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September 23, 1994

Here's a working principle:

Everything we do for the case $I = I^2$ holds more generally for I essentially idempotent, i.e. $I^n = I^{n+1} = \cdots$, and is likely to hold in general provided [redacted] we introduce the pro object I^∞.

Thus we should have adjoint functors

$$\begin{tikzcd} \mathrm{null}(R,I) \arrow[r, "\iota_*" description] & \mathrm{mod}(R) \arrow[l, bend right, "\iota^*"'] \arrow[l, bend left, "\iota^!"] \arrow[r, "j^*" description] & M_u(R,I) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

$$\iota^*(M) = R/I^\infty \otimes_R M = M/I^\infty M$$

$$\iota^!(M) = \mathrm{Hom}_R(R/I^\infty, M) = {}_{I^\infty}M$$

$$j_!(M) = I^{(\infty)} \otimes_R M$$

$$j_*(M) = \mathrm{Hom}_R(I^{(\infty)}, M)$$

when I is essentially idempotent, but in general $\iota^*, j_!$ have values which are pro- [redacted] modules and $\iota^!, j_*$ ________________ ind- modules.

Now ask what is the analogue of the excision [redacted] properties $L\iota^*(\iota_* M) = M$ or [redacted] $R\iota^!(\iota_* M) = M$, for all null modules M. The former amounts to $I^\infty \overset{L}{\otimes}_R R/I = 0$ and the latter probably reduces to $R/I \overset{L}{\otimes}_R I^\infty = 0$.

This means that I^∞ is left and right firm in some derived category sense.

Let's examine what it means for I^∞ to be right firm:

$$I^\infty \otimes_R I \xrightarrow{\sim} I^\infty \otimes_R R = I^\infty$$

This means we have a functor $M \mapsto I^\infty \otimes_R M$ from modules to pro-modules which inverts null isomorphisms.

Let's show that for I^∞ to be right firm is independent of R. We follow the $I = I^2$ proof:

$$\begin{tikzcd} I \otimes_{\tilde{I}} M \arrow[rr, "\sim"] \arrow[dr] & & I \otimes_R M \arrow[dl] \\ & M & \end{tikzcd}$$

where the top arrow is an isom. since

$$(a_1 a_2) r \otimes_{\tilde{I}} m = a_1 \otimes_{\tilde{I}} a_2 r m = a_1 a_2 \otimes_{\tilde{I}} r m$$

Similarly we have a commutative diagram

$$\begin{tikzcd} I^n \otimes_{\tilde{I}} M \arrow[r] & I^n \otimes_R M \\ I^{n+1} \otimes_{\tilde{I}} M \arrow[u] \arrow[r] & I^{n+1} \otimes_R M \arrow[u] \arrow[ul, dashed, "\varphi"'] \end{tikzcd}$$

where $\varphi(x \otimes_R m) = x \otimes_{\tilde{I}} m$ for $x \in I^{n+1}$, $m \in M$.

φ is well-defined because

$$(x'a) r \otimes_{\tilde{I}} m = x' \otimes_{\tilde{I}} a r m = x'a \otimes_{\tilde{I}} m \qquad \begin{matrix} x' \in I^n \\ a \in I \end{matrix}$$

This establishes

$$\boxed{I^\infty \otimes_{\tilde{I}} M \xrightarrow{\sim} I^\infty \otimes_R M}$$

for any R-module M. Consequently

$$\boxed{I^\infty \otimes_{\tilde{I}} I \xrightarrow{\sim} I^\infty \iff I^\infty \otimes_R I \xrightarrow{\sim} I^\infty}$$

Next I would like to show $I^{(\infty)} \xrightarrow{\sim} I^\infty$ using the diagram

$$\begin{tikzcd}
I^\infty \otimes_R I^{(\infty)} \arrow[rr, "(1)"] \arrow[d, "(2)"'] & & I^\infty \otimes_R R \arrow[d, "="] \\
R \otimes_R I^{(\infty)} = I^{(\infty)} \arrow[rr] & & I^\infty
\end{tikzcd}$$

The map (1) [redacted] should be an isomorphism because I^∞ is right firm and $I^{(\infty)} \to R$ is a null isomorphism. The map (2) should be an isom. because $I^{(\infty)}$ is firm and $I^\infty \to R$ is a null isomorphism. The problem is to show that the pro-objects work properly.

[redacted]

<u>September 24:</u>

First record

$$\mathrm{Hom}_{R^{op}}(V \otimes_R I, W) = \mathrm{Hom}_{R^{op}}(V, \mathrm{Hom}_{R^{op}}(I, W))$$

so we can deal with right modules.

September 24, 1994 (cont)

Assume $I^\infty \otimes_R I \xrightarrow{\sim} I^\infty \otimes_R R = I^\infty$, i.e.

$$\varinjlim_n \mathrm{Hom}_{R^{op}}(I^n \otimes_R I, -) \xleftarrow{\sim} \varinjlim_n \mathrm{Hom}_R(I^n, -)$$

Applying this to [redacted] $\mathrm{Hom}_{R^{op}}(I^{(m)}, -)$ and using the preceding adjunction formula we have

$$\varinjlim_n \mathrm{Hom}_{R^{op}}(I^n \otimes_R I^{(m+1)}, -) \xleftarrow{\sim} \varinjlim_n \mathrm{Hom}_{R^{op}}(I^n \otimes_R I^{(m)}, -)$$

whence $\forall m \geqslant 0$

$$\varinjlim_n \mathrm{Hom}_{R^{op}}(I^n \otimes_R I^{(m)}, -) \xleftarrow{\sim} \varinjlim_n \mathrm{Hom}_{R^{op}}(I^n, -)$$

and so

$$\varinjlim_m \varinjlim_n \mathrm{Hom}_{R^{op}}(I^n \otimes_R I^{(m)}, -) \xleftarrow{\sim} \varinjlim_n \mathrm{Hom}_{R^{op}}(I^n, -)$$

This proves that $I^\infty \otimes_R I^{(\infty)} \xrightarrow{\sim} I^\infty$.

On the other hand because $I^n \longrightarrow R$ is a null-isomorphism we have

$$\begin{array}{ccc} \varinjlim_m \mathrm{Hom}_{R^{op}}(I^n \otimes_R I^{(m)}, -) & \xleftarrow{\sim} & \varinjlim_m \mathrm{Hom}_{R^{op}}(I^{(m)}, -) \\ \| & & \| \\ \mathrm{Hom}_{m_n(R^{op}, I^{op})}(I^n, -) & \xleftarrow{\sim} & \mathrm{Hom}_{m_n(R^{op}, I^{op})}(R, -) \end{array}$$

hence

$$\varinjlim_n \varinjlim_m \mathrm{Hom}_{R^{op}}(I^n \otimes_R I^{(m)}, -) \xleftarrow{\sim} \varinjlim_m \mathrm{Hom}_R(I^{(m)}, -)$$

proving that $I^\infty \otimes_R I^{(\infty)} \xrightarrow{\sim} I^{(\infty)}$.

Remark that [redacted] the isomorphisms $I^\infty \otimes_R I^{(\infty)} \xrightarrow{\sim} I^\infty$ or $I^{(\infty)}$ seem to [redacted] be outside the Artin-Rees category. One can interpret $I^\infty \otimes_R I^{(\infty)}$ as any of the isomorphic systems $(I^n \otimes_R I^{(n+i)})$ in the AR category, but to have the inverse map $I^\infty \longrightarrow I^\infty \otimes_R I^{(\infty)}$ seems to require doubling the indices. This is clearer in the case of $I^{(\infty)} \otimes_R I^{(\infty)} \xrightarrow{\sim} I^{(\infty)}$.

[redacted] The next step is the version with derived categories. What we want are adjoint functors in some pro-object or ind-object sense:

$$\begin{tikzcd} D(\text{null}) \arrow[r, "\iota_*" description] & D(R) \arrow[l, dashed, shift right=3, "L\iota^*"'] \arrow[l, dashed, shift left=3, "R\iota^!"] \arrow[r, "j^*" description] & D(\mathcal{M}_n) \arrow[l, dashed, shift right=3, "Lj_!"'] \arrow[l, dashed, shift left=3, "Rj_*"] \end{tikzcd}$$

Here $L\iota^*(X) = R/I^\infty \overset{L}{\otimes}_R X \qquad R\iota^!(X) = R\,\mathrm{Hom}_R(R/I^\infty, X)$

$Lj_!(X) = [I \overset{L}{\otimes}_R]^\infty X \qquad Rj_*(X) = R\,\mathrm{Hom}_R([I \overset{L}{\otimes}_R]^\infty I, X)$

should be the formulas to expect.

It's probably more interesting to do things when I^∞ is Dfirm: $I^\infty \overset{L}{\otimes} R/I = 0$, equivalently $I^\infty \overset{L}{\otimes}_R I \longrightarrow I^\infty \overset{L}{\otimes}_R R = I^\infty$ is a quis in a suitable proobject sense. Formally one should then have $Lj_!j^*X) = I^\infty \overset{L}{\otimes}_R X$ (i.e. $[I \overset{L}{\otimes}_R]^\infty(-) \simeq I^\infty \overset{L}{\otimes}_R -$).

and $Rj_*j^*X) = R\,\mathrm{Hom}_R(I^\infty, X)$ and so we have distinguished Δ's

$$I^\infty \overset{L}{\otimes}_R X \longrightarrow X \longrightarrow R/I^\infty \overset{L}{\otimes}_R X \longrightarrow \cdots$$

$$R\mathrm{Hom}_R(R/I^\infty, X) \to X \longrightarrow R\mathrm{Hom}_R(I^\infty, X) \longrightarrow \cdots$$

which can be written respectively

$$Lj_!(j^*X) \longrightarrow X \longrightarrow \iota_* L\iota^*(X) \longrightarrow \cdots$$

$$\iota_* R\iota^!(X) \longrightarrow X \longrightarrow Rj_*(j^*X) \longrightarrow \cdots$$

[redacted] Before embarking on this pro-derived-category stuff let's try to prove the condition $R/I \overset{L}{\otimes}_R I^\infty$ [redacted] $= 0$ is independent of R. The precise statement is that the proobjects $\mathrm{Tor}_j^R(R/I, I^\infty) = \{n \mapsto \mathrm{Tor}_j^R(R/I, I^n)\}$ vanish $\forall j$. This is automatic for $j=0$ and for $j=1$ we see using

$$0 \to \mathrm{Tor}_1^R(R/I, I^n) \longrightarrow I \otimes_R I^n \longrightarrow I^n \longrightarrow I^n/I^{n+1} \to 0$$

that it says $I \otimes_R I^\infty \xrightarrow{\sim} I^\infty$, i.e. I^∞ is left firm.

Assume $R/I \overset{L}{\otimes}_R I^\infty = 0$. Let $E \to \tilde{I}/I$ be a projective $\tilde{I}^{op}$-module resolution of $\tilde{I}/I$, let $F \to I^n$ be a projective R-module resolution of I^n. Then

$$\tilde{I}/I \overset{L}{\otimes}_{\tilde{I}} I^n \simeq E \otimes_{\tilde{I}} I^n \simeq E \otimes_{\tilde{I}} R \otimes_R F$$

Better: Consider the double complex $E \otimes_{\tilde{I}} R \otimes_R F$. Then

$$H_p^h H_q^v = H_p(E \otimes_{\tilde{I}} H_q(F)) = \begin{cases} \mathrm{Tor}_p^{\tilde{I}}(\tilde{I}/I, I^n) & q=0 \\ 0 & q \neq 0 \end{cases}$$

$$H^v_p H^h_q = H_p\left(H_q(E \otimes_{\tilde{I}} R) \otimes_R F\right)$$

$$= \mathrm{Tor}^R_p\left(\mathrm{Tor}^{\tilde{I}}_q(\tilde{I}/I, R), I^n\right)$$

so we get a spectral sequence of inverse systems in n:

$$(*) \quad E^2_{pq} = \mathrm{Tor}^R_p\left(\mathrm{Tor}^{\tilde{I}}_q(\tilde{I}/I, R), I^n\right) \Rightarrow \mathrm{Tor}^{\tilde{I}}_*(\tilde{I}/I, I^n)$$

Now $\underbrace{\mathrm{Tor}^{\tilde{I}}_*(\tilde{I}/I, R)}_{= H_*(E \otimes_{\tilde{I}} R)} \cdot I = 0$ because

$$E \otimes_{\tilde{I}} R \longrightarrow E \longrightarrow E \otimes_{\tilde{I}} R$$

$$\xi \otimes r \longmapsto \xi(ra) \longmapsto \xi(ra) \otimes 1 = (\xi \otimes r)a$$

so right multiplication by a on $H_*(E \otimes_{\tilde{I}} R)$ factors through $H_*(E) = 0$ except in degree 0, where $H_0(E \otimes_{\tilde{I}} R) = \tilde{I}/I \otimes_{\tilde{I}} R = R/IR = R/I$ is killed by I^{op}.

Remains to show

Lemma: If $\mathrm{Tor}^R_*(R/I, I^\infty) = 0$, then for any null module M for (R^{op}, I^{op}) we have $\mathrm{Tor}^R_*(M, I^\infty) = 0$.

Assume shown that $\mathrm{Tor}^R_j(M, I^\infty) = 0$ for all such M and $j < p$. Then $\mathrm{Tor}^R_p(-, I^\infty)$ is right exact and it vanishes for free $(R/I)^{op}$ modules, so it vanishes for all $(R/I)^{op}$ modules, hence also for extensions of these, hence $\mathrm{Tor}^R_p(-, I^\infty)$ vanishes for all null (R^{op}, I^{op}) modules.

Now return to $(*)$ and think of $(*)$ as a spectral sequence with values in pro abelian groups.

The E^2 term is zero, hence we have $\mathrm{Tor}_*^{\tilde{I}}(\tilde{I}/I, I^\infty) = 0$.

Conversely assume $\tilde{I}/I \overset{L}{\otimes}_R I^\infty = 0$. Choose $E \to R/I$ a projective R^{op} resolution $F \to I^n$ a projective $\tilde{I}$-resolution.

Consider the double complex $E \otimes_R I \otimes_{\tilde{I}} F$. Then

$$H_p^h H_q^v = H_p\left(E \otimes_R H_q(I \otimes_{\tilde{I}} F)\right)$$

$$= \mathrm{Tor}_p^R\left(R/I, \mathrm{Tor}_q^{\tilde{I}}(I, I^\infty)\right)$$

$$H_p^v H_q^h = H_p\left(H_q(E \otimes_R I) \otimes_{\tilde{I}} F\right)$$

$$= \mathrm{Tor}_p^{\tilde{I}}\left(\mathrm{Tor}_q^R(R/I, I), I^\infty\right)$$

Now $\mathrm{Tor}_q^R(R/I, I) = \begin{cases} I/I^2 & q = 0 \\ \mathrm{Tor}_{q+1}^R(R/I, R/I) & q \geq 1 \end{cases}$

satisfies $\mathrm{Tor}_q^R(R/I, I) \cdot I = 0$. Or we can argue as before:

$$\begin{array}{ccccc} E \otimes_R I & \longrightarrow & E & \longrightarrow & E \otimes_R I \\ \xi \otimes a' & \longmapsto & \xi a' & \longmapsto & \xi a' \otimes a = (\xi \otimes a')a \end{array}$$

Now by the above lemma for $\tilde{I}$ in place of R we see $\mathrm{Tor}_p^{\tilde{I}}(\mathrm{Tor}_q^R(R/I, I), I^\infty) = 0$ for all p, q.

Also

$$\mathrm{Tor}_q^{\tilde{I}}(I, I^\infty) \xrightarrow{\sim} \mathrm{Tor}_q^{\tilde{I}}(\tilde{I}, I^\infty) = \begin{cases} I^\infty & q = 0 \\ 0 & q \neq 0 \end{cases}$$

so the first spectral sequence degenerates leading to

$$\mathrm{Tor}_p^R(R/I, I^\infty) = 0.$$

September 25, 1994

I propose to show that $R/I^\infty \overset{L}{\otimes}_R I^\infty = 0$ implies $R/I \overset{L}{\otimes}_R I^\infty = 0$. In Tor terms this means showing

$$\underbrace{\operatorname{Tor}_j^R(R/I^\infty, I^\infty) = 0}_{\forall j \le k} \Longrightarrow \underbrace{\operatorname{Tor}_j^R(R/I, I^\infty) = 0}_{\forall j \le k}$$

and I want to use induction on k. Recall the formulas

$$\operatorname{Tor}_j^R(R/I, J) = \begin{cases} J/IJ & j=0 \\ \operatorname{Ker}\{I \otimes_R J \to IJ\} & j=1. \end{cases}$$

Thus $\operatorname{Tor}_0^R(R/I, I^\infty) = I^\infty / I I^\infty = (I^n/I^{n+1})$ which we know is zero. Thus true for $k=0$.

Now assuming true for $k-1$ we have from

$$\begin{tikzcd}[column sep=small] \operatorname{Tor}_k^R(R/I^\infty, I^\infty) \arrow[r] \arrow[d, equal] & \operatorname{Tor}_k^R(R/I, I^\infty) \arrow[r] & \operatorname{Tor}_{k-1}^R(I/I^\infty, I^\infty) \arrow[d, equal] \\ 0 & & 0 \end{tikzcd}$$

The last term is zero by induction and the implication $\Longleftarrow$ which we already know.

I need to make these pro object arguments convincing. Consider $k=1$

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] & \\ \operatorname{Tor}_1^R(R/I^\infty, I^\infty) \arrow[r] \arrow[d] & \operatorname{Tor}_1^R(R/I, I^\infty) \arrow[r] \arrow[d] & I/I^\infty \otimes_R I^\infty \arrow[d, equal] \\ I^\infty \otimes_R I^\infty \arrow[r] \arrow[d] & I \otimes_R I^\infty \arrow[r] \arrow[d] & I/I^\infty \otimes_R I^\infty \\ I^\infty \arrow[r, equal] \arrow[d] & I^\infty \arrow[d] & \\ 0 & 0 & \end{tikzcd}$$

The point is that $I/I^\infty \otimes_R I^\infty = 0$
and this you should be able to check:

$\beta = a_1 \cdots a_n \otimes a_{n+1} \cdots a_{n+m}$

$$\begin{tikzcd}
I^n \otimes_R I^m \arrow[r] & I \otimes_R I^m \arrow[r] & I/I^n \otimes_R I^m \arrow[r] & 0 \\
 & I \otimes_R I^{n-1+m} \arrow[r] \arrow[u] & I/I^n \otimes_R I^{n-1+m} \arrow[u] &
\end{tikzcd}$$

$\alpha = a_1 \otimes a_2 \cdots a_{n+m}$

Note: α, β becomes equal in $I \otimes_R I^m$ so
is the zero map.

Actual if $MI^k = 0$, then the map
$M \otimes_R I^{k+n} \longrightarrow M \otimes_R I^n$ is zero.

Observation: Recall

$$\mathrm{Hom}_{Pro}\left(\{X_i\}_{i\in I}, \{Y_j\}_{j\in J}\right) = \varprojlim_j \varinjlim_i \mathrm{Hom}(X_i, Y_j)$$

$$= \varprojlim_j \mathrm{Hom}_{Pro}\left(\{X_i\}_{i\in I}, Y_j\right)$$

Thus $\boxed{\{Y_j\}_{j\in J} \xrightarrow{\ \sim\ } \varprojlim_j Y_j \text{ in the Pro category}}$

Consequence

$$\{Y_{jk}\}_{(j,k)\in J\times K} \simeq \varprojlim_{(j,k)} Y_{jk}$$

$$\simeq \varprojlim_j \varprojlim_k Y_{jk} \simeq \varprojlim_j \{Y_{jk}\}_{k\in K}$$

In particular

$$I^{\infty}\otimes_R I^{(\infty)} = \varprojlim_n I^n \otimes_R I^{(\infty)}$$ [redacted]

$$\xrightarrow{\sim} \varprojlim_n R\otimes_R I^{(\infty)} = I^{(\infty)}$$

and if $I^\infty \otimes_R I \xrightarrow{\sim} I^\infty$, then

$$I^\infty \otimes_R I^{(\infty)} = \varprojlim_m I^\infty \otimes_R I^{(m)} = I^\infty$$

constant inverse system

Let's review the equivalence

$$R/I \overset{L}{\otimes}_R I^\infty = 0 \iff \tilde{I}/I \overset{L}{\otimes}_{\tilde{I}} I^\infty = 0$$

but this time using standard resolutions.

[redacted]

$(\Rightarrow)$: [redacted] Consider the bicomplex

$$\tilde{I}/I \otimes_{-\tau} \mathcal{B}(I) \otimes_\tau R \otimes_{-\tau} \mathcal{B}(\bar{R}) \otimes_\tau I^\infty$$

$\tilde{I}/I \overset{L}{\otimes}_{\tilde{I}} R \overset{L}{\otimes}_R I^\infty$

where I write as if R is augmented. (Thus $\mathcal{B}(\bar{R}) = k \oplus \bar{R} \oplus \bar{R}^{\otimes 2} \oplus \cdots$ is not a well-defined complex [redacted] in general, but $R \otimes_{-\tau} \mathcal{B}(\bar{R}) \otimes_\tau I^\infty$ is a complex.) This gives a spectral sequence

$$E^2_{pq} = \mathrm{Tor}^R_p\Big(\underbrace{\mathrm{Tor}^{\tilde{I}}_q(\tilde{I}/I, R)}_{\substack{I^{op}\text{-null by spectral}\\ \text{argument}}}, I^\infty\Big) \Rightarrow \mathrm{Tor}^{\tilde{I}}_*(\tilde{I}/I, I^\infty)$$

$$R/I \overset{L}{\otimes}_R I^\infty = 0 \Rightarrow \mathrm{Tor}^R_*(I^{op}\text{null}, I^\infty) = 0 \Rightarrow \underset{=0}{\mathrm{Tor}^{\tilde{I}}_*(\tilde{I}/I, I^\infty)}$$

$(\Leftarrow)$: Consider the bicomplex

$$R/I \otimes_{-\tau} \mathcal{B}(\bar{R}) \otimes_\tau R \otimes_{-\tau} \mathcal{B}(I) \otimes_\tau I^\infty$$

$R/I \overset{L}{\otimes}_R R \overset{L}{\otimes}_{\tilde{I}} I^\infty$

This gives a spectral sequence

$$E^2_{pq} = \mathrm{Tor}^R_p\left(R/I, \mathrm{Tor}^{\tilde{I}}_q(R, I^\infty)\right) \Rightarrow \mathrm{Tor}^{\tilde{I}}_*(R/I, I^\infty)$$

Assuming $\tilde{I}/I \overset{L}{\otimes}_{\tilde{I}} I^\infty = 0$, we have

$$\mathrm{Tor}^{\tilde{I}}_q(R, I^\infty) \xleftarrow{\sim} \mathrm{Tor}^{\tilde{I}}_q(I, I^\infty) \xrightarrow{\sim} \mathrm{Tor}^{\tilde{I}}_q(\tilde{I}, I^\infty) = \begin{cases} I^\infty & q=0 \\ 0 & q \neq 0 \end{cases}$$

so $\mathrm{Tor}^R_p(R/I, I^\infty) = \mathrm{Tor}^R_p(\underbrace{R/I}_{I^{op}\ \text{null}}, I^\infty) = 0$

The reason for the review is to prepare for dealing with Hochschild homology. Recall the proof of $I \overset{L}{\otimes}_R \simeq I \overset{L}{\otimes}_{\tilde{I}}$ when I is h-unital, i.e. $I \overset{L}{\otimes}_{\tilde{I}} I \xrightarrow{\sim} I$ which we know is equivalent to $I \overset{L}{\otimes}_R I \xrightarrow{\sim} I$. This goes:

$$I \overset{L}{\otimes}_R \simeq I \overset{L}{\otimes}_{\tilde{I}} I \overset{L}{\otimes}_R \simeq I \overset{L}{\otimes}_R I \overset{L}{\otimes}_{\tilde{I}} \simeq I \overset{L}{\otimes}_{\tilde{I}} .$$

We want the same proof to work for I^∞. The idea then is we have

$$I^\infty \overset{L}{\otimes}_R \simeq I^\infty \overset{L}{\otimes}_{\tilde{I}^k} I^\infty \overset{L}{\otimes}_R \simeq I^\infty \overset{L}{\otimes}_R I^\infty \otimes_{\tilde{I}^k} \simeq I^\infty \overset{L}{\otimes}_{\tilde{I}^k}$$

and we take $\text{"}\varprojlim_k\text{"}$ to get $I^\infty \overset{L}{\otimes}_R \simeq I^\infty \overset{L}{\otimes}_{\tilde{I}^\infty}$.

Here $\text{"}\varprojlim_k\text{"}$ means we take the limit in the pro cat.

Now I want to make this precise enough to believe. Consider the bicomplex

$$I^\infty \otimes_{-\tau} B(I^k) \otimes_\tau I^\infty \otimes_{-\tau} B(\tilde{R}) \otimes_\tau$$

One spectral sequence has

$$E^2_{pq} = H_p\Big(R, \underbrace{\mathrm{Tor}^{I^k}_q(I^\infty, I^\infty)}_{0 \text{ if } q \neq 0 \text{ and } I^\infty \text{ if } q=0}\Big)$$

The other has

$$E^2_{pq} = H_p(\tilde{I^k}, \mathrm{Tor}^R_q(I^\infty, I^\infty)) = \begin{cases} 0 & q \neq 0 \\ H_p(\tilde{I}^k, I^\infty) & q = 0 \end{cases}$$

Thus we get

$$H_p(R, I^\infty) \simeq H_p(\tilde{I^k}, I^\infty)$$

Another way to state the result is that we have an isom in the Pro derived cat.

$$I^\infty \otimes_{-\tau} \mathcal{B}(\bar{R}) \otimes_\tau \xrightarrow{\sim} I^\infty \otimes_{-\tau} \mathcal{B}(I^k) \otimes_\tau$$

compatible as k varies.

Let $k \to \infty$ and then we get

$$H_p(R, I^\infty) \simeq H_p(\tilde{I}^\infty, I^\infty)$$

a pro object isomorphism.

Suppose R comm. noetherian. Then I claim that $R/I \overset{L}{\otimes}_R I^\infty = 0$. This is equivalent to $R/I \overset{L}{\otimes}_R R/I^\infty \xleftarrow{\sim} R/I$. In fact we shall show $M \xrightarrow{\sim} M \overset{L}{\otimes}_R R/I^\infty$ for any finitely generated module M. Let's shift from left to right so that we want $M \xrightarrow{\sim} R/I^\infty \overset{L}{\otimes}_R M$. The point is that $R/I^\infty \otimes_R -$ is an exact functor on the abelian category of fg R modules.

$$0 \to M' \to M \to M'' \to 0$$

$$0 \to \underbrace{\frac{M' \cap I^n M}{I^n M'}}_{\text{essentially zero by AR lemma.}} \to M'/I^n M' \to M/I^n M \to M''/I^n M'' \to 0$$

September 29, 1994

Let $\mathcal{A}$ be an abelian category, $\mathcal{T}$ a Serre subcategory. Let $D(\mathcal{A})_{\mathcal{T}}$ be the full subcategory of $D(\mathcal{A})$ consisting of complexes whose homology is in $\mathcal{T}$. Then $D(\mathcal{A})_{\mathcal{T}}$ is a <u>thick</u> triangulated subcategory of $D(\mathcal{A})$, i.e. for any map $f: X \to Y$ in $D(\mathcal{A})$ such that $\mathrm{Cone}(f) \in D(\mathcal{A})_{\mathcal{T}}$ and such f factors through a complex in $D(\mathcal{A})_{\mathcal{T}}$, one has X and $Y \in D(\mathcal{A})_{\mathcal{T}}$. Consequently, there is defined a quotient triangulated category $D(\mathcal{A})/D(\mathcal{A})_{\mathcal{T}}$ (Verdier Etat 0).

The claim is that the canonical map

$$D(\mathcal{A})/D(\mathcal{A})_{\mathcal{T}} \longrightarrow D(\mathcal{A}/\mathcal{T})$$

is an equivalence of triangulated categories. This has been proved by a Japanese mathematicians (J Alg 141?) Miyachi.

My idea for the proof is that both sides should be localizations of $C(\mathcal{A})$, then check that a map in $C(\mathcal{A})$ becomes an isomorphism in one iff it does so in the other. This gives an equivalence of categories, which has to respect the triangles.

Take a map $X \xrightarrow{f} Y$ in $C(\mathcal{A})$ and let $Z = \mathrm{Cone}(f)$. Then we know f becomes an isom. in $D(\mathcal{A})/D(\mathcal{A})_{\mathcal{T}}$ iff $Z \in D(\mathcal{A})_{\mathcal{T}}$, i.e. $H_p(Z) \in \mathcal{T}$, $\forall p$. Also f becomes an isom. in $D(\mathcal{A}/\mathcal{T})$ iff $H_p(j^*(Z)) = j^*(H_p(Z)) = 0$, i.e. $H_p(Z) \in \mathcal{T}$.

Now suppose we have a functor $F: C(\mathcal{A}) \to \mathcal{C}$ which inverts maps $f: X \to Y$ which are $\mathcal{T}$-quis or quis mod $\mathcal{T}$, i.e. the cone on f has homology in $\mathcal{T}$. Then F inverts quis, so it induces $\bar{F}: D(\mathcal{a}) \to \mathcal{C}$. Now take a map g in $D(\mathcal{a})$ whose cone lies in $D(\mathcal{a})_{\mathcal{T}}$. Up to isomorphism the map g in $D(\mathcal{a})$ lifts to a map $f: X \to Y$ in $C(\mathcal{a})$. To prove $\bar{F}(g)$ invertible it suffices to show $F(f)$ is invertible. But the cone on g and the cone on f are isomorphic, so f is a $\mathcal{T}$-isom. so $F(f)$ is invertible. Thus $\bar{F}$ induces

$\bar{\bar{F}}: D(\mathcal{a})/D(\mathcal{a})_{\mathcal{T}} \longrightarrow \mathcal{C}$.

Next, let F be the same (inverting $\mathcal{T}$-quis). I want to prove that F descends to $C(\mathcal{a}/\mathcal{T})$. Assume for the moment

$$(*) \qquad C(\mathcal{a})/C(\mathcal{T}) \xrightarrow{\sim} C(\mathcal{a}/\mathcal{T}).$$

Then $C(\mathcal{a}/\mathcal{T})$ is obtained from $C(\mathcal{a})$ by inverting $C(\mathcal{T})$-isomorphisms, i.e. a map of complexes $f: X \to Y$ whose kernel + cokernel are in $C(\mathcal{T})$. Now Factor f:

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow X \longrightarrow \mathrm{Im}(f) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Im}(f) \longrightarrow Y \longrightarrow \mathrm{Coker}(f) \longrightarrow 0$$

Then the cones on $X \to \mathrm{Im}(f)$ and $\mathrm{Im}(f) \to Y$ have the homology of $\mathrm{Ker}(f)[1]$ and $\mathrm{Coker}(f)$, resp., so these maps are $\mathcal{T}$-quis, inverted by F, hence f is inverted by F. Thus $(*) \Rightarrow F$ descends to $\bar{F}: C(\mathcal{a}/\mathcal{T}) \longrightarrow \mathcal{C}$.

Now we have to show $\bar{F}$ inverts any quis in $C(\mathcal{a}/\mathcal{T})$. By $(*)$ any map g in $C(\mathcal{a}/\mathcal{T})$ up to isomorphism comes from a map $f: X \to Y$ in $C(\mathcal{a})$. g is a quis iff f is a $\mathcal{T}$-quis. Assuming g is a quis, then f is a $\mathcal{T}$-quis, so $F(f) = \bar{F}(g)$ is invertible. Thus $\bar{F}$ descends to $\bar{\bar{F}}: D(\mathcal{a}/\mathcal{T}) \to \mathcal{C}$.

It remains now to check $\boxed{C(\mathcal{a})/C(\mathcal{T}) \xrightarrow{\sim} C(\mathcal{a}/\mathcal{T})}$

The point is that we have an exact functor $j^*: C(\mathcal{a}) \longrightarrow C(\mathcal{a}/\mathcal{T})$ whose "kernel" is $C(\mathcal{T})$, and we want conditions for $C(\mathcal{a})/C(\mathcal{T}) \longrightarrow C(\mathcal{a}/\mathcal{T})$ to be an equivalence.

So consider in general an exact functor $F: \mathcal{a} \longrightarrow \mathcal{B}$ between abelian categories such that $F(A) = 0 \implies A = 0$. Then F is faithful, since

$0 \neq f: A \longrightarrow A' \implies A \twoheadrightarrow f(A) \hookrightarrow A' \implies F(A) \twoheadrightarrow F(f(A)) \hookrightarrow F(A')$

with $F(f(A)) \neq 0$, so $F(f) \neq 0$.

Assume that any injection $B' \hookrightarrow B$ in $\mathcal{B}$ is isomorphic to F applied to $A' \hookrightarrow A$ in $\mathcal{a}$. It's equivalent to say that any short exact sequence in $\mathcal{B}$ is isomorphic to one coming from a short exact sequence in $\mathcal{a}$.

Given $u: B_1 \longrightarrow B_2$ consider the graph

$$\begin{tikzcd} B_1 \arrow[r, "{(1,u)}"] \arrow[dr, equal] & B_1 \times B_2 \arrow[d] \\ & B_1 \end{tikzcd}$$

Up to isomorphism we can suppose $B_i = F(A_i) \quad i = 1, 2$.

Then by our assumption there is a subobject $A' \subset A_1 \times A_2$ such that $F(A') = \Gamma_u \subset F(A_1 \times A_2) = B_1 \times B_2$.
Since

$$\begin{tikzcd} A' \arrow[r, phantom, "\subset"] \arrow[dr] & A_1 \times A_2 \arrow[d] \\ & A_1 \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} \Gamma_u \arrow[r, phantom, "\subset"] \arrow[dr, "\sim"] & B_1 \times B_2 \arrow[d] \\ & B_1 \end{tikzcd}$$

it follows that $A' \xrightarrow{\sim} A_1$, so A' is the graph of a map $A_1 \to A_2$ carried by F into u.
This proves:

<u>Prop.</u> An exact functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories such that $F(A) = 0 \Rightarrow A = 0$ is an equivalence iff every pair $B' \subset B$ in $\mathcal{B}$ lifts up to isomorphism to a pair $A' \subset A$ in $\mathcal{A}$. imprecise see p. 29

<u>Cor:</u> Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian category and let $\mathcal{S}$ be the Serre subcategory of $\mathcal{A}$ such that $F(A) = 0$. Then the induced functor $\mathcal{A}/\mathcal{S} \to \mathcal{B}$ is an equivalence iff every pair $B' \subset B$ in $\mathcal{B}$ lifts up to isomorphism to a pair $A' \subset A$ in $\mathcal{A}$. see p29

The corollary follows by applying the proposition to $\mathcal{A}/\mathcal{S} \to \mathcal{B}$.

Now we apply this corollary to $j^* : C(\mathcal{A}) \to C(\mathcal{A}/\mathcal{T})$. First we lift objects, and afterward subobjects. Let $Y \in C(\mathcal{A}/\mathcal{T})$ and choose pairs $(X_n, j^* X_n \simeq Y_n)$ for all n.

The differential $Y_1 \xrightarrow{d_1^Y} Y_0$ is represented by

$$\begin{tikzcd} & X_1' \arrow[dl, "s"'] \arrow[dr] & \\ X_1 & & X_0 \end{tikzcd}$$

where s is a $\mathcal{T}$-isomorphism. Replacing X_1 by X_1' we can arrange d_1^Y to come from $d_1^X : X_1 \to X_0$. Next we can similarly modify X_2 so that d_2^Y comes from a map $X_2 \xrightarrow{f} X_1$. Then $d_1^X f : X_2 \to X_0$ has image in $\mathcal{T}$, so we can modify X_2 again by shrinking to obtain $d_2^X : X_2 \to X_1$ covering d_2^Y and such that $d_1^X d_2^X = 0$. Repeating, we construct the complex $\to X_n \to \cdots \to X_0$ in the positive direction.

Next the differential $Y_0 \xrightarrow{d_0^Y} Y_{-1}$ is represented by

$$\begin{tikzcd} X_0 \arrow[dr] & & X_{-1} \arrow[dl, "s"'] \\ & X_{-1}' & \end{tikzcd}$$

, so replacing X_{-1} by X_{-1}' we can assume d_0^Y lifts to $X_0 \xrightarrow{f} X_{-1}$. Then $f d_1^X : X_1 \to X_0 \to X_{-1}$ has image in $\mathcal{T}$, so by dividing X_{-1} by this image we arrange to lift d_0^Y to $d_0^X : X_0 \to X_{-1}$ such that $d_0^X d_1^X = 0$. Now repeating this we construct the complex $X_0 \to X_{-1} \to \cdots$ in the negative direction.

Suppose now given a subcomplex $Y' \subset Y$. We can assume $Y = j^* X$. Then Y_n' is equivalent to a $\mathcal{T}$-commensurability family of $X_n' \subset X_n$. Starting with X_0' pick an X_1'. Then the image of $X_1' \xrightarrow{d} X_0 / X_0'$ is in $\mathcal{T}$, so by shrinking X_1' to a co-$\mathcal{T}$ subobject we can arrange $d(X_1') \subset X_0'$. Then repeat to do the positive side.

Finally pick X'_{-1} lifting Y'_{-1}, then the image of $X'_0 \xrightarrow{d_0} X_{-1}/X'_{-1}$ is in $\mathcal{J}$ so by enlarging X'_{-1} by this image we can arrange that $d_0(X'_0) \subset X'_{-1}$. Then repeat to do the negative direction.

This concludes the proof of $\boxed{C(\mathcal{a})/C(\mathcal{J}) \xrightarrow{\sim} C(\mathcal{a}/\mathcal{J})}$.

September 30, 1994

Consider ideals $J \subset I$ in R. Let's use the notation $\mathrm{mod}(R/I^\infty)$ for $\mathrm{null}(R,I)$, and let's also write* $\mathcal{M}(I)$ for $\mathcal{M}_n(R,I)$, which is allowed since we know the later depends only on I. We have [redacted] a "short exact sequence" of abelian categories

$$\begin{tikzcd}
\mathrm{mod}(R/J^\infty)/\mathrm{mod}(R/I^\infty) \arrow[r, hook] & \mathrm{mod}(R)/\mathrm{mod}(R/I^\infty) \arrow[r, two heads] \arrow[d, "\wr"] & \mathrm{mod}(R)/\mathrm{mod}(R/J^\infty) \arrow[d, equal] \\
& \mathcal{M}(I) \arrow[r, "\text{restriction}", "\text{of scalars}"'] & \mathcal{M}(J)
\end{tikzcd}$$

* better: $\mathcal{M}(I) = \mathcal{M}_n(\tilde{I}, I)$, then use $\mathcal{M}_n(R,I) \xrightarrow{\sim} \mathcal{M}(I)$

[redacted] Assume that $J = J^2$. Then the "kernel" is $\mathrm{mod}(R/J)$ modulo R-modules killed by J and some power of I, i.e. the kernel is

$$\mathrm{mod}(R/J)\Big/\mathrm{mod}(R/J+I^\infty) = \mathcal{M}_n(R/J, I/J)$$
$$\simeq \mathcal{M}(I/J)$$

Here we have used $(I/J)^n = J+I^n/J$ so

$$R/J\,\Big/\,(I/J)^n = R/J+I^n$$

Thus we have an "exact sequence"

$$\boxed{\mathcal{M}(I/J) \hookrightarrow \mathcal{M}(I) \twoheadrightarrow \mathcal{M}(J) \quad \text{if } J=J^2}$$

In general consider the functor

$$\mathrm{mod}(R/J^n) \longrightarrow \mathrm{mod}(R)\big/\mathrm{mod}(R/I^\infty)$$

which is exact. The kernel is $\mathrm{mod}(R/J^n+I^\infty)$.

I would like to identify the image with $\operatorname{mod}(R/J^n)/\operatorname{mod}(R/J^n+I^\infty) = \mathcal{M}(I/J^n)$.

It suffices to check that given an object in the image, i.e. an R-module M killed by J^n, and given a subobject of M in $\mathcal{M}_n(R,I)$, i.e. a submodule $M' \subset M$ determined up to I-null commensurability, that then this subobject comes from a subobject of M in $\operatorname{mod}(R/J^n)$. But $J^n M' = 0$, so this is obvious.

Thus it seems we have

$$\operatorname{mod}(R/J^\infty)/\operatorname{mod}(R/I^\infty) = \bigcup_n \mathcal{M}(I/J^n)$$

yielding the exact sequence

$$\boxed{\bigcup_n \mathcal{M}(I/J^n) \hookrightarrow \mathcal{M}(I) \twoheadrightarrow \mathcal{M}(J)}$$

in general. Observe that this checks if I happens to be unital:

$$\bigcup_n \operatorname{mod}(I/J^n) \hookrightarrow \operatorname{mod}(I) \longrightarrow \mathcal{M}(J)$$

I now want to understand Joachim's excision proof for bivariant periodic cyclic cohomology.

The proof is basically the same as the argument I understood last year. However pro-objects are a new technical tool, and instead of the cyclic complex Joachim uses the X-complex models, specifically $\mathcal{X}^\infty = \{X(TA)/F^n_{IA}X(TA)\}$ and $\mathcal{F}^\infty = \{F^n_{IA}X(TA)\}$

The basic structure aside from the tools used is the same. Namely, in order to handle an extension $A = B/I$ (everything is now nonunital) one chooses a quasi-free algebra T mapping onto B, whence $B = T/K$, $A = T/J$, $I = J/K$. One has four extensions

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & J \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & T \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
 & & A \arrow[r, equal] & A &
\end{tikzcd}$$

The point is that K, J are "approximately h-unital" so [redacted] excision holds for the three extensions with K, J as ideal. It formally follows that excision holds for that remaining extension. (In fact choosing T free, the 6 term exact sequence for $I \to B \to A$ is the same up to degree shift as the 6 term exact sequence for $K \to J \to I$.)

The key point where "approximately unital" enters is to show that

$$\begin{tikzcd}
C(K) \arrow[r] \arrow[d] & C(K/K^\infty) \arrow[d] \\
C(J) \arrow[r] & C(J/K^\infty)
\end{tikzcd}$$

is homotopy cartesian, where C is the appropriate Hochschild complex. Thus the map $(K, K^\infty) \subset (J, K^\infty)$

induces an isomorphism on relative Hochschild homology. Unfortunately I don't know the Goodwillie-Feigin-Tysgan result for Hochschild homology, but one should be able to work with the cyclic complex. In this case the kernel of $C^\lambda(J) \longrightarrow C^\lambda(J/K^\infty)$ is built out of complexes $\left[K^\infty \overset{L}{\otimes}_{\tilde{J}}\right]^{(p)}$.

Now for K "approx h-unital" this homology depends only on K and is independent of the ring J containing K as ideal.

It would be nice to link the excision in per. cyc. homology result to the way approx h-unital arises in connection with derived categories associated to module categories for non unital rings.

The idea is to assign to an extension $A = B/I$ the 'short exact sequence'

$$\mathcal{M}(B/I^\infty) \hookrightarrow \mathcal{M}(B) \twoheadrightarrow \mathcal{M}(I)$$

$$D(\mathcal{M}(B/I^\infty)) \hookrightarrow D(\mathcal{M}(B)) \twoheadrightarrow D(\mathcal{M}(I))$$

where the latter requires I to be approx h-unital. Unfotnately $\mathcal{M}(B/I^\infty)$ is not $\mathcal{M}(B/I)$, so this doesn't seem to work.

<u>October 2, 1994</u>

Let's review pro-objects in an abelian category $\mathcal{A}$. Ignoring set theory problems these are the same as left exact functors from $\mathcal{A}$ to $\mathcal{A}b$, the correspondence being

$$(X_i) \longmapsto \operatorname{Hom}((X_i), -) = \varinjlim_i \operatorname{Hom}(X_i, -)$$

Thus we have an equivalence

$$(\operatorname{pro} \mathcal{A})^{op} \xrightarrow{\sim} \operatorname{lex}(\mathcal{A}, \mathcal{A}b)$$

Now $\operatorname{pro}\mathcal{A}$ is an abelian category. We give two arguments.

1) Given a map $(X_i) \xrightarrow{f} (Y_j)$ of pro-objects one knows there is a common cofinal indexing so that the map is represented by a map of inverse systems $(f_i : X_i \to Y_i)$. Then we can factor each f_i to obtain

$$0 \to (\operatorname{Ker}(f_i)) \to (X_i) \to (\operatorname{Im}(f_i)) \to 0$$

$$0 \to (\operatorname{Im}(f_i)) \to (Y_i) \to (\operatorname{Coker}(f_i)) \to 0$$

two short exact sequences of inverse systems. If these are both left + right exact in $\operatorname{pro}\mathcal{A}$, then we see $\operatorname{Ker}(f) = (\operatorname{Ker}(f_i))$, similarly for Im and Coker, so $\operatorname{pro}\mathcal{A}$ is abelian.

Thus we reduce to showing that a short ex seq $0 \to (X_i') \to (X_i) \to (X_i'') \to 0$ of inverse systems is both left + right exact in $\operatorname{pro}\mathcal{A}$. Given $Y = (Y_j)$ then $0 \to \operatorname{Hom}(Y_j, X_i') \to \operatorname{Hom}(Y_j, X_i) \to \operatorname{Hom}(Y_j, X_i'')$ is exact, so taking $\varinjlim_j$ and then $\varprojlim_i$ gives

$$0 \to \mathrm{Hom}((Y_j),(X_i')) \to \mathrm{Hom}((Y_j),(X_i)) \to \mathrm{Hom}((Y_j),(X_i''))$$

is exact. On the other hand

$$0 \to \mathrm{Hom}(X_i'', Y_j) \to \mathrm{Hom}(X_i, Y_j) \to \mathrm{Hom}(X_i', Y_j)$$

is exact, so taking $\varinjlim_i$ then $\varprojlim_j$ gives

$$0 \to \mathrm{Hom}((X_i''),(Y_j)) \to \mathrm{Hom}((X_i),(Y_j)) \xrightarrow{d} \mathrm{Hom}((X_i'),(Y_j))$$

is exact.

2) (Gabriel's thesis). Consider add$(\mathcal{a}, \mathcal{ab})$, which is a Grothendieck category, the full subcategory $\mathcal{E}$ of effaceable functors (i.e. $\forall$ pair $(A, \xi \in F(A))$ $\exists$ injection $A \hookrightarrow A'$ killing ξ). Then $\mathcal{E}$ is a Serre subcategory closed under direct sums.

We now identify the 'solid' subcategory $\mathcal{E}^\perp$ with lex$(\mathcal{a}, \mathcal{ab})$. First let I be an $\mathcal{E}$-free injective functor in add$(\mathcal{a}, \mathcal{ab})$. If

$$0 \to A' \to A \to A'' \to 0$$

is exact in $\mathcal{a}$, then one has the exact sequence

$$* \qquad 0 \to h^{A''} \to h^A \to h^{A'} \to h^{A'}/\mathrm{Im}\, h^A \to 0$$

in add$(\mathcal{a}, \mathcal{ab})$, where is effaceable:

$$\begin{tikzcd} A' \arrow[r, hook] \arrow[d, "\xi"'] \arrow[dr, phantom, "\text{push out}"] & A \arrow[d] \\ B \arrow[r, hook] & B_1 \end{tikzcd}$$

shows that any $\xi \in (h^{A'}/\mathrm{Im}\, h^A)(B)$ is killed by some injection $B \hookrightarrow B_1$. Thus applying Hom$(-, I)$ to $*$ gives $0 \to K \to I(A') \to I(A) \to I(A'') \to 0$.

where $K = \mathrm{Hom}(h^{A'}/\mathrm{Im}\, h^{A}, I) = 0$ as I is $\mathcal{E}$-free. Thus an $\mathcal{E}$-free injective is an exact functor.

[redacted] Any $F \in \mathcal{E}^{\perp}$ is a kernel [redacted] of a map $I^0 \to I^1$ of $\mathcal{E}$-free injectives, and hence is left exact.

Finally, if F is left exact, then in particular it is $\mathcal{E}$-free, so $\exists\ F \hookrightarrow I^0$ with I^0 $\mathcal{E}$-free injective. Then one checks $F' = I^0/F$ is $\mathcal{E}$-free (respects injections) as F is left exact, so $I^0/F \hookrightarrow I^1$ and $F = \mathrm{Ker}(I^0 \to I^1)$ showing $F \in \mathcal{E}^{\perp}$.

Then one invokes localization theory in Grothendieck categories to get the adjoint functors

$$\begin{tikzcd}
\mathcal{E} \arrow[r, shift left, "\iota"] & \mathrm{add}(\mathcal{A}, \mathcal{A}b) \arrow[l, shift left, "\iota^!"] \arrow[r, shift left, "j^*"] \arrow[d, shift left, "R^0"] & \mathrm{add}(\mathcal{A}, \mathcal{A}b)/\mathcal{E} \arrow[l, shift left, "j_*"] \arrow[dl, "\sim"] \\
& \mathrm{lex}(\mathcal{A}, \mathcal{A}b) \arrow[u, shift left] &
\end{tikzcd}$$

so $\mathrm{lex}(\mathcal{A}, \mathcal{A}b)$ is a Grothendieck category in particular.

But $(\mathrm{pro}\,\mathcal{A})^{op} = \mathrm{lex}(\mathcal{A}, \mathcal{A}b)$, so $\mathrm{pro}\,\mathcal{A}$ is a dual Grothendieck category, hence it has enough projectives; (I should say 'should have', since I'm ignoring the set theory issues).

It seems to be hard to write down a projective pro-object. Let's first look [redacted] injective pro-objects. If Y is injective in $\mathcal{A}$, then Y is an injective in $\mathrm{pro}\,\mathcal{A}$. To see this recall that any short exact sequence in $\mathrm{pro}\,\mathcal{A}$ may be represented [redacted] a short exact sequence [redacted] $0 \to (X_i') \to (X_i) \to (X_i'') \to 0$ of inverse systems with the same index directed set say. But

$$\varinjlim_i \mathrm{Hom}((-)_i, Y)$$

is clearly exact when Y is injective in $\mathcal{A}$. We can take a direct product of these functors and get an exact functor. Thus if $\{Y_s, s \in S\}$ is a family of injectives and σ ranges over finite subsets of S and $Y_\sigma = \prod_{s \in \sigma} Y_s$, then

$$\prod_S \mathrm{Hom}((X_i), Y_s) = \varprojlim_\sigma \mathrm{Hom}((X_i), Y_\sigma)$$

$$= \mathrm{Hom}((X_i), (Y_\sigma))$$

is an exact functor of (X_i), showing that (Y_σ) is an injective pro-object.

In particular for a countable family $\{Y_k\}$ of injectives the pro-object ([redacted] $\prod_{k \le n} Y_k)_n$ is injective in pro $\mathcal{A}$.

Joachim claims that $(W_n) = \left(\bigoplus_{k \ge n} E_k\right)$, where E_k is a family of vector spaces, is a projective pro-vector space. Picture:

$$\cdots \hookrightarrow \prod_{k \ge 2} E_k \hookrightarrow \prod_{k \ge 1} E_k \hookrightarrow \prod_{k \ge 0} E_k$$

Here seems to be a counterexample. NO Take E_k to be 1-dimensional with basis element ξ_k. Then

$$\mathrm{Hom}_{\mathbb{C}}(W_n, V) = \prod_{k \ge n} V \xi_k^* \simeq V[[x]] \Big/ \sum_{k < n} V x^k$$

where we identify the dual basis element ξ_k^* with x^k. Thus

$$\mathrm{Hom}_{\mathbb{C}}((W_n), V) \simeq \varinjlim_n V[[x]] / \sum_{k<n} V x^k$$

$$= V[[x]]/V[x]$$

Now consider the adic inverse system $R_j = R/I^{j+1}$. Then we have a surjection $R \longrightarrow R_j$ of inverse systems, [redacted] and

$$\mathrm{Hom}_{\mathbb{C}}((W_n), (R_j)) = \varprojlim \left(R_j[[x]] / R_j[x] \right)$$

$$= \hat{R}[[x]] / \varprojlim_j R_j[x]$$

restricted power series $\sum r_n x^n \in \hat{R}[[x]]$, $r_n \to 0$ in $\hat{R}$

Here we've used the exact sequence

$$0 \longrightarrow R_j[x] \longrightarrow R_j[[x]] \longrightarrow R_j[[x]]/R_j[x] \longrightarrow 0$$

of inverse systems, where the maps from j to $j-1$ are surjective, to conclude that

$$0 \longrightarrow \varprojlim R_j[x] \longrightarrow \overbrace{\varprojlim R_j[[x]]}^{\hat{R}[[x]]} \longrightarrow \varprojlim \left(R_j[[x]]/R_j[x] \right) \longrightarrow 0$$

is exact, [redacted] since $R^1\varprojlim$ vanishes for [redacted] surjective inverse systems.

Now

$$\mathrm{Hom}_{\mathbb{C}}((W_n), R) = R[[x]]/R[x]$$

does not need to map onto $\mathrm{Hom}_{\mathbb{C}}((W_n), (R_j)) = \hat{R}[[x]] / \varprojlim R_j[x]$, e.g. if $r \in \hat{R}$ does not come from R, then $\sum r x^n$ is not restricted. NO One has $R[[x]] + \varprojlim R_j[x] = \hat{R}[[x]]$, namely given $\sum r_n x^n \in \hat{R}[[x]]$, choose r_n' coming from R such that $r_n - r_n' \to 0$.

Review: We have seen

Y <u>injective</u> <u>in</u> $\mathcal{A} \Rightarrow Y$ <u>injective</u> <u>in</u> pro $\mathcal{A}$.

Proof above uses $\mathrm{Hom}((X_i), Y) = \varinjlim \mathrm{Hom}(X_i, Y)$ is exact in (X_i) and any sh ex seq in pro $\mathcal{A}$ is represented by a sh. ex seq of inverse systems.

Second proof by showing h^Y is projective in $\mathrm{lex}(\mathcal{A}, \mathcal{Ab})$: A sh ex seq in lex is given by a left exact sequence in $\mathrm{add}(\mathcal{A}, \mathcal{Ab})$

$$0 \to F' \to F \to F''$$

such that $C = \mathrm{Coker}(F \to F'')$ is effaceable.

If Y injective and C effaceable, then $C(Y) = 0$, since Y is a summand of Z whenever $Y \hookrightarrow Z$.

Thus $0 \to F'(Y) \to F(Y) \to F''(Y) \to 0$ is exact, showing $F \mapsto \mathrm{Hom}(h^Y, F) = F(Y)$ is exact.

Both proofs allow further construction of injectives $\{Y_\sigma = \prod_{j \in \sigma} Y_j\}$ or $\varinjlim_\sigma h^{Y_\sigma} = \bigoplus_j h^{Y_j}$.

Correction to p.17. <u>Suppose given</u> $F : \mathcal{A} \to \mathcal{B}$ <u>an exact functor between abelian categories such that</u> $F(A) = 0 \Rightarrow A = 0$. <u>Assume</u> 1) $\forall B \in \mathcal{B}$ $\exists A \in \mathcal{A}$ <u>s.t.</u> $F(A) \simeq B$, 2) $\forall A \in \mathcal{A}$ <u>and every subobject</u> $B' \subset F(A)$ <u>there exists a subobject</u> $A' \subset A$ <u>such that</u> $F(A') = B'$; <u>(more precisely</u> $\forall$ <u>injection</u> $B' \hookrightarrow F(A)$, <u>there is an injection</u> $A' \hookrightarrow A$ <u>and an isom.</u> $F(A') \xrightarrow{\sim} B'$ <u>such that</u>

$$\begin{tikzcd} F(A') \arrow[rr, "\sim"] \arrow[dr, hook] & & B' \arrow[dl, hook] \\ & F(A) & \end{tikzcd}$$

<u>commutes.</u> Then $F : \mathcal{A} \to \mathcal{B}$ <u>is an equivalence of categories.</u>

The point is that to prove F is fully faithful I need to know that any subobject B' of $F(A_1 \times A_2) \simeq F(A_1) \times F(A_2)$ projecting isomorphically onto $F(A_1)$ (these are in 1-1 correspondence with maps $F(A_1) \to F(A_2)$) actually comes from a subobject $A' \subset A_1 \times A_2$. It's not enough to have an isomorphism

$$\begin{array}{ccc} F(A') & \subset & F(A) \\ \wr| & & \wr| \\ B' & \subset & F(A_1 \times A_2) \end{array}$$

because I need a subobject of $A_1 \times A_2$.

Observe 2) alone implies $F : \mathcal{A} \to \mathcal{B}$ is fully faithful.

October 4, 1994

Here's how to understand why the inverse system $n \mapsto W_n = \bigoplus_{k \geq n} E_k$ is projective as pro-object. Consider the abelian category $\mathrm{fun}(\mathbb{N}^{op}, \mathcal{A})$ where $\mathcal{A}$ is an abelian category. Then we have an exact functor

$$\mathrm{fun}(\mathbb{N}^{op}, \mathcal{A}) \longrightarrow \mathrm{pro}_{\mathbb{N}} \mathcal{A}$$

$\mathbb{N}$ means countable directed sets for indexing

whose kernel is the Serre subcategory of inverse systems (M_n) which are zero as pro objects, i.e. $\forall n \ \exists n' \geq n$ such that $M_{n'} \to M_n$ is zero.

Let's check [redacted] for any (M_n) and subobject of (M_n) in pro $\mathcal{A}$, that this subobject comes from a subinverse system of (M_n). A monic $(M'_n) \to (M_n)$ in $\mathrm{pro}_{\mathbb{N}} \mathcal{A}$ is represented by an injection after [redacted] reindexing $M'_{n'_k} \hookrightarrow M_{n_k}$, where we can suppose [redacted] (n'_k), (n_k) are strictly increasing. We can replace n'_k by k. Then we have

$$\begin{array}{ccccccc}
\longrightarrow M_{n_1} \longrightarrow \cdots & \longrightarrow & M_{n_0} & \longrightarrow & M_1 & \longrightarrow & M_0 \\
\cup & & \cup & & & & \\
M'_1 & & M'_0 & & & &
\end{array}$$

and we can clearly fill in the gaps by the images of M'_k in M_n for $n_k \geq n > n_{k-1}$.

Thus it's clear that

$$\mathrm{fun}(\mathbb{N}^{op}, \mathcal{A}) / \text{null systems} \xrightarrow{\ \sim\ } \mathrm{pro}_{\mathbb{N}} \mathcal{A}$$

Now (W_n), where $W_n = \bigoplus_{k \geq n} E_k$, is the direct sum of the inverse systems $E_n(\leq n)$, which are evidently projective, since $\mathrm{Hom}_{\mathrm{fun}}((E_n(\leq n)), (M_n)) = \mathrm{Hom}(E_n, M_n)$.

[redacted] Joachim's argument uses this fact and the fact that any surjection [redacted] in $\mathrm{pro}_{\mathbb{N}}\,\mathcal{A}$ is represented by a surjection of inverse systems to conclude that (W_n) is projective. Another ingredient is that (W_n) when reindexed has the same form and so stays projective.

This is the essential point, which [redacted] can be used as follows. Use the formula for maps in the quotient abelian category, namely a map $(W_n) \dashrightarrow (M_n)$ in $\mathrm{pro}_{\mathbb{N}}\,\mathcal{A}$ should be represented by a diagram.

$$\begin{tikzcd} & (W_n') \arrow[dl, "s"'] \arrow[dr] & \\ (W_n) & & M_n \end{tikzcd}$$

where s is a null-isomorphism. There should be a cofinality argument that the category of such s with target (W_n) admits the cofinal family [redacted] $(W_{\varphi(n)}) \longrightarrow (W_n)$, where φ: $\varphi(0) < \varphi(1) < \cdots$ is any order-preserving injection from $\mathbb{N}$ to itself.

So instead of the Artin-Rees pro-category where one uses the translations $\varphi(n) = n+k$, we use the monoid of order-preserving injections. Another possibility to consider would be to use linear φ - might this be related to Dixmier trace? In any case linear φ come up in Goodwillie's theorem.

I checked the cofinality lemma i.e. $M \longrightarrow M'$ is a null-isomorphism iff $\exists\, \varphi$ and a compatible dotted arrow for the square

$$\begin{tikzcd} \varphi^* M \arrow[r] \arrow[d] & \varphi^* M' \arrow[d] \arrow[dl, dashed] \\ M \arrow[r] & M' \end{tikzcd}$$

The same proof works since φ^* is exact (only right exactness is needed). You need to check that $M=(M_n)$ is null iff $\exists\, \varphi$ such that $\varphi^* M \to M$ is zero.

Another angle is the relation to left exact functors. Suppose we ~~restrict~~ to countable dimensional vector spaces for our abelian category $\mathcal{A}$. Then all short exact sequences in $\mathcal{A}$ split and every object is both projective and injective. If we choose $V_0 = k^{(\mathbb{N})}$ and let $R = \mathrm{End}_k(V_0)$, then $\mathcal{A}$ should be equivalent to the ~~[illegible]~~ category $\mathcal{P}(R^{op})$ of f.g. projective R-modules, (which happens to be abelian in this case). So $\mathrm{fun}(\mathcal{A}, \mathcal{Ab})$ is the category of R-modules. But also, since exact sequences in $\mathcal{A}$ split, $\mathrm{lex}(\mathcal{A}, \mathcal{Ab}) = \mathrm{fun}(\mathcal{A}, \mathcal{Ab})$, so pro $\mathcal{A}$ should be opposite to the category of R-modules.

The projective R-module R should correspond to V_0 considered as pro-object, which we know is injective. ~~[illegible]~~ A free R-module will correspond to the direct sum sort of injective pro-object encountered before ($\{Y_0\}$).

Apparently $\prod_{k\geq 0} \mathrm{Hom}(E_k, V_0) \Big/ \bigoplus_{k\geq 0} \mathrm{Hom}(E_k, V_0)$ is an injective R-module if everything above is OK.

October 5, 1994

Another correction: Let $\overbrace{F}^{:\mathcal{A}\to\mathcal{B}}$ be an exact functor between abelian categories with kernel 0. Then the condition

$$* \quad \forall A \in \mathcal{A},\ \forall B' \subset F(A) \quad \exists A' \subset A \quad \text{s.t.} \quad F(A') = B'$$

is sufficient that F be fully faithful but it is not necessary.

An example is the inclusion of constant sheaves in all sheaves on a connected space. The example I encountered yesterday is the inclusion of $\mathrm{pro}_{\mathbb{N}}\mathcal{A}$, countable pro-objects, in $\mathrm{pro}\,\mathcal{A}$, all pro-objects.

<u>October 7, 1994</u>

Problem: To show

$$D^b(\mathrm{mod}(R/I^\infty)) \longrightarrow D^b(\mathrm{mod}(R))_{\mathrm{mod}(R/I^\infty)}$$

is an equivalence of triangulated categories when I is approximately h-unital.

First I have to understand $D^b(\mathcal{A})$ for an abelian category $\mathcal{A}$. There are two main candidates which are equivalent (Verdier SGA $4\frac{1}{2}$ Etat 0). One is the quotient

$$\underbrace{K^b(\mathcal{A})}_{\substack{\text{bdd complexes} \\ \text{up to htpy}}} / \underbrace{K^{b,0}(\mathcal{A})}_{\substack{\text{bdd acyclic} \\ \text{complexes}}}$$

The other is the full subcategory of $D(\mathcal{A})$ consisting of complexes with bounded homology. There's an obvious map from the former to the latter.

Let's prove this map is an equivalence. First it's essentially surjective since if X is a complex with bounded homology, the Postnikov filtration gives quis

$$\begin{tikzcd}[column sep=small]
{} \arrow[r] & X_{q+1} \arrow[r] \arrow[d, equal] & X_q \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & X_1 \arrow[r] & X_0 \arrow[r] \arrow[d, equal] & X_{-1} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & X_{-p+1} \arrow[r] \arrow[d, equal] & X_{-p} \arrow[r] \arrow[d, phantom, "\cup"] & \cdots \\
{} \arrow[r] & X_{q+1} \arrow[r] \arrow[d] & X_q \arrow[r] \arrow[d] & \cdots \arrow[r] & X_1 \arrow[r] \arrow[d, equal] & X_0 \arrow[r] \arrow[d, equal] & X_{-1} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & X_{-p+1} \arrow[r] \arrow[d, equal] & Z_{-p} \arrow[r] \arrow[d, equal] & 0 \cdots \\
{} & 0 \arrow[r] & X_q/B_q \arrow[r] & \cdots \arrow[r] & X_1 \arrow[r] & X_0 \arrow[r] & X_{-1} \arrow[r] & \cdots \arrow[r] & X_{-p+1} \arrow[r] & Z_{-p} \arrow[r] & 0
\end{tikzcd}$$

So it remains to prove fully-faithful. Start with $X, Y \in K^b(\mathcal{A})$ and the formula

$$\underbrace{R^0 \mathrm{Hom}_{\mathcal{A}}(X, Y)}_{\mathrm{Hom}_{D(\mathcal{A})}(X,Y)} = \varinjlim_{X' \xrightarrow{s} X} H^0 \mathrm{Hom}(X', Y)$$

where the limit is taken over the filtering category Quis/X consisting of quis $X' \xrightarrow{s} X$ in $K(\mathcal{a})$. The problem is that X' is not bounded but has only bounded homology. Take $\xi \in R^0\mathrm{Hom}(X,Y)$ and represent it

$$\begin{tikzcd} X' \arrow[d, "s"] \arrow[dr, "f"] & \\ X & Y \end{tikzcd}$$

As above we can replace X' by a quasi-isomorphic right-bounded sub complex, so assume our representation of ξ is such that X' is right-bounded. The next point is that because Y is bounded we have

$$H^0\mathrm{Hom}(X',Y) = H^0\mathrm{Hom}(X'/X'',Y)$$

where $X'' : \to X_{n+1} \to Z_n \to 0$ with $n \ll 0$. Similarly for X in place of Y. Thus we can replace X' by a quasi-isomorphic bounded quotient complex, so we can assume our representation of ξ takes place in $K^b(\mathcal{a})$. This shows the surjectivity of

$$\mathrm{Hom}_{K^b(\mathcal{a})/K^{b,0}(\mathcal{a})}(X,Y) \longrightarrow \mathrm{Hom}_{D(\mathcal{a})}(X,Y)$$

Finally the injectivity: Let ξ in the former be represented by the maps

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

in $K^b(\mathcal{a})$. For ξ to go to zero in $R^0\mathrm{Hom}_{\mathcal{a}}(X,Y)$

means $\exists$ quis $X'' \xrightarrow{s_1} X'$ such that $f s_1 = 0$ with $X'' \in K(\mathcal{a})$. Again the same process works – we can truncate X'' to a quis $\underbrace{}_{}$bounded ~~subquotient~~ complex X''', and obtain a quis $X''' \xrightarrow{s_2} X''$ such that $f s_2 = 0$.

This proves injectivity and proves the result that

$$D^b(\mathcal{a}) = K^{b,b}(\mathcal{a}) / K^{b,0}(\mathcal{a}) \longrightarrow K^{\infty,\infty}(\mathcal{a}) / K^{\infty,0}(\mathcal{a}) = D(\mathcal{a})$$

is fully-faithful.

Let's apply this in the case of $\mathrm{mod}(R/I^\infty) = \mathrm{mll}(R,I)$. Then any bounded complex in $\mathrm{mod}(R/I^\infty)$ is a bounded complex of R/I^n-modules for some n:

$$K^b(\mathrm{mod}(R/I^\infty)) = \bigcup_n K^b(\mathrm{mod}(R/I^n))$$

Here we use $\mathrm{Hom}^\cdot_R(X,Y) = \mathrm{Hom}^\cdot_{R/I^n}(X,Y)$ if X, Y are complexes of R/I^n modules. Then it's clear for X, Y in $D^b(\mathrm{mod}(R/I^\infty))$ that

$$\varinjlim_n R^0\mathrm{Hom}_{R/I^n}(X,Y) = \varinjlim_n \varinjlim_{X' \xrightarrow{s} X} H^0\mathrm{Hom}_{R/I^n}(X',Y) \qquad X' \text{ over } R/I^n$$

$$= \varinjlim_n \varinjlim_{X' \xrightarrow{s} X} H^0\mathrm{Hom}_R(X',Y) \qquad X' \text{ over } R/I^n$$

$$= \varinjlim_{X' \xrightarrow{s} X} H^0\mathrm{Hom}_R(X',Y) \qquad X' \text{ bdd complex in } \mathrm{mod}(R/I^\infty).$$

$$= R^0\mathrm{Hom}_{\mathrm{mod}(R/I^\infty)}(X,Y)$$

At this point we understand maps in $D^b(\mathrm{mod}(R/I^\infty))$. We want to prove $D^b(\mathcal{s}) \to D^b(\mathcal{a})_{\mathcal{s}}$

is an equivalence, where $\mathcal{A} = \mathrm{mod}(R)$, $\mathcal{B} = \mathrm{mod}(R/I^\infty)$.
Fully-faithful amounts to

$$\varinjlim_n R^0\mathrm{Hom}_{R/I^n}(X,Y) \xrightarrow{\sim} R^0\mathrm{Hom}_R(X,Y)$$

where X, Y are bdd complexes of null modules.

October 8, 1994

Assuming I is approx h-unital we want to prove that

$$\iota_* : D^b(\mathrm{mod}(R/I^\infty)) \longrightarrow D^b(\mathrm{mod}(R))$$

is fully faithful and the essential image is the full subcategory of complexes with null homology.

Let's now prove the fully faithful part, i.e.

$$(1) \qquad \varinjlim_k R^j \mathrm{Hom}_{R/I^k}(X, Y) \xrightarrow{\sim} R^j \mathrm{Hom}_R(X, Y)$$

if X and Y are bounded complexes of null modules. Using Postnikov filtrations one reduces to the case where X, Y are null modules $\underbrace{}_{M,N}$ in degree zero. We have a standard spectral sequence associated to

$$R\mathrm{Hom}_R(X, Y) = R\mathrm{Hom}_{R/I^k}(R/I^k \overset{L}{\otimes}_R X, Y)$$

namely

$$(2) \quad E_2^{pq} = \mathrm{Ext}^p_{R/I^k}(\mathrm{Tor}^R_q(R/I^k, M), N) \Rightarrow \mathrm{Ext}^{p+q}_R(M, N)$$

Now since I is approx h-unital and M is null we have

$$\{\mathrm{Tor}^R_q(R/I^k, M)\}_k \simeq \begin{cases} M & q = 0 \\ 0 & q \neq 0 \end{cases}$$

Thus looking at the spectral sequence in the category of ind abelian grps. we have

$$E_2^{pq} = \begin{cases} (\mathrm{Ext}^p_{R/I^k}(M, N))_k & q = 0 \\ 0 & q \neq 0 \end{cases}$$

so the spectral sequence degenerates yielding $\left(\mathrm{Ext}^j_{R/I^k}(M, N)\right)_k \simeq \mathrm{Ext}^j_R(M, N)$

which is stronger than (1).

Now the problem is to understand concretely what's going on. First recall the derivation of the spectral sequence (2). We choose a projective R-resolution $P \to M$ and an injective R/I^k-resolution $N \to E$, then look at the double complex

$$\mathrm{Hom}_R(P, E) = \mathrm{Hom}_{R/I^k}(P/I^kP, E).$$

It seems essential to use the injective resolution E in addition to P.

Note that the complex of inverse systems $(P/I^kP)_k$ has ess. zero homology in degrees ≥ 1, and it resolves M modulo ess. zero inverse systems. One might hope to reindex and maybe use different ~~quotients~~ of P to get a complex of inverse systems (P/F_kP) which resolves the constant inverse system M. But this does not seem to work as we shall now see.

Let's start with an element of $\mathrm{Ext}^j_R(M,N)$ represent it by a map $P_j \to N$. Put $X^n_j = P_j/I^nP_j$. Then we have for large n

$$\begin{tikzcd} & N & \\ X^n_{j+1} \arrow[r] \arrow[ur, dashed, "0"] & X^n_j \arrow[u] \arrow[r] & X^n_{j-1} \cdots \end{tikzcd} \qquad \text{i.e.} \qquad \begin{tikzcd} N & & \\ X^n_j/B^n_j \arrow[u] \arrow[r] & X^n_{j-1} \arrow[r] & X^n_{j-2} \end{tikzcd}$$

First look at $j=2$. Then we have exactness at

$$X^n_2 \longrightarrow X^n_1 \longrightarrow X^n_0 \longrightarrow \hat{M} \longrightarrow 0$$

We have

$$\begin{tikzcd} & & N & & \\ H_2^n \arrow[r, hook] & X_2^n/B_2^n \arrow[ur] \arrow[r, two heads] & B_1^n \arrow[u] & \subset Z_1^n & \subset X_1^n \end{tikzcd}$$

$\exists$ for large n because $(X_2^n/B_2^n)_n \to (Z_1^n)_n$ is an isom of proobjects

Because $H_1(P/I^kP)$ is ess. zero there is an $n' \geq n$ such that $Z_1^{n'}/B_1^{n'} \to Z_1^n/B_1^n$ is zero, i.e. such that $Z_1' \to Z_1^n$ has image in B_1^n. Thus we get the picture

$$\begin{tikzcd} & N & & & & \\ 0 \arrow[r] & Z_1^{n'} \arrow[u] \arrow[r] & X_1^{n'} \arrow[r] & X_0^{n'} \arrow[r] & M \arrow[r] & 0 \\ \arrow[r] & P_2 \arrow[u] \arrow[r] & P_1 \arrow[u] \arrow[r] & P_0 \arrow[u] \arrow[r] & M \arrow[u, equal] & \end{tikzcd}$$

which means that we have managed to lift our class in $\operatorname{Ext}^2_R(M,N)$ back to $\operatorname{Ext}^2_{R/I^{n'}}(M,N)$.

Notice that the complex of null modules we have used is not obviously a ~~quotient~~ of P.

Here's a general construction

Start with

$$\begin{tikzcd} & N & \\ X_{j+1}^n \arrow[ur, dashed, "0"] \arrow[r] & X_j^n \arrow[u] \arrow[r] & X_{j-1}^n \end{tikzcd} \qquad \text{i.e.} \qquad \begin{tikzcd} & N & & \\ H_j^n \arrow[r, hook] & X_j^n/B_j^n \arrow[u] \arrow[r, two heads] & B_{j-1}^n \arrow[ul, dashed] & \subset Z_{j-1}^n \end{tikzcd}$$

By enlarging n we can assume $H_j^n \to N$ is zero, hence the dotted arrow $B_{j-1}^n \to N$ exists, and by enlarging n further we can assume this arrow extends to Z_{j-1}^n. Then we have

$$\begin{tikzcd} & N & \\ 0 \arrow[r] & Z_{j-1}^n \arrow[u] \arrow[r] & X_{j-1}^n \end{tikzcd}$$

and we can pushout:

$$\begin{tikzcd} & N \arrow[r] & Q^0 \arrow[r] & N^1 \arrow[r] & 0 \\ 0 \arrow[r] & Z^n_{j-1} \arrow[r] \arrow[u] & X^n_{j-1} \arrow[r] \arrow[u] & B^n_{j-2} \arrow[r] \arrow[u, "s"'] & 0 \end{tikzcd}$$

Repeat this process, i.e. enlarge n so that $B^n_{j-2} \longrightarrow N^1$ extends to Z^n_{j-2}, then pushout:

$$\begin{tikzcd} N^1 \arrow[r] & Q^1 \\ Z^n_{j-2} \arrow[u] \arrow[r, phantom, "\subset"] & X^n_{j-2} \arrow[u] \end{tikzcd}$$

to obtain N^2. What this gives is a resolution

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] & Q^0 \arrow[r] & \cdots\cdots \arrow[r] & Q^{j-1} \arrow[r] & N^j \arrow[r] & 0 \\ {} \arrow[r] & P_j \arrow[r] \arrow[u] & P_{j-1} \arrow[r] \arrow[u] & \cdots\cdots \arrow[r] & P_0 \arrow[r] \arrow[u] & M \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

October 9, 1994

First recall truncations of a complex, equivalently the Postnikov system. First restrict to chain complexes, i.e. $\to X_n \to X_{n-1} \to \cdots$ with $X_n = 0$ for $n < 0$.

Let X be a chain complex and Y a complex [redacted] with support in $[t, 0]$. Then any map from X to Y factors

$$\begin{tikzcd}
\arrow[r] & X_{t+1} \arrow[r] & X_t \arrow[r] \arrow[d] & X_{t-1} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & X_0 \arrow[d, equal] \\
\arrow[r] & 0 \arrow[r] & X_t/B_t \arrow[r] \arrow[d] & X_{t-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & X_0 \arrow[d] \\
& 0 \arrow[r] & Y_t \arrow[r] & Y_{t-1} \arrow[r] & \cdots \arrow[r] & Y_0
\end{tikzcd}$$

denote this $\mathrm{post}_{\leq t}(X)$ as it has the same homology as X in degrees $\leq n$.

Thus $$Z^0 \mathrm{Hom}_{C(a)}(X, Y) = Z^0 \mathrm{Hom}_{C(a)}(\mathrm{post}_{\leq t}(X), Y)$$

Also $$\mathrm{Hom}^{-1}_{C(a)}(X, Y) = \mathrm{Hom}^{-1}_{C(a)}(\mathrm{post}_{\leq t}(X), Y)$$

so $$\mathrm{Hom}_{K(a)}(X, Y) = \mathrm{Hom}_{K(a)}(\mathrm{post}_{\leq t}(X), Y).$$

Similarly for maps in the derived category

$$\begin{tikzcd}
X' \arrow[rr] \arrow[d, "s"'] \arrow[dr] & & Y \\
& \mathrm{post}_{\leq t}(X') \arrow[ur] \arrow[d, "s"] & \\
X \arrow[r] & \mathrm{post}_{\leq t}(X) &
\end{tikzcd}$$

so $$\mathrm{Hom}_{D(a)}(X, Y) = \varinjlim \mathrm{Hom}_{K(a)}(X', Y) \qquad (X' \xrightarrow{s} X)$$

$$= \varinjlim \mathrm{Hom}_{K(a)}(\mathrm{post}_{\leq t}(X'), Y) \qquad \text{"}$$

which gives a map

$$\mathrm{Hom}_{D(a)}(X,Y) \longrightarrow \mathrm{Hom}_{D(a)}(\mathrm{post}_{\le t}(X), Y)$$

But there's an obvious map the other way, given by composition in $D(a)$. These should be inverse.

We can check this when there are enough projectives as follows. Suppose P is a projective chain complex together with qis $P \to X$. Then

$$\mathrm{Hom}_{D(a)}(X,Y) = \mathrm{Hom}_{K(a)}(P,Y)$$

$$= \mathrm{Hom}_{K(a)}(\mathrm{post}_{\le t}(P), Y)$$

Now $\mathrm{post}_{\le}(P)$: $\quad P_t/dP_{t+1} \to P_{t-1} \to \cdots \to P_0$ is complex supported in $[t,0]$, projective except in degree t, and equipped with a qis $\mathrm{post}_{\le t}(P) \to \mathrm{post}_{\le t}(X)$. Choosing a projective resolution

$$\to P'_{t+2} \to P'_{t+1} \to P_t \to P_t/dP_{t+1} \to 0$$

we get a projective complex $P' : \to P'_{t+1} \to P_t \to P_{t-1} \to \cdots$ [redacted] equipped with ~~a~~ qis $P' \to \mathrm{post}_{\le t}(X)$, and such that $\mathrm{post}_{\le t}(P) = \mathrm{post}_{\le t}(P')$. Thus

$$\mathrm{Hom}_{D(a)}(\mathrm{post}_{\le t}(X), Y) = \mathrm{Hom}_{K(a)}(\underbrace{\mathrm{post}_{\le t}(P')}_{=\mathrm{post}_{\le t}(P)}, Y)$$

showing that

$$(*) \quad \mathrm{Hom}_{D(a)}(X,Y) = \mathrm{Hom}_{D(a)}(\mathrm{post}_{\le t}(X), Y)$$

Now consider a chain complex P of projective R-modules and a complex Y of null modules supported in $[t, 0]$. To save writing abbreviate $\mathrm{post}_{\leq t}$ to $P_{\leq t}$. Let k be such that $I^k Y = 0$. Then

$$R^0\mathrm{Hom}_R(P, Y) = H^0\mathrm{Hom}_R(P, Y)$$

$$= H^0\,\mathrm{Hom}_{R/I^k}(P/I^kP, Y)$$

$$= R^0\mathrm{Hom}_{R/I^k}(P/I^kP, Y) \qquad P/I^kP \text{ is proj over } R/I^k$$

$$= R^0\mathrm{Hom}_{R/I^k}(P_{\leq t}(P/I^kP), Y) \qquad \text{by } \circledast \text{ on p 44}$$

Note incidently that

$$P_{\leq t}(P/I^kP) = P_{\leq t}(P)/I^k P_{\leq t}(P)$$

In degree t the former is $P_t/I^kP_t + \mathrm{Im}(P_{t+1})$ and the latter is $P_t/\mathrm{Im}(P_{t+1}) + I^kP_t$.

I haven't used that P is a chain complex, only that it is right bounded. So we really have the formula

$$R^0\mathrm{Hom}_R(X, Y) = R^0\mathrm{Hom}_{R/I^k}(P_{\leq t}(R/I^k \overset{L}{\otimes}_R X), Y)$$

for X right bounded and Y supported in degree $\leq t$ and killed by I^k. So

$$\boxed{R^0\mathrm{Hom}_R(X, Y) = \varinjlim_{t,k} R^0\mathrm{Hom}_{R/I^k}(P_{\leq t}(R/I^k \overset{L}{\otimes}_R X), Y)}$$

for any complex Y in $D^b(\mathrm{null})$.

This means the functor

$$\iota_*: D^b(\mathrm{null}) \longrightarrow D^b(R)$$

has a pro left adjoint. It would be better to say one has

$$\mathrm{pro}\, D^b(\mathrm{null}) \underset{\iota_*}{\overset{L\iota^*}{\rightleftarrows}} \mathrm{pro}\, D^b(R)$$

where $L\iota^*(X) = \left(P_{\leq t}(R/I^k \overset{L}{\otimes}_R X) \right)_{t,k}$

<u>October 10, 1994</u>

The first point to discuss is when is an inverse system $(X_n)_n$ in a category essentially constant, i.e. isomorphic to as pro-object to a constant inverse system $(X)_n$. This means one has an isomorphism

$$\varinjlim_n \mathrm{Hom}(X_n, Y) \simeq \mathrm{Hom}(X, Y)$$

of functors of Y. The map $\rightarrow$ is given (via Yoneda's lemma) by a compatible family of map $\varphi_n : X_n \longrightarrow X$. The map $\leftarrow$ is represented by a map $\psi : X_{n_0} \rightarrow X$ for some n_0. For thes composition $\circlearrowright$ to be the identity means $X \xrightarrow{\varphi_{n_0}} X_{n_0} \xrightarrow{\psi} X$ is the identity. For the composition $\circlearrowleft$ to be the identity means what? First without changing the pro object we can restrict n to be $\geq n_0$. Then we want the composition

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d, dotted] & X \arrow[r] \arrow[d, dotted, equal] & X_n \arrow[d, dotted] \\
X_{n_0+1} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & X_{n_0+1} \arrow[d] \\
X_{n_0} \arrow[r, "\psi"] & X \arrow[r] & X_{n_0}
\end{tikzcd}$$

to be the identity map as a map of pro-objects. This means $\forall n$ the composition $X_n \xrightarrow{\psi \pi^n_{n_0}} X \xrightarrow{\varphi_n} X_n$ becomes the same as the identity $X_n \rightarrow X_n$ when pulled back via $\pi^{n'}_n : X_{n'} \rightarrow X_n$ for some $n' \geq n$.

So for each $n \geq n_0$ we have a map $X_n \xrightarrow{\psi \pi^n_{n_0}} X \xrightarrow{\varphi_n} X_n$ which is idempotent: $\psi \pi^n_{n_0} \varphi_n = \psi \varphi_{n_0} = 1$ on X.

Moreover, denoting this idempotent e_n, the pair $e_n, 1_{X_n}$ are equalized by some $\pi^{n'}_n : X_{n'} \to X_n$, and hence for all sufficient large n'. In fact since we have $X \xrightarrow{\varphi_{n'}} X_{n'}$ such that $\pi^{n'}_n \varphi_{n'} = \varphi_n = \overbrace{(\varphi_n \psi \pi^n_{n_0})}^{e_n} \varphi_n$, we see that $\varphi_n : X \to X_n$ gives an isomorphism of X with the image of e_n which is also the kernel of the pair $1, e_n$.

Let's repeat this. Suppose we have an isomorphism of pro-objects

$$(X_n)_n \simeq (X)_n$$

The map $\leftarrow$ is given by $\varphi = (\varphi_n)$, $\varphi_n : X \to X_n$ compatible. The map $\to$ is represented by a map $\psi : X_{n_0} \to X$ for some n_0, more precisely it is given by

$$(X_n)_n \xrightarrow{p_{n_0}} X_{n_0} \xrightarrow{\psi} X$$

Then $X \xrightarrow{\varphi} (X_n)_n \xrightarrow{p_{n_0}} X_{n_0} \xrightarrow{\psi} X$ is 1_X.

Notice that we then have a projector $p_{n_0} \varphi \psi = \varphi_{n_0} \psi$ on X_{n_0}. We have also ~~an isom~~ that the map

$$(X_n)_n \xrightarrow{p_{n_0}} X_{n_0} \xrightarrow{\psi} X \xrightarrow{\varphi} (X_n)_n$$

is the identity. Thus $\psi p_{n_0}, \varphi$ are inverse isomorphisms of pro-objects.

Formally it follows that $X \xrightarrow{\varphi_{n_0}} X_{n_0}$ is the image of the projector $\varphi_{n_0}\psi$ on X_{n_0}.

Now this can be done at any level $n \geq n_0$ using $\psi_n = \psi p^n_{n_0} : X_n \longrightarrow X$. We have a projector $X_n \xrightarrow{\psi_n} X \xrightarrow{\varphi_n} X_n$ whose image is X since $\psi_n \varphi_n = \psi p^n_{n_0} \varphi_n = \psi \varphi_{n_0} = 1$.

Let's try to describe the picture without assume X exists. One is given an inverse system $(X_n)_n$ such that $(h^{X_n})_n$ is essentially constant. This implies for some n_0 we have a map $h^{X_{n_0}} \longrightarrow (h^{X_n})_n$, i.e. $X_{n_0} \xrightarrow{f} (X_n)_n$ such that $X_{n_0} \longrightarrow X_n \longrightarrow X_{n_0}$ is idempotent for all $n \geq n_0$. Also

October 13, 1994

Summary of pro derived category stuff.

1) $\varinjlim_m D^b(R/I^m) = D^b(nil)$ so

$$R^i \operatorname{Hom}_{nil}(Y, Y_1) = \varinjlim_m R^i \operatorname{Hom}_{R/I^m}(Y, Y_1)$$

2) $$R^0 \operatorname{Hom}_R(X, Y) = R^0 \operatorname{Hom}_{R/I^n}(R/I^n \overset{L}{\otimes}_R X, Y)$$ standard if $I^n Y = 0$

$$= \varinjlim_n R^0 \operatorname{Hom}_{R/I^n}(R/I^n \overset{L}{\otimes}_R X, Y)$$

$$= \varinjlim_{m \geq n} R^0 \operatorname{Hom}_{R/I^m}(R/I^n \overset{L}{\otimes}_R X, Y)$$ cofinal

$$= \varinjlim_n R^0 \operatorname{Hom}_{nil}(R/I^n \overset{L}{\otimes}_R X, Y)$$

shows $Li^*(X) \overset{def}{=} R/I^\infty \overset{L}{\otimes}_R X$ is a left-pro-adjoint for $i_*: D^b(nil) \longrightarrow D^b(R)$. (add here the inverse system of truncations)

3) $i_*: D^b(nil) \xrightarrow{\sim} D^b(R)_{nil}$ when I approx h-unital.

two arguments: Postnikov system for $X \in D^b(R)_{nil}$ and

$I^\infty \overset{L}{\otimes}_R X = 0 \Rightarrow X \simeq R/I^\infty \overset{L}{\otimes}_R X$ in pro $D^b(R)$

$\Rightarrow X$ [redacted] is a retract of $R/I^n \overset{L}{\otimes}_R X$ [redacted] in the cat $D^b(R)$ for some n. Then i_* fully faithful $\Rightarrow$ the corresp. projector on $R/I^n \overset{L}{\otimes}_R X$ comes from a projector [redacted] in $D^b(R)_{nil}$, which comes from a projector on $R/I^n \overset{L}{\otimes}_R X$ in $D^b(R/I^m)$ for some $m \geq n$. The image Y of this projector exists in $D^b(R/I^m)$, and then $i_* Y \simeq X$.

Next we construct $Lj_!$

4) Start with Miyachi's result

$$D^b(\mathcal{M}) = D^b(R) / D^b(R)_{nil}$$

consequences:

(i) $R^i \mathrm{Hom}_{\mathcal{M}}(j^*X, j^*X_1) = \varinjlim_{X'} R^i \mathrm{Hom}_R(X', X_1)$

when the limit is taken over all $X' \to X$ which are nil-quis (cone lies in $D^b(R)_{nil}$)

(ii) Assuming I approx. h-unital there is a functor $Lj_! : D^b(\mathcal{M}) \longrightarrow \text{pro } D^b(R)$ defined by

$$Lj_!(j^*X) = I^\infty \overset{L}{\otimes}_R X$$

5) $Lj_!$ is left pro-adjoint to j^*.

$$
\begin{array}{ccc}
R^i \mathrm{Hom}_R(Lj_!(j^*X), X_1) = \varinjlim_{n} R^i \mathrm{Hom}_R(I^n \overset{L}{\otimes}_R X, X_1) \\
\swarrow \simeq \quad \text{since } I^\infty \overset{L}{\otimes}_R (-) \text{ inverts nil-quis} \\
\varinjlim_{X'} \varinjlim_{n} R^i \mathrm{Hom}_R(I^n \overset{L}{\otimes}_R X', X_1) \\
\uparrow \simeq \quad \text{cofinality} \\
R^i \mathrm{Hom}_{\mathcal{M}}(X, X_1) = \varinjlim_{X'} R^i \mathrm{Hom}_R(X', X_1)
\end{array}
$$

October 20, 1994

Return to $\mathcal{A}$ abelian, $\mathcal{S}$ Serre subcategory and let's go over the result that

$$D(\mathcal{A})_{\mathcal{S}} \hookrightarrow D(\mathcal{A}) \twoheadrightarrow D(\mathcal{A}/\mathcal{S})$$

is exact.

Lemma 1. Let $F: \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories, $\mathcal{S} = $ "Ker"(F). Assume $\forall A$ in $\mathcal{A}$ and subobject $B' < F(A)$, $\exists$ a subobject $A' \subset A$ such that $F(A') = B'$. Then the induced functor $\bar{F}: \mathcal{A}/\mathcal{S} \to \mathcal{B}$ is fully faithful.

Proof. $\bar{F}$ is exact and "Ker"$\bar{F} = 0$, so $\bar{F}$ is faithful. (Any nonzero map f factors into a surjection onto a nonzero object followed by an injection; these are preserved by $\bar{F}$ so $\bar{F}(f) \neq 0$.)

To show essentially surjective, let $u: F(A_1) \to F(A_2)$ be a map, let $B' = \Gamma_u \subset F(A_1) \oplus F(A_2) = F(A_1 \oplus A_2)$. By hypothesis $\exists$ $A' \subset A_1 \oplus A_2$ such that $F(A') = \Gamma_u$. This implies that the projection $A' \to A_1$ becomes an isomorphism in $\mathcal{B}$. Thus $A_1 \xleftarrow{s} A' \longrightarrow A_2$ is a map from A_1 to A_2 in $\mathcal{A}/\mathcal{S}$ carried by $\bar{F}$ into u.

Lemma 2: $C(\mathcal{A}/\mathcal{S}) = C(\mathcal{A})/C(\mathcal{S})$.

The canonical functor $\mathcal{A} \xrightarrow{Q} \mathcal{A}/\mathcal{S}$ with "Ker" $= \mathcal{S}$ induces an exact functor $C(\mathcal{A}) \xrightarrow{Q} C(\mathcal{A}/\mathcal{S})$ with "kernel" $C(\mathcal{S})$. Let $X \in C(\mathcal{A})$ let Y be a subcomplex of $Q(X) \in C(\mathcal{A}/\mathcal{S})$. Lift each Y^n to $W^n \subset X^n$.

Put $X'^n = W^n + dW^{n-1}$. Then $Q(X'^n) = Y^n + dY^{n-1} = Y^n$ for all n, and $dX'^n = dW^n \subset X'^{n+1}$, so X' is a subcomplex of X such that $Q(X') = Y$.

By lemma 1 we know $\bar{Q} : C(\mathcal{A})/C(\mathcal{S}) \to C(\mathcal{A}/\mathcal{S})$ is fully-faithful. Let $Y \in C(\mathcal{A}/\mathcal{S})$ and observe that Y is a subcomplex of the cone on itself equipped with zero differential

$$\begin{tikzcd}
{} \arrow[r] & Y^{n-1} \arrow[r, "d"] \arrow[d, "\binom{1}{d}"] & Y^n \arrow[r, "d"] \arrow[d, "\binom{1}{d}"] & Y^{n+1} \arrow[r] \arrow[d, "\binom{1}{d}"] & {} \\
{} \arrow[r, "1"] & Y^{n-1}\oplus Y^n \arrow[r, "1"] & Y^n \oplus Y^{n+1} \arrow[r, "1"] & Y^{n+1}\oplus Y^{n+2} \arrow[r, "1"] & {}
\end{tikzcd}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ d \end{pmatrix} = \begin{pmatrix} d \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ d \end{pmatrix} d$$

Picking W^n such that $Q(W^n) = Y^n$ for all n, it's clear that Y is a subcomplex of $Q(W \oplus W[1])$ where W has $d = 0$. Thus by the above $Y = Q(X)$, where X is a subcomplex of $W \oplus W[1]$, so $\bar{Q}$ is essentially surjective.

I now want to check carefully that

$$D(\mathcal{A})/D(\mathcal{A})_{\mathcal{S}} \xrightarrow{\sim} D(\mathcal{A}/\mathcal{S})$$

We have the exact functor $\mathcal{A} \xrightarrow{j^*} \mathcal{A}/\mathcal{S}$ which induces $D(\mathcal{A}) \longrightarrow D(\mathcal{A}/\mathcal{S})$, a map of Δated categories, whose kernel is $D(\mathcal{A})_{\mathcal{S}}$. (The kernel consists of X in $D(\mathcal{A})$ such that $j^* X \simeq 0$ in $D(\mathcal{A}/\mathcal{S})$, which means $H_q(j^*X) = j^* H_q(X) = 0 \quad \forall q$, i.e. $H_q(X) \in \mathcal{S}$ for all q.) Thus there is an induced map of triangulated categories

$$D(\mathcal{A})/D(\mathcal{A})_{\mathcal{S}} \longrightarrow D(\mathcal{A}/\mathcal{S})$$

which will be an equivalence of Δated cats $\Longleftrightarrow$ it's an equivalence of cats.

The way to prove this is to show both categories are localizations of $C(\mathcal{A})$ with respect to the family of ~~$\mathcal{S}$qui~~, i.e. a map $f: X \longrightarrow X'$ in $C(\mathcal{A})$ such that $H_q(f)$ is an $\mathcal{S}$-isom. $\forall q$. Start with $D(\mathcal{A}/\mathcal{S})$. Clearly the canonical functor

$$C(\mathcal{A}) \longrightarrow C(\mathcal{A}/\mathcal{S}) \longrightarrow D(\mathcal{A}/\mathcal{S})$$

carries $\mathcal{S}$-quis in $C(\mathcal{A})$ into quis in $C(\mathcal{A}/\mathcal{S})$ which are carried into isos. in $D(\mathcal{A}/\mathcal{S})$.

Conversely suppose given $F: C(\mathcal{A}) \to \mathcal{C}$ inverting $\mathcal{S}$-quis. Let $f: X \to X'$ be a $C(\mathcal{S})$ isom. in $C(\mathcal{A})$. Factor f:

$$0 \longrightarrow \underset{C(\mathcal{S})}{\underset{\cap}{\mathrm{Ker}(f)}} \longrightarrow X \longrightarrow \mathrm{Im}(f) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Im}(f) \longrightarrow X' \longrightarrow \overset{C(\mathcal{S})}{\overset{\cup}{\mathrm{Coker}(f)}} \longrightarrow 0$$

Clearly $X \to \mathrm{Im}(f)$, $\mathrm{Im}(f) \to X'$ are $\mathcal{S}$-quis, hence so is f. Thus F inverts $C(\mathcal{S})$-isos, [redacted] hence descends to

$$C(\mathcal{A})/C(\mathcal{S}) \xrightarrow{\sim} C(\mathcal{A}/\mathcal{S})$$

[This is [redacted] not an isomorphism of categories because $C(\mathcal{A})$ and $C(\mathcal{A}/\mathcal{S})$ do not have the same objects[redacted]. [redacted] So instead we must argue 2-universally?)

To show F then descends to $D(\mathcal{A}/\mathcal{S})$ consider a quis $g: Y \to Y'$ in $C(\mathcal{A}/\mathcal{S})$. Up to isomorphism in $C(\mathcal{A}/\mathcal{S})$ [redacted] we can assume g comes from a map $f: X \to X'$ in $C(\mathcal{A})$, and then f must be an $\mathcal{S}$-quis since $H_q(j^*(f)) = j^* H_q(f) = 0$. This means that F inverts g. Thus F descends to $D(\mathcal{A}/\mathcal{S})$.

Next look at $D(\mathcal{A})/D(\mathcal{A})_{\mathcal{S}}$. Since $F: C(\mathcal{A}) \to \mathcal{C}$ inverts $\mathcal{S}$-quis, it inverts quis in $C(\mathcal{A})$ and hence descends to $D(\mathcal{A})$. To see it descends to $D(\mathcal{A})/D(\mathcal{A})_{\mathcal{S}}$ we must show it inverts any map $g: X \to X'$ in $D(\mathcal{A})$ whose cone has homology groups in $\mathcal{S}$. Up to isom in $D(\mathcal{A})$ we can suppose g lifts to a map $f: X \to X'$ in $C(\mathcal{A})$ which is necessarily an $\mathcal{S}$-quis, hence inverted by F.

What really happens with the equivalence

$$C(\mathcal{A})/C(\mathcal{S}) \xrightarrow{\sim} C(\mathcal{A}/\mathcal{S})$$

is that the former has complexes in $\mathcal{A}$ for its objects. Thus I should replace [redacted] $C(\mathcal{A}/\mathcal{S})$ and

$D(\mathcal{A}/\mathcal{S})$ by the equivalent categories having the same objects as $C(\mathcal{A})$. When this is done I should have an isomorphism of categories: $D(\mathcal{A})/D(\mathcal{A})_{\mathcal{S}} = D(\mathcal{A}/\mathcal{S})$ which are localizations of $C(\mathcal{A})$.

So now let us consider the problem of $Lj_!$. We have $j_! : \mathcal{A}/\mathcal{S} \longrightarrow \mathrm{Pro}(\mathcal{A})$ formal left adjoint for $j^* : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{S}$. Consider

$$\begin{array}{ccc} K(\mathcal{A}/\mathcal{S}) & \xrightarrow{j_!} & K(\mathcal{A}) \\ \downarrow{\scriptstyle Q_{\mathcal{A}/\mathcal{S}}} & & \downarrow{\scriptstyle Q_{\mathcal{A}}} \\ D(\mathcal{A}/\mathcal{S}) & \overset{Lj_!}{\dashrightarrow} & D(\mathcal{A}) \end{array}$$

By $Lj_!$ one means a dotted arrow above together with a universal map $Lj_! \circ Q_{\mathcal{A}/\mathcal{S}} \longrightarrow Q_{\mathcal{A}} \circ j_!$. The way you get $Lj_!$ is to use the formal left adjoint for $Q_{\mathcal{A}/\mathcal{S}}$ which is

$$Y \longmapsto \underset{\substack{Y' \xrightarrow[\text{Qis}]{} Y \\ \text{in } K(\mathcal{A}/\mathcal{S})}}{\text{“}\varprojlim\text{”}} Y' \qquad \text{then followed by } Q_{\mathcal{A}} \circ j_!$$

This gives

$$Lj_!(Y) = \underset{\substack{Y' \xrightarrow[\text{Qis}]{} Y \\ \text{in } K(\mathcal{A}/\mathcal{S})}}{\text{“}\varprojlim\text{”}} Q_{\mathcal{A}} j_! Y'$$

whence

$$R^0 Hom_a(Lj_!(Y), X_1)$$
$$= \varinjlim R^0 Hom_a(j_! Y', X_1)$$

limits taken over $Y' \xrightarrow[Qis]{} Y$ in $K(a/s)$

see p 58

$$= \varinjlim_{Y'} \varinjlim_{X_1'} H^0 Hom_{a/s}(Y', j^*X_1')$$

$X_1 \to X_1'$ Qis in K

$$= \varinjlim_{X_1'} R^0 Hom_{a/s}(Y, j^*X_1') = R^0 Hom_{a/s}(Y, j^*X_1).$$

Thus $Lj_!$ [redacted] defined this way is a formal left adjoint for $j^*: D(a) \longrightarrow D(a/s)$.

Now using $D(a/s) = D(a)/D(a)_s$ we earlier obtained another description of this adjoint:

$$* \qquad Lj_!(j^*X) = \text{"}\varprojlim_{\substack{X' \to X \\ s\text{-quis} \\ \text{in } D(a)}}\text{"} X'$$

namely:

$$R^0 Hom_a(Lj_!(j^*X), X_1) = \varinjlim R^0 Hom_a(X', X_1)$$
$$= R^0 Hom_{a/s}(j^*X, j^*X_1)$$

This effectively checks that $*$ coincides with the formal left derived functor of $j_!$

One problem is whether $K(a/s)$ is the quotient triangulated category of $K(a)$ by the kernel of $K(a) \longrightarrow K(a/s)$, i.e. complexes in a which become contractible in a/s. It seems that we have formal adjoints

$$K(a) \underset{Rj_*}{\overset{Lj_!}{\leftleftarrows\!\!\rightarrow}} K(a/s)$$

Actually you weren't careful about $j_!$ above, e.g. $j_! : a/s \longrightarrow \mathrm{Pro}\, a$ induces $j_! : K(a/s) \longrightarrow K(\mathrm{Pro}\, a)$, but is there a map $K(\mathrm{Pro}\, a) \longrightarrow \mathrm{Pro}\, K(a)$?

The good way to look at this issue is as follows. Recall $C(a/s) = C(a)/C(s)$ so that we have a formal adjoint $j_! : C(a/s) \longrightarrow \mathrm{pro}\, C(a)$ which sends Y to the category of (X, ξ), where $\xi : Y \xrightarrow{\sim} j^*(X)$ in $C(a/s)$. ??

---

correction to p57, more details actually. Start with the <u>definition</u>

$$Lj_!(Y) = \text{"}\varprojlim_{\substack{Y' \xrightarrow{\text{quis}} Y \\ \text{in } K(a/s)}}\text{"}\ Q_a j_! Y'$$

$X_1 \to X_1'$ quis in $K(a)$

Then

$$\mathrm{Hom}_{D(a/s)}(Y, j^*(X_1)) = \varinjlim_{X_1'} \mathrm{Hom}_{D(a/s)}(Y, j^*(X_1'))$$

$$= \varinjlim_{X_1'} \varinjlim_{Y'} \mathrm{Hom}_{K(a/s)}(Y', j^*(X_1'))$$

$$= \varinjlim_{Y'} \varinjlim_{X_1'} \mathrm{Hom}_{K(a)}(j_! Y', X_1')$$

$$= \varinjlim_{Y'} \mathrm{Hom}_{D(a)}(j_! Y', X_1) = \mathrm{Hom}_{D(a)}(Lj_!(Y), X_1)$$

October 21, 1994

Suppose A is a C^*-algebra. Can we identify the category $\mathcal{M}(A)$ with something simpler than the categories firm(A), sol(A), or nonf(A)? We know that A is flat as either left or right module over itself, and that $A = A^2$. Thus A is firm flat.

Is it possible that $AM = M \Rightarrow A \otimes_{\tilde{A}} M \xrightarrow{\sim} M$?

We know that $\operatorname{Tor}_1^{\tilde{A}}(\tilde{A}/A, M) = \operatorname{Ker}(A \otimes_{\tilde{A}} M \to AM)$ vanishes for all M iff $\tilde{A}/A$ is right $\tilde{A}$ flat iff A has local left identities. $A = C_0((0,1])$ does not have local left identitils, so for this A there exists a module M such that

$$0 \longrightarrow \underset{\substack{\shortparallel \\ 0}}{\operatorname{Tor}_1^A(\tilde{A}/A, M)} \longrightarrow A \otimes_A M \longrightarrow AM \longrightarrow 0$$

But then applying $A \otimes_{\tilde{A}} -$ we get

$$\begin{array}{ccc} A \otimes_{\tilde{A}} A \otimes_{\tilde{A}} M & \xrightarrow{\sim} & A \otimes_{\tilde{A}} AM \\ \simeq\downarrow & & \downarrow \\ A \otimes_{\tilde{A}} M & \longrightarrow & AM \end{array}$$

from which it follows that $\operatorname{Ker}(A \otimes_{\tilde{A}} AM \to AM) \neq 0$.

Example: Take $M = \tilde{A}/\tilde{A}f$, where $f = x$.

Then $\operatorname{Tor}_1^{\tilde{A}}(\tilde{A}/A, \tilde{A}/\tilde{A}f) = A \cap \tilde{A}f / A(\tilde{A}f) = \tilde{A}f/Af$.

Now $f \in Af \Rightarrow f = af \Rightarrow a = 1$ in this case. This contradiction shows $\operatorname{Tor}_1^A(\tilde{A}/A, \tilde{A}/\tilde{A}f) \neq 0$, so $A(\tilde{A}/\tilde{A}f) = A/\tilde{A}f$ is conil-free but not firm.

A is not usually solid since $\mathrm{Hom}_A(A, A)$ is the algebra of right multipliers, something like the functions on the Stone-Cech compactification.

Finally it seems that $\mathrm{ncuf}(A)$ is the best model for $\mathcal{M}(A)$. Note that A is nil-free because it has approximate identities.

October 23, 1994

Return to $\mathcal{A}$ abelian, $\mathcal{S}$ Serre subcategory, $j^*: \mathcal{A} \to \mathcal{A}/\mathcal{S}$ the quotient abelian cat. We have seen (p52) that $j^*: C(\mathcal{A}) \to C(\mathcal{A}/\mathcal{S})$ can be identified with the quotient $C(\mathcal{A}) \to C(\mathcal{A})/C(\mathcal{S})$ up to equivalence. Thus

$$\mathrm{Hom}_{C(\mathcal{A}/\mathcal{S})}(j^*X, j^*X_1)$$

$$= \varinjlim_{X' \to X \ C(\mathcal{S})\text{-isom}} \mathrm{Hom}_{C(\mathcal{A})}(X', X_1)$$

$$= \varinjlim_{\substack{X_1 \to X_1' \\ C(\mathcal{S})\text{-isom.}}} \mathrm{Hom}_{C(\mathcal{A})}(X, X_1')$$

which yields formal left & right adjoints

$$j_!(j^*X) = \underset{X' \to X \ C(\mathcal{S})\text{-isom.}}{\text{"}\varprojlim\text{"}} X'$$

$$j_*(j^*X) = \underset{X_1 \to X_1' \ C(\mathcal{S})\text{-isom}}{\text{"}\varinjlim\text{"}} X_1'$$

for $j^*: C(\mathcal{A}) \to C(\mathcal{A}/\mathcal{S})$.

Next I want to extend this to the level of $K = \mathrm{Ho}\,C$ categories. Recall that

$$\mathrm{Hom}_{C(\mathcal{A})}(X, X_1) = Z^0 \mathrm{Hom}^{\cdot}_{\mathcal{A}}(X, X_1).$$

Let ΓX be the obvious contractible complex mapping onto X:

$$
\begin{tikzcd}
\arrow[r] & X^n \oplus X^{n-1} \arrow[r, "\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}"] \arrow[d, "(1\ d)"'] & X^{n+1} \oplus X^n \arrow[r] \arrow[d, "(1\ d)"'] & {} \\
\arrow[r] & X^n \arrow[r, "d"] & X^{n+1} \arrow[r] & {}
\end{tikzcd}
$$

$$(1\ \ d)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = (d\ \ 0) = (1\ \ d)\,d.$$

Notice also that we have the autom

$$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} : \begin{matrix} X^n \\ \oplus \\ X^{n-1} \end{matrix} \longrightarrow \begin{matrix} X^n \\ \oplus \\ X^{n-1} \end{matrix}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad\qquad \begin{pmatrix} d & 0 \\ 1 & -d \end{pmatrix}$$

i.e. $\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & -d \end{pmatrix} = \begin{pmatrix} d & 0 \\ 1 & -d \end{pmatrix}$

so that ΓX is the h-fibre of the identity of X.

We have

$$Z^0 \operatorname{Hom}^0_a(W, \Gamma X) = \operatorname{Hom}^{-1}_a(W, X):$$

$$
\begin{tikzcd}
\arrow[r, "d"] & W^0 \arrow[r, "d"] \arrow[d, "\begin{pmatrix} gd \\ g \end{pmatrix}"] & W^1 \arrow[r, "d"] \arrow[d, "\begin{pmatrix} gd \\ g \end{pmatrix}"] & {} \\
\arrow[r] & X^0 \oplus X^{-1} \arrow[r, "\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}"] & X^1 \oplus X^0 \arrow[r] & {}
\end{tikzcd}
$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} gd \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ gd \end{pmatrix} = \begin{pmatrix} gd \\ g \end{pmatrix} d$$

such that $Z^0 \operatorname{Hom}^0_a(W, \Gamma X) \longrightarrow Z^0 \operatorname{Hom}^0_a(W, X)$

given by $\begin{pmatrix} gd \\ g \end{pmatrix} \longmapsto (1 \ \ d)\begin{pmatrix} gd \\ g \end{pmatrix} = [d,g]$

is $[d,-] : \mathrm{Hom}^{-1}_a(W,X) \longrightarrow Z^0\mathrm{Hom}^{\bullet}_a(W,X)$.

Combine the above:

$$\begin{tikzcd}
\varinjlim_{X'} Z^0\mathrm{Hom}^{\bullet}_a(X',X_1) \arrow[r, "\sim"] & Z^0\mathrm{Hom}^{\bullet}_{a/s}(j^*X, j^*X_1) \\
\varinjlim_{X'} Z^0\mathrm{Hom}^{\bullet}_a(X',\Gamma X_1) \arrow[r, "\sim"] \arrow[u, "{[redacted]}"'] \arrow[d, equal] & Z^0\mathrm{Hom}^{\bullet}_{a/s}(j^*X, \underbrace{j^*\Gamma X_1}_{\Gamma(j^*X_1)}) \arrow[u, "{[redacted]}"'] \arrow[d, equal] \\
\varinjlim_{X'} \mathrm{Hom}^{-1}_a(X',X_1) \arrow[r] \arrow[uu, bend left, dashed, "{[d,-]}"] & \mathrm{Hom}^{-1}_{a/s}(j^*X, j^*X_1)
\end{tikzcd}$$

where X' ranges over the filtering category of $C(s)$-quis $X' \to X$ in $C(a)$. Conclude

$$* \quad \boxed{\varinjlim_{\substack{X' \to X \\ C(s)\text{ quis in } C(a)}} \mathrm{Hom}_{K(a)}(X',X_1) \xrightarrow{\sim} \mathrm{Hom}_{K(a/s)}(j^*X, j^*X_1)}$$

It seems now that we can prove that $j^* : K(a) \longrightarrow K(a/s)$ [redacted] induces an equivalence of Δated cats $K(a)/\mathcal{T} \xrightarrow{\sim} K(a/s)$, where $\mathcal{T}$ is the thick subcategory "Ker j^*" of X such that $j^*X \simeq 0$ in $K(a/s)$. We have [redacted] by $*$ above a functor $K(a/s) \longrightarrow K(a)/\mathcal{T}$ induced by [redacted] the formal adjoint $j_! : C(a/s) \longrightarrow \text{pro } C(a)$ and the fact that any $C(s)$-isom in [redacted] $C(a)$ becomes a

$\mathcal{T}$-isomorphism in $K(\mathcal{A})$.

Let's check $K(\mathcal{A})/\mathcal{T} \longrightarrow K(\mathcal{A}/\mathcal{S})$ is fully-faithful:

$$\begin{tikzcd}
\overbrace{\varinjlim_{\substack{X' \to X \text{ in } K(\mathcal{A}) \\ \mathcal{T}\text{-isom.}}} \operatorname{Hom}_{K(\mathcal{A})}(X', X_1)}^{\operatorname{Hom}_{K(\mathcal{A})/\mathcal{T}}(X, X_1)} \arrow[r] & \operatorname{Hom}_{K(\mathcal{A}/\mathcal{S})}(j^*X, j^*X_1) \\
\varinjlim_{\substack{X'' \to X \text{ in } C(\mathcal{A}) \\ C(\mathcal{S})\text{-isom.}}} \operatorname{Hom}_{K(\mathcal{A})}(X'', X_1) \arrow[u] \arrow[ur, "\sim"] &
\end{tikzcd}$$

Take an element α of $\underbrace{\operatorname{Hom}_{K(\mathcal{A})/\mathcal{T}}(X, X_1)}_{\text{going to zero in } K(\mathcal{A}/\mathcal{S})}$, represent it by

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "\mathcal{T}\text{-isom}"] & X_1 \\
X &
\end{tikzcd}
\quad \text{such that} \quad
\begin{tikzcd}
j^*X' \arrow[r] \arrow[d, "\cong"] & j^*X_1 \\
j^*X \arrow[ur, "0"'] &
\end{tikzcd}$$

Thus $j^*X \longrightarrow j^*X_1$ is zero, so we know by * that there is [redacted] $X'' \longrightarrow X'$, a $C(\mathcal{S})$-isom in $C(\mathcal{A})$ such that $X'' \longrightarrow X' \longrightarrow X_1$ is zero in $K(\mathcal{A})$. Since $X'' \longrightarrow X'$ is a $\mathcal{T}$-isomorphism in $K(\mathcal{A})$, it follows that the original map α is zero.

<u>Conclusion</u>: $K(\mathcal{A}/\mathcal{S})$ is the ~~quotient~~ of $K(\mathcal{A})$ by the thick subcategory $\operatorname{Ker}(K(\mathcal{A}) \to K(\mathcal{A}/\mathcal{S}))$ consisting of complexes which become contractible in $\mathcal{A}/\mathcal{S}$.

October 26, 1994

$\mathcal{A}$, $\mathcal{S}$, $\mathcal{A}/\mathcal{S}$ in general

$\mathcal{T}$ the thick subcat $\mathrm{Ker}\left(j^*: K(\mathcal{A}) \to K(\mathcal{A}/\mathcal{S})\right)$.

We have seen that j^* induces an equivalence

$$* \qquad \boxed{K(\mathcal{A})/\mathcal{T} \xrightarrow{\sim} K(\mathcal{A}/\mathcal{S})}$$

Consequently

$$\underbrace{K(\mathcal{A})_{\mathcal{S}}}_{\substack{\text{means homology}\\ \text{in } \mathcal{S}}}/\mathcal{T} \xrightarrow{\sim} K(\mathcal{A}/\mathcal{S})_0$$

$$\Longrightarrow \quad K(\mathcal{A})/K(\mathcal{A})_{\mathcal{S}} \xrightarrow{\sim} K(\mathcal{A}/\mathcal{S})/K(\mathcal{A}/\mathcal{S})_0 = D(\mathcal{A}/\mathcal{S})$$

also

$$K(\mathcal{A})/K(\mathcal{A})_0 = D(\mathcal{A})$$

$$K(\mathcal{A})_{\mathcal{S}}/K(\mathcal{A})_0 = D(\mathcal{A})_{\mathcal{S}}$$

$$\Longrightarrow \quad K(\mathcal{A})/K(\mathcal{A})_{\mathcal{S}} \simeq D(\mathcal{A})/D(\mathcal{A})_{\mathcal{S}}$$

so again we obtain

$$D(\mathcal{A})/D(\mathcal{A})_{\mathcal{S}} \xrightarrow{\sim} D(\mathcal{A}/\mathcal{S})$$

What is the meaning of $*$? We've calculated that

$$\mathrm{Hom}_{K(\mathcal{A}/\mathcal{S})}(j^*X, j^*X_1) = \varinjlim_{\substack{X' \to X \ \ C(\mathcal{S})\text{-isom} \\ \text{in } C(\mathcal{A})}} \mathrm{Hom}_{K(\mathcal{A})}(X', X_1)$$

and we have (pretty much by defn.) that

$$\mathrm{Hom}_{K(\mathcal{A})/\mathcal{T}}(X, X_1) = \varinjlim_{\substack{X' \to X \ \ \mathcal{T}\text{-isom} \\ \text{in } K(\mathcal{A})}} \mathrm{Hom}_{K(\mathcal{A})}(X', X_1)$$

So the equivalence $*$ amounts to the functor

$$\left(\begin{matrix} X' \to X \ \ C(\mathcal{S})\text{-isom} \\ \text{in } C(\mathcal{a}) \end{matrix}\right) \longrightarrow \left(\begin{matrix} X' \to X \ \ \mathcal{T}\text{-isom} \\ \text{in } K(\mathcal{a}) \end{matrix}\right)$$

[redacted] between filtering categories being cofinal. Here's a direct check:

Let $W \to X$ be a $\mathcal{T}$-isom in $K(\mathcal{a})$. We have to construct a commutative diagram

$$\begin{tikzcd} & X' \arrow[dl] \arrow[d, "\bar{f}"] \\ W \arrow[r] & X \end{tikzcd}$$

in $K(\mathcal{a})$ where the map $X' \xrightarrow{\bar{f}} X$ comes from ~~an~~ $C(\mathcal{S})$-isomorphism $X' \xrightarrow{f} X$ in $C(\mathcal{a})$. Up to isom. in $K(\mathcal{a})$ we can assume $W \to X$ comes from an epimorphism $W \xrightarrow{g} X$ which is split in each degree. ~~We then have a triangle~~ ~~$W \xrightarrow{g} X \to$ in $K(\mathcal{a})$~~ Consider $j^*W \xrightarrow{j^* g} j^*X$ in $C(\mathcal{a}/\mathcal{S})$. This is an epimorphism which is split in each degree, so the kernel of $j^* g$ is heq to the h-fibre, which is contractible by the assumption that $W \to X$ is a $\mathcal{T}$-isom. Thus the obstruction to a section of $j^* g$ existing vanishes. Since $C(\mathcal{a}/\mathcal{S}) = C(\mathcal{a})/C(\mathcal{S})$ a section of $j^* g$ is represented by a diagram

$$\begin{tikzcd} & X' \arrow[dl] \arrow[d, "f"] \\ W \arrow[r, "g"] & X \end{tikzcd}$$

where f is a $C(\mathcal{S})$-isomorphism. This completes the proof.

Observation. A diagram

$$\begin{tikzcd} U' \arrow[d, phantom, "\cap" description] \arrow[ddrr, "f"] & & \\ U & & V \arrow[d, two heads] \\ & & V' \end{tikzcd}$$

is equivalent to a submodule $Z \subset U \oplus V$. In effect, given Z let $U' = pr_1(Z) \simeq Z+V/V$, let $V' = V/Z \cap V$, let f be the map

$$U' = {}^{Z+V}/_{V} \xleftarrow{\sim} Z/Z\cap V \xrightarrow{pr_2} V/Z\cap V = V'$$

In other words $f(u) = v$ (mod $Z \cap V$) iff $(u, v) \in Z$. Conversely given the diagram above let $Z = U' \times_{V'} V = \{(u,v) \mid f(u) = \text{image of } v\}$.

Another check: suppose U, V fd hermitian inner product vector spaces over $\mathbb{C}$. Then we have the Cayley transform picture of a subspace $Z \subset U \oplus V$ which is a unitary $g = F\varepsilon$ inverted by ε; here $\varepsilon = 1$ on U, $\varepsilon = -1$ on V and F is the involution $= 1$ on Z. Then the -1 eigenspace of g corresponds to the choice of U', V' and f to the other eigenvalues.

So now we can identify a map $j^*U \longrightarrow j^*V$ in A/S with an equivalence class of correspondences $Z \subset U \oplus V$ such that $pr_1 : Z \to U$ is an S-isomorphism.

To prove $\iota_* : D^b(S) \longrightarrow D^b(R)_S$ is fully faithful in the approx h-unital situation. The argument in the h-unital case I want to generalize goes as follows.

$$\begin{tikzcd}
R^0\mathrm{Hom}_R(Y, Y_1) \arrow[r, phantom, "="] & R^0\mathrm{Hom}_{R/I}(R/I \overset{L}{\otimes}_R Y, Y_1) \\
& R^0\mathrm{Hom}_{R/I}(Y, Y_1) \arrow[u, "\alpha^*"'] \arrow[ul, "\text{effect of } \iota_*"]
\end{tikzcd}$$

It suffices to show $\alpha : R/I \overset{L}{\otimes}_R Y \longrightarrow Y$ is an isomorphism in $D(R/I)$ and this we do by calculating homology, or what amounts to roughly the same thing, we apply ι_* (which is conservative by homology considerations), hence look at $R/I \overset{L}{\otimes}_R Y \to Y$ in $D(R)$. The point then is

$$\underbrace{I \overset{L}{\otimes}_R Y}_{\substack{0 \text{ by} \\ I \text{ h-unital} \Rightarrow}} \longrightarrow Y \underset{\iota_*\alpha}{\overset{\text{quis}}{\rightleftarrows}} R/I \overset{L}{\otimes}_R Y \longrightarrow$$

The generalization to the approximate h-unital case goes as follows. First in elementary terms. Choose a R-projective resolution $P \to Y$. Then

$$\begin{tikzcd}
R^*\mathrm{Hom}_R(Y, Y_1) \arrow[r, phantom, "="] & \varinjlim_n R^*\mathrm{Hom}_S(P/I^nP, Y_1) & \varinjlim_n \mathrm{Ext}^p_S(H_q(P/I^nP), Y_1) \arrow[l, Rightarrow] \\
& R^*\mathrm{Hom}_S(Y, Y_1) \arrow[u] \arrow[ul] & \varinjlim_n \mathrm{Ext}^p_S(H_q(Y), Y_1) \arrow[l, Rightarrow] \arrow[u]
\end{tikzcd}$$

so it suffices to see that $H_q(P/I^\infty P) = H_q(R/I^\infty \overset{L}{\otimes}_R Y)$ is isom. to $H_q(Y)$ for all q, and this comes from $H_q(I^\infty \overset{L}{\otimes}_R Y) = 0$ by approximate h-unitality of I.

October 29, 1994

Transitivity result. Given ideals $I \subset J \subset R$ in R unital we have seen that there is an exact sequence of abelian categories

$$\frac{nil(R,I)}{nil(R,J)} \hookrightarrow \mathcal{M}(R,J) \twoheadrightarrow \mathcal{M}(R,I)$$
$$\|$$
$$\bigcup_n \mathcal{M}(R/I^n, \; J/I^n)$$

I wish to show now that if I is approx. h-unital then there is an exact sequence of Δated cats

$$\begin{array}{ccccc} \varinjlim_n D^b(\mathcal{M}(R/I^n, J/I^n)) & \hookrightarrow & D^b(\mathcal{M}(R,J)) & \twoheadrightarrow & D^b(\mathcal{M}(R,I)) \\ \| & & \| & & \| \\ D^b\left(\frac{nil(R,I)}{nil(R,J)}\right) & & D^b(R)/D^b(R)_{nil(R,J)} & & D^b(R)/D^b(R)_{nil(R,I)} \end{array}$$

Notice that $D^b(R)_{nil(R,J)} \subset D^b(R)_{nil(R,I)}$ are thick subcategories of $D^b(R)$ with quotients $D^b(\mathcal{M}(R,J))$, $D^b(\mathcal{M}(R,I))$ resp. Veridier proves transitivity (SGA $4\frac{1}{2}$ C.D. 1.2.4.3), that one has an exact sequence

$$\frac{D^b(R)_{nil(R,I)}}{D^b(R)_{nil(R,J)}} \hookrightarrow D^b(\mathcal{M}(R,J)) \twoheadrightarrow D^b(\mathcal{M}(R,I))$$

On the other hand I approx h-unital $\Longrightarrow$ one has an equivalence

$$D^b(nil(R,I)) \xrightarrow{\sim} D^b(R)_{nil(R,I)}$$

What is the thick subcategory $\mathcal{T} \subset D^b(nil(R,I))$

corresponding under this equivalence to $D^b(R)_{nil(R,J)} \subset D^b(R)_{nil(R,I)}$?

It suffices to identify $\mathcal{T}$ with $D^b(nil(R,I))_{nil(R,J)} \subset D^b(nil(R,I))$, for then we have

$$\begin{array}{ccc} \dfrac{D^b(nil(R,I))}{D^b(nil(R,I))_{nil(R,J)}} & \xrightarrow{\sim} & \dfrac{D^b(R)_{nil(R,I)}}{D^b(R)_{nil(R,J)}} \\ \wr\| & & \\ D^b\left(\dfrac{nil(R,I)}{nil(R,J)}\right). & & \end{array}$$

Let $X \in \mathcal{T}$, i.e. X is a ~~bounded~~ complex of $nil(R,I)$-modules whose image in $D^b(R)$ has homology groups in $nil(R,J)$. Thus $X \in D^b(nil(R,I))_{nil(R,J)}$ by definition of the latter, and conversely. $\square$.