1, 2, \ldots$$

and the annihilation operators should [redacted] be

$$\frac{1}{\sqrt{p}} \rho(z^{+p}) = \frac{1}{\sqrt{p}} \sum_{m\in\mathbb{Z}}' a_{+p+m}^* a_m$$

So the 1-boson state [redacted] of energy 1 is

$$\rho(z)|0\rangle = z^{-1}\wedge z^{1}\wedge z^{2}\wedge\cdots = a_{-1}^* a_0 |0\rangle$$

and the 1-boson state of energy 2 is

$$\frac{1}{\sqrt{2}} \rho(z^2)|0\rangle = \frac{1}{\sqrt{2}} (a_{-2}^* a_0 + a_{-1}^* a_1)|0\rangle$$

December 7, 1981

Japanese soln. to Riemann-Hilbert problem.

Again consider $V = L^2(S^1) = z^{-1}H_- \oplus H_+$ and the associated Clifford module $\Lambda(V; H_+) = \Lambda(H_+^\perp) \otimes \Lambda(H_+^*)$. On this Clifford module we have operators a_n $n \in \mathbb{Z}$ which kill z^n and a_n^* which creates z^n. Maybe the good thing to say is that for each $v \in V$ we have a creation operator $e(v)$ and its adjoint $i(v^*)$. Especially interesting are the operators

$$\psi_z^* = e(\delta_z) = \sum z^{-n} a_n^*$$

$$\psi_z = i(\delta_z^*) = \sum z^n a_n$$

which satisfy the anti-commutation relations

$$\{\psi_z^*, \psi_{z'}^*\} = \{\psi_z, \psi_{z'}\} = 0 \qquad \{\psi_z, \psi_{z'}^*\} = \sum_n z^{-n}(z')^n = \delta_{z,z'}$$

The ground state is $|0\rangle = z^0 \wedge z^1 \wedge z^2 \wedge \cdots$ and is killed by $a_{-1}, a_{-2}, \cdots, a_0^*, a_1^* \cdots$. Thus

$$\psi_z|0\rangle = \sum_{n \geq 0} z^n a_n |0\rangle \quad \text{is analytic} \quad |z| < 1$$

and

$$\langle 0|\psi_z = \langle 0|\sum_{n<0} z^n a_n = \sum_{n<0} z^n \langle 0|a_n$$

is analytic for $|z| > 1$.

Now suppose we are given $f: S^1 \to S^1$ of degree 0 and we lift it to a unitary operator $\tilde{f}$ such that

$$\tilde{f}\, e(v)\, \tilde{f}^{-1} = e(fv)$$

hence

$$\tilde{f}\, \psi_z^*\, \tilde{f}^{-1} = f(z)\, \psi_z^*$$

$$\tilde{f}\, \psi_z\, \tilde{f}^{-1} = \overline{f(z)}\, \psi_z \qquad f(z)\, \tilde{f}\psi_z = \psi_z \tilde{f}$$

Let us consider now the quantity

$$g_-(z) = \langle 0|\psi_z \psi_{z_0}^* \tilde{f}|0\rangle$$

which should be [redacted] a power series in z^{-1} with constant term. Then

$$g_-(z) = \langle 0|\psi_z\psi^*_{z_0}\tilde f|0\rangle = \delta_{z,z_0}\langle 0|\tilde f|0\rangle - \langle 0|\psi^*_{z_0}\psi_z\tilde f|0\rangle$$

$$= \delta_{z,z_0}\langle 0|\tilde f|0\rangle - f(z)\underbrace{\langle 0|\psi^*_{z_0}\tilde f\psi_z|0\rangle}_{g_+(z)}$$

where $g_+(z)$ is a power series in z.

Check for $\tilde f = id$. Then

$$g_-(z) = \langle 0|\psi_z\psi^*_{z_0}|0\rangle = \sum_{n<0} z^n\langle 0|a_n a_n^*|0\rangle z_0^{-n}$$

$$= \frac{z^{-1}z_0}{1-z^{-1}z_0} \quad \text{converges for } |z_0| \le 1,\ |z| \ge 1$$

and
$$g_+(z) = \langle 0|\psi^*_{z_0}\psi_z|0\rangle = \sum_{n\ge 0} z_0^{-n}\langle 0|a_n^* a_n|0\rangle z^n$$

$$= \frac{1}{1-z_0^{-1}z} \quad \text{converges for } |z_0|\ge 1,\ |z|<1.$$

and extended to distributions on S^1 one has

$$g_-(z) - g_+(z) = \delta_{z_0,z}$$

[redacted] This $\delta_{z_0,z}$ is interesting. In the Japanese paper one takes $f(z)$ to be a function on S^1 which is constant except for jumps. Then it's better to think of S^1 as being replaced by the real line. One gives points $a_1 < a_2 < \cdots < a_k$ on $\mathbb{R}$ and then f is constant off these points starting with $f=1$ for $x<a_1$. For such an f the degree is meaningless, so I am confused. But in any case we can look at what (g_-, g_+) constitutes. It is an analytic function off the real axis [redacted] such that across the interval (a_i, a_{i+1}), g_- and $-g_+$ fit together up to the scalar, [redacted] given by f on this interval. Thus one gets a multiple-valued analytic function with prescribed monodromy.

One still has to see about the actual singularities of g and see that these are algebraic, i.e.

$$g(z) = (z-a_i)^k h(z)$$

with h holomorphic invertible at $z = a_i$.

An interesting question is whether such a discontinuous f actually can be implemented by a unitary operator.

December 9, 1981

Let's review the representation of the central extension of the Lie algebra. An element of the Lie algebra is $f = \sum c_n z^n$ where c_n is an $r \times r$ matrix. As an operator it becomes

$$\sum a^*_{n,i} \langle z^n e_i | f | z^m e_j \rangle a_{m,j}$$

or simply

$$\sum a^*_n \langle z^{n-m} | f \rangle a_m = \sum a^*_{m+p} c_p a_m .$$

Thus it is a combination of the operators

$$\circledast \qquad f_p = \sum_p a^*_{m+p} a_m$$

where to simplify I will use $r=1$. We calculated for $|0\rangle = z^0 \wedge z^1 \wedge \cdots$ case that f_p as such makes sense for $p \neq 0$ and $[f_p, f_{-p}] = p$

whereas the other brackets are all zero.

The formula $\circledast$ reminds me of density operators for an electron gas. ~~[illegible]~~ Recall that we put things in a box of volume V, so that (forgetting spin) the 1-particle basis consists of the states

$$\frac{1}{\sqrt{V}} e^{-ikx} \qquad k \in \frac{2\pi}{L} \mathbb{Z}^d$$

hence

$$\langle x| = \sum_k \langle x|k\rangle \langle k|$$

and

$$\psi(x) = \frac{1}{\sqrt{V}} \sum_k e^{ikx} a_k$$

$$\psi(x)^* = \frac{1}{\sqrt{V}} \sum_{k'} e^{-ik'x} a^*_{k'}$$

and the density is

$$\rho(x) = \psi(x)^* \psi(x) = \frac{1}{V} \sum e^{-i(k'-k)x} a^*_{k'} a_k$$

$$= \frac{1}{V} \sum_q e^{-iqx} \underbrace{\sum_k a^*_{k+q} a_k}_{\rho_q}$$

Normally we work with energy $\varepsilon_k = k^2/2m - \mu$ so the Fermi-sea state has finitely many particles in it. We won't see anything like the relation $[f_p, f_{-p}] = p$ unless possibly we pass to the infinite-volume limit.

December 10, 1981

Continuous version.

$$\langle x| = \int \frac{dk}{2\pi} \overbrace{\langle x|k\rangle}^{e^{ikx}} \langle k| \qquad\qquad \langle k|l\rangle = 2\pi\,\delta(k-l)$$

$$\psi(x) = \int \frac{dk}{2\pi} e^{-ikx}\, a_k \qquad\qquad \{a_k, a_l^*\} = \langle k|l\rangle$$

$$\rho(x) = \psi(x)^*\psi(x) = \int \frac{dq}{2\pi} e^{-iqx} \underbrace{\int \frac{dk}{2\pi}\, a^*_{q+k}\, a_k}_{\rho_q}$$

Now we compute $[\rho_q, \rho_{q'}]$. This is confusing but one might try to proceed as in the discrete case.

$$\left[\int_{|k|\leq M} \frac{dk}{2\pi}\, a^*_{q+k}\, a_k ,\ \int_{|k'|\leq M} \frac{dk'}{2\pi}\, a^*_{q'+k'}\, a_{k'} \right]$$

$$= \int_{\substack{|k|\leq M \\ |k'|\leq M}} \frac{dk\,dk'}{(2\pi)^2} \left[a^*_{q+k}\, \delta_{k,q'+k'}\, a_{k'} - a^*_{q'+k'}\, \delta_{q+k,k'}\, a_k \right]$$

? too hard.

Let's go back to the discrete case

$$\rho_q = \sum_k a^*_{k+q}\, a_k \qquad\qquad k \in \frac{2\pi}{L}\mathbb{Z}$$

and we know already that

$$[\rho_q, \rho_{q'}] = q\,\delta_{q,-q'}$$

which means that we get standard boson

creation and annihilation operators

$$b_q = \frac{1}{\sqrt{q}} \rho_q \qquad q > 0.$$

So now suppose we take a quadratic operator in the b_q and call it the Hamiltonian. We have

$$\rho(x) = \frac{1}{V} \sum_q e^{-iqx} \rho_q$$

So for example one might try

$$\int dx\, \rho(x)^2 = \int_V dx \frac{1}{V^2} \sum_q e^{-iqx} \rho_q \sum_{q'} e^{-iq'x} \rho_{q'}$$

$$= \frac{1}{V} \sum_q \rho_q \rho_{-q}$$

$$= \frac{1}{V} \sum_{q>0} q(b_q b_q^* + b_q^* b_q) \qquad + \frac{1}{V}\rho_0^2$$

$$= \frac{1}{V} \sum_{q>0} 2q\, b_q^* b_q \qquad + \left(\frac{1}{V}\sum_{q>0} q + \frac{1}{V}\rho_0^2\right)$$

where we drop the infinite constant and make the definition

$$H = \frac{1}{2} :\int dx\, \rho(x)^2: \;=\; \frac{1}{V} \sum_{q>0} q\, b_q^* b_q$$

So it's clear the equation of motion for this Hamiltonian is easy to solve. It corresponds to each oscillator b_q having frequency q, and I know that corresponds to a_k having frequency k.

Let's review carefully the above. I start with $L^2(S^1)$ which I think of as periodic functions with period 2π. Better: let's shift to the line with periodic functions of period L. Then we get the orthonormal basis $\frac{1}{\sqrt{L}} e^{-inx} \qquad n \in \frac{2\pi}{L}\mathbb{Z}$

and we can form the exterior algebra with operators a_n, a_n^* relative to this basis. Then

$$\langle x| = \sum_n \langle x|n\rangle\langle n|$$

yields

$$\psi(x) = \frac{1}{\sqrt{L}}\sum e^{inx}\, a_n$$

$$\psi^*(x) = \frac{1}{\sqrt{L}}\sum e^{-inx}\, a_n^*$$

so that

$$\{\psi(x), \psi^*(y)\} = \frac{1}{L}\sum_{n,m} e^{inx} e^{-imy} \underbrace{\{a_n, a_m^*\}}_{\delta_{n,m}}$$

$$= \frac{1}{L}\sum_n e^{in(x-y)} = \delta(x-y)$$

In order to make the connection with work on S^1 I should recall that

$$\psi(z) = \sum z^n a_n$$

$$\psi(z)^* = \sum z^{-n} a_n^*$$

$$\{\psi(z), \psi(z')^*\} = \sum \left(z(z')^{-1}\right)^n = \delta_{z,z'}$$

because I have used the measure $\frac{d\theta}{2\pi}$ on S^1.

Now before I was concerned with the problem of extending functions on the circle to operators, and thereby obtaining a central extension. So I wrote

$$f \longmapsto \sum a_n^* \langle z^n|f|z^m\rangle a_m$$

$$= \sum_p \langle z^p|f\rangle \sum_m a_{p+m}^* a_m$$

The analogous thing here is

$$f \mapsto \sum_p \left(\frac{1}{L}\int e^{-ipx} f(x)\,dx\right) \sum_m a_{p+m}^* a_m$$

$$= \int dx\, f(x) \underbrace{\left[\frac{1}{L}\sum_p e^{-ipx} \sum_m a_{p+m}^* a_m\right]}_{\psi(x)^*\psi(x)}$$

So what one is doing formally is to write
$$f = \int dx \ |x\rangle f(x) \langle x| \qquad \text{on } L^2(S^1)$$
and to send it to the operator
$$\rho(f) = \int dx \ f(x)\, \psi^*(x)\psi(x).$$
But now we know that in the present situation the definition of the operator $\rho(x) = \psi(x)^*\psi(x)$ involves a few subtleties. We actually define it by defining carefully the operators
$$\rho_p = \sum_m a^*_{p+m} a_m$$
on the exterior algebra. For $p \neq 0$, this makes sense as it stands, but for $p=0$ one must remove an infinite scalar. In fact one writes it in normal product form. Thus relative to $|0\rangle = z^0 \wedge z^1 \wedge z^2 \wedge \dots$ one has
$$\rho_0 = \sum_{m<0} a^*_m a_m - \sum_{m\geq 0} a_m a^*_m$$

I am admitting for the moment the commutation relations
$$[\rho_p, \rho_q] = p\,\delta_{p,-q}$$
but [redacted] I should work out a good proof. These commutation relations are equivalent to
$$[\rho(f), \rho(g)] = \operatorname{Res}(g\,df)$$
or
$$[\rho(x), \rho(y)] = i\,\delta'(x-y)$$
In effect
$$[\rho(x), \rho(y)] = \sum_{m,n} [e^{-imx}\rho_m, e^{-iny}\rho_n] = \sum_{m,n} e^{-imx-iny}\, m\,\delta_{m,-n}$$
$$= \sum_m e^{-im(x-y)}\, m = i\,\delta'(x-y)$$

Now we know that the operators

$$b_n = \frac{1}{\sqrt{n}}\rho_n \qquad b_n^* = \frac{1}{\sqrt{n}}\rho_{-n} \qquad n>0$$

form a collection of boson creation and annihilation operators. Furthermore any of the states $z^{k} \wedge z^{k+1} \wedge \cdots$ are killed by the b_n for $n>0$, [redacted] so that in each degree [redacted] for the exterior algebra we have a ground state. In fact we get a nice irreducible repn. [redacted] in each degree by the Jacobi identity.

The next stage is to form from the operators b_n, b_n^* a quadratic Hamiltonian. [redacted] The simplest thing is

$$\int \frac{1}{2}\rho(x)^2 \frac{dx}{2\pi} = \int \frac{1}{2}\left(\sum e^{-inx}\rho_n\right)^2 \frac{dx}{2\pi}$$

$$= \frac{1}{2}\sum \rho_n\rho_{-n}$$

$$= \frac{1}{2}\sum_{n>0} n(\underbrace{b_n b_n^*}_{b_n^* b_n + 1} + b_n^* b_n) + \frac{1}{2}\rho_0^2$$

[redacted]

$$= \sum_{n>0} n b_n^* b_n + \underbrace{\frac{1}{2}\left(\sum_{n>0} n + \rho_0^2\right)}_{\text{infinite constant which becomes 0 if we normal order.}}$$

Thus $$\boxed{H = :\int \frac{1}{2}\rho(x)^2 \frac{dx}{2\pi}: \; = \sum_{n>0} n b_n^* b_n}$$

Now in [redacted] each exterior degree we have an [redacted] a vector killed by the $b_n = \frac{1}{\sqrt{n}}\sum a_{n+k}^* a_n$ $n>0$ namely $z^l \wedge z^{l+1} \wedge z^{l+2} \wedge \cdots$, [redacted] so that in each exterior degree I have the symmetric algebra. The time-evolution described by the above [redacted] Hamiltonian assigns the energy n to the boson created by b_n^*. Same as the normal time-evolution.

December 12, 1981

change operator to L

Review KdV: Start with operator $A = \partial_x^2 + q$ and you look for a skew-adjoint operator B such that the Lax equation $\dot{A} = [B, A]$ gives a differential equation inside the family of Schroedinger operators. For example $B = \partial_x$ leads to $\dot{q} = [\partial_x, \partial_x^2 + q] = q'$ which has translation $q_t(x) = q(x+t)$ for its solutions.

Next try $B = \partial_x^3 + a\partial_x + \partial_x a = \partial_x^3 + 2a\partial_x + a'$

and compute: $[B, A] = [\partial_x^3 + 2a\partial_x + a', \partial_x^2 + q]$

$$= [\partial_x^3, q] + 2[a, \partial_x^2]\partial_x + [a', \partial_x^2] + 2a[\partial_x, q]$$

$$= 3q'\partial_x^2 + 3q''\partial_x + q''' + 2(-1)[2a'\partial_x + a'']\partial_x - [2a''\partial_x + a'''] + 2aq'$$

$$= [3q' - 4a']\partial_x^2 + [3q'' - 4a'']\partial_x + \underbrace{[q''' - a''' + 2aq']}$$

So we take $a = \frac{3}{4}q$ and then $= \frac{1}{4}q''' + \frac{3}{2}qq'$

and so we the KdV equation of motion:

$$\boxed{\partial_t q = \frac{1}{4}q''' + \frac{3}{2}qq'}$$

Now we suppose that we have a family of potentials q_t satisfying KdV such that scattering analysis can be applied: Let $\varphi_k(x)$ be the solution of $(\partial_x^2 + q)u = -k^2 u$ with

$$e^{-ikx} \longleftrightarrow A(k)e^{-ikx} + B(k)e^{ikx}$$

Differentiate $A\varphi_k = -k^2\varphi_k$ wrt. t and you get

$$\underbrace{\dot{A}\varphi_k} + A\dot{\varphi}_k = -k^2\dot{\varphi}_k$$

$$[BA - AB]\varphi_k = -k^2 B\varphi_k - AB\varphi_k ,$$

so
$$A(\dot{\varphi}_k - B\varphi_k) = -k^2(\dot{\varphi}_k - B\varphi_k)$$

which shows that $\dot{\varphi}_k - B\varphi_k$ is a solution of the Schroedinger equation. Compute its asymptotic behavior and you get

$$-(-ik)^3 e^{-ikx} \longleftrightarrow [\mathring{A} - (-ik)^3 A] e^{-ikx} + [\mathring{B} - (ik)^3 B] e^{ikx}$$

which show $\quad \mathring{\varphi}_k - B\varphi_k = -(-ik)^3 \varphi_k$. So subtract from the asymptotic behavior and you get

$$0 \longleftrightarrow \mathring{A} e^{-ikx} + [\mathring{B} - \underbrace{(ik)^3}_{-i^3} B + \underbrace{(-ik)^3}_{-i^3} B] e^{ikx}$$

which gives the equations of motion

$$\boxed{\mathring{A} = 0 \qquad \mathring{B} = -2ik^3 B}$$

which can be integrated to given $A(k,t) = A(k,0)$
$B(k,t) = e^{-2ik^3 t} B(k,0)$.

Now A has an analytic extension to the UHP and its zeroes there occur at the bound states. Hence these energies $-k^2$, $k = iK$ don't change in time. At such a bound state we can define $B(iK)$ by

$$\varphi_{iK} = B(iK)\psi_{iK}$$

$$e^{Kx} \longleftrightarrow B(iK) e^{-Kx}$$

and the same argument as before shows that

$$\mathring{B}(iK) = -2i(iK)^3 B(iK)$$
$$= -2K^3 B(iK)$$

(One has to be careful not to assume $B(k)$ is defined in the UHP. It's only defined in general for k real, usually in a strip around the real axis, and at the bound state energies. For example, there are the reflection-less potentials which have $B(k) = 0$ for k real, yet the $B(iK) \neq 0$.)

Lastly one has

$$G_k(x,x') = \langle x | \frac{1}{k^2 + A} | x' \rangle = \frac{\varphi_k(x_<)\,\psi_k(x_>)}{W(\varphi_k, \psi_k)} \qquad W = 2ik\,A(k)$$

$$\operatorname*{Res}_{k=iK} G_k(x,x') = \frac{\varphi_{iK}(x_<)\,\psi_{iK}(x_>)}{-2K\,A'(iK)} = \frac{B(iK)}{(-2K)A'(iK)}\ \varphi_{iK}(x)\,\varphi_{iK}(x')$$

which relates the number <u>$B(iK)$</u> to the L^2 norm of $\varphi_{iK}(x)$.

Next project is to generalize this to the Dirac-type equations, namely

$$\underbrace{\begin{pmatrix} \frac{1}{i}\partial_x & p \\ \tilde{p} & i\partial_x \end{pmatrix}}_{L} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = k \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Write $\quad L = \alpha\partial_x + q \qquad \alpha = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \qquad q = \begin{pmatrix} 0 & p \\ \tilde{p} & 0 \end{pmatrix}$

and try for $B = \alpha\partial_x^2 + a\partial_x + b$ (coeff. of ∂_x^2 must commute with α, hence $\sim \alpha$)

$$[B, L] = [\alpha\partial_x^2 + a\partial_x + b, \alpha\partial_x + q]$$

$$= [\alpha\partial_x^2, q] + [a\partial_x, \alpha\partial_x] + [a\partial_x, q] + [b, \alpha\partial_x] + [b, q]$$

$$= [\alpha, q]\partial_x^2 + \alpha[\partial_x^2, q] + [a, \alpha\partial_x]\partial_x + \cancel{a[\partial_x, \alpha\partial_x]} + [a, q]\partial_x + aq' + [b, \alpha]\partial_x + \alpha(-b') + [b, q]$$

$$= [\alpha, q]\partial_x^2 + \alpha(2q'\partial_x + q'') + [a, \alpha]\partial_x^2 + \alpha\underbrace{[a, \partial_x]}_{-a'}\partial_x + \ldots$$

$$= \{[\alpha, q] + [a, \alpha]\}\partial_x^2 + \{\alpha 2q' - \alpha a' + [a, q] + [b, \alpha]\}\partial_x + \{\alpha q'' + aq' - \alpha b' + [b, q]\}$$

So we want $\quad a = q \quad$ to kill the coeff of ∂_x^2. Then the coeff of ∂_x is

$$\alpha 2q' - \alpha q' + [b, \alpha].$$

Now the natural candidate for b is $\frac{q'}{2}$ because in the self-adjoint case $(\tilde{p} = \bar{p})$ then B will be skew-adjoint:

$$B = \alpha\partial_x^2 + q\partial_x + \tfrac{1}{2}q' = \alpha\partial_x^2 + \tfrac{1}{2}q\partial_x + \tfrac{1}{2}\partial_x q.$$

Then the coeff of ∂_x is

$$\alpha q' + \left[\tfrac{q'}{2}, \alpha\right] = \alpha q' + \underbrace{\tfrac{q'}{2}\alpha}_{-\alpha\frac{q'}{2}} - \alpha\tfrac{q'}{2} = 0.$$

So the equation of motion is so far

$$\dot{q} = [B, L] = \alpha q'' + qq' - \alpha\tfrac{q''}{2} + \left[\tfrac{q'}{2}, q\right].$$

It looks like we must add a diagonal term $f\alpha$ to b. Thus $\quad b = \tfrac{1}{2}q' + f\alpha$, and then

$$\dot{q} = \alpha q'' + qq' - \alpha\left(\tfrac{1}{2}q'' + f'\alpha\right) + \left[\tfrac{1}{2}q' + f\alpha, q\right]$$

$$= \tfrac{1}{2}\alpha q'' + \tfrac{1}{2}(q'q + qq') + f' + f(\alpha q - q\alpha)$$

So we arrange the diagonal terms to cancel

$$\frac{1}{2}(q'q+qq') + f' = 0$$

by taking $f = -\frac{1}{2}q^2$. Then we get

$$\dot{q} = \frac{1}{2}\alpha q'' - \frac{1}{2}q^2(2\alpha q)$$

$$= \frac{1}{2}\alpha q'' - \alpha q^3$$

Recall that $q = \begin{pmatrix} 0 & p \\ \tilde{p} & 0 \end{pmatrix}$ we get the equations

$$\begin{cases} i\dot{p} = \frac{1}{2}p'' - p\tilde{p}p \\ -i\dot{\tilde{p}} = \frac{1}{2}\tilde{p}'' - \tilde{p}p\tilde{p} \end{cases}$$

which are call the non-linear Schroedinger equation.

Formulas $L = \alpha\partial_x + q = \begin{pmatrix} \frac{1}{i}\partial_x & p \\ \tilde{p} & i\partial_x \end{pmatrix}$

$$B = \alpha\partial_x^2 + q\partial_x + \frac{1}{2}q' - \frac{1}{2}\alpha q^2$$

So now let's do the scattering. Take soln of $Lu = ku$

$$e^{-ikx}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \xleftrightarrow{\phi_k} \begin{pmatrix} Be^{ikx} \\ Ae^{-ikx} \end{pmatrix}$$

As before we know that $-\dot{\phi}_k + B\phi_k$ is again a solution whose asymptotic behavior is

$$\alpha\partial_x^2 \; e^{-ikx}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \longleftrightarrow -\begin{pmatrix} \dot{B}e^{ikx} \\ \dot{A}e^{-ikx} \end{pmatrix} + \alpha\partial_x^2\begin{pmatrix} Be^{ikx} \\ Ae^{-ikx} \end{pmatrix}$$

$$e^{-ikx}\begin{pmatrix} 0 \\ i \end{pmatrix} \longleftrightarrow - \frac{1}{(-ik)^2}\begin{pmatrix} \dot{B}e^{ikx} \\ \dot{A}e^{-ikx} \end{pmatrix} + \begin{pmatrix} \frac{1}{i}Be^{ikx} \\ iAe^{-ikx} \end{pmatrix}$$

$$e^{-ikx}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \longleftrightarrow + \frac{i}{(-ik)^2}\begin{pmatrix} \dot{B}e^{ikx} \\ \dot{A}e^{-ikx} \end{pmatrix} + \begin{pmatrix} -Be^{ikx} \\ Ae^{-ikx} \end{pmatrix}$$

So you conclude $\dot{A} = 0$ $\quad \frac{i\dot{B}}{-k^2} = 2B$ or

$$\boxed{\dot{A} = 0, \; \dot{B} = 2ik^2B}$$

December 14, 1981

Let's start with an $S : S^1 \longrightarrow S^1$.
I would like to compute $|\langle 0 | \tilde{S} | 0 \rangle|$ which is some sort of determinant. Here $|0\rangle$ belongs to the subspace H_+, and $\tilde{S}|0\rangle$ belongs to SH_+. Relative to the decomposition $L^2(S^1) = z^{-1}H_- \oplus H_+$ we have

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad\qquad B^*B + D^*D = I$$

and I will suppose D is invertible since otherwise $\langle 0 | \tilde{S} | 0 \rangle = 0$. Then

$$SH_+ = \{ Bx + Dx \mid x \in H_+ \} = \{ BD^{-1}x + x \mid x \in H_+ \}$$
$$= \text{graph of } T = BD^{-1} : H_+ \longrightarrow z^{-1}H_-$$

and hence I know that

$$\tilde{S}|0\rangle = \text{const } (x_1 + Tx_1) \wedge (x_2 \wedge Tx_2) \wedge \cdots$$

where the constant has to make this of norm 1. Now

$$\| (x_1 + Tx_1) \wedge (x_2 \wedge Tx_2) \wedge \cdots \|^2 = \det(1 + T^*T)$$

and hence the constant is $\det(1+T^*T)^{-1/2}$ in absolute value. Thus

$$|\langle 0 | \tilde{S} | 0 \rangle| = \det(1 + T^*T)^{-1/2} = (\det D^*D)^{1/2}$$

Note that the assumption that D is invertible implies that S has degree zero, since the degree = index of $D = P_+ S P_+$. Another point is that if we use the factorization

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix}$$

to formally define $\tilde{S}$, then

$$\langle 0 | \tilde{S} | 0 \rangle = \det(D)$$

except that this isn't meaningful because $\log \det(D)$ has an infinite imaginary part.

Now let us look carefully at the equation

$$|\langle 0|\tilde{S}|0\rangle| = \det(1+T^*T)^{-1/2} = \det(D^*D)^{1/2}$$

and see if I can get to the orthogonal projection stuff. You want to use $D^*D = I - B^*B$ where $B = P_- S P_+$ and this operator is related to the orthogonal projection. How? ~~We know that $H_- \cap SH_+ = z(H_+)^\perp \cap SH_+$ is one-dimensional. Let's try to project 1 on this subspace, and call e the result.~~

~~$$1 = e + \text{something in } \left(z(H_+)^\perp \cap SH_+\right)^\perp = zH_+ + S(H_+)^\perp$$~~

~~$$z = ze + f_+ + zSg_- \qquad f_+ \in H_+,\ g_- \in (H_+)^\perp$$~~

We have $L^2 = (H_+)^\perp \oplus SH_+$ (not orthogonal) hence $1 + g_- = Sf_+$ with $g_- \in (H_+)^\perp$ and $f_+ \in H_+$. This is just the factorization of $S = f_-/f_+$ with f_- normalized. Thus we have

$$g_- = P_- S f_+ \qquad f_+ = P_+ \bar{S}(1+g_-)$$

or that

$$1 + g_- = [redacted]\ 1 + P_- S P_+ \bar{S}(1+g_-)$$

$$\therefore \quad 1 + g_- = \frac{1}{1 - P_- S P_+ \bar{S}}\ [redacted]\ 1$$

So therefore we see that if we set [redacted] up an integral equation to compute the factorization of S, then $|\langle 0|\tilde{S}|0\rangle|$ is the Fredholm determinant for the integral equation. [redacted]

Still the fundamental problem remains: how to compute this number $|\langle 0|\tilde{S}|0\rangle| = \det(1 - B^*B)^{1/2}$ in this simple case of a degree 0 map $S: S^1 \to S^1$.

December 15, 1981

Let $S: S^1 \longrightarrow S^1$ be a map of degree zero. Then we can compute $|\langle 0|\tilde{S}|0\rangle|$ in the following way. We know that $\tilde{S}$, when viewed in the boson representation, is given by a translation operator, i.e. one of the form

$$e^{-\frac{1}{2}\|\gamma\|^2} \; e^{a^*(\bar{\gamma})} \; e^{-a(\gamma)}$$

so that $|\langle 0|\tilde{S}|0\rangle| = e^{-\frac{1}{2}\|\gamma\|^2}$. Let's work this out carefully. Let us begin with the Lie algebra formulas.

The Lie algebra when complexified has the basis z^n which is lifted to the operator

$$\rho_n = \sum_{k\in\mathbb{Z}}{}' \, a^*_{n+k} a_k$$

with appropriate definitions when $n=0$. (This gives a representation of the central extension of the Lie algebra) We have the commutation relations

$$[\rho_n, \rho_m] = n\,\delta_{n,-m} = \operatorname{res}(z^m dz^n).$$

Now all we have to do is exponentiate. A typical element in our central extension lying over S looks how? $\log(S) = \sum c_n z^n = g_- + c_0 + g_+$ where $g_+ = \sum_{n>0} c_n z^n$ and $\overline{g_+} = -g_-$, $c_0 \in i\mathbb{R}$. A typical $\tilde{S}$ will be of the form $\zeta\, e^{g_-} e^{g_+}$ with a suitable scalar ζ. Now

$$g_+ = \sum c_n z^n \longmapsto \sum c_n \rho_n = \sum_{n\geq 1} c_n \sqrt{n}\, b_n$$

$$g_- = -\sum \bar{c}_n z^{-n} \longmapsto \sum{}' -\bar{c}_n \rho_{-n} = \sum -\bar{c}_n \sqrt{n}\, b_n^*$$

Call the former operator $b(g_+)$ and the latter $b^*(g_-)$. Then $\tilde{S} = \zeta\, e^{b^*(g_-)} e^{b(g_+)}$ and $\tilde{S}$ will be unitary provided that

$$\tilde{S}\tilde{S}^* = |\zeta|^2 e^{b^*(g_-)} e^{b(g_+)} e^{b(g_+)^*} e^{[b^*(g_-)]^*} e^{-b^*(g_-)} e^{-b(g_+)}$$

$$= |\zeta|^2 e^{[b(g_+),\, -b^*(g_-)]}$$

$$= |\zeta|^2 e^{\sum_1^\infty |c_n|^2 n}$$

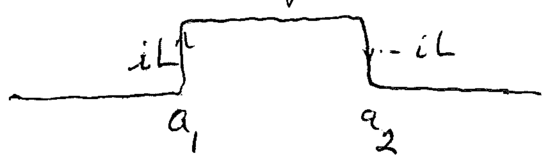
So therefore I see that

$$|\langle 0|\tilde{S}|0\rangle| = |\zeta| = e^{-\frac{1}{2}\sum_1^\infty |c_n|^2 n}.$$

~~[illegible]~~

Example: Let's consider an S occurring in the Japanese solution to RH problem:

$\log S$:



$$\delta(x-a) = \sum' e^{in(x-a)} \qquad \text{wrt } \frac{dx}{2\pi}$$

$$\frac{d}{dx}\log S = iL\left(\sum' e^{in(x-a_1)} - \sum e^{in(x-a_2)}\right)$$

$$\log S = L\sum_{n\neq 0}' \frac{e^{inx}\left(e^{-ina_1} - e^{-ina_2}\right)}{n}$$

so that $c_n = L\dfrac{e^{-ina_1} - e^{-ina_2}}{n}$. Take $a_1 = 0$ and then one sees that

$$\sum_1^\infty |c_n|^2 n = L^2 \sum_1^\infty \frac{|1 - e^{-ina_2}|^2}{n}$$

is logarithmically divergent.

Next calculation is to compute $\langle 0|\tilde{S}|0\rangle$ infinitesimally if possible. Let's begin with the idea that the tangent space to the Grassmannian of subspaces L of index 0 relative to L_0 at the point L_0 is $\mathrm{Hom}(L_0, V/L_0)$ where one looks a continuous homomorphisms, that is, which vanish on one of the given subspaces $L' \subset L_0$. Let's identify the image of this tangent space in $\Lambda(V; L_0)$. We can exterior multiply $e(v)|0\rangle$ and this depends only on $v \bmod L_0$. We can also interior multiply by $i(\lambda)$, and $i(\lambda)|0\rangle$ depends only on the restriction of λ to L_0. So the map we want is

$$V/L_0 \otimes L_0^* \longrightarrow \Lambda^0(V; L_0)/\text{line gen. by } |0\rangle$$

$$(v \otimes \lambda) \longmapsto e(v)\,i(\lambda)|0\rangle$$

and this is well-defined because $\;= -i(\lambda)e(v)|0\rangle + \langle\lambda|v\rangle|0\rangle$.

Let's compute $|\langle 0|\tilde{s}|0\rangle|$ infinitesimally. We have

$$|\langle 0|\tilde{s}|0\rangle| = \det(1 - B^*B)^{1/2}$$
$$= -\tfrac{1}{2}\,\mathrm{tr}(B^*B) \quad \cdots\cdots$$

where $B = P_- S P_+$. Suppose $S = 1 + \underbrace{\log S}_{\sum c_n z^n} + \cdots$ and we suppose the c_n are small. Then to the first order in the c_n, the matrix for S is

$$\left(\begin{array}{cc|ccc}
 & & c_{-3} & & \\
 & c_{-1} & c_{-2} & c_{-3} & \\
 & 1+c_0 & c_{-1} & c_{-2} & c_{-3} \\
\hline
c_2 & c_1 & 1+c_0 & c_{-1} & \\
 & c_2 & c_1 & 1+c_0 &
\end{array}\right)$$

where B is the upper right corner. Clearly

$$\mathrm{tr}(B^*B) = |c_{-1}|^2 + 2|c_{-2}|^2 + 3|c_{-3}|^2 + \cdots$$
$$= \sum_{1}^{\infty} n\,|c_n|^2$$

which checks the earlier formula.

December 16, 1981

Let's try the factorization of the whole S-matrix

$$S = \begin{pmatrix} T & R \\ \tilde{R} & T \end{pmatrix} \qquad \tilde{R} = -\frac{\bar{R}}{\bar{T}}\, T .$$

We want $\quad I + g_- = S g_+ \quad$ with $g_+ \in H_+$, $g_- \in H_+^\perp$

Assuming this holds we get the equations

$$\begin{aligned} g_- &= P_- S g_+ \\ g_+ &= P_+ S^*(I + g_-) \\ &= P_+(S^*) + \underbrace{P_+ S^* P_- S g_+}_{(P_+ S^* P_- S P_+) g_+} \end{aligned}$$

Now

$$P_- S P_+ = \begin{pmatrix} 0 & P_- R P_+ \\ P_- \tilde{R} P_+ & 0 \end{pmatrix} \qquad \text{since } T(H_+) = H_+ .$$

$$P_+ S^* P_- = \begin{pmatrix} 0 & P_+ \bar{\tilde{R}} P_- \\ P_+ \bar{R} P_- & 0 \end{pmatrix}$$

So therefore,

$$P_+ S^* P_- S P_+ = \begin{pmatrix} P_+ \bar{\tilde{R}} P_- \tilde{R} P_+ & 0 \\ 0 & P_+ \bar{R} P_- R P_+ \end{pmatrix}$$

Now when $|R| < 1$ on S^1 this is a contraction operator. Call it K, and we have

$$g_+ = \frac{1}{1-K}\, \underbrace{P_+(S^*)}_{\begin{pmatrix} P_+ \bar{T} & P_+ \bar{\tilde{R}} \\ P_+ \bar{R} & P_+ \bar{T} \end{pmatrix}} .$$

What's interesting here is that $P_+ \bar{T}$ is the constant $\overline{T(0)} = \int \bar{T} \frac{d\theta}{2\pi}$. So?

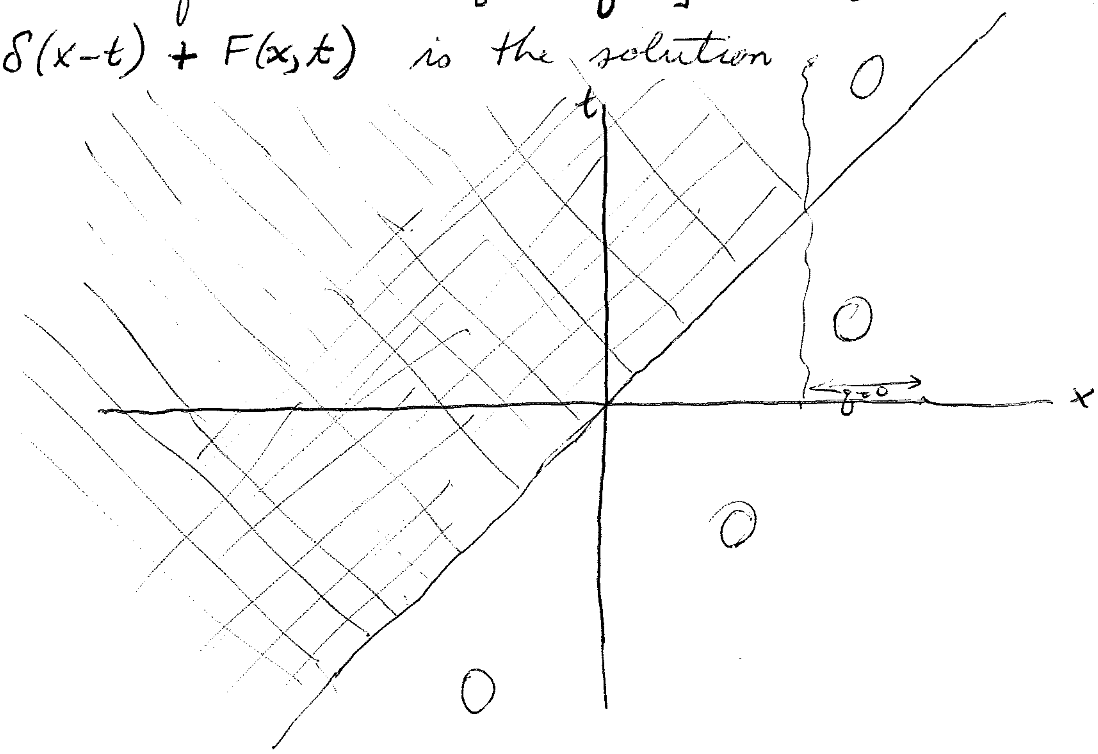
Let's review the Marchenko equation method.
The Schroedinger equation is $[\partial_x^2 + q(x)]u = -k^2 u$.
Let $f_k(x)$ denote the solution $\sim e^{ikx}$ as $x \to +\infty$.
It is a solution of the Volterra integral equation

$$f_k(x) = e^{ikx} + \int_x^\infty \frac{\sin k(x-y)}{k}(-q(y))\, f_k(y)\,dy$$

which one can solve by iteration. Putting in e^{iky} for $f_k(y)$ one gets exponentials $e^{ik(x-y)+iky}$, $e^{-ik(x-y)+iky} = e^{ik(2y-x)}$ and as $y > x$ one gets exponentials e^{iky} with $y > x$ upon iteration. Thus

$$(1) \qquad f_k(x) = e^{ikx} + \int_x^\infty F(x,y)e^{iky}\,dy$$

Another way to understand $F(x,y)$ is to use the Fourier transform to relate the Schroedinger equation to the wave equation $[\partial_x^2 + q(x)]u = \partial_t^2 u$. In particular $\delta(x-t) + F(x,t)$ is the solution



By differentiating (1) wrt x and using that F satisfies the wave equation, one can estalish the formula

$$\boxed{q(x) = +2\frac{d}{dx}F(x,x)}$$

This should follow directly from the fact $\delta(x-t) + F(x,t)$ is a solution of the wave equation.

The Marchenko equation results from the following. Let the scattering be

$$T e^{-ikx} \longleftrightarrow e^{-ikx} + R e^{+ikx}$$

or more precisely

$$T \phi_k = f_{-k} + R f_k$$

Recall that T is a linear combination of e^{iky} $y>0$ and $\phi_k(x)$ involves e^{-iky} $y<x$, hence $T\phi_k$ is a linear combination of $e^{-ikx}e^{iku}$ with $u>0$. Therefore

$$f_{-k}^{(x)} + R f_k(x) \in e^{-ikx}(1+H_+)$$

~~Or if we put $f_k(x) = e^{ikx} F_k(x)$ we have~~

$$\cancel{F_{-k}(x) + R e^{2ikx} F_k(x) \in 1 + H_+}$$

or putting $f_k(x) = e^{ikx}(1+F_x(k))$ we get

$$1 + \bar{F}_x + R e^{2ikx}(1+F_x) \in 1 + H_+$$

or

$$\boxed{\bar{F}_x + P_- R e^{2ikx}(1+F_x) = 0}$$

which is essentially the Marchenko equation. Here

$$F_x(k) = \int_x^\infty F(x,y)\, e^{-ik(y-x)}\, dy = \int_0^\infty F(x,x+u)\, e^{iku}\, du$$

and one knows $F(x,y)$ is real so that $\overline{F_x(k)} = F_x(-k)$.

Recall the relation between $(\partial_x^2 + g)u = -k^2 u$ and the Dirac style equation

$$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik & p \\ \bar{p} & -ik \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

When $p = \bar{p}$,
$$\left(\frac{d}{dx} - p\right)(u_1+u_2) = ik(u_1-u_2)$$
$$\left(\frac{d}{dx} + p\right)(u_1-u_2) = ik(u_1+u_2)$$

so that
$$\left(\frac{d}{dx}+p\right)\left(\frac{d}{dx}-p\right)(u_1+u_2) = -k^2(u_1+u_2)$$

$$= \frac{d^2}{dx} + (-p' - p^2)(u_1 + u_2)$$

so that $\quad -q = +p' + p^2.$

Asymptotics: Looking for an asymptotic solution of Dirac of the form

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = e^{ikx} \begin{pmatrix} a_0 + a_1/k + \cdots \\ b_0 + b_1/k + \cdots \end{pmatrix} \quad \text{yields}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = e^{ikx} \begin{pmatrix} 1 + \frac{1}{2ik} \int^x |p|^2 + \cdots \\ \frac{1}{2ik} \bar{p}(x) + \cdots \end{pmatrix}$$

This means that if we construct the solution $\vec{f}_k \sim e^{ikx}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as $x \to \infty$ by solving the Volterra equation:

$$\vec{f}_k(x) = e^{ikx}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \int_x^\infty \begin{pmatrix} e^{ik(x-x')} & 0 \\ 0 & e^{-ik(x-x')} \end{pmatrix} \begin{pmatrix} 0 & p(x') \\ \bar{p}(x') & 0 \end{pmatrix} \vec{f}_k(x')\, dx'$$

$$= e^{ikx}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_x^\infty \vec{F}(x,y)\, e^{iky}\, dy$$

then we can find $\quad + \frac{1}{2}\bar{p}(x) = -F_2(x,x)$

$$\frac{1}{2}\int_{+\infty}^{x} |p|^2 = -F_1(x,x)$$

which agrees with $-q = +p' + p^2 = 2\frac{d}{dx} F(x,x).$

December 17, 1981

For Schroedinger equation

$$f_k(x) = e^{ikx}\Big(1 + \underbrace{\int_0^\infty F(x,x+u)\,e^{iku}\,du}_{F_x(k)}\Big)$$

$$f_{-k}(x) = \overline{f_k(x)} = e^{-ikx}\big(1 + \overline{F_x(k)}\big)$$

and

$$\underbrace{T(k)\,\phi_k(x)}_{\in\, e^{-ikx}(1+H_+)} = f_{-k}(x) + R(k) f_k(x)$$

so that

$$1 + \overline{F_x} + \overbrace{Re^{2ikx}}^{R_x}(1+F_x) \in 1 + H_+$$

leading to the Marchenko equation

$$\boxed{\overline{F_x} + P_- R_x(1+F_x) = 0}$$

For the Dirac equation

$$\vec{f}_k(x) = e^{ikx}\left(\begin{pmatrix}1\\0\end{pmatrix} + \underbrace{\int_0^\infty \begin{pmatrix}F_1(x,x+u)\\F_2(x,x+u)\end{pmatrix} e^{iku}\,du}_{\begin{pmatrix}F_{1x}(k)\\F_{2x}(k)\end{pmatrix}}\right)$$

$$\widetilde{\vec{f}_k(x)} = e^{-ikx}\left(\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}\overline{F_{2x}}\\\overline{F_{1x}}\end{pmatrix}\right)$$

[redacted] and

$$T_{[redacted]}(k)\,\vec{\phi}_k\,[redacted] = \tilde{\vec{f}}_k + R(k)\vec{f}_k$$

so that we get the equations

$$\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}\overline{F_{2x}}\\\overline{F_{1x}}\end{pmatrix} + R_x\begin{pmatrix}1+F_{1x}\\F_{2x}\end{pmatrix} \in \begin{pmatrix}0\\1\end{pmatrix} + H_+$$

or

$$\boxed{\begin{aligned}\overline{F_{2x}} + P_- R_x(1+F_{1x}) &= 0\\ \overline{F_{1x}} + P_- R_x(F_{2x}) &= 0\end{aligned}}$$

[redacted] Somehow the solution of these equations is connected with a factorization of the S-matrix.

So let the S-matrix be

$$S = \begin{pmatrix} R & T \\ T & \tilde{R} \end{pmatrix} \qquad \tilde{R} = -\frac{\bar{R}}{\bar{T}}T$$

so that

$$P_- S P_+ = \begin{pmatrix} P_- R P_+ & 0 \\ 0 & P_- \tilde{R} P_+ \end{pmatrix} \qquad S^* = \begin{pmatrix} \bar{R} & \bar{T} \\ \bar{T} & \bar{\tilde{R}} \end{pmatrix}$$

The factorization results from the fact the Hilbert space is a direct sum:

$$V = H_- \oplus SH_+ \qquad H_- = (H_+)^\perp$$

Hence I can take $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and decompose it.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + g_- = S g_+ \qquad g_- \in (H_+)^\perp,\ g_+ \in H_+$$

This gives

$$g_- = P_- S g_+$$

$$g_+ = \underbrace{P_+ S^* \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\begin{pmatrix} P_+\bar{R} \\ P_+\bar{T} \end{pmatrix}} + P_+ S^* g_-$$

$$(I - P_+ S^* P_- S) g_+ = \begin{pmatrix} P_+\bar{R} \\ P_+\bar{T} \end{pmatrix}$$

$$P_+ S^* P_- P_- S P_+ = \begin{pmatrix} P_+ \bar{R} P_- R P_+ & 0 \\ 0 & P_+ \bar{\tilde{R}} P_- \tilde{R} P_+ \end{pmatrix}$$

Hence

$$g_+ = \begin{pmatrix} P_+\bar{R} + P_+\bar{R}P_-RP_+\bar{R} + \cdots\cdots \\ P_+\bar{T} + P_+\bar{\tilde{R}}P_-\tilde{R}P_+\bar{T} + \cdots\cdots \end{pmatrix}$$

So take equations for $\vec{F} = \vec{F}_x$ when $x = 0$.

$$\bar{F}_2 + P_- R(1 + F_1) = 0 \implies F_2 + P_+\bar{R}(1 + \bar{F}_1) = 0$$

$$\bar{F}_1 + P_- R F_2 = 0$$

$$\bar{F}_1 = P_- R(-F_2) \qquad (-F_2) = P_+\bar{R} + P_+\bar{R}(\bar{F}_1)$$

Consequently

$$-F_2 = P_+\bar{R} + P_+\bar{R}\,P_-R\,P_+\bar{R} + \cdots = (g_+)_1$$

Now further work along these lines gets confusing because of how to interpret 1 in the continuous case. In the above I regarded $P_+1 = 1$.

Possible approach is to do things to first order in R. Solve the LS equation $[k^2 + \partial_x^2]\psi = q\psi$

Change q to −q

$$\psi_k = e^{-ikx} + \int \frac{e^{ik|x-x'|}}{2ik}(+q(x'))\,\psi_k(x')\,dx'$$

by iterating once. Get

$$\psi_k = e^{-ikx} + \frac{+1}{2ik}\int_{-\infty}^{\infty} e^{ik|x-x'|}\,q(x')e^{-ikx'}\,dx'$$

$$e^{-ikx}\left(1 + \frac{1}{2ik}\int_{-\infty}^{\infty} q(x')dx'\right) \longleftrightarrow e^{-ikx} + e^{ikx}\left(+\frac{1}{2ik}\int e^{-2ikx'}q(x')\,dx'\right)$$

so to first order in the potential q we get

$$T(k) = 1 + \frac{1}{2ik}\int_{-\infty}^{\infty} q(x')\,dx'$$

$$R(k) = +\frac{1}{2ik}\int_{-\infty}^{\infty} e^{-2ikx'}q(x')\,dx'$$

Notice: This seems to imply the S matrix is singular at $k=0$. [redacted] First notice that we can factor: $q = p' + p^2$ when there is a non-vanishing solution u of $(-\partial_x^2 + q)u = 0$, in which case $p = u'/u$. Assuming q decays very fast, the asymptotic [redacted] behavior of ϕ is linear as x [redacted] $\to \pm\infty$, hence p decays like $1/x$. If I want p to decay fast, it is necessary, that the solution $\phi \sim 1$ as $x \to -\infty$ also [redacted] be asymptotically constant as $x \to +\infty$. In this case there is a unique p.

Conclusion: The class of Schroedinger equations $(-\partial_x^2+q)u=k^2u$ which can be effectively reduced to Dirac systems with p decaying fast is quite small. First of all the spectrum has to be ≥ 0, so that we can find [redacted] solutions ϕ of $(-\partial_x^2+q)\phi=0$ such that $\phi>0$ always, but then this ϕ has to be unique which means that if $\phi \sim$ Constant as $x\to-\infty$, [redacted] then also $\phi\sim$ const as $x \longrightarrow +\infty$. For example no $q(x)\geq 0$ except $q\equiv 0$ is in this class.

Somehow it is clear that all the action takes place with the bound states, and that you really ought to first understand how to bring these into the picture.

For the moment let's finish up the analysis when $p'+p^2=q$. We work to first order in p. Then

$$\int q = \int p' = 0.$$

$$\int e^{-2ikx} q(x)\,dx = \int e^{-2ikx} p'(x)\,dx$$

$$= 2ik\int e^{-2ikx} p(x)\,dx$$

so we have

$$\begin{cases} T(k) = 1 & (O(p^2)) \\ R(k) = \displaystyle\int_{-\infty}^{\infty} e^{-2ikx} p(x)\,dx & \left(O(p^2)\right) \end{cases}$$

Thus

$$-F_{20}(k) = P_+\overline{R} = P_+\int_{-\infty}^{\infty} e^{2ikx}\,\overline{p(x)}\,dx$$

$$= \int_0^{\infty} e^{2ikx}\,\overline{p(x)}\,dx \qquad \sim -\frac{1}{2ik}\,\overline{p(0)}$$

New idea: Use the parameter $\lambda = k^2$ instead of k. For each λ the space of solutions of $(-\partial^2 + q)u = \lambda u$ is 2-diml. We therefore get a rank 2 holomorphic vector bundle over $\mathbb{C}$, which can be trivialized by using Cauchy data at some point. [redacted] Over $\{\lambda \notin \mathbb{R}_{\geq 0}\}$ there are 2 sections of this bundle f_λ, g_λ defined as follows. f_λ is the solution decaying as $x \longrightarrow +\infty$ and g_λ is the solution decaying as $x \longrightarrow -\infty$. Thus

$$f_\lambda(x) \sim e^{ikx} \quad x\to\infty \quad \text{where } k = \sqrt{\lambda} \text{ in the UHP}$$

$$g_\lambda(x) \sim e^{-ikx} \quad x \to -\infty$$

We have the following requirements. [redacted] At each bound ~~state~~ f_λ, g_λ become proportional to each other. Along $\mathbb{R}_{\geq 0}$ one has four sections $f_\lambda^\pm, g_\lambda^\pm$ defined by analytic continuation, and these four sections are related by the S-matrix. It is simplest to describe things when $R=0$.

Suppose $\lambda > 0$. Then we have

$$T(k)\, e^{-ikx} \longleftrightarrow e^{-ikx}$$

or simply $\boxed{T(+\sqrt{\lambda})\, g_\lambda^+ = f_\lambda^-}$. In effect f_λ^- is the solution [redacted] either $\sim e^{ikx}$ or $\sim e^{-ikx}$ which when λ and hence $+\sqrt{\lambda} = k$ is pushed into the LHP decays. Hence $f_\lambda^- \sim e^{-ikx}$. Similarly

$$\boxed{T(-\sqrt{\lambda})\, g_\lambda^- = f_\lambda^+}$$

Now we can pose this as a problem. Given $T(\sqrt{\lambda})$ and some non-zero numbers c_λ at the bound states, [redacted] consider a pair of analytic functions f_λ^+, g_λ^+ in the closed UHP and a pair f_λ^-, g_λ^- in the closed LHP, except possibly $\lambda = 0$, which satisfy the required properties. Thus $f_\lambda^- = f_\lambda^+$ for $\lambda < 0$, and $g_\lambda^{\pm} = c_\lambda f_\lambda^{\pm}$ at bound states, etc.

We get a solution of this problem for each x.

Perhaps if one works in growth conditions for a solution, where these depend on x, we get back the functions $f_\lambda^{\pm}(x)$, $g_\lambda^{\pm}(x)$

December 20, 1981

Review Marchenko equation with bound states.

$$G_\lambda(x,x') = \frac{\phi_k(x_<)\, f_k(x_>)}{W(\phi_k, f_k)} \qquad k = \sqrt{\lambda} \text{ in UHP}$$

$$e^{-ikx} \xleftarrow{\ \phi_k\ } \xrightarrow{\quad} A(k)e^{-ikx} + B(k)e^{ikx}$$

$$f_k(x) \sim e^{ikx} \quad \text{as } x \to +\infty$$

Thus $\phi_k = Af_{-k} + Bf_k$ so One has $W(\phi_k, f_k) = A(k)\underbrace{W(e^{-ikx}, e^{ikx})}_{2ik}$

$$\begin{cases} f_k(x) = e^{ikx} + \displaystyle\int_x^\infty F(x,y)e^{iky}\,dy \\ e^{ikx} = f_k(x) + \displaystyle\int_x^\infty \tilde{F}(x,y) f_k(y)\,dy \end{cases}$$

obtained by solving Volterra integral equations.

Completeness relation

$$\delta(x,x') = \frac{1}{2\pi i}\int G_\lambda(x,x')\,d\lambda = \frac{-1}{2\pi i}\int \frac{\phi_k(x_<)\, f_k(x_>)}{2ik\,A(k)}\,2k\,dk$$

$$= \sum_K \frac{\phi_{iK}(x_<)\, f_{iK}(x_>)}{iA'(iK)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\phi_k(x_<)\, f_k(x_>)}{A(k)}\,dk$$

Here $k = iK$ runs over the bound state energies, i.e. points in UHP where $A(k) = 0$. At these points B is defined: $\phi_{iK}(x) = B(iK) f_{iK}(x)$. Also with $R = \frac{B}{A}$ we thus get

$$\delta(x,x') = \sum_K \underbrace{\frac{B(iK)}{iA'(iK)}}\, f_{iK}(x) f_{iK}(x') + \frac{1}{2\pi}\int_{-\infty}^{\infty} (f_{-k} + Rf_k)(x_<)\, f_k(x_>)\,dk$$

and one notes that $\smile$ is the inverse of $\|f_{iK}(x)\|^2$. Call it c_K.

Now we will suppose $x' > x$ and use that $e^{ikx'}$ is a linear combination of the $f_k(y)$ with $y > x$ to get

$$0 = \sum_K c_K f_{iK}(x) e^{-Ky} + \frac{1}{2\pi}\int_{-\infty}^{\infty} (f_{-k} + Rf_k)(x) e^{+iky}\, dk \qquad y > x$$

Finally put in $f_k(x) = e^{ikx} + \int_x^{\infty} F(x,z) e^{ikz} dz$ and you get

$$0 = \sum_K c_K e^{-K(x+y)} + \int_x F(x,z) \sum_K c_K e^{-K(z+y)} dz$$

$$+ \int_x^{\infty} dz\, F(x,z) \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikz} e^{iky} dk}_{\delta(z-y)} + \hat{R}(x+y)$$

$$+ \int_x^{\infty} dz\, F(x,z) \int \frac{dk}{2\pi} R(k) e^{ik(z+y)}$$

So the integral equation becomes

$$(*) \qquad F(x,y) + H(x+y) + \int_x^{\infty} dz\, F(x,z) H(z+y) = 0 \qquad y > x$$

where
$$H(x) = \sum_K c_K e^{-Kx} + \int_{-\infty}^{\infty} \frac{dk}{2\pi} R(k) e^{ikx}$$

Let's rewrite the integral equation in the form

$$F(x, x+\hat{y}) + H(2x+\hat{y}) + \int_0^{\infty} d\hat{z}\, F(x, x+\hat{z}) H(2x+\hat{z}+\hat{y}) = 0$$

It's then clear that solving $(*)$ for $F(x,y)$ is the same as solving it when $x = 0$, but with $H(y)$ replaced by $H(2x+y)$. Thus c_K gets changed to $c_K e^{-2Kx}$ and $R(k)$ to $R(k)e^{2ikx}$

Now the integral equation $(*)$ is not too useful. What we want to concentrate on is the function $k \xrightarrow{f} f_k(0) = 1 + \int_0^{\infty} F(0,z) e^{ikz} dz$ which is in $1 + H_+$ and hence is an analytic function of k in the UHP. We want $\bar{f} + Rf$ to be analytic in UHP except for simple poles at the points iK and the residues is to be $c_K f_{iK}$ essentially.

$\bar{f}$ means $k \mapsto f_{-k}$

Let's do this directly from the DE. We are after $f : k \longrightarrow f_k(0)$ which is ~~to~~ be in $1+H_+$. We have:

$$\frac{\phi_k(0)}{A(k)} = f_{-k}(0) + \frac{B(k)}{A(k)} f_k(0)$$

and this has a simple pole at $k = iK$ where $A(iK) = 0$. the residue at the pole is

$$\operatorname*{Res}_{k \to iK} \frac{\phi_k(0)}{A(k)} = \frac{\phi_{iK}(0)}{A'(iK)} = \underbrace{\frac{B(iK)}{A'(iK)}}_{i c_K} f_{iK}(0)$$

Therefore $f \in 1 + H_+$ and we want

$$\bar{f} + Rf \in 1 + \sum_K \frac{i c_K f_{iK}(0)}{k - iK} + H_+$$

Next let's do some examples when $R = 0$.

Suppose we have 1 bound state at iK. We want $f \in 1 + H_+$ such that $\bar{f} \in 1 + \frac{i c f(iK)}{k - iK} + H_+$. Try

$$f = 1 + \frac{ia}{k + iK} \qquad f(iK) = 1 + \frac{ia}{2iK}$$

Then

$$\bar{f} = 1 + \frac{ia}{-k + iK} = 1 + \frac{ic}{k - iK}\left(1 + \frac{ia}{2iK}\right) + g \in H_+$$

so that

$$-ia = ic\left(1 + \frac{ia}{2iK}\right) = ic + \frac{ica}{2K}$$

$$-ic = ia\left(1 + \frac{c}{2K}\right)$$

$$\boxed{a = \frac{-c}{1 + \frac{c}{2K}}}$$

In general let's put $f = 1 + \sum_K \frac{i a_K}{k + iK}$. Then

$$f_{-k} = \cancel{1} + \sum_K \frac{i a_K}{-k + iK} = \cancel{1} + \sum_K \frac{i c_K}{k - iK}\left(1 + \sum_{K'} \frac{i a_{K'}}{iK + iK'}\right)$$

$$\boxed{-a_K = c_K\left(1+\sum_{K'} \frac{a_{K'}}{K+K'}\right)}$$

Go back to a single K and let's work out the potential. We know that

$$f_k(x) = e^{ikx}\left(1+\frac{1}{k+iK}\cdot\frac{-ice^{-2Kx}}{1+\frac{ce^{-2Kx}}{2K}}\right)$$

But in general for $(-\partial_x^2+q)u = k^2u$ we have

$$f_k(x) \sim e^{ikx}\left(1+\frac{1}{2ik}\int_\infty^x q + \cdots\right)$$

so therefore

$$\boxed{\int_\infty^x q = 2i\,\frac{-ice^{-2Kx}}{1+\frac{ce^{-2Kx}}{2K}} = \frac{2c}{e^{2Kx}+\frac{c}{2K}}}$$

Check this against the factorization approach

$$L_0+\beta^2 = -\partial_x^2+q_0+\beta^2 = -(\partial_x+p)(\partial_x-p)$$

where $p=\phi'/\phi$ and $(L_0+\beta^2)\phi=0$ with ϕ non-vanishing. Then

$$L_1+\beta^2 = -(\partial_x-p)(\partial_x+p)$$

$$q_1+\beta^2 = p^2-p' = p^2+p'-2p' = q_0+\beta^2-2p'.$$

so the new potential is

$$q_1 = q_0-2p'$$
$$= q_0 - 2\frac{d^2}{dx^2}\log\phi$$

So consequently if we take $q_0=0$ and

$$\phi = e^{\beta x}+\gamma e^{-\beta x}$$

$$p = \beta\,\frac{e^{\beta x}-\gamma e^{-\beta x}}{e^{\beta x}+\gamma e^{-\beta x}} = \beta - \frac{2\beta\gamma e^{-\beta x}}{e^{\beta x}+\gamma e^{-\beta x}}$$

$$p = \beta - \frac{2\beta\gamma}{e^{2\beta x}+\gamma} \qquad \therefore \quad \boxed{q_1 = \frac{d}{dx}\,\frac{4\beta\gamma}{e^{2\beta x}+\gamma}}$$

which agrees with the above: $\beta = K$, $\gamma = \frac{c}{2K}$, $4\beta\gamma = 4K\frac{c}{2K} = 2c.$

Let's now consider a Dirac style system

$$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik & p \\ \tilde{p} & -ik \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where, in order to have bound states, I will want to keep $\tilde{p} \neq \bar{p}$. On the right $x \longrightarrow \infty$ we obtain solutions with the asymptotic behavior

$$f_k(x) \sim \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix} \qquad \tilde{f}_k(x) \sim \begin{pmatrix} 0 \\ e^{-ikx} \end{pmatrix}$$

f_k is analytic in the UHP, $\tilde{f}_k$ in the LHP. Similarly as $x \longrightarrow -\infty$ we get solutions

$$\phi_k(x) \sim \begin{pmatrix} 0 \\ e^{-ikx} \end{pmatrix} \qquad \tilde{\phi}_k \sim \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix} \qquad x \to -\infty.$$

What about scattering? Two A coefficients, one for the upper and the other for the lower half plane

$$\begin{cases} \phi_k = A(k)\tilde{f}_k + B(k) f_k \\ \tilde{\phi}_k = \tilde{A}(k) f_k + \tilde{B}(k)\tilde{f}_k \end{cases}.$$

These equations make sense along the real axis, where both $f, \tilde{f}$ are defined. However A extends to the upper half plane because it is the Wronskian $\det(f_k, \phi_k)$. Also the values for B are defined at points in the UHP where $A = 0$; similarly values for $\tilde{B}$ are defined at points in the LHP where $\tilde{A} = 0$.

So now I want to assume that the reflection is zero i.e. $B = \tilde{B} = 0$ on $\mathbb{R}$, and I would like to see if the bound state information is enough to pin down $f_k, \tilde{f}_k$ at $x = 0$. Let's assume there is a bound state at iK in the UHP and $-iL$ in the LHP. Then

$$\frac{\phi_k}{A(k)} = \tilde{f}_k$$

is by construction analytic in the LHP, but $\frac{\phi}{A}$

gives an analytic extension to the UHP except for the pole at $k = iK$ where we know that

$$\operatorname*{res}_{k=iK} \tilde{\phi} = \frac{\phi_{iK}}{A'(iK)} = \frac{B(iK)}{A'(iK)} f_{iK} .$$

Similarly $\frac{\tilde{\phi}}{\tilde{A}} = f$ is analytic with a pole at $-iL$ where

$$\operatorname*{res}_{k=-iL} f = \frac{\tilde{B}(-iL)}{\tilde{A}'(-iL)} \tilde{f}_{-iL}$$

So now we should be able to solve this for $f \in 1 + H_+$, $\tilde{f} \in 1 + H_-$. Try to get

$$f = 1 + \frac{a}{k+iL} \qquad \tilde{f} = 1 + \frac{b}{k-iK}$$

to satisfy

$$f = 1 + \frac{d}{k+iL} \tilde{f}(-iL)$$

$$\tilde{f} = 1 + \frac{c}{k-iK} f(iK)$$

Here c, d are given and you want to find a, b.

Let's consider a Dirac system which is self-adjoint:

$$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik & p \\ \bar{p} & -ik \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Then we have the scattering relations

$$\phi = A\tilde{f} + Bf$$
$$\tilde{\phi} = \tilde{B}\tilde{f} + \tilde{A}f$$

or

$$(\tilde{\phi} \;\; \phi) = (f \;\; \tilde{f})\begin{pmatrix} \tilde{A} & B \\ \tilde{B} & A \end{pmatrix}$$

Taking Wronskians we get

$$\tilde{A}A - \tilde{B}B = 1 .$$

In this situation where $\tilde{p} = \bar{p}$ we have the symmetry

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, k \longmapsto \begin{pmatrix} \bar{u}_2 \\ \bar{u}_1 \end{pmatrix} \quad \bar{k}$$

and so $\qquad \tilde{A}(k) = \overline{A(\bar{k})}\ , \quad \tilde{B}(k) = \overline{B(\bar{k})}$.

Now the scattering matrix expresses the incoming basis $\tilde{f}, \tilde{\phi}$ (which are analytic in the LHP) in terms of the outgoing basis f, ϕ (which are analytic in UHP):

$$\tilde{f} = \frac{1}{A}\phi - \frac{B}{A} f$$

$$\tilde{\phi} = \tilde{B}\left(\frac{1}{A}\phi - \frac{B}{A} f\right) + \tilde{A} f = \frac{1}{A} f + \frac{\tilde{B}}{A}\phi$$

or
$$\begin{pmatrix}\tilde{f} & \tilde{\phi}\end{pmatrix} = \begin{pmatrix} f & \phi\end{pmatrix}\begin{pmatrix} -\frac{B}{A} & \frac{1}{A} \\ \frac{1}{A} & \frac{\tilde{B}}{A}\end{pmatrix}$$

This shows very simply what is involved when one solves the inverse spectral problem. Namely at $x = 0$ we want $f \in \binom{1}{0} + H_+$ $\quad \phi \in \binom{0}{1} + H_+$ and $\tilde{f} = \binom{0}{1} + H_-$, $\tilde{\phi} = \binom{1}{0} + H_-$. It seems that the above is valid even when $\tilde{p} \neq \bar{p}$.

December 25, 1981

Let's review a little the situation with the DE

$$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik & p \\ \tilde{p} & -ik \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where $p, \tilde{p}$ decay fast as $|x| \longrightarrow \infty$. For k in the closed UHP we can define "small" solutions f, ϕ with asymptotic behavior

$$f \sim e^{ikx}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad x \longrightarrow +\infty$$

$$\phi \sim e^{-ikx}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad x \longrightarrow -\infty$$

and similarly for k in the closed LHP we have $\tilde{f}, \tilde{\phi}$ with

$$\tilde{f} \sim e^{-ikx}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad x \longrightarrow +\infty$$

$$\tilde{\phi} \sim e^{ikx}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad x \longrightarrow -\infty .$$

On the real axis we have functions $A(k), B(k), \tilde{A}(k), \tilde{B}(k)$ such that

$$\phi = A\tilde{f} + Bf$$

$$\tilde{\phi} = \tilde{B}\tilde{f} + \tilde{A}f$$

$$(\tilde{\phi}\ \phi) = (f\ \tilde{f})\begin{pmatrix} \tilde{A} & B \\ \tilde{B} & A \end{pmatrix}$$

Clearly $A = W(f, \phi)$ extends to UHP and $\tilde{A} = W(\tilde{\phi}, \tilde{f})$ extends to LHP. Furthermore at points iK in the UHP where $A(iK) = 0$, the solutions f, ϕ are linearly dependent so we have a non-zero number $B(iK)$ defined so that $\phi_{iK} = B(iK) f_{iK}$. Similarly for LHP. Since the Wronskian of two solutions is constant we have

$$\begin{vmatrix} \tilde{A} & B \\ \tilde{B} & A \end{vmatrix} = \tilde{A}A - \tilde{B}B = 1$$

Now solve for the incoming solutions $\tilde{f}, \tilde{\phi}$ in terms of the outgoing solutions f, ϕ to get the scattering matrix

$$(\tilde{f}\ \tilde{\phi}) = (f\ \phi)\begin{pmatrix} -\frac{B}{A} & \frac{1}{A} \\ \frac{1}{A} & \frac{\tilde{B}}{A} \end{pmatrix}$$

This gives us a factorization of the S-matrix into parts holomorphic in the upper and lower HPs.

In order to take into account growth conditions we use that

$$(f\ \ \phi) \in \left(e^{ikx}\left[\begin{pmatrix}1\\0\end{pmatrix}+H_+\right]\quad e^{-ikx}\left[\begin{pmatrix}0\\1\end{pmatrix}+H_+\right]\right)$$

$$= \left(\begin{pmatrix}1&0\\0&1\end{pmatrix}+H_+\right)\begin{pmatrix}e^{ikx}&0\\0&e^{-ikx}\end{pmatrix}$$

$$(\tilde{\phi}\ \ \tilde{f}) \in \left(\begin{pmatrix}1&0\\0&1\end{pmatrix}+H_-\right)\begin{pmatrix}e^{ikx}&0\\0&e^{-ikx}\end{pmatrix}$$

so therefore

$$\underset{\in I+H_-}{\left(e^{-ikx}\tilde{\phi}\ \ e^{ikx}\tilde{f}\right)} = \underset{\in I+H_+}{\left(e^{-ikx}f\ \ e^{ikx}\phi\right)}\begin{pmatrix}\frac{1}{A} & -\frac{B}{A}e^{2ikx}\\ \frac{\tilde{B}}{A}e^{-2ikx} & \frac{1}{A}\end{pmatrix}$$

Under suitable conditions such a factorization should exist and be unique. However the ~~interesting~~ cases appear to be when there is no reflection: $B=\tilde{B}=0$, but where there is the bound state information needed to pin down the solution.

Let's notice first of all that the S-matrix approaches I as $k \to \pm\infty$ and that an algebraic function in H_+ is holomorphic on the closed UHP and it vanishes at ∞. Thus what we are trying to do is to take a matrix fu. $k \mapsto S(k)$, which we can think of as a map $S^1 \to GL_2$ with value 1 at $z=1$ and factor it into pieces holom. inside and outside normalized so that they are 1 at $z=1$. Here $z=1$ belongs to $k=\infty$: $z=\frac{k-i}{k+i}$.

Now we know this factorization takes place when S has degree 0 and is not too far from the identity. But

$$\det S = \frac{1+\tilde{B}B}{A^2} = \frac{\tilde{A}A}{A^2} = \frac{\tilde{A}}{A}$$

$$= \frac{1}{A^2} \text{ when there is no reflection.}$$

Then the degree of S will be $(-2) \times$ number of zeroes of A in the UHP.

So the rank 2 vector bundle over $\mathbb{P}_1$ defined by the S-matrix is going to be of positive degree

December 27, 1981

$$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik & p \\ \tilde p & -ik \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

asymptotic solutions

$$e^{ikx}\begin{pmatrix} 1 + \frac{1}{2ik}\int^x p\tilde p + \cdots \\ \frac{1}{2ik}\tilde p(x) + \cdots \end{pmatrix} \qquad e^{-ikx}\begin{pmatrix} -\frac{1}{2ik}\, p(x) + \cdots \\ 1 - \frac{1}{2ik}\int^x \tilde p p + \cdots \end{pmatrix}$$

$$f_k(x) \sim \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix} \text{ as } x \to +\infty \qquad f_k(x) \doteq e^{ikx}\begin{pmatrix} 1 + \frac{1}{2ik}\int_\infty^x p\tilde p + \cdots \\ \frac{1}{2ik}\tilde p(x) \end{pmatrix}$$

$$\tilde f_k(x) \sim \begin{pmatrix} 0 \\ e^{-ikx} \end{pmatrix} \quad x \to +\infty \qquad \tilde f_k(x) \doteq e^{-ikx}\begin{pmatrix} -\frac{1}{2ik}\, p(x) + \cdots \\ 1 - \frac{1}{2ik}\int_\infty^x p\tilde p + \cdots \end{pmatrix}$$

$$\phi_k(x) \sim \begin{pmatrix} 0 \\ e^{-ikx} \end{pmatrix} \quad x \to -\infty \qquad \phi_k(x) \doteq e^{-ikx}\begin{pmatrix} -\frac{1}{2ik}\tilde p(x) \\ 1 - \frac{1}{2ik}\int_{-\infty}^x p\tilde p \end{pmatrix}$$

$$\tilde\phi_k(x) \sim \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix} \quad x \to -\infty \qquad \tilde\phi_k(x) \doteq e^{ikx}\begin{pmatrix} 1 + \frac{1}{2ik}\int_{-\infty}^x p\tilde p \\ \frac{1}{2ik}\tilde p(x) \end{pmatrix}$$

$$\begin{aligned} \phi &= A\tilde f + Bf \\ \tilde\phi &= \tilde B\tilde f + \tilde A f \end{aligned} \qquad \begin{pmatrix} \tilde\phi & \tilde f \end{pmatrix} = \begin{pmatrix} f & \phi \end{pmatrix}\begin{pmatrix} \frac{1}{A} & -\frac{B}{A} \\ \frac{\tilde B}{A} & \frac{1}{A} \end{pmatrix}$$

Thus $\quad A \sim 1 - \frac{1}{2ik}\int_{-\infty}^{\infty} p\tilde p + \cdots \qquad B \sim 0$ because B decays exponentially

One concludes that provided one looks at the reflectionless case, the asymptotic expansions might contain the useful information. So let's suppose $B = \tilde B = 0$ (on $\mathbb{R}$). Then $\tilde A A = 1$ so

$\tilde A = \frac{1}{A}$ fn. N poles iK in UHP, N zeroes $-iL$ in LHP

and $= 1$ at $k = \infty$.

Hence A is a rational fn. $\qquad A = \prod \frac{k+iL}{k-iK}$

Now if we have $\tilde{\phi} = \frac{1}{A} f$ with $\tilde{\phi} \in 1 + H_-$, $f \in 1 + H_+$

we have
$$\Pi(k - iK)\tilde{\phi} = \Pi(k + iL) f = \binom{1}{0} k^N + c_1 k^{N-1} + \dots + c_N$$
Here $x = 0 \Rightarrow$

because it holom. in both half planes and $\sim k^N$ at ∞.

Similarly
$$\Pi(k - iK)\tilde{f} = \Pi(k + iL)\phi = \binom{0}{1} k^N + \tilde{c}_1 k^{N-1} + \dots + \tilde{c}_N$$

so that we have **4N** constants to determine. But we have the bound state conditions

Here $x=0$

$$\phi_{iK} = B(iK) f_{iK} \qquad \text{at the } N \text{ points } iK \text{ in UHP}$$

$$\tilde{\phi}_{-iL} = \tilde{B}(-iL) \tilde{f}_{-iL} \qquad \text{——————} \; -iL \text{ in LHP.}$$

~~[illegible]~~

Now one wants to rewrite this completely without reference to the original DE. The point is really that A is fixed and the bound state conditions, that is, the numbers $B(iK)$, $\tilde{B}(-iL)$ depend upon x. How?

$$f_k(x) \in e^{-ikx}\left(\binom{1}{0} + H_+\right), \quad \phi_k(x) \in e^{-ikx}\left(\binom{0}{1} + H_+\right)$$

Thus $B(iK)$ becomes $B(iK)e^{-2Kx}$ when you want the value of ϕ_k at x.

Let's take a slightly more abstract view. We have that $\tilde{A} = \frac{1}{A} = \Pi\left(\frac{k + iL}{k - iK}\right)$ is fixed, and that it gives rise to a vector bundle over $\mathbb{P}^1$ isomorphic to $\mathcal{O}(N) \oplus \mathcal{O}(N)$.

We have sections of this bundle

$$s_1 = \left(e^{-ikx}\tilde{\phi}(x), e^{-ikx} f(x)\right) \quad \text{has value } \binom{1}{0} \text{ at } k = \infty$$

$$s_2 = \left(e^{ikx}\tilde{f}(x), e^{ikx}\phi(x)\right) \quad \text{has value } \binom{0}{1} \text{ at } k = \infty.$$

These sections are pinned down by the bound state conditions

$$* \quad \begin{cases} s_2(iK) = B(iK)e^{-2Kx} s_1(iK) \\ s_1(-iL) = \tilde{B}(-iL)e^{-2Lx} s_2(-iL) \end{cases}$$

But notice that the separate components are governed by the same equations. Look at the first components. They are pairs of ~~sections~~ s_1', s_2' of $O(N)$ satisfying the equations * such that $s_1' = 1$, $s_2' = 0$ at $k = \infty$. Moreover

$$s_2' \sim -\frac{1}{2ik} p(x) \qquad \text{as } k \longrightarrow \infty,$$

so we recover the ~~potential~~.

Now it should be possible to reformulate everything in terms of line bundles on a singular curve.

December 29, 1981

Consider $[-\partial_x^2 + q]u = k^2 u$ in the case where there is no reflection. Then we have

$$\phi_k = A(k) f_{-k} \quad \text{on the real axis}$$
$$A(-k)A(k) = 1 \qquad k \in \mathbb{R}.$$
$$\phi_{iK} = B(iK) f_{iK} \qquad \text{at the points } iK \text{ in UHP where } A(iK) = 0.$$

Notice first that $A(-k) = \frac{1}{A(k)}$ gives a meromorphic extension of $A(k)$ to the LHP having simple poles at the points $-iK$. So if we use the asymptotic expansion

$$A(k) = 1 + O(\tfrac{1}{k})$$

we see that A is meromorphic on $\mathbb{P}^1$, hence it rational:

$$A(k) = \prod_K \left(\frac{k - iK}{k + iK} \right)$$

Then from

$$f_{-k} = \frac{1}{A(k)} \phi_k$$

we see that f_k, which is analytic in the UHP, has a meromorphic extension to the LHP with simple poles at the points $k = -iK$, where we have

$$\operatorname{Res}_{k=-iK} f_k = \lim_{k \to -iK} (k + iK) f_k = \lim_{k \to iK} (-k + iK) f_{-k}$$
$$= \lim_{k \to iK} (-k + iK) \frac{1}{A(k)} \phi_k$$
$$= -\frac{B(iK)}{A'(iK)} f_{iK}$$

December 31, 1981

Let's see if we can work out completely the theory of soliton solutions of the KdV equation. Let us begin with a Schroedinger equation

$$[-\partial_x^2 + q(x)]u = k^2 u$$

where $q(x)$ is smooth. We can solve this equation formally

$$u_k(x) = e^{-ikx}\left(a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \cdots\right)$$

where the a_n are functions of x. Putting $u = e^{ikx} v$ the DE becomes

$$2ik\,\partial_x v = (-\partial_x^2 + q)v$$

leading to the recursion relations.

$$2i\,a_0' = 0 \qquad \Rightarrow a_0 = \text{const} \qquad \text{say } a_0 = 1$$

$$2i\,a_1' = (-\partial^2 + q)a_0 \qquad \Rightarrow a_1' = \frac{q}{2i}$$

$$2i\,a_2' = (-\partial^2 + q)\,a_1$$

which give a formal solution

$$e^{ikx}\left(1 + \frac{1}{2ik}\int^x q + \cdots\right).$$

If the potential q decays sufficiently fast then we can consider the formal solutions of the sort

$$f_k \sim e^{ikx} \qquad \text{as } x \to +\infty$$

$$\phi_{-k} \sim e^{-ikx} \qquad \text{as } x \to -\infty.$$

Hence

$$f_k = e^{-ikx}\left(1 + \frac{1}{2ik}\int_{\infty}^{x} q + \cdots\right)$$

$$\phi_{-k} = e^{ikx}\left(1 + \frac{1}{2ik}\int_{-\infty}^{x} q + \cdots\right).$$

and one has a formal transmission coefficient satisfying

$$\phi_{-k} = A(-k)\,f_{-k} \quad \text{, so} \quad A(-k) = 1 + \frac{1}{2ik}\int_{-\infty}^{\infty} q + \cdots$$

Note that from this formal point of view there is no reflection coefficient.

Now under the assumption that q decays fast one can obtain a unique solution $f_k(x)$ of the Schroedinger equation $\sim e^{ikx}$ as $x \to \infty$ defined for $\mathrm{Im}(k) \geq 0$ by solving the integral equation of Volterra type

$$f_k(x) = e^{ikx} - \int_x^\infty \frac{\sin k(x-x')}{k} q(x') f_k(x') dx'$$

This should be analytic for k in the UHP, and have the above described formal behavior as $|k| \to \infty$ in the UHP. Similarly we have $\phi_k(x) \sim e^{-ikx}$ as $x \to -\infty$ defined and analytic for $\mathrm{Im}\, k \geq 0$. For k real $\neq 0$ one has

$$\phi_k = A(k) f_{-k} + B(k) f_k$$

$$\phi_{-k} = B(-k) f_{-k} + A(-k) f_k$$

and

$$\begin{vmatrix} A(k) & B(k) \\ B(-k) & A(-k) \end{vmatrix} = 1$$

because of Wronskians. Also we have

$$W(\phi_k, f_k) = A(k)\, W(f_{-k}, f_k) = 2ik\, A(k)$$

will extend analytically into the UHP because ϕ_k, f_k do. Finally $A(k)$ has the above described formal expansion, and $B(k)$ has formal expansion 0.

[One of the fascinating formulas related to the above, which I have never quite understood as well as I would like, is the following

$$\det(1 - G_k^{0+} q) = A(k)$$

where $$G_k^{0+}(x, x') = \langle x | \frac{1}{k^2 + \partial^2} | x' \rangle = \frac{e^{ik|x-x'|}}{2ik} \qquad \mathrm{Im}\, k \geq 0$$

This is connected with solving the LS integral equation

$$\psi_k^+ = e^{-ikx} + \int_{-\infty}^{\infty} G_k^{0+}(x,x')\, q(x')\psi_k^+(x')\,dx'$$

which gives the solution of the Schroedinger equation with

$$T(k)e^{-ikx} \xleftrightarrow{\ \psi_k^+\ } e^{-ikx} + R(k)e^{ikx}$$

Then ψ_k^+ is analytic in the UHP except for poles where $A(k)$ vanishes.]

When $A(iK) = 0$ for some point iK in the UHP one has constants $B(iK) \neq 0$ such that

$$\phi_{iK} = B(iK)\, f_{iK}$$

Another general fact is that $A(k)$ has at most simple zeroes in the UHP.

Next discuss the KdV equation. Start with the operator $L = -\partial^2 + q$ where now $q = q(x,t)$ and with the KdV equation written in Lax form:

$$\mathring{L} = [B, L]$$

where $B = \partial^3 + a\partial + \partial a$, and a is a suitable fn. of q chosen so that $[B, L]$ is of degree 0. Then differentiating

$$L f_k = k^2 \cancel{f_k}$$

wrt t gives

$$[B, L] f_k + L\mathring{f}_k = k^2 \mathring{f}_k$$

$$B k^2 f_k - LBf_k + L\mathring{f}_k = k^2 \mathring{f}_k$$

or

$$L(\mathring{f}_k - Bf_k) = k^2(\mathring{f}_k - Bf_k)$$

and so $\mathring{f}_k - Bf_k$ must be a linear comb. of f_k, f_{-k}.

Look at $x \to +\infty$ and use $B \sim \partial^3$, $f_k \sim e^{ikx}$ which is independent of t. Thus