July 16, 1994

Let $\mathcal{A}$ be a small abelian category and consider

$$\underset{\text{left exact functors}}{Lex(\mathcal{A}, \mathcal{Ab})} \hookrightarrow \underset{\text{additive functors}}{Add(\mathcal{A}, \mathcal{Ab})}$$

According to Gabriel $Lex(\mathcal{A}, \mathcal{Ab})$ is an abelian category, and $F \mapsto R^0F$ is left adjoint to the above inclusion.

The observation is that this fits into the framework of localizing subcategories of Grothendieck categories. Call $F : \mathcal{A} \to \mathcal{Ab}$ effaceable if given $\xi \in F(M)$, $\exists\ M \overset{i}{\hookrightarrow} N$ st ξ is killed by i. The effaceable functors form a Serre subcategory of $Add(\mathcal{A}, \mathcal{Ab})$ closed under direct sums.

Let $M \subset N$ in $\mathcal{A}$. Then we have in $Add(\mathcal{A}, \mathcal{Ab})$ an exact sequence

$$* \qquad 0 \to h^{N/M} \to h^N \to h^M \to h^M/\operatorname{Im} h^N \to 0$$

where $h^M/\operatorname{Im} h^N$ is effaceable. Indeed, let $\xi \in (h^M/\operatorname{Im} h^N)(X) = \operatorname{Hom}(M, X)/\operatorname{Im} \operatorname{Hom}(N, X)$. Define $X \hookrightarrow Y$ by pushout

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d, "\xi"'] & N \arrow[d] \\ X \arrow[r, hook] & Y \end{tikzcd}$$

Then ξ goes to zero in $(h^M/\operatorname{Im} h^N)(Y)$.

Let E be an injective functor, which is effaceable-free. Then applying $\mathrm{Hom}(-,E)$ to $*$ we get

$$0 \to E(M) \to E(N) \to E(N/M) \to 0$$

showing that E is exact.

It then follows that the kernel of any map between injective effaceable-free functor is left exact.

Conversely given F left exact, it's effaceable-free (this is equivalent to preserving monos), so there is a monic $F \hookrightarrow E$ with E injective effaceable-free. Then one checks E/F is effaceable free, so it follows that F is the kernel of a map between injective effaceable-free functors.

Some remaining points to be clarified:

Why is the localization functor given by

$$R^0F(M) = \varinjlim_{M\subset N} \mathrm{Ker}\big(F(N) \to F(N/M)\big) \;?$$

Is there any link of this to the fact that

$$\mathrm{Lex}(\mathcal{A}, \mathcal{A}b) = \mathrm{Pro}\,\mathcal{A}\;?$$

i.e. the left exact functors are of the form $F(M) = \varinjlim \mathrm{Hom}(X_\alpha, M)$ for some filtered inverse system $\{X_\alpha\}$ in $\mathcal{A}$.

Consider now R, I with R noetherian commutative. Then we have coherent sheaves on $\mathcal{U} = Sp(R) - Sp(R/I)$

$$\begin{tikzcd}
\dfrac{fg\,mod(R)}{fg\,null(R,I)} \arrow[r, hook] \arrow[dr, hook] & \dfrac{mod(R)}{null(R,I)} \arrow[d, two heads] \\
& \dfrac{mod(R)}{tors(R,I)}
\end{tikzcd}$$

call this $\mathcal{A}$

then A is the ind category $Lex(\mathcal{A}^{op}, \mathcal{Ab})$ of $\mathcal{A}$

quasi-coherent sheaves on $\mathcal{U}$

Some natural questions

1. exactness of embedding

$$\frac{mod(R)}{tors(R,I)} \simeq \mathcal{C}ofirm(R,I) \hookrightarrow \frac{mod(R)}{null(R,I)}$$

also the restriction to $\dfrac{fg\,mod(R)}{fg\,null(R,I)}$

2. We have an embedding

$$firm(R,I) \hookrightarrow Lex\left(\frac{mod(R)}{null(R,I)}, \mathcal{Ab}\right)$$

$$M \longmapsto Hom(M, -).$$

Does the fact that Lex is abelian (mod set theory) help?

Observation: If M firm, or more generally $M = IM$, then any quotient N of M which is fin gen. also satisfies $N = IN$ and this means $\exists\, a \in A$ such that $(1-a)N = 0$. The support of N is closed in $Sp(R)$ and disjoint from $Sp(R/I)$. Not very interesting.

It seems unfruitful to consider $Hom_R(M, -)$ for M firm. In the noetherian comm. situation the thing to consider is $\mathcal{A} = \dfrac{fg\,mod(R)}{fg\,null(R,I)}$ its ind category $\dfrac{mod(R)}{I\text{-tors}}$ and its pro category chez Deligne.

Suppose A left ideal in R unital. Then we have the Morita context $\begin{pmatrix} \tilde{A} & R \\ A & R \end{pmatrix}$ which leads to the equivalence

$$\begin{array}{ccc} M & \longmapsto & A \otimes_{\tilde{A}} M \\ N = R \otimes_R N & \longleftarrow & N \end{array}$$ for firm modules

the equivalence

$$\begin{array}{ccc} M & \longmapsto & \mathrm{Hom}_{\tilde{A}}(R, M) \\ \mathrm{Hom}_R(A, N) & \longleftarrow & N \end{array}$$ for solid modules

the equivalence
$$M \longmapsto \mathrm{Im}\left\{ A \otimes_{\tilde{A}} M \longrightarrow \mathrm{Hom}_{\tilde{A}}(R, M) \right\}$$
for null + conull free modules.

I thought it should be true that M simple non null $\Longrightarrow$ the corresponding n+cnf R-module N is naturally isomorphic to M and hence also a simple R-module. This is true when A is an ideal (see p 587) but the argument breaks down if A is only a left ideal. Let's go over the argument.

Let $M \in \mathrm{mod}(\tilde{A}) \ni M/AM = {}_A M = 0$. The corresponding null + conull-free R-module wrt the ideal AR is the image of φ in:

$$\varphi(a \otimes m)(r) = (ra)m$$

$$\begin{tikzcd} & 0 \arrow[d] \\ & \mathrm{Hom}_{\tilde{A}}(R/A, M) \arrow[d] \\ A \otimes_{\tilde{A}} M \arrow[r, "\varphi"] \arrow[d, "\text{because } M/AM=0"'] & \mathrm{Hom}_{\tilde{A}}(R, M) \arrow[d] \\ M \arrow[r, hook] & \mathrm{Hom}_{\tilde{A}}(A, M) \end{tikzcd}$$

because ${}_A M = 0$

φ is an R-module map, the other maps are only $\tilde{A}$-module maps in general. When A is an ideal however $A(R/A) = 0$ so that $\mathrm{Hom}_{\tilde{A}}(R/A, M) = 0$ and we can identify the image of φ with M itself.

In fact we can use the Morita context $\begin{pmatrix} \tilde{A} & A \\ A & R \end{pmatrix}$.

Let's shift notation to the ideal $I \subset R$. Then we have the Morita context $\begin{pmatrix} \tilde{I} & I \\ I & R \end{pmatrix}$ with $QP = I^2 \subset \tilde{I}$ and $PQ = I^2 \subset R$. Thus we have equivalences of categories

$$\mathrm{firm}(\tilde{I}, I) \xrightarrow{\sim} \mathrm{firm}(R, I)$$
$$M \longmapsto I \otimes_{\tilde{I}} M \simeq M$$
$$N \simeq I \otimes_R N \longleftarrow N$$

The second functor is restriction of scalars for $\tilde{I} \to R$.

$$\mathrm{solid}(\tilde{I}, I) \xrightarrow{\sim} \mathrm{solid}(R, I)$$
$$M \longmapsto \mathrm{Hom}_{\tilde{I}}(I, M) \simeq M$$
$$N \simeq \mathrm{Hom}_R(I, N) \longleftarrow N$$

Again the second functor is restriction of scalars.

$$\mathrm{nfcuf}(\tilde{I}, I) \xrightarrow{\sim} \mathrm{nfcuf}(R, I)$$

$$M \longmapsto \mathrm{Im}\left(\begin{tikzcd} I \otimes_{\tilde{I}} M \arrow[rr, "\varphi"] \arrow[dr] & & \mathrm{Hom}_{\tilde{I}}(I, M) \\ & M \arrow[ur] & \end{tikzcd} \right)$$

$$\mathrm{Im}\left(\begin{tikzcd} I \otimes_R N \arrow[rr] \arrow[dr] & & \mathrm{Hom}_R(I, N) \\ & N \arrow[ur] & \end{tikzcd} \right) \longleftarrow N$$

More on Morita invariance examples

The problem with $A \subset B$, $ABA \subset A^2$, $B = \tilde{B}A\tilde{B}$ is that this is not the appropriate thing to reduce to.

Suppose A is a subring of R unital such that $ARA \subset A^2$, e.g. $RA \subset A$ or $AR \subset A$. Then one has the Morita context

$$\begin{pmatrix} \tilde{A} & AR \\ RA & R \end{pmatrix} \qquad \begin{aligned} QP &= ARA = A^2 \\ PQ &= RA^2R = (RAR)^2 \end{aligned}$$

whence a Morita equivalence $A \sim RAR$.

If you want to do this non-unitally, working with $B = RAR$, then possible Morita contexts are

$$\begin{pmatrix} A & AB \\ BA & B \end{pmatrix} \subset \begin{pmatrix} A & A\tilde{B} \\ \tilde{B}A & B \end{pmatrix}$$

Ideals in the smaller are

$$QP = AB^2A = A(\cancel{R}A\cancel{R})(RA\cancel{R})A = A^4$$

$$PQ = BA^2B = (RA\cancel{R})A^2(\cancel{R}AR) = RA^4R = B^4$$

Ideals in the larger are

$$QP = A\tilde{B}A = A^2$$
$$PQ = \tilde{B}A^2\tilde{B} = (\tilde{B}A\tilde{B})^2$$

But notice that
$$\begin{aligned} \tilde{B}A\tilde{B} &= A + BA + AB + BAB \\ &= A + RA\cancel{R}A + A\cancel{R}AR + RA\cancel{R}A\cancel{R}AR \end{aligned}$$

so $\tilde{B}A\tilde{B} \supset B^3$. Thus you don't get the situation $B = \tilde{B}A\tilde{B}$.

Idea. Let P be a projective R-module. Then one gets a Jansian torsion theory where the torsion modules are $M \ni \mathrm{Hom}_R(P,M)=0$. Choose generators $p_i \in P$, $\iota \in \Lambda$, whence a surjection $R^{(\Lambda)} \longrightarrow P$, and choose a lifting

$$P \hookrightarrow R^{(\Lambda)}, \quad p \longmapsto (f_i(p)) \quad \text{with } f_i \in \mathrm{Hom}_R(P,R).$$

Then
$$p = \sum_{\Lambda} f_i(p)\, p_i \quad \text{and} \quad f = \sum_{\Lambda} f_i\, f(p_i)$$
so the f_i generator the dual $\mathrm{Hom}_R(P,R)$.

If M is an R-module we have

$$\mathrm{Hom}_R(P,M) \underset{\hookrightarrow}{\twoheadleftarrow} \mathrm{Hom}_R(R^{(\Lambda)},M) = M^{(\Lambda)}$$

$$p \mapsto \sum f_i(p)\, m_i \longleftarrow (m_i)$$

so
$$\mathrm{Hom}_R(P,M) = 0 \iff \Big(\sum_{\Lambda} f_i(P)\Big) M = 0.$$

Thus
$$I = \sum f_i(P) = \sum_{\text{all } f} f(P) = \mathrm{Im}\Big\{ P \otimes_{\mathbb{Z}} \mathrm{Hom}_R(P,R) \longrightarrow R \Big\}$$
also called $tr(P)$

is the idempotent ideal corresponding to this torsion theory. (It's idempotent because $f(p) = \sum_{\Lambda} f_i(p) f(p_i) \Rightarrow I \subset I^2$.)

You need to look up the Bergman theorem in Golan's book, which says under some countability hypotheses that an idempotent ideal is the trace of a projective module. (see p733) Notice that $p = \sum_{\Lambda} f_i(p)\, p_i \Longrightarrow P = IP$ so P is a firm projective module. This means that $I \not\subset \mathrm{Jac}(R)$ otherwise one has a contradiction of the (generalized) Kaplansky theorem. Recall Kaplansky's thm.

says any projective module over a local ring is free, hence it's trace is the whole ring if the module is $\neq 0$. So Bergman's thm. must have $I \not\subset Jac(R)$ as hypothesis.

$$\mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(X, Q)) = \mathrm{Hom}_{\mathbb{Z}}(X \otimes_R M, Q)$$

so

X R^{op} flat, Q $\mathbb{Z}$-injective $\Rightarrow$ $\mathrm{Hom}_{\mathbb{Z}}(X,Q)$ R-injective

$$\mathrm{Hom}_R(R/I, \mathrm{Hom}_{\mathbb{Z}}(X, Q)) = \mathrm{Hom}_{\mathbb{Z}}(X/XI, Q)$$

so $X = XI \Longrightarrow \mathrm{Hom}_{\mathbb{Z}}(X,Q)$ torsion-free

Thus X firm flat, Q $\mathbb{Z}$-inj $\Rightarrow$ $\mathrm{Hom}_{\mathbb{Z}}(X,Q)$ solid inj

Suppose M torsion-free R-module, let $0 \neq m \in M$. There exists a sequence $\alpha = (a_n)$ in I such that $F(\alpha) \xrightarrow{1 \otimes m} F(\alpha) \otimes_R M$ is $\neq 0$, hence $\exists$ $F(\alpha) \otimes_R M \longrightarrow \mathbb{Q}/\mathbb{Z}$ s.t. the composition with $1 \otimes m$ is $\neq 0$. Thus we have an R-module map $M \longrightarrow \mathrm{Hom}_R(F(\alpha), \mathbb{Q}/\mathbb{Z})$ s.t. $m \not\mapsto 0$. This means that there are enough solid injectives of the form $\mathrm{Hom}_R(F(\alpha), \mathbb{Q}/\mathbb{Z})$.

Review the adjoint functors

$$\mathrm{tors}(R,I) \underset{\iota^!}{\overset{\iota_*}{\rightleftarrows}} \mathrm{mod}(R) \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathrm{mod}(R)/\mathrm{tors}(R,I)$$

Here ι_* is the inclusion, j^* the canonical functor to the quotient category. $\iota^!(M)$ is the (largest) torsion submodule of M; it exists because $\mathrm{tors}(R,I)$ is a Serre subcategory closed under $\oplus$'s. This also implies $M/i^!(M)$ is torsion-free.

Let's construct j_*, i.e. for each module N we will produce a map $N \to j_* N$ s.t.

$$\mathrm{Hom}_{\mathcal{M}_t}(M, N) \simeq \mathrm{Hom}_R(M, j_* N) \quad \forall M$$

Here $\mathcal{M}_t = \mathrm{mod}(R)/\mathrm{tors}(R,I)$.

Recall that

1. An injective module Q is solid iff it is torsion-free:

$$0 \to \underset{\substack{\| \\ {}_I Q}}{\mathrm{Hom}_R(R/I, Q)} \longrightarrow \underset{\substack{\| \\ Q}}{\mathrm{Hom}_R(R, Q)} \longrightarrow \mathrm{Hom}_R(I, Q) \to 0$$

2. The injective hull of a torsion-free module is torsion-free, hence a module is torsion-free iff it can be embedded in a solid injective.

3. A module is solid iff it is the kernel of a map between solid injectives.

Now observe that if Q is a solid injective, then $\mathrm{Hom}_R(-, Q)$ from $\mathrm{mod}(R)$ to $\mathcal{Ab}$ is exact and it kills $\mathrm{tors}(R,I)$, thus it descends to the quotient category. Using

$$\mathrm{Hom}_{\mathcal{M}_t}(M, Q) = \varinjlim \mathrm{Hom}_R(M', Q)$$

where the limit is taken over the cat of arrows $M' \xrightarrow{s} M$ which have target M and which are torsion isoms., we obtain

$$\operatorname{Hom}_{\mathcal{M}_t}(M,Q) \xleftarrow{\sim} \operatorname{Hom}_R(M,Q)$$

If N is solid, then choosing a copresentation

$$0 \longrightarrow N \longrightarrow Q^0 \longrightarrow Q^1$$

with Q^i solid injective, we see the above implies

$$\operatorname{Hom}_{\mathcal{M}_t}(M,N) \xleftarrow{\sim} \operatorname{Hom}_R(M,N)$$

Now if N is arbitrary we construct a resolution modulo torsion by solid injectives.

$$0 \longrightarrow N/i^!N \longrightarrow Q^0 \longrightarrow N^1 \longrightarrow 0$$

$$0 \longrightarrow N^1/i^!N^1 \longrightarrow Q^1 \longrightarrow N^2 \longrightarrow 0.$$

~~so that~~

$$\cancel{0 \longrightarrow N \longrightarrow Q^0 \longrightarrow Q^1}$$

~~is exact mod tors(R,I). Let j_*N be the kernel of $Q^0 \longrightarrow Q^1$, whence we have a map $N \longrightarrow j_*N$~~

and define j_*N to be the kernel of $Q^0 \longrightarrow Q^1$.
Then j_*N is solid.
There is an obvious map $N \longrightarrow j_*N$ whose kernel is $i^!N$ and whose cokernel is $i^!N^1$:

$$\begin{tikzcd}
 & & & i^!N^1 \arrow[d] & \\
0 \arrow[r] & N/i^!N \arrow[r] \arrow[d] & Q^0 \arrow[r] \arrow[d, equal] & N^1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & j_*N \arrow[r] & Q^0 \arrow[r] & Q^1 &
\end{tikzcd}$$

Thus $N \longrightarrow j_*N$ is an isom mod torsion where

$j_* N$ is solid. So we have

$$\mathrm{Hom}_{\mathcal{M}_t}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{M}_t}(M, j_* N) \cong \mathrm{Hom}_R(M, j_* N)$$

(see Golan's book cotorsion free chapter)

Bergman's thm. says in a ring such that every countable ~~left~~ ideal is projective that any idempotent ideal is the trace of a projective module.

July 18, 1994

From Prest's book

A *preradical* is a subfunctor τ of the identity functor on modules. Define

$\mathcal{T}_\tau = \{M \mid \tau M = M\}$ $\quad$ τ-torsion

$\mathcal{F}_\tau = \{M \mid \tau M = 0\}$. $\quad$ τ-torsion-free

Then $\mathcal{T}_\tau$ is closed under quotients and $\oplus$'s:

$$\begin{tikzcd} M \arrow[r, two heads] \arrow[d, equal] & M'' \arrow[d, phantom, "\cup"] \\ \tau M \arrow[r] & \tau M'' \end{tikzcd} \Rightarrow \tau M'' = M'' \qquad \begin{tikzcd} \oplus \tau M_i \arrow[r, equal] \arrow[dr] & \oplus M_i \arrow[d, phantom, "\cup"] \\ & \tau(\oplus M_i) \end{tikzcd} \text{ must be } =$$

Dually $\mathcal{F}_\tau$ is closed under subobjects and Π's

Conversely suppose given a class $\mathcal{T}$ of modules closed under quotients and $\oplus$'s. Define

$$\tau_{\mathcal{T}} M = \sum_{\substack{(N,f) \\ N \in \mathcal{T}}} \operatorname{Im}(f : N \to M) \subset M$$

Note that $\tau_{\mathcal{T}} M$ is a quotient of a direct sum of modules in $\mathcal{T}$ so $\tau_{\mathcal{T}} M \in \mathcal{T}$. Clearly $\tau_{\mathcal{T}}$ is a subfunctor of the identity, i.e. a preradical. $\tau_{\mathcal{T}} M$ is the largest submodule of M belonging to $\mathcal{T}$. It's clear also that

$$\tau_{\mathcal{T}_\tau} M = \sum_{\substack{N,f \\ \tau N = N}} \operatorname{Im}(N \xrightarrow{f} M) \subset \tau M$$

$$M \in \mathcal{T}_{\tau_{\mathcal{T}}} \iff \tau_{\mathcal{T}} M = M \iff M \in \mathcal{T}$$

Dually given a class $\mathcal{F}$ closed under subobjects and quotients put

$$\tau_{\mathcal{F}} M = \bigcap_{\substack{(N,f) \\ N \in \mathcal{F}}} \operatorname{Ker}\{M \xrightarrow{f} N\}$$

This is a subfunctor of the identity.

Note that $M/\tau_{\mathcal{F}} M$ embeds in a direct product of members of $\mathcal{T}$, hence $M/\tau_{\mathcal{F}} M \in \mathcal{F}$. $\tau_{\mathcal{F}} M$ is the smallest submodule of M such that the quotient belongs to $\mathcal{F}$.

$$\tau_{\mathcal{F}_\tau} M = \bigcap_{\substack{(N,f) \\ \tau N = 0}} \operatorname{Ker}\{M \xrightarrow{f} N\} \supset \tau M$$

$$M \in \mathcal{F}_{\tau_{\mathcal{F}}} \iff \tau_{\mathcal{F}} M = 0 \iff M \in \mathcal{F}$$

A preradical τ is a <u>radical</u> when $\tau(M/\tau M) = 0$

A preradical τ is <u>idempotent</u> when $\tau(\tau M) = \tau M$.

If τ is a radical then $\mathcal{T}_\tau$ is closed under extensions:

Given $0 \to M' \to M \to M'' \to 0$ with $\tau M' = M'$ and $\tau M'' = M''$, then $M' = \tau M' \subset \tau M$, so $M'' = M/M'$ maps onto $M/\tau M$. But $\tau M'' = M''$ and $\tau(M/\tau M) = 0$ imply the map $M'' \to M/\tau M$ is zero, hence $M = \tau M$.

Conversely suppose τ is a preradical such that $\mathcal{T}_\tau$ is closed under extensions. Given M, let M' be the inverse image in M of $\tau(M/\tau M)$. We then have an extension

$$0 \longrightarrow \tau M \longrightarrow M' \longrightarrow \tau(M/\tau M) \longrightarrow 0$$

If τ is idempotent, then τM and $\tau(M/\tau M) \in \mathcal{T}_\tau$, so $M' \in \mathcal{T}_\tau$, i.e. $\tau M' = M'$. Then $M' = \tau M' \subset \tau M$ implies $M' = \tau M$ so that $\tau(M/\tau M) = 0$.

Thus the converse isn't so clear.

If τ is an idempotent preradical, then $\mathcal{F}_\tau$ is closed under extensions.

Given $0 \to M' \to M \to M'' \to 0$ with $\tau M'' = 0 = \tau M'$. Then τM goes to zero in M'', so $\tau M \subset M'$ and $\tau M = \tau\tau M \subset \tau M' = 0$.

I guess the good situation is when τ is an idempotent radical. Then

$\mathcal{T}_\tau$ is closed under quotients, extensions, $\oplus$'s

$\mathcal{F}_\tau$ ——————— subobjects, extensions, Π's

and probably the three things $\tau, \mathcal{T}_\tau, \mathcal{F}_\tau$ are equivalent.

Also $\mathcal{T}_\tau$ is closed under subobjects $\iff$ $\mathcal{F}_\tau$ is closed under injective hulls $\iff$ τ is left exact.

Now suppose τ, τ' are idempotent radicals such that $\mathcal{T}_\tau = \mathcal{F}_{\tau'}$. This is called TTF, a torsion torsionfree theory. Then $\mathcal{T}_\tau$ is a Serre subcategory closed under products, so it's a jansian torsion theory: there is a unique idempotent ideal I in R such that

$$\mathcal{T}_\tau = \{M \mid IM = 0\}.$$

$$\tau M = \mathrm{Hom}_R(R/I, M) = {}_I M$$

Now what is τ'? $\tau' M$ should be the kernel of all maps from M to an I-null module, hence $\tau' M = IM$. Thus $\mathcal{T}_{\tau'} = \{M \mid M = IM\}$. This is not closed under submodules in general,* however it is closed under $\oplus$'s, quotients, and extensions:

$$\begin{tikzcd}
 & I \otimes_R M' \arrow[r] \arrow[d, two heads] & I \otimes_R M \arrow[r] \arrow[d] & I \otimes_R M'' \arrow[r] \arrow[d, two heads] & 0 \arrow[d] \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow I \otimes_R M \twoheadrightarrow M.$

What I missed:

equivalence between

$\mathcal{T}$ closed under quotients, $\oplus$'s and idempotent preradicals

$\mathcal{F}$ closed under subobjects, $\prod$'s and radicals.

Also in the situation $\mathcal{T}_\tau = \mathcal{F}_{\tau'}$, the ~~idempotent~~ radical $\tau' M = IM$ commutes with filtered limits, hence τ' is a finite type torsion theory (i.e. if $R/\mathfrak{a}$ is torsion: $I(R/\mathfrak{a}) = R/\mathfrak{a}$ with $\mathfrak{a}$ a left ideal then the same is true for some finitely generated left ideal $\subset \mathfrak{a}$. This is obvious: $IR + \mathfrak{a} = R \Leftrightarrow \exists a \in \mathfrak{a}: 1 - a \in IR$ $\Rightarrow I(R/Ra) = R/Ra$.

* see November 22, 1997

July 17, 1994

Question: Suppose M conullfree: $M = IM$ and N is nullfree: ${}_I N = 0$; (nullfree = torsionfree). Is $\mathrm{Hom}_R(M,N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{M}_t}(M,N)$?

No. Take $(R,I) = (\mathbb{Z}, p\mathbb{Z})$ whence $\mathcal{M}_t = \mathrm{mod}(\mathbb{Z}[\frac{1}{p}])$, take $M = \mathbb{Z}[\frac{1}{p}]$, $N = \mathbb{Z}$. Then $\mathrm{Hom}_{\mathbb{Z}}(M,N) = 0$ but M and N became isomorphic in $\mathcal{M}_t$.

Go back to the example $R = k[x,y]$, $I = (x,y)$. Recall that we have the Cousin resolution

$$0 \to R \to \underbrace{E(R)}_{\substack{\text{quotient}\\ \text{field of } R}} \to \bigoplus_{\substack{p \\ \text{height 1} \\ \text{primes}}} E(R/p) \to E(k) \to 0$$

Then if F is flat and firm we get the resolution

$$0 \to F \to F \otimes_R E(R) \to \bigoplus_p F \otimes_R E(R/p) \to 0$$

since $E(k)$ is I-torsion $\Rightarrow$ $F \otimes_R E(k) = 0$.
This allows us to replace any complex of flat firm modules by a complex of solid injectives.

Check that flat $\otimes$ injective is injective over a noetherian comm. ring. Reason is that a flat module is ~~an~~ filtered inductive limit of f.g. free modules, so it suffices to check a filtered inductive limit of injectives is injective. But any ideal $\mathfrak{a}$ in R is fin. presented so

$$\mathrm{Hom}(R, \varinjlim_\alpha Q_\alpha) = \varinjlim_\alpha \mathrm{Hom}(R, Q_\alpha)$$
$$\downarrow$$
$$\mathrm{Hom}(\mathfrak{a}, \varinjlim Q_\alpha) = \varinjlim \mathrm{Hom}(\mathfrak{a}, Q_\alpha)$$

so it's clear.

Next if Q is injective one has the canonical filtration

$$0 \subset \Gamma_I(Q) \subset \bigoplus_p \Gamma_p(Q) \subset Q$$

by injective submodules. When Q is solid injective $\Gamma_I(Q) = 0$, and we get a canonical exact sequence

$$0 \to \bigoplus_p \Gamma_p(Q) \to Q \to E(R) \otimes_R Q \to 0$$

which splits.

Thus if Q is a solid injective complex it appears as the h-fibre of a map

$$\underbrace{E(R) \otimes_R Q}_{\substack{\text{complex of}\\ \text{vector spaces over}\\ \text{the quotient}\\ \text{field of } R}} \longrightarrow \bigoplus_p \underbrace{\Gamma_p(Q)}_{\substack{\text{complex of}\\ \text{injective torsion}\\ \text{modules over the}\\ \text{disc. valn. ring } R_p}}$$

Put $K = E(R)$ = quotient field of R

We've seen that any firm flat module F is the kernel of a surjection:

$$F \otimes_R K \longrightarrow \bigoplus_p F \otimes_R E(p)$$

of solid injectives where the first is a K-vector space and the second is a direct sum of injective hulls $E(R/p)$ of primes of height one.

Conversely, note K is flat and that

$E(R/p)$ has flat dimension ≤ 1:

$$0 \longrightarrow R_p \longrightarrow K \longrightarrow E(R/p) \longrightarrow 0$$

Thus it should follow that any firm flat module F has a unique up to canonical isomorphism representation as the kernel of a surjection from a K vector space to a height one injective.

Summarize:

firm flat = kernel of a surjection from a height 0 injective to a height one injective

solid injective = extension of height 0 injective by a height 1 injective

These are canonical descriptions and apply to complexes.

July 21, 1994

For a general torsion theory on mod(R):

$$\mathrm{tors} \underset{\iota^!}{\overset{\iota_*}{\rightleftarrows}} \mathrm{mod} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathrm{mod}/\mathrm{tors}$$

when do we have a triangle in $D^+(\mathrm{mod})$

$$\iota_* R\iota^!(M) \longrightarrow M \longrightarrow Rj_*(j^*M) \longrightarrow \; ?$$

Recall that a torsion theory is called <u>stable</u> when the injective hull of any torsion module is torsion. ~~This is equivalent to E injective $\Rightarrow$ $\iota^! E$ is an injective module.~~

~~Check: If E is injective, then the injective hull $E(\iota^! E)$ of the torsion submodule is a direct summand of E.~~

Let Q be any injective module. The injective hull $E(\iota^! Q)$ of the torsion submodule $\iota^! Q$ is a summand of Q. Assuming the torsion theory is stable, $E(\iota^! Q)$ is torsion, hence contained in $\iota^! Q$. Thus $E(\iota^! Q) = \iota^! Q$, showing that $\iota^! Q$ is injective, and that Q splits into the direct sum of a torsion injective $\iota^! Q$ and a torsion-free injective.

On the other hand, suppose we assume $\iota^! Q$ is injective for every injective Q. If M is a torsion module, take Q to be $E(M)$. Then $\iota^! Q$ is an injective submodule of Q containing M, so $\iota^! Q = Q$, since Q is a minimal injective containing M. Thus the injective hull of any torsion module is torsion. Thus we have proved

A torsion theory is stable iff the torsion submodule of any injective module is injective iff any injective is the direct sum of a torsion injective and a torsion-free injective.

Now assuming that our torsion theory is stable consider M in $D^+(\mathrm{mod})$, and replace it by a quasi-isomorphic injective complex Q. Then we have an exact sequence

$$0 \longrightarrow i^! Q \longrightarrow Q \longrightarrow Q/i^! Q \longrightarrow 0$$

Now $i^! Q = i_* R i^!(M)$ and $Q/i^! Q$ is a complex of torsion-free injectives which is quasi-isomorphic modulo torsion to M, hence $Q/i^! Q = Rj_*(j^* M)$. Thus we get the desired Δ in $D^+(\mathrm{mod})$.

Consider now the jansian case $I = I^2$

$$\begin{tikzcd} \mathrm{mod}(R/I) \arrow[r, "i_*"] & \mathrm{mod}(R) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{M} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

$$i_* i^*(M) = M/IM \qquad j_! j^*(M) = I^g \otimes_R M$$

$$i_* i^!(M) = {}_I M \qquad j_* j^*(M) = \mathrm{Hom}_R(I^g, M)$$

Notice the confusion of notation when you use $i^! M$ for the torsion submodule. Then $i^!$ has the right adjoint i_* which is exact, hence $i^!$ carries injectives to injectives, i.e. it looks like any torsion theory is stable. Instead we probably ought to use τM for the torsion submodule $\iota_*\iota^!(M)$. Note that

$$\mathrm{Hom}_R(M, {}_I N) = \mathrm{Hom}_R(M/IM, N).$$

i.e. $\mathrm{Hom}_R(M, \iota_*\iota^!(N)) = \mathrm{Hom}_R(\iota_*\iota^*(M), N)$

so that $\iota_*\iota^!$ is left adjoint to $\iota_*\iota^*$.

Then stability means that $\iota_*\iota^!$ respects injectives, and this is equivalent to the left adjoint $\iota_*\iota^*(M) = M/IM$ being exact, i.e. to R/I being right flat.

Question: You know firm(R,I) is abelian when I is right R-flat. But firm(R,I) depends only on I, not R.

Can you, given an arbitrary (R,I), find a Morita equivalence firm$(R,I) \simeq$ firm(S,J) where J is S^{op}-flat?

Are there conditions on I guaranteeing that I can be embedded as a right flat ideal in some unital ring R?

July 23, 1994

There might be a derived category version of the construction $\mathcal{M}(R,I) = \text{mod}(R)/\text{null}$, namely, let $D(\text{mod}(R))_{\text{null}}$ be the full subcat of $D(\text{mod}(R))$ consisting of complexes whose homology groups are [redacted] null. This is a triangulated subcategory, so I believe there is a quotient triangulated category

$$\mathcal{D}(R,I) = D(\text{mod}(R))\big/D(\text{mod}(R))_{\text{null}}$$

(modulo set theory problems). One can ask whether this is equivalent to $D(\text{mod}(R)/\text{null})$. [redacted] (Since Franke has constructed an abelian category whose derived category is "outside the universe" the set theory problems may be real.)

One might hope (see p. 709) that

$$\begin{aligned}\text{Hom}_{\mathcal{D}}(M,N) &= \varinjlim \text{Hom}_D\left([I\overset{L}{\otimes}_R]^n M, N\right) \\ &= \varinjlim \text{Hom}_D\left(M, R\text{Hom}_R\left([I\overset{L}{\otimes}_R]^{(n-1)} I, N\right)\right)\end{aligned}$$

(In any case the argument on p. 709 seems to show that the inverse system $[I\overset{L}{\otimes}_R]^n M$ is locally essentially zero when M is null.)

It [redacted] seems that the firm and solid subcategories [redacted] should be resp.

$R/I \overset{L}{\otimes}_R M \simeq 0$ equiv. $M \sim F$ firm flat

$R\text{Hom}_R(R/I, M) \simeq 0$ equiv. $M \sim Q$ solid injective

Note that earlier arguments using $\mathrm{Hom}_{\mathbb{Z}}(R/I, \mathbb{Q}/\mathbb{Z})$ are obsolete. For example F firm flat, $IM = 0 \implies$ $\mathrm{Ext}^*_R(F, M) = 0$ is proved as follows. Let $P \to F$ be a projective R-module resolution, $M \to Q$ an injective resolution of R/I-modules. Then

$$R\mathrm{Hom}_R(F, M) \simeq \mathrm{Hom}_R(P, M) \simeq \mathrm{Hom}_R(P, Q)$$
$$= \mathrm{Hom}_{R/I}(P/IP, Q)$$

is acyclic because $P/IP \simeq F/IF \simeq 0$ and Q is injective.

But in fact things are even simpler namely, Q is unnecessary:

$$R\mathrm{Hom}_R(F, M) \simeq \mathrm{Hom}_R(P, M)$$
$$= \mathrm{Hom}_{R/I}(P/IP, M)$$

and this is homotopy to zero since P/IP is an acyclic projective complex of R/I-modules

July 24, 1994

Morita invariance for the firm derived category seems to be a consequence of the fact that it is equivalent to the derived category for the exact category of firm flat modules which we know is Morita invariant. (Understood here is the restriction to complexes bdd below for the lower indexing.) I would like to give a direct proof in the spirit of my work the past few days (specifically the work which is in the notes for the "paper", where the firm derived category is described as consisting of M in $D_+(R)$ such that $R/I \overset{L}{\otimes}_R M = 0$).

So given a Morita context $\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$ with ideals $I = QP$, $J = PQ$ say, I want to prove that firm $D_+(R, I) \xrightarrow{\sim}$ firm $D_+(S, J)$, $M \mapsto P \overset{L}{\otimes}_R M$ is an equivalence of categories.

First show this functor is well-defined. Suppose M projective, whence $P \overset{L}{\otimes}_R M$ quis $P \otimes_R M$. To show $S/J \overset{L}{\otimes}_S (P \otimes_R M)$ quis 0.

To do this we can ignore the left S-module structure and choose a projective right S-module resolution $E \to S/J$. Then

$$S/J \overset{L}{\otimes}_S (P \otimes_R M) = (E \otimes_S P) \otimes_R M$$

and it suffices to show that $E \otimes_S P$ is right I-null, because then since $R/I \overset{L}{\otimes}_R M = 0$ we have $- \overset{L}{\otimes}_R M$ kills all complexes of R^{op}-modules which are I^{op}-null.

Take a generator gp for I. We have

the commutative diagram

$$\begin{tikzcd}[column sep=small]
E\otimes_S P \arrow[rr, "\xi\otimes p_1 \,\mapsto\, (\xi\otimes p_1)gp \,=\, \xi(p_1 g)\otimes p"] \arrow[dr, "{\xi\otimes p_1 \,\mapsto\, \xi(p_1 g)}"'] & & E\otimes_S P \\
& EJ \subset E \arrow[ur, "\xi \,\mapsto\, \xi\otimes p"'] &
\end{tikzcd}$$

Since E is a resolution of S/J, the inclusion $EJ \subset E$ induces the zero map on homology. Thus right mult. by gp on $H_*(E\otimes_S P)$ is zero, and we win.

Now we know that $M \mapsto P\otimes^L_R M$ from $D_+(R)$ to $D_+(S)$ carries firm $D_+(R,I)$ into firm $D_+(S,J)$. Similarly $Q\otimes^L_S -$ gives a triangulated functor in the opposite direction. We next want to see these functors are quasi-inverse to each other.

Take $M \in D_+(R)$ projective, let $U \to P\otimes_R M$ be a projective resolution, whence $Q\otimes^L_S(P\otimes^L_R M) \simeq Q\otimes_S U$. One has an obvious map

$$Q\otimes_S U \longrightarrow Q\otimes_S P\otimes_R M \longrightarrow M$$

which we want to be a quis. Again we can ignore the left R-module structure and choose an S^{op} projective resolution $E \to Q$. Then we have

$$\begin{tikzcd}
E\otimes_S U \arrow[r, "\substack{\text{quis as } E \text{ is } S^{op} \text{ proj} \\ \text{and } U\to P\otimes_R M \\ \text{is a quis}}"] \arrow[d, "\substack{\text{quis} \\ \text{as } U \ S\text{-proj} \\ \text{and } E\to Q \\ \text{quis}}"'] & E\otimes_S P\otimes_R M \arrow[d] \\
Q\otimes_S U \arrow[r] & Q\otimes_S P\otimes_R M \arrow[r] & M
\end{tikzcd}$$

So all we need to do is show that $E \otimes_S P \longrightarrow R$ is an I^{op}-null quis, i.e. the homology of the cone is I^{op}-null. This amounts to

$$H_n(E \otimes_S P) = \mathrm{Tor}_n^S(Q, P)$$

being I^{op}-null for $n > 0$ and

$$Q \otimes_S P \longrightarrow R$$

being an I^{op}-null isom. The latter we know already: the kernel + cokernel are killed by I.

Thus we want to show that the quis $E \to Q$ of S^{op}-modules goes into $E \otimes_S P \to Q \otimes_S P$, which is a I^{op}-null quis. This amounts to exactness of the functor

$$\mathrm{mod}(S^{op}) \longrightarrow \mathrm{mod}(R^{op}) \longrightarrow \mathrm{mod}(R^{op})/\mathrm{null}(R^{op}, I^{op})$$

and this follows from the fact that $- \otimes_S P$ gives an equivalence $\mathcal{M}(S^{op}, J^{op}) \simeq \mathcal{M}(R^{op}, I^{op})$.

Direct proof. Let N be an acyclic complex of S^{op}-modules. Then we claim the homology of $N \otimes_S P$ is killed by I^{op}. In effect take a generator qp for I. Then

$$\nu \otimes p_1 \longmapsto (\nu \otimes p_1) qp = \nu(p_1 q) \otimes p$$

$$\begin{tikzcd} N \otimes_S P \arrow[rr] \arrow[dr] & & N \otimes_S P \\ & N \arrow[ur] & \end{tikzcd}$$

$$\nu \otimes p_1 \longmapsto \nu(p_1 q) \qquad \nu \longmapsto \nu \otimes p$$

commutes and N has zero homology so it's clear.

July 28, 1994

Here's a derived category version of Morita equivalence. Suppose $\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$ a Morita context. Then we have a triangulated functor

(1) $\quad M \longmapsto P \otimes_R M \qquad K(R) \longrightarrow K(S)$

where K here means the homotopy category of complexes. Let $I \subset R$, $J \subset S$ be ideals such that $QJP \subset I \subset QP$, $PIQ \subset J \subset PQ$ as usual. Then for $p \in P$, $a \in I$, $q \in Q$ one has a comm. diag

(2)
$$\begin{tikzcd} P \otimes_R M \arrow[r, "paq\cdot"] \arrow[d, "q\cdot"'] & P \otimes_R M \\ M \arrow[r, "a\cdot"] & M \arrow[u, "p\otimes -"'] \end{tikzcd} \qquad \begin{tikzcd} p_1 \otimes m \arrow[r, mapsto] \arrow[d, mapsto] & paqp_1 \otimes m = p \otimes a(qp_1)m \\ (qp_1)m \arrow[r, mapsto] & a(qp_1)m \arrow[u, mapsto] \end{tikzcd}$$

so that $I \cdot H_*(M) = 0 \implies PIQ \cdot H_*(P \otimes_R M) = 0$.

This means that $P\otimes_R -$ carries I-null complexes into **J-null** complexes, hence induces a functor on the quotient triangulated categories (assuming these exist.)

Next check that [redacted] $M \mapsto \mathrm{Hom}_S(Q, M)$ gives an isomorphism functor on the quotient categories.

(I forgot in (2) above to point out that if I is any ideal such that $I H_*(M) = 0$, then $PIQ \cdot H_*(P \otimes_R M) = 0$. In particular if $H_*(M) = 0$, then PQ kills $H_*(P \otimes_R M)$.)

In general if $\varphi : X \longrightarrow Y$ is a map of complexes its h-fibre is $F_n = X_n \oplus Y_{n+1}$ with $d_F = \begin{pmatrix} d_X & 0 \\ \varphi & -d_Y \end{pmatrix}$. If $\psi : Y \to X$ is a map of complexes, then

$$\left[\begin{pmatrix} d_X & \\ \varphi & -d_Y \end{pmatrix}, \begin{pmatrix} 0 & \psi \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} \psi\varphi & d_X\psi - \psi d_Y \\ 0 & \varphi\psi \end{pmatrix} = \begin{pmatrix} \psi\varphi & 0 \\ 0 & \varphi\psi \end{pmatrix}$$

showing that $\begin{pmatrix} \psi\varphi & 0 \\ 0 & \varphi\psi \end{pmatrix} \sim 0$ on F.

We apply this to

$$\begin{tikzcd} P\otimes_R M \arrow[r, "\varphi"] \arrow[d, "\psi_{p,q}\varphi"'] & \mathrm{Hom}_R(Q,M) \arrow[d, "\varphi\psi_{p,q}"] \arrow[dl, "\psi_{p,q}"'] \\ P\otimes_R M \arrow[r, "\varphi"] & \mathrm{Hom}_R(Q,M) \end{tikzcd}$$

$$p_1\otimes m \overset{\varphi}{\longmapsto} (q_1 \mapsto (q_1p_1)m)$$
$$p\otimes f(q) \overset{\psi_{p,q}}{\longleftarrow\!\shortmid} f$$

Then $\psi_{p,q}\varphi : p_1\otimes m \longmapsto \psi_{p,q}(q_1 \mapsto (q_1p_1)m) = p\otimes (qp_1)m$
$= pq(p_1\otimes m)$

and $\varphi\psi_{p,q} : f \longmapsto \varphi(p\otimes f(q)) \longmapsto (q_1 \mapsto q_1p\, f(q))$
$= (q_1 \mapsto f(q_1pq))$.
$= pq\cdot f$

This shows the cone on φ is killed by PQ

Another application

$$\begin{tikzcd} Q\otimes_S P\otimes_R M \arrow[r, "\varphi"] \arrow[d] & M \arrow[d] \arrow[dl, "\psi_{q,p}"'] \\ Q\otimes_S P\otimes_R M \arrow[r, "\varphi"] & M \end{tikzcd}$$

$$q_1\otimes p_1\otimes m \overset{\varphi}{\longmapsto} (q_1p_1)m$$
$$q\otimes p\otimes m \overset{\psi_{q,p}}{\longleftarrow\!\shortmid} m$$

$$\psi_{q,p}\varphi(q_1\otimes p_1\otimes m) = q\otimes p\otimes (q_1p_1)m = q\otimes pq_1p_1\otimes m$$
$$= qp(q_1\otimes p_1\otimes m)$$

$$\varphi\psi_{q,p}(m) = \varphi(q\otimes p\otimes m) = (qp)\,m$$

Thus the cone on $Q\otimes_S P\otimes_R M \longrightarrow M$ is killed by QP.

How to obtain quasi-coherent sheaves over $\mathbb{P}_n$ from a nonunital ring.

Consider $S(V) = \bigoplus_{n\geq 0} S_n(V)$, $\dim(V) < \infty$ over a field k.

A graded module $M = \bigoplus_{n\geq 0} M_n$ over $S(V)$ has operators e_n = projection on M_n and multiplication by elements of V: $m \mapsto vm$. These satisfy the relations

$$e_n e_m = \begin{cases} e_n & n=m \\ 0 & n\neq m \end{cases} \qquad e_n v = \begin{cases} v e_{n-1} & n\geq 1 \\ 0 & n=0 \end{cases}$$

Let's choose a basis $V = \sum_{i=1}^{d} k x_i$, whence we have the basis x^α for $S(V)$. The operators e_n and x_i generate an algebra spanned by $x^\alpha e_n$, x^α for all $n \geq 0$ and multi-indices α. The multiplication of these "monomials" is determined by

$$e_n x^\alpha = \begin{cases} x^\alpha e_{n-|\alpha|} & \text{if } |\alpha| \leq n \\ 0 & \text{if } |\alpha| > n \end{cases}$$

So we obtain a twisted tensor product algebra

$$S(V) \otimes \Big(k \oplus \bigoplus_{n\geq 0} k e_n\Big)$$

$$= S(V) \oplus \bigoplus_{n\geq 0} \underbrace{S(V) e_n}_{\simeq S(V)\otimes k e_n}$$

If we tried to use the monomials $e_n x^\alpha$ instead, i.e. the twisted tensor product in the opposite order, then $e_n S(V) \not\simeq S(V)$ since $e_n x^\alpha = 0$ for $|\alpha| > n$. However we do get a different description

$$S(V) \oplus \bigoplus_{n\geq 0} e_n S_{\leq n}(V)$$ for this algebra.

Now put $R = S(V)\otimes\left(k \oplus \bigoplus_{n\geq 0} k e_n\right)$ with this twisted multiplication.

An R-module M is an $S(V)$ module equipped with projectors e_n, $n\geq 0$ such that the relations $e_n v = \begin{cases} v e_{n-1} & n>0 \\ 0 & n=0 \end{cases}$ hold. Thus we have a diagram

$$\begin{tikzcd} & M \arrow[dr] & \\ \bigoplus_{n\geq 0} e_n M \arrow[ur] \arrow[rr, hook] & & \prod_{n\geq 0} e_n M \end{tikzcd}$$

of $S(V)$ modules.

Actually I should probably do things in analogy with sheaf theory. Recall the diagrams of exact sequences when R/I is R^{op} flat (equiv. $\forall a_1,\dots,a_n\in I\ \exists a\in I:\ (1-a)a_i=0$):

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] \\ & & j_!j^*M \arrow[r, equal] \arrow[d] & j_!j^*j_*j^*M \arrow[d] \\ 0 \arrow[r] & i_*i^!M \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & j_*j^*M \arrow[d] \\ 0 \arrow[r] & i_*i^*(i_*i^!M) \arrow[r] & i_*i^*M \arrow[r] \arrow[d] & i_*i^*j_*j^*M \arrow[d] \\ & & 0 & 0 \end{tikzcd}$$

In the module case

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] \\ & & I\otimes_R M \arrow[r, equal] \arrow[d] & I\otimes_R \mathrm{Hom}_R(I,M) \arrow[d] \\ 0 \arrow[r] & {}_I M \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & \mathrm{Hom}_R(I,M) \arrow[d] \\ 0 \arrow[r] & R/I\otimes_R({}_I M) \arrow[r] & (R/I)\otimes_R M \arrow[r] \arrow[d] & R/I\otimes_R \mathrm{Hom}_R(I,M) \arrow[d] \\ & & 0 & 0 \end{tikzcd}$$

In the case $R = S(V) \otimes \left(\overrightarrow{\oplus k e_n}\right)$

$I = S(V) \otimes (\oplus k e_n)$, we check R/I is R^{op}-flat.

We know $I = \bigoplus_{n \geq 0} k e_n \otimes S_{\leq n}(V)$ so $\left\{ \sum_{n=0}^{N} e_n \right\}$ is an approximate left (also right) identity.

The corresponding diagram for an R-module M is

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & \oplus e_n M \arrow[r, equal] \arrow[d] & \oplus e_n M \arrow[d] \\
0 \arrow[r] & \bigcap_n (e_n M) \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & \prod e_n M \arrow[d] \\
0 \arrow[r] & \cdot \arrow[r] & M/\oplus e_n M \arrow[r] \arrow[d] & (\prod/\oplus)(e_n M) \arrow[d] \\
 & & 0 & 0
\end{tikzcd}
$$

So what seems best is to write the following diagram

$$
\begin{tikzcd}
 & & M \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart.}"] & M/\Sigma e_n M \arrow[d] \\
0 \arrow[r] & \oplus e_n M \arrow[r] & \prod e_n M \arrow[r] & (\prod/\oplus)(e_n M)
\end{tikzcd}
$$

linking the triangular diagram on the previous page to description of modules via triples.

So far we have described $\mathbb{N}$-graded modules over $S(V)$, and I **guess** the picture I have so far really amounts to the choice of generators $\{S(V) \otimes u_n, n \geq 0\}$, where u_n has degree n.

About Munkholm's talk. He defines an action of [redacted] the poset $\mathbb{N}$ on a category $\mathcal{A}$ [redacted] to be [redacted] a functor $\mathbb{N} \times \mathcal{A} \longrightarrow \mathcal{A}$.

Such a functor is equivalent to either

i) a functor $\mathbb{N} \longrightarrow \underline{\mathrm{Hom}}_{cat}(\mathcal{A}, \mathcal{A})$, i.e. a sequence of functors and maps of functors from $\mathcal{A}$ to itself of the form

$$F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots$$

ii) a functor $\mathcal{A} \longrightarrow \underline{\mathrm{Hom}}_{cat}(\mathbb{N}, \mathcal{A})$, i.e. functor sending each X in $\mathcal{A}$ to an inductive system

$$F_0(X) \longrightarrow F_1(X) \longrightarrow F_2(X) \longrightarrow \cdots$$

depending functorially in X.

It seems that this is not what [redacted] Munkholm means by an action of $\mathbb{N}$ on $\mathcal{A}$, rather there are some extra "usual axioms" to be satisfied. However, before trying to decipher this, let's discuss $\underline{\mathrm{Hom}}_{~~cat~~}(\mathbb{N}, \mathcal{A})$ a bit. This is the category of sequential inductive systems $\vec{X} = (X_0 \to X_1 \to X_2 \to \cdots)$ in $\mathcal{A}$.

There is an "Artin-Rees" quotient category $\mathcal{Q}$ with maps

$$\mathrm{Hom}_{\mathcal{Q}}(\vec{X}, \vec{Y}) = \varinjlim_n \mathrm{Hom}_{\mathcal{A}^{\mathbb{N}}}(\vec{X}, \vec{Y}(n))$$

where if $\vec{Y}$ [redacted] $= (Y_0 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow \cdots)$ then

$$\vec{Y}(n) = (Y_n \longrightarrow Y_{n+1} \longrightarrow Y_{n+2} \longrightarrow \cdots)$$

It should be true that $\mathcal{Q}$ is obtained from $\mathcal{A}^{\mathbb{N}}$ by inverting the canonical arrow $\vec{X} \longrightarrow \vec{X}(1)$ for every $\vec{X}$.

In the case where $\mathcal{A}$ is abelian it looks

looks like we are dividing by the Serre subcategory of $\vec{X}$ such that $\exists n$ s.t. $\vec{X} \longrightarrow \vec{X}(n)$ is zero, i.e. $X_k \longrightarrow X_{k+n}$ is zero for all k.

Here seems to be the sort of thing Munkholm considers. Suppose $F : \mathcal{A} \longrightarrow \mathcal{A}$ is a functor and $\eta : id \longrightarrow F$ is a map of functors such that the two maps $F\cdot\eta, \eta\cdot F : F \longrightarrow F^2 = F\circ F$ coincide, i.e. $\forall X$ in $\mathcal{A}$ the two arrows

$$F(X) \underset{\eta_{F(X)}}{\overset{F(\eta_X)}{\rightrightarrows}} F(F(X))$$

coincide. This seems to be equivalent to what he calls an action of $\mathbb{N}$ on $\mathcal{A}$.

The localization $\mathbb{N}^{-1}\mathcal{A}$ has the same objects as $\mathcal{A}$ with

$$\mathrm{Hom}_{\mathbb{N}^{-1}\mathcal{A}}(X, Y) = \varinjlim_n \mathrm{Hom}_{\mathcal{A}}(X, F^n(Y))$$

Thus $\mathbb{N}^{-1}\mathcal{A}$ should be the category obtained by inverting the arrow $\eta_X : X \longrightarrow F(X)$ for all X.

Let's check his claim that if $\mathcal{A}$ is abelian and F is left exact, then $\mathbb{N}^{-1}\mathcal{A}$ is abelian. It suffices to show that for any short exact sequence $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ goes into one which is both left + right exact in $\mathbb{N}^{-1}\mathcal{A}$

Now $\mathrm{Hom}_{\mathcal{A}}(-, F^n Y)$ transforms this short exact sequence in $\mathcal{A}$ to a left exact sequence and taking $\varinjlim_n$ yields a left exact sequence. On the other hand, because F is left exact $\varinjlim_n \mathrm{Hom}_{\mathcal{A}}(Y, F^n(-))$ transforms the given short exact sequence into a left exact one, so we win.

As a check, we should see that the objects Y which becomes zero in $N^{-1}\mathcal{A}$ form a Serre subcategory. The functor $\varinjlim_n \operatorname{Hom}_{\mathcal{A}}(-, F^n(Y))$ is zero iff $\exists n$ such that $Y \longrightarrow F^n(Y)$ is zero (in which case $F^k(Y) \longrightarrow F^{k+n}(Y)$ is zero $\forall k$).

Consider

$$\begin{tikzcd}
0 \arrow[r] & Y' \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] \arrow[dl, dashed, "\exists"'] & Y'' \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & F^n(Y') \arrow[r] \arrow[d, "0"] & F^n(Y) \arrow[r] \arrow[d] & F^n(Y'') & \\
0 \arrow[r] & F^{p+n}(Y') \arrow[r] & F^{p+n}(Y) & &
\end{tikzcd}$$

so it's clear.

July 31, 1994 (back from Paris)

Here's what Mankholm means by an $\mathbb{N}$-action on a category $\mathcal{A}$. Consider $\mathbb{N}$ as a monoid object in cat with operation given by addition. Thus addition gives a map of posets

$$\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$$

which is associative and commutative. Also the functor $pt \longrightarrow \mathbb{N}$ sending the unique object to 0 is an identity for this operation.

On the other hand for any category $\mathcal{A}$ $\underline{Hom}_{cat}(\mathcal{A}, \mathcal{A})$ is a monoid object in cat with the operation given by composition. (To be rigorous one should suppose $\mathcal{A}$ small.) A $\mathbb{N}$-action on $\mathcal{A}$ is a functor

$$\mathbb{N} \longrightarrow \underline{Hom}_{cat}(\mathcal{A}, \mathcal{A})$$

compatible with the product and identity objects. Thus $0 \longmapsto id_{\mathcal{A}}$, and if $1 \longmapsto F$, then $n \longmapsto F^n$. Suppose we write $n * X$ for the action of n on X, so $n * X = F^n(X)$. Let

$$\begin{array}{ccc} \eta_X : X & \longrightarrow & F(X) \\ \parallel & & \parallel \\ 0 * X & \longrightarrow & 1 * X \end{array}$$

be the map corresponding to the arrow $0 \to 1$ in $\mathbb{N}$. Then we have

these are the same

$$\begin{tikzcd} 1 * X \arrow[d] \arrow[r, phantom, "="] & 0 * (1 * X) \arrow[d, "\text{induced by } 0<1"'] \arrow[r, phantom, "="] & F(X) \arrow[d, "\eta_{F(X)}"] \\ 2 * X \arrow[r, phantom, "="] & 1 * (1 * X) \arrow[r, phantom, "="] & F(F(X)) \\ 1 * X \arrow[u] \arrow[r, phantom, "="] & 1 * (0 * X) \arrow[u, "\text{induced by } 0<1"] \arrow[r, phantom, "="] & F(X) \arrow[u, "F(\eta_X)"'] \end{tikzcd}$$

showing $\eta \cdot F = F \cdot \eta$.

To avoid assuming $\mathcal{A}$ small, one writes the action as a functor $\mathbb{N} \times \mathcal{A} \xrightarrow{\mu} \mathcal{A}$ such that one has

i) associativity:
$$\begin{tikzcd} \mathbb{N} \times \mathbb{N} \times \mathcal{A} \arrow[r, "1 \times \mu"] \arrow[d, "+ \times 1"'] & \mathbb{N} \times \mathcal{A} \arrow[d, "\mu"] \\ \mathbb{N} \times \mathcal{A} \arrow[r, "\mu"] & \mathcal{A} \end{tikzcd}$$
commutes

ii) identity
$$\begin{tikzcd} x \arrow[d, mapsto] & \mathcal{A} \arrow[d] \arrow[dr, "1"] & \\ (0,x) & \mathbb{N} \times \mathcal{A} \arrow[r, "\mu"] & \mathcal{A} \end{tikzcd}$$

Another point is that if $\mathbb{N}$ acts on $\mathcal{A}$ small, and $\mathcal{C}$ is another category, then $\mathbb{N}$ acts on $\underline{\mathrm{Hom}}_{\mathrm{cat}}(\mathcal{A}, \mathcal{C})$. This is because one has a map $\mathbb{N} \longrightarrow \underline{\mathrm{Hom}}_{\mathrm{cat}}(\mathcal{A}, \mathcal{A})$ preserving identity + product, and $\underline{\mathrm{Hom}}_{\mathrm{cat}}(\mathcal{A}, \mathcal{A})$ acts on $\underline{\mathrm{Hom}}_{\mathrm{cat}}(\mathcal{A}, \mathcal{C})$.

Consider now some of his examples.

If X is a metric space, let $\mathcal{B}$ be the poset given by $(n, x) \in \mathbb{N} \times X$ with the ordering $(n,x) \leq (n', x') \iff d(x,x') \leq n' - n$. Then $y \in B(n,x)$, i.e. $d(x,y) \leq n$, $\implies d(x',y) \leq d(x,x') + d(x,y) \leq n'-n+n = n'$, so $B(n,x) \subset B(n',x')$. Thus one can roughly think of $\mathcal{B}$ as the poset of balls with integral radius in X.

There's an obvious action of $\mathbb{N}$ on $\mathcal{B}$, namely $n' * (n, x) = (n'+n, x)$.

Now consider $\underline{\mathrm{Hom}}_{cat}(\mathcal{B}, \mathcal{Ab})$. Objects are functors from $\mathcal{B}$ to abelian groups. The $\mathbb{N}$-action on $\mathcal{B}$ induces one on these functors. Then $\mathbb{N}^{-1}\,\underline{\mathrm{Hom}}_{cat}(\mathcal{B}, \mathcal{Ab})$ is by defn. the category of $\mathbb{Z}\mathcal{B}$ modules. Thus one is considering abelian group valued functors on the poset of balls modulo those functors $(n, x) \longmapsto M(n, x)$ such that $\exists\, n_0$ s.t. for all (n, x) the map $M(n, x) \longrightarrow M(n_0+n, x)$ is zero.

August 1, 1994

Puzzle: Let A be a left ideal in R unital. We have seen that the Morita context

$$\begin{pmatrix} \tilde{A} & R \\ A & R \end{pmatrix} \qquad \begin{array}{c} M \longmapsto A\otimes_{\tilde{A}} M \\ N = R\otimes_R N \longleftarrow N \end{array}$$

yields an equivalences $\mathcal{M}(\tilde{A},A) \sim \mathcal{M}(R,AR)$, firm$(\tilde{A},A) \sim$ firm(R,AR), etc. If we restrict to firm modules however we can use extension of scalars: $M \mapsto R\otimes_{\tilde{A}} M$ instead of $A\otimes_{\tilde{A}} M$. In effect the inclusion $A \hookrightarrow R$ is an isomorphism modulo null$(\tilde{A}^{op}, A^{op})$, since $(R/A)\cdot A = {}^{RA+A/A}=0$.
Thus $A\otimes_{\tilde{A}} M \xrightarrow{\sim} R\otimes_{\tilde{A}} M$ for M in firm$(\tilde{A},A)$.

This is strange because $M \mapsto R\otimes_{\tilde{A}} M$ doesn't seem to be part of a Morita context, although this is true for a (two-sided) ideal.

Why this arose: I observed in constructing flat firm modules

$$F(\alpha) = \varinjlim \left(\tilde{A}^{n_0} \xrightarrow{\alpha^1} \tilde{A}^{n_1} \xrightarrow{\alpha^2} \cdots \right)$$
$$\parallel$$
$$F(\alpha) = \varinjlim \left(R^{n_0} \xrightarrow{\alpha^1} R^{n_1} \xrightarrow{\alpha^2} \cdots \right)$$

that obviously $R\otimes_{\tilde{A}} F(\alpha) \xrightarrow{\sim} F(\alpha)$. Since any firm module is a cokernel of a map between direct sums of $F(\alpha)$'s, one has $R\otimes_{\tilde{A}} M = M$ for M firm

August 3, 1994

It turns out that once we leave the $I = I^2$ situation there are equivalences of module categories (in the generalized sense) which are not Morita equivalences (i.e. obtained from a Morita context).

Consider a Morita context $\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$, which is unital as usual, such that $PQ = S$. Then we have $(QP)^2 = QSP = QP$, so the ideal QP in R is idempotent. Also we know that $P \in \mathcal{P}(R^{op})$, $Q \in \mathcal{P}(R)$ are dual f.g. projective modules, and that $S = P \otimes_R Q = \operatorname{Hom}_{R^{op}}(P,P) = \operatorname{Hom}_R(Q,Q)$. This is the picture of a Morita equivalence [redacted] with a unital ring S.

Next consider $R = \mathbb{Z}$, $I = p\mathbb{Z}$, or more generally a commutative ring R and $I = Rf$ where f is a nonzero divisor. Then

$$\text{solid}(R, Rf) \simeq \text{mod}(R)/\text{tors}(R, Rf)$$

is equivalent to $\text{mod}(R_f)$, where $R_f = R[f^{-1}]$ is the localization obtained by inverting f. The ideal I is not idempotent, so we don't have a Morita equivalence as above.

Suppose we ~~[illegible]~~ try to write the equivalence $\text{solid}(R, Rf) \simeq \text{mod}(R_f)$ using bimodules:

$$M \longmapsto \operatorname{Hom}_R(Q, M)$$

$$\operatorname{Hom}_{R_f}(P, N) \longleftarrow N$$

Then P must be R_f with obvious left R_f - right R bimodule structure. Q as left R-module must be such that $R_f \otimes_R Q = R_f$. Since R_f must also act on the right of Q it's fairly clear that after killing any f-torsion, we must have $Q = R_f$. But then $QP = R_f R_f = R_f \not\subset R$.

Another example: Consider first the category of graded modules $M = \bigoplus_{n\in\mathbb{Z}} M_n$ over the Laurent polynomial alg $k[x,x^{-1}]$, where $\deg(x)=1$. This category is equivalence to mod(k), the equivalence being given by functors $M \longmapsto M_0$ and $N \longmapsto k[x,x^{-1}]\otimes_k N$. This situation is an example of a Morita equivalence

$$\begin{pmatrix} R & Re \\ eR & eRe \end{pmatrix} = \begin{pmatrix} Re_0R & k[x,x^{-1}]e_0 \\ e_0k[x,x^{-1}] & k \end{pmatrix} \qquad e=e_0$$

where $R = k[x,x^{-1}] \,\tilde{\otimes}\, \left(\bigoplus_{n\in\mathbb{Z}} ke_n\right)^{\sim}$ has the k-basis x^p, x^pe_n with $p,n\in\mathbb{Z}$ and multiplication given by $e_nx = xe_{n-1}$. Thus

e_0R has basis x^pe_0, $p\in\mathbb{Z}$.

Re_0 has basis e_0x^p, $p\in\mathbb{Z}$

And $Re_0\otimes e_0R$ has basis $x^pe_0\otimes e_0x^q$, $p,q\in\mathbb{Z}$ which map to $x^pe_0x^q = x^{p+q}e_{-q}$, whence

$$Re_0\otimes e_0R \xrightarrow{\sim} Re_0R = \bigoplus_{m,n\in\mathbb{Z}} kx^me_n$$

On the other hand suppose we try to obtain the same module category (graded $k[x,x^{-1}]$-modules) starting from $\mathbb{Z}$-graded $k[x]$-modules.

August 8, 1994

Expore the derived category picture.

Assume $I = I^2$ and let's start with the solid side, where we have injective resolutions, in order to treat first the simplest situation. Notation

$$\mathcal{M} = \mathrm{mod}(R)/\mathrm{mod}(R/I)$$

$$D^+(R) = D^+(\mathrm{mod}(R))$$

full Δated subcats of $D^+(R)$
$$\left\{\begin{array}{l} D^+(R)_{null} = \{ M \in D^+(R) \mid I H^*(M) = 0 \} \\ D^+(R)_{sol} = \{ \text{———} \mid R\mathrm{Hom}_R(R/I, M) = 0 \} \end{array}\right.$$

We have functors

$$\begin{tikzcd} D^+(R)_{null} \arrow[r, hook] & D^+(R) \arrow[r] & D^+(\mathcal{M}) \\ & D^+(R)_{sol} \arrow[u, hook] & \end{tikzcd}$$

Claim $D^+(R)_{sol} \longrightarrow D^+(\mathcal{M})$ is an equivalence of Δated cats. Why? $\mathcal{M}$ is a Grothendieck cat, so if $\mathcal{M}_{inj}$ is the full subcategory of injectives in $\mathcal{M}$ one [redacted] has an equivalence of Δated cats $C^+(\mathcal{M}_{inj}) \xrightarrow{\sim} D^+(\mathcal{M})$, where C^+ is homotopy category of complexes. Next, recall that if $M \longrightarrow E$ is a minimal injective resolution of M and $R\mathrm{Hom}_R(R/I, M) = 0$, then we know that E is a complex of solid injectives. Thus $D^+(R)_{sol}$ is equivalent to the full subcat of $D^+(R)$ consisting of solid injective complexes, [redacted] and as these are injective complexes the maps are just homotopy classes of maps between them. Thus we have an equiv. of Δated cats:

$$C_+(\text{sol inj}) \xrightarrow{\sim} D^+(R)_{sol}$$

Finally we have an equivalence of solid injective R-modules with injectives in $\mathcal{M}$. Thus have

$$\begin{tikzcd} D^+(R)_{sol} \arrow[r] & D^+(\mathcal{M}) \\ C^+(\text{sol inj}) \arrow[u, "\sim"] \arrow[r, "\sim"] & C^+(\mathcal{M}_{inj}) \arrow[u, "\sim"'] \end{tikzcd}$$

proving the claim.

As far as I can see the hypothesis that $I = I^2$ has not been used except in identifying I-null modules with I-torsion modules.

What should happen in general is that we have equivalences

$$D^+(R)_{sol} \xrightarrow{\sim} D^+(R)/D^+(R)_{tors} \xrightarrow{\sim} D^+(\mathcal{M})$$

adjoint functors

$$\begin{tikzcd} D^+(R)_{tors} \arrow[r, hook, shift left] & D^+(R) \arrow[l, shift left, "\tau"] \arrow[r, shift left, "j^*"] & D^+(\mathcal{M}) \arrow[l, shift left, "Rj_*"] \end{tikzcd}$$

and a canonical functorial Δ

$$\tau M \longrightarrow M \longrightarrow Rj_*(j^*M) \longrightarrow$$

To see this put $\mathcal{D} = D^+(R)/D^+(R)_{tors}$ and recall that maps in this category are calculated via fractions

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}(M,N) &= \varinjlim_{M' \xrightarrow{s} M} \mathrm{Hom}_{D^+(R)}(M', N) \\ &= \varinjlim_{N \xrightarrow{s} N'} \mathrm{Hom}_{D^+(R)}(M, N') \end{aligned}$$

But a basic fact is that

$$R\,Hom_R(T, Q) = 0 \quad \text{if} \begin{cases} T \in D^+(R)_{tors} \\ Q \in D^+(R)_{sol} \end{cases}$$

In effect we can suppose Q solid injective whence $R\,Hom_R(T, Q) = Hom_R(T, Q)$ and we can use the increasing Postnikov system of T

$$\begin{tikzcd}
T^{-1} \arrow[r] \arrow[d, equal] & Z^0 \arrow[r] \arrow[d, phantom, "\cap"] & 0 & & \\
T^{-1} \arrow[r] \arrow[d, equal] & T^0 \arrow[r] \arrow[d, equal] & Z^1 \arrow[r] \arrow[d, phantom, "\cap"] & 0 & \\
T^{-1} \arrow[r] & T^0 \arrow[r] & T^1 \arrow[r] & Z^2 \arrow[r] & 0
\end{tikzcd}$$

to reduce to the case where T is a torsion module sitting in degree zero, in which case $Hom_R(T, Q) = 0$ (actually $= 0$).

Using this one has for $N \in D^+(R)_{sol}$

$$Hom_{D^+(R)}(M, N) \xrightarrow{\sim} Hom_{D^+(R)}(M', N)$$

if $M' \xrightarrow{s} M$ [redacted] has cone in $D^+(R)_{tors}$. Thus

$$Hom_{D^+(R)}(M, N) \xrightarrow{\sim} Hom_{\mathcal{D}}(M, N) \quad \text{if } N \in D^+(R)_{sol}$$

follows from the first formula for $\uparrow$. But it also follows from the second formula, since given $N \xrightarrow{s} N'$ one has $Hom_{D^+(R)}(N', N) \xrightarrow{\sim} Hom_{D^+(R)}(N, N)$, thus the identity $N \xrightarrow{1} N$ is cofinal in the filtering cat of $N \xrightarrow{s} N'$.

At this point we know $D^+(R)_{sol} \longrightarrow D^+(R)/D^+(R)_{tors}$ is fully-faithful. As we also know [redacted] $\forall M$ $\exists\ M \longrightarrow Q$ torsion quis, Q solid injective, it follows this functor is an equivalence of category.

I think the rest should be straightforward. One defines ΓM as the fibre of the adjunction arrow $M \longrightarrow R_{j*}(j^*M)$. It's then clear that

$$\mathrm{Hom}_{D^+(R)}(T, \Gamma M) \xrightarrow{\sim} \mathrm{Hom}_{D^+(R)}(T, M) \qquad T \in D^+(R)_{tors}$$

since $\mathrm{Hom}_{D^+(R)}\big(T, \underbrace{R_{j*}(j^*M)}_{solid}\big) = 0$.

Everything so far in the torsion + solid context seems to work in general. The real issue is then somehow to understand $D^+(R)_{tors}$, e.g. how is this related to $D^+(\mathrm{tors}(R,I))$?

In the $I = I^2$ these coincide iff the h-unitality condition $I \otimes^L_R I \xrightarrow{\sim} I$ holds.

In general one might try to describe $D^+(R)_{tors}$ as DG modules over something. In the case $I = I^2$ then torsion complexes should be something like complexes of R/I module with extra operations. If $R = \tilde{I}$ and we are over a field k : $\tilde{I} = k \oplus I$, then the bar construction is a DG coalgebra whose homology is $\mathrm{Tor}_*^{\tilde{I}}(k,k)$. One can look at DG comodules over the bar construction.

Question: In the commutative noetherian case is it true that $D^+(R)_{tors} = D^+(\mathrm{tors}(R,I))$?

Take M in $D^+(R)_{tors}$. Up to qis we can suppose M is a minimal injective complex, i.e. M^n = injective hull of Z^n for all n. Let's proceed by induction on n to show that M^n is torsion for all n, this being obvious for $n \ll 0$. Assuming

M^{n-1} is torsion we see from the exact sequence

$$\underbrace{M^{n-1}}_{\text{tors}} \longrightarrow Z^n \longrightarrow \underbrace{H^n(M)}_{\text{tors}} \longrightarrow 0$$

that Z^n is torsion. But for a stable torsion theory the injective hull of a torsion module is torsion, so M^n is torsion. Thus M is a complex of torsion modules.

To finish note that for $\text{tors} = \text{tors}(R, I)$, the injective objects are injective R-modules. So we have an equivalence

$$C^+(\text{tors inj}) \xrightarrow{\sim} D^+(\text{tors})$$

Now $C^+(\text{tors inj})$ is a full subcat of $C^+(\text{inj}) \xrightarrow{\sim} D^+(R)$, so one has a full embedding $C^+(\text{torsinj}) \hookrightarrow D^+(R)_{\text{tors}}$. On the other hand we have seen that any M in $D^+(R)_{\text{tors}}$ is quis to a torsion injective complex. Thus one has

$$D^+(R)_{\text{tors}} \xleftarrow{\sim} D^+(\text{tors}).$$

when the I-torsion theory on $\text{mod}(R)$ is <u>stable</u>, e.g. R comm + noetherian.

August 9, 1994

Let's summarize some of yesterday's work about the solid picture and then proceed to the firm picture.

Let's begin with mod = mod(R), the full subcats

$$\text{tors} = \text{tors}(R,I)$$
$$\text{sol} = \text{solid}(R,I)$$

of I-torsion and I-solid R-modules, and $\mathcal{M}_t = \text{mod}/\text{tors}$.

Key points:

i) N solid $\implies$ $\text{Hom}_R(-,N)$ inverts tors isos.

ii) $\forall M$ $\exists$ torsion isom $M \longrightarrow M^\#$ with $M^\#$ solid.

These imply $\text{Hom}_R(M,N) \xleftarrow{\sim} \text{Hom}_R(M^\#,N)$ for all solid N, hence the inclusion sol $\hookrightarrow$ mod has the left adjoint $M \longmapsto M^\#$.

Since $\text{Hom}_{\mathcal{M}_t}(M,N) = \varinjlim\limits_{M' \xrightarrow{s} M} \text{Hom}_R(M',N)$

i) implies $\text{Hom}_R(M,N) \xrightarrow{\sim} \text{Hom}_{\mathcal{M}_t}(M,N)$ if N solid, in particular sol $\longrightarrow \mathcal{M}_t$ is fully faithful.

ii) implies this functor is essentially surjective, whence one has an equivalence sol $\simeq \mathcal{M}_t$

Next consider D^+R, the full ~~subcats~~ dated subcats

$$D^+R_{\text{tors}} = \{M \mid H_*(M) \text{ torsion}\}$$

$$D^+R_{\text{sol}} = \{M \mid R\text{Hom}_R(R/I, M) \simeq 0\}$$

Key points

i) $M \in D^+R_{\text{tors}}$, $N \in D^+R_{\text{sol}} \implies R\text{Hom}_R(M,N) \simeq 0$.

ii) $\forall M \in D^+R$ $\exists$ ~~tors~~-quis $M \longrightarrow M^\#$ st $M^\# \in D^+_{\text{sol}}$.

In fact we know that $M^\# = Rj_*(j^*M)$, specifically

$M^\# = j_*(\mathcal{I})$, where $j^* M \to \mathcal{I}$ is an injective resolution of complexes over $\mathcal{M}_t$. Now i), ii) imply

$$\mathrm{Hom}_{D^+R}(M, N) \xleftarrow{\sim} \mathrm{Hom}_{D^+R}(M^\#, N)$$

for $N \in D^+R_{sol}$, hence $M \mapsto M^\# = Rj_*(j^*M)$ is left adjoint to the inclusion $D^+R_{sol} \subset D^+R$. It's clear we also have an equivalence

$$D^+R_{sol} \xrightarrow{\sim} D^+R / D^+R_{tors} ;$$

~~We then have adjoint functors~~

$$\cancel{D^+R_{tors} \rightleftarrows D^+R \underset{Rj_*}{\overset{j^*}{\rightleftarrows}} D^+\mathcal{M}_t}$$

by the same sort of formal arguments using i), ii).

There's an extra point here, namely the equivalence

$$* \qquad D^+R / D^+R_{tors} \xrightarrow{\sim} D^+\mathcal{M}_t .$$

This is perhaps true quite generally, maybe restricting to bdd complexes, namely

$$D(\mathcal{A}/\mathcal{S}) = D\mathcal{A} / \underbrace{D\mathcal{A}_{\mathcal{S}}}$$

complexes with homology in the Serre subcategory $\mathcal{S}$.

One can see $*$ holds because D^+R_{sol} and $D^+\mathcal{M}_t$ each have equivalent injective complex subcategories, and these subcats are equivalent.

Now I want to assume $I = I^2$, whence toro = null, and I want to consider the firm picture.

Consider D_+R and the full Δated subcats
$$D_+R_{null} = \{M \mid I H_*(M) = 0\}, \quad D_+R_{firm} = \{M \mid R/I \overset{L}{\otimes}_R M \simeq 0\}.$$

The key points are

i) If $M \in D_+R_{firm}$, $N \in D_+R_{null}$, then $R\mathrm{Hom}_R(M,N) \simeq 0$.

ii) For any $M \in D_+R$, $\exists$ null quis $M^\# \to M$ with $M^\# \in D_+R_{firm}$

Proof of i): Can suppose M projective, whence $R\mathrm{Hom}_R(M,N) = \mathrm{Hom}_R(M,N)$. Consider the Postnikov system of N:

$$\begin{tikzcd}
0 \arrow[r] & N_1/B_1 \arrow[r] \arrow[d] & N_0 \arrow[r] \arrow[d] & N_{-1} \arrow[d, equal] & \\
& 0 \arrow[r] & N_0/B_0 \arrow[r] \arrow[d] & N_{-1} \arrow[r] \arrow[d] & {} \\
& & 0 \arrow[r] & N_{-1}/B_{-1} \arrow[r] & {}
\end{tikzcd}$$

fibre here is $N_1/B_1 \to B_0$ which is $H_1(N)[1]$

This is an inverse system of ~~quotients~~ $N^{(p)}$ of N such that $N = \varprojlim N^{(p)}$ and $\mathrm{Ker}(N^{(p)} \to N^{(p-1)})$ is $H_p(N)[p]$. Since a surjective sequential inverse system of acyclic complexes is acyclic (Milnor exact sequence), it suffices to show $\mathrm{Hom}_R(M,N) \simeq 0$ when N is a null module in degree zero.

By hypothesis $R/I \overset{L}{\otimes}_R M = M/IM$ is acyclic. As M/IM is a proj complex of R/I-modules it is homotopic to zero. Thus $\mathrm{Hom}_R(M,N) = \mathrm{Hom}_R(M/IM, N)$ is acyclic.

(Note that this argument does not work in the general torsion context since it is possible to have nonzero maps from a firm flat module to a torsion module.)

ii) follows from the existence of flat resolutions in $\mathcal{M}$ which after lifting via $j_!$ become firm flat complexes.

Formally it should follow from i) + ii) that the inclusion $D_+R_{firm} \subset D_+R$ has a right adjoint: $M \mapsto M^\# = Lj_!(j^*M)$. Moreover we should have an equivalence of categories

$$D_+R_{firm} \xrightarrow{\sim} D_+R / D_+R_{null}$$

I want now to check the extra point:

$$D_+R_{firm} \xrightarrow{\sim} D_+\mathcal{M}$$

Because of the existence of enough flat objects in $\mathcal{M}$, we should be able to construct $Lj_! : D_+\mathcal{M} \longrightarrow D_+R_{firm}$. This seems to be a 'resolution' theorem, going from all complexes in $\mathcal{M}$ to complexes in the exact category $\mathcal{M}_{flat}$. The idea then is

$$\begin{tikzcd} D_+R_{firm} \arrow[r] & D_+\mathcal{M} \\ D_+(\text{firm flat}) \arrow[u, "\simeq"] \arrow[r, phantom, "="] & D_+(\mathcal{M}_{flat}) \arrow[u, "\simeq"'] \end{tikzcd}$$

August 10, 1994

I want to check claims about flat resolutions. Consider $\mathcal{M} = \mathcal{M}(R, I)$, $\mathcal{M}_{flat}$ the full subcategory of firm flat modules. I want to check that one has an equivalence of Δated cats

$$\begin{array}{ccc} D_+(\mathcal{M}_{flat}) & \xrightarrow{\sim} & D_+(\mathcal{M}) \\ \| \text{def} & & \| \text{def} \\ C_+(\mathcal{M}_{flat})/\text{acyc.} & & C_+(\mathcal{M})/\text{acyc.} \end{array}$$

First check fully-faithful. We have

$$\operatorname{Hom}_{D_+(\mathcal{M}_{flat})}(F, G) = \varinjlim_{F' \xrightarrow{s} F} \operatorname{Hom}_{C_+(\mathcal{M}_{flat})}(F', G)$$

s means qis

$$\operatorname{Hom}_{D_+(\mathcal{M})}(F, G) = \varinjlim_{M' \xrightarrow{s} F} \operatorname{Hom}_{C_+(\mathcal{M})}(M', G)$$

These agree because $C_+(\mathcal{M}_{flat}) \to C_+(\mathcal{M})$ is fully faithful (these are homotopy categories), provided for any M' $\exists$ qis $F' \xrightarrow{s} M'$ with F' flat, since then the limit over $M' \xrightarrow{s} F$ can be taken over the cofinal category of $F' \xrightarrow{s} F$. The same condition ($\forall M$ $\exists F \xrightarrow{s} M$ with F flat) implies the functor $D_+(\mathcal{M}_{flat}) \to D_+(\mathcal{M})$ is essentially surjective.

Let's prove this condition holds by constructing flat Cartan-Eilenberg resolutions. Given $M \in D_+(\mathcal{M})$ consider its Postnikov system. To simplify suppose $M_n = 0$ for $n < 0$.

Choose a flat resolution $F(H_0) \to H_0$. Then consider $M_0 \times_{H_0} F(H_0)$ and ~~[illegible]~~ note that

$$\begin{tikzcd} M_0 \times_{H_0} F(H_0) \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & F(H_0) \arrow[d, "\text{(surj) qis}"] \\ M_0 \arrow[r, "\text{surj}"] & H_0 \end{tikzcd}$$

implies pr_1 is a (surj) qis and pr_2 is surjective. Choose ~~[illegible]~~ $F(M_0)$ to be a flat complex with a surj. qis $F(M_0) \longrightarrow M_0 \times_{H_0} F(H_0)$, and let $F(B_0) =$ Kernel of $F(M_0) \longrightarrow F(H_0)$. Then we have

$$\begin{tikzcd} 0 \arrow[r] & F(B_0) \arrow[r] \arrow[d] & F(M_0) \arrow[r] \arrow[d, "\text{(surj) qis}"] & F(H_0) \arrow[r] \arrow[d, "\text{(surj) qis}"] & 0 \\ 0 \arrow[r] & B_0 \arrow[r] & M_0 \arrow[r] & H_0 \arrow[r] & 0 \end{tikzcd}$$

and because $F(M_0)$ maps onto the fibre product we conclude $F(B_0) \longrightarrow B_0$ is a (surjective) qis. (The surjective in parantheses is obvious when B_0 is a single module, and even in the case of a complex is probably not essential.) The important point is that $F(B_0)$ being the kernel of a surjection of flat complexes is flat. ~~[illegible]~~

~~Now repeat this construction: Choose $F(M_1) \xrightarrow{\text{qis}} M_1 \times_{B_0} F(B_0)$:~~

$$\begin{tikzcd} 0 \arrow[r] & F(Z_1) \arrow[r] \arrow[d] & F(M_1) \arrow[r] \arrow[d] & F(B_0) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Z_1 \arrow[r] & M_1 \arrow[r] & B_0 \arrow[r] & 0 \\ 0 \arrow[r] & {} & & & \\ 0 \arrow[r] & B_1 \arrow[r] & {} & & \end{tikzcd}$$

Now continue this process following the Postnikov system of M:

$$\begin{tikzcd} 0 \arrow[r] & M_1/B_1 \arrow[r] \arrow[d] & M_0 \arrow[d, equal] & \\ 0 \arrow[r] & M_1/Z_1 \arrow[r] \arrow[d, phantom, "\shortparallel"] & M_0 \arrow[d] & \\ & B_0 & 0 \arrow[r] & M_0/B_0 \\ & & & H_0 \arrow[u, phantom, "\shortparallel"] \end{tikzcd}$$

Thus we construct

$$\begin{tikzcd} 0 \arrow[r] & F(H_1) \arrow[r] \arrow[d] & F(M_1/B_1) \arrow[r] \arrow[d, "\text{quis}"] & F(B_0) \arrow[r] \arrow[d, "\text{quis}"] & 0 \\ 0 \arrow[r] & H_1 \arrow[r] & M_1/B_1 \arrow[r] & B_0 \arrow[r] & 0 \end{tikzcd}$$

by choosing $F(M_1/B_1)$ flat mapping $\overbrace{\text{to}}^{\text{via a surj quis}}$ $(M_1/B_1) \times_{B_0} F(B_0)$

Then

$$\begin{tikzcd} 0 \arrow[r] & F(B_1) \arrow[r] \arrow[d] & F(Z_1) \arrow[r] \arrow[d] & F(H_1) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B_1 \arrow[r] & Z_1 \arrow[r] & H_1 \arrow[r] & 0 \end{tikzcd}$$

and Then

$$\begin{tikzcd} 0 \arrow[r] & F(H_2) \arrow[r] \arrow[d] & F(M_2/B_2) \arrow[r] \arrow[d] & F(B_1) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H_2 \arrow[r] & M_2/B_2 \arrow[r] & B_1 \arrow[r] & 0 \end{tikzcd}$$

But we haven't got $F(M_1)$?

So construct:

$$\begin{tikzcd} F(M_1) \arrow[r, two heads] \arrow[d, "\text{quis}"] & F(M_1/B_1) \arrow[r, two heads] \arrow[d, "\text{quis}"] & F(B_0) \arrow[r, two heads] \arrow[d, "\text{quis}"] & 0 \\ M_1 \arrow[r, two heads] & M_1/B_1 \arrow[r, two heads] & B_0 \arrow[r, two heads] & 0 \end{tikzcd}$$

Then define

$$F(B_1) = \operatorname{Ker}\{F(M_1) \to F(M_1/B_1)\}$$

$$F(Z_1) = \operatorname{Ker}\{F(M_1) \to F(B_0)\}$$

$$F(H_1) = \operatorname{Ker}\{F(M_1/B_1) \to F(B_0)\}$$

whence we have the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & F(B_1) \arrow[r] \arrow[d, "\text{qis}"] & F(Z_1) \arrow[r] \arrow[d, "\text{qis}"] & F(H_1) \arrow[r] \arrow[d, "\text{qis}"] & 0 \\ 0 \arrow[r] & B_1 \arrow[r] & Z_1 \arrow[r] & H_1 \arrow[r] & 0. \end{tikzcd}$$

This seems to proceed to yield the required C.E. resolution.

[redacted] I don't think it's essential for the derived category purposes to have C.E. resolutions. One can construct a qis $F \to M$ with F flat stupidly, by first expressing M as the quotient of a nearly contractible flat complex:

$$\begin{tikzcd} \arrow[r] & F_3 \oplus F_2 \arrow[r] \arrow[d] & F_2 \oplus F_1 \arrow[r] \arrow[d] & F_1 \oplus F_0 \arrow[r] \arrow[d] & 0 \\ \arrow[r] & M_2 \arrow[r] & M_1 \arrow[r] & M_0 \arrow[r] & 0 \end{tikzcd}$$

then doing the same for the kernel, etc.

Beilinson theory about sheaves on $\mathbb{P}^n$. Apparently Beilinson's short paper (~1978) is about a derived category correspondence between certain modules and certain sheaves. This is some sort of tilting business. A Polish mathematician has some improvements + complements, and his papers are reviewed by Jeremy Rickard. Here is what I have pieced together from the reviews.

Let $R = S(V)$, $I = V \cdot S(V)$, $k = R/I$. We have the Koszul resolution

$$0 \longrightarrow S(V) \otimes \Lambda^d V \longrightarrow \cdots\cdots \longrightarrow S(V) \otimes V \longrightarrow S(V) \longrightarrow k \to 0$$

for computing $\mathrm{Tor}^R_*(k, -)$ and $\mathrm{Ext}^*_R(k, -)$.

I'm going to concentrate on the Ext. This is because $\mathrm{Ext}^*_R(k, M)$ is a graded module over $\mathrm{Ext}^*_R(k,k) = \Lambda(V^*)$. Otherwise I would be working with $\mathrm{Tor}^R_*(k, M)$ which is a graded comodule over the coalgebra $\mathrm{Tor}^R_*(k,k) = \Lambda(V)$. One has a canonical isomorphism

$$\mathrm{Tor}^R_*(k, M) = \mathrm{Ext}^{d-*}_R(k, M) \otimes \Lambda^d V$$

Given an R-module M we get a d.g. $\Lambda(V^*)$ module $\Lambda(V^*) \otimes M$ with differential $\sum v_i^* \otimes v_i$ which computes $\mathrm{Ext}^*_R(k, M)$. Obviously this extends to a functor from complexes of R-modules, i.e. d.g. R-modules, to d.g. $\Lambda(V^*)$ modules. It should be clear that it descends to a functor between bdd derived categories.

Conversely a d.g. $\Lambda(V^*)$ module N gives rise to a d.g. R-module $S(V) \otimes N$ with differential $1 \otimes d + \sum_i v_i \otimes v_i^*$, and this also should descend to a functor between bdd derived categories.

When we compose these functors

$$M \longmapsto \Lambda(V^*) \otimes M \longmapsto S(V) \otimes \Lambda(V^*) \otimes M$$

one can't expect this to give the identity on $D^b(R)$, because $R\mathrm{Hom}_R(k, -)$ sees nothing away from the origin. In fact $R\mathrm{Hom}_R(k, -)$ vanishes exactly on the ferm = solid subcategory of $D^b(R)$.

One expects that for M perfect the total homology of $\Lambda(V^*) \otimes M$ should be finite diml. This is clear because M perfect $\iff$ M quis a finitely generated free R-module complex (recall f g projectives over $S(V)$ are ~~stably~~-free (Serre) and even free (Serre conjecture). Thus this assertion reduces to the case $M = R$, + clear.

Since $\Lambda(V^*)$ is finite-dimensional a bdd d.g. $\Lambda(V^*)$ module with f.d. homology should be quis a d.g. $\Lambda(V^*)$ module which is f.g. Clearly a f.g. d.g. $\Lambda(V^*)$ module N gives rise to a perfect R-module complex $S(V) \otimes N$.

~~This means that the derived category of perfect complexes over $S(V)$ should be equivalent~~

So far I haven't paid any ~~attention~~ to the grading on $S(V)$. So now restrict to graded modules over $S(V)$, which means that d.g. $S(V)$-modules are bigraded in some way, also d.g. $\Lambda(V^*)$-modules.

Then one can expect the functors between perfect complexes of graded $S(V)$-modules and f.d. (bi)graded $\Lambda(V^*)$-modules to give an equivalence of derived categories.

Finally under this equivalence perfect complexes whose homology is finite-dimensional correspond to free d.g. f.d. $\Lambda(V^*)$-modules. Thus one has an equivalence between the derived cat of perfect complexes over $\mathbb{P}(V^*)$ and the derived cat of bigraded d. f.d. $\Lambda(V^*)$-modules mod free such modules. This seems to be Beilinson's No* theorem, although his correspondence is perhaps slightly different (maybe related to [redacted] Mumford's regular sheaves).

What's needed here is to check the claims, say in the case $d=1$, just to be certain of the convergence. The problem [redacted] of interest for me concerns where the firm = solid derived category might come in. The firm = solid derived category is equivalent to the derived category of quasi-coherent sheaves on $\mathbb{P}(V^*)$, so the question is how to bring in the perfect complexes. You need some link - how to recognize when a solid complex corresponds to a perfect complex on $\mathbb{P}(V^*)$.

No*: Bernstein - Gelfand + Gelfand prove this result: derived category of f.g. graded $S(V)$ modules, and stable category (kill projectives) of suitable bigraded f.d. $\Lambda(V^*)$ modules, and derived cat of coherent sheaves on $\mathbb{P}(V^*)$ are equivalent.

Beilinson gets a true tilting example: $\exists$ fin. diml. algebra A such that bdd f.g. bounded derived cat of A-modules $\simeq$ derived cat of coh sheaves on $\mathbb{P}_n$.

August 11, 1994

Problems:

1) In the comm. noetherian case, where $\mathcal{M}_t$ is the category of quasi-coherent sheaves on $U = Sp(R) - Sp(R/I)$, one has the notion of perfect complex in $D^b(\mathcal{M}_t)$. In fact Grothendieck in SGA6 has described perfect complexes intrinsically as objects in the derived category which are of finite presentation in a suitable sense. You want to develop this idea, e.g. find out whether there is an intrinsic notion of perfect complex in general, whether it depends only on I as prering, and whether it is Morita invariant.

2) Suppose R quasi-free, does it follow for any ideal I that the excision result holds; namely that any complex with torsion homology is quasi-isomorphic to a complex of torsion modules. Maybe null instead of torsion.

August 12, 1994

Beilinson thm. First do $\mathbb{P}^1$. Let $T = \mathcal{O} \oplus \mathcal{O}(-1)$. Then T generates the derived category of coherent sheaves on $\mathbb{P}^1$. In effect

$$0 \to \Lambda^2 V \otimes \mathcal{O}(-2) \to V \otimes \mathcal{O}(-1) \to \mathcal{O} \to 0$$

$$0 \to \Lambda^2 V \otimes \mathcal{O}(-3) \to V \otimes \mathcal{O}(-2) \to \mathcal{O}(-1) \to 0$$

$$\vdots$$

shows $\mathcal{O}(-1), \mathcal{O}(-2), \ldots$ lie in the Δated subcategory generated by $\mathcal{O}, \mathcal{O}(-1)$. Similarly

$$0 \to \Lambda^2 V \otimes \mathcal{O}(-1) \to V \otimes \mathcal{O} \to \mathcal{O}(1) \to 0$$

shows $\mathcal{O}(1), \mathcal{O}(2), \ldots$ lie in this Δated subcat.

Next $\mathrm{Ext}^*(T,T)$.

$$\mathcal{H}om(T,T) = \begin{pmatrix} \mathcal{O} \\ \mathcal{O}(-1) \end{pmatrix} \otimes \begin{pmatrix} \mathcal{O} & \mathcal{O}(1) \end{pmatrix} \simeq \begin{pmatrix} \mathcal{O} & \mathcal{O}(1) \\ \mathcal{O}(-1) & \mathcal{O} \end{pmatrix}$$

$$\Gamma(\mathcal{H}om(T,T)) = \begin{pmatrix} k & V \\ 0 & k \end{pmatrix} \qquad \text{call this } A$$

$$H^1(\mathcal{H}om(T,T)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{as } H^1(\mathcal{O}(-1)) = 0$$

Thus
$$\mathrm{Ext}^n(T,T) = H^n(\mathcal{H}om(T,T)) = \begin{cases} A & n = 0 \\ 0 & n \neq 0 \end{cases}$$

so by ~~tilting~~ theory we should have an equivalence between $D^b_{coh}(\mathbb{P}^1)$ and $D^b_{fg}(A)$

In general for $\mathbb{P}(V^*)$ the same thing works. Let $T = \mathcal{O} \oplus \mathcal{O}(-1) \oplus \cdots \oplus \mathcal{O}(-d+1)$ $\quad d = \dim V$ and recall $H^q(\mathbb{P}(V^*), \mathcal{O}(n)) = 0$ unless $\begin{matrix} q = 0 & n \geq 0 \\ q = d-1 & n \leq -d \end{matrix}$

We have the basic exact sequence

$$0 \to \Lambda^d V \otimes \mathcal{O}(-d) \to \cdots\cdots \to V \otimes \mathcal{O}(-1) \to \mathcal{O} \to 0$$

which represents a canonical generator of

$$H^{d-1}(\mathbb{P}(V^*), \mathcal{O}(-d)) \otimes \Lambda^d V$$

and the rest comes from Serre duality.

We have

$$\mathcal{H}om(T, T) = \begin{pmatrix} \mathcal{O} \\ \\ \mathcal{O}(-d+1) \end{pmatrix} \otimes_{\mathcal{O}} \begin{pmatrix} \mathcal{O} & \cdots & \mathcal{O}(d-1) \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{O} & \mathcal{O}(1) & \cdots & \mathcal{O}(d-1) \\ \mathcal{O}(-1) & \mathcal{O} & & \\ & & & \\ \mathcal{O}(-d+1) & \cdots & \cdots & \mathcal{O} \end{pmatrix}$$

so that

$$\mathrm{Ext}^q(T, T) = \begin{cases} 0 & q \neq 0 \\ \begin{pmatrix} k & V & S^2V & \cdots & S^{d-1}V \\ & k & V & & \vdots \\ & 0 & k & & \vdots \\ & & & \ddots & k \end{pmatrix} \end{cases}$$

It might be better instead of T to take the thing that occurs with Severi-Brauer varieties. This means using $\Lambda^j V \otimes \mathcal{O}(-j)$ for $j = 0, \ldots, d-1$.

Thus for $d = 2$:

$$A = \begin{pmatrix} k & V \otimes V^* \\ 0 & k \end{pmatrix}$$

Q: Is A of finite global dim, i.e. homological dimension?

for $d = 3$:

$$A = \begin{pmatrix} k & V \otimes V^* & S^2V \otimes \Lambda^2 V^* \\ & k & V \otimes V^* \\ & & k \end{pmatrix}$$

August 14, 1994

Geigle-Lenzing paper on perpendicular categories J-Alg 144 (1991)

Basic definition: Given a class $\mathcal{S}$ of objects in an abelian category $\mathcal{A}$ one defines the right + left $\perp$ categories by

$$\mathcal{S}^{\perp} = \{ M \mid \mathrm{Hom}(T,M) = \mathrm{Ext}^1(T,M) = 0 \quad \forall T \in \mathcal{S}\}$$

$$^{\perp}\mathcal{S} = \{ M \mid \mathrm{Hom}(M,T) = \mathrm{Ext}^1(M,T) = 0 \quad \forall T \in \mathcal{S}\}$$

Consider $\mathcal{A} = \mathrm{mod}(R)$, $\mathcal{S} = \mathrm{mod}(R/I)$. Then

$$\mathcal{S}^{\perp} = \text{solid}(R,I)$$

$$^{\perp}\mathcal{S} = \text{firm}(R,I)$$

Check: Given a module M, choose $0 \to K \to P \to M \to 0$ with P projective. Then

$$0 \to \mathrm{Hom}_R(M,T) \to \underset{\substack{\| \\ \mathrm{Hom}_{R/I}(P/IP,T)}}{\mathrm{Hom}_R(P,T)} \to \underset{\substack{\| \\ \mathrm{Hom}_{R/I}(K/IK,T)}}{\mathrm{Hom}_R(K,T)} \to \mathrm{Ext}^1_R(M,T) \to 0$$

so $M \in {}^{\perp}\mathcal{S} \iff K/IK \xrightarrow{\sim} P/IP$, which by

$$0 \to \mathrm{Tor}_1^R(R/I,M) \to K/IK \to P/IP \to M/IM \to 0$$

is equivalent to $M \in \text{firm}(R,I)$.

Similarly if $0 \to M \to Q \to C \to 0$ with Q injective, one has

$$0 \to \operatorname{Hom}_R(T,M) \to \underset{\substack{\| \\ \operatorname{Hom}_{R/I}(T,{}_IQ)}}{\operatorname{Hom}_R(T,Q)} \to \underset{\substack{\| \\ \operatorname{Hom}_{R/I}(T,{}_IC)}}{\operatorname{Hom}_R(T,C)} \to \operatorname{Ext}^1_R(T,M) \to 0$$

so $M \in \mathcal{S}^\perp \iff {}_IQ \xrightarrow{\sim} {}_IC$, which is equivalent to $\operatorname{Ext}^j_R(R/I, M) = 0$ for $j = 0, 1$. i.e. to $M \in \mathrm{sol}(R,I)$.

Consider a homomorphism $R \to U$.

Suppose restriction of scalars $\mathrm{mod}(U) \to \mathrm{mod}(R)$ is fully faithful: for all U-modules M, N we have

$$\operatorname{Hom}_U(M,N) \xrightarrow{\sim} \operatorname{Hom}_R(M,N)$$
$$\operatorname{Hom}_R(M,N) = \operatorname{Hom}_U(U\otimes_R M, N)$$

Since this holds $\forall N$, we must have $U\otimes_R M \xrightarrow{\sim} M$ in particular $U\otimes_R U \xrightarrow{\sim} U$.

Conversely $U\otimes_R U \xrightarrow{\sim} U \overset{-\otimes_U M}{\Longrightarrow} U\otimes_R M \xrightarrow{\sim} M$
$\Longrightarrow$ $\mathrm{mod}(U) \to \mathrm{mod}(R)$ fully faithful.

Suppose that $R \to U$ is an epimorphism in the category of rings, i.e. $\operatorname{Hom}_{\text{rings}}(U,S) \to \operatorname{Hom}_{\text{rings}}(R,S)$ is injective for all S. Recall that the U-bimodule $\Omega^1_R U = \operatorname{Ker}\{U\otimes_R U \to U\}$ is universal for derivations $D: U \to B$ (vanishing on R), where B is a U-bimodule.

If $\Omega^1_R U \neq 0$, then we have a nontrivial derivation vanishing on R

$$u \xmapsto{d} 1\otimes u - u\otimes 1 \in \Omega^1_R U$$

hence two homomorphisms $1, 1+d : U \to U \oplus \Omega^1_R U$ which agree on R. Thus $R \to U$ an epim. of rings $\Longrightarrow U\otimes_R U \xrightarrow{\sim} U$.

Conversely if $f, g : U \rightrightarrows S$ are two $\neq$ homomorphisms agreeing on R, then

$$f-g : U \longrightarrow S$$

is a $\neq 0$ derivation from U to S considered as U bimodule via f on one side and g on the other. This implies $\Omega^1_R U \neq 0$, hence $U\otimes_R U \to U$ is not an isom.

Thus we have

Prop. TFAE for a homom. $R \to U$.

1) restriction of scalars $\mathrm{mod}(U) \to \mathrm{mod}(R)$ is fully faithful.

2) $U \otimes_R U \xrightarrow{\sim} U$

2') $\Omega^1_R U = 0$

3) $R \to U$ is an epimorphism in the category of rings.

In Geigle-Lenzing there is the notion of homological epimorphisms $R \to U$ of rings involving $U \overset{L}{\otimes}_R U \xrightarrow{\sim} U \iff D(U) \to D(R)$ fully faithful.

You've examined the case $U = R/I$.

August 25, 1994

$\mathcal{A}$ abelian category, $\mathcal{T}$, $\mathcal{U}$ full subcategories such that

1) $\mathrm{Ext}^j(T, U) = 0 \quad j=0,1 \quad T \in \mathcal{T} \quad U \in \mathcal{U}$

2) $\forall M$ in $\mathcal{A}$ $\exists \tilde{M}$ in $\mathcal{U}$ and $\varepsilon_M : M \to \tilde{M}$ such that $\mathrm{Ker}(\varepsilon_M)$, $\mathrm{Coker}(\varepsilon_M)$ are in $\mathcal{T}$.

Given $0 \to M' \to M \to M'' \to 0$ we get

$$(*) \qquad 0 \to \mathrm{Hom}(M'', U) \to \mathrm{Hom}(M, U) \to \mathrm{Hom}(M', U) \to \mathrm{Ext}^1(M'', U)$$

Thus M' in $\mathcal{T} \Rightarrow \mathrm{Hom}(M'', U) \xrightarrow{\sim} \mathrm{Hom}(M, U)$

M'' in $\mathcal{T} \Rightarrow \mathrm{Hom}(M, U) \xrightarrow{\sim} \mathrm{Hom}(M', U)$

so $\mathrm{Hom}(-, U)$ inverts any map with Ker + Coker in $\mathcal{T}$. In particular ε_M so

$$(**) \qquad \varepsilon_M^* : \mathrm{Hom}(\tilde{M}, U) \xrightarrow{\sim} \mathrm{Hom}(M, U) \qquad \forall U \text{ in } \mathcal{U}$$

This implies $(\tilde{M}, \varepsilon_M)$ unique up to canon. isom., also that $M \mapsto \tilde{M}$ is left adjoint to $\mathcal{U} \subset \mathcal{A}$.

Next TFAE for M in $\mathcal{A}$

1) $\Downarrow$ (1) $M \in \mathcal{T}$

(2) $\mathrm{Hom}(M, U) = 0 \quad \forall U \in \mathcal{U}$

$(**)$ $\Updownarrow$ (3) $\tilde{M} = 0$

Finally $\tilde{M} = 0 \Rightarrow M \to 0$ has kernel in $\mathcal{T}$ so $M \in \mathcal{T}$.

Returning to $(*)$ one then sees that M in $\mathcal{T}$ $\Leftrightarrow$ M' and M'' in $\mathcal{T}$. (note M in $\mathcal{T} \Rightarrow \mathrm{Hom}(M, U) = 0$ $\forall U \Rightarrow \mathrm{Hom}(M'', U) = 0$ $\forall U \Rightarrow M'' \in \mathcal{T} \Rightarrow \mathrm{Ext}^1(M'', U) = 0$, so then $\mathrm{Hom}(M', U) = 0$ $\forall U \Rightarrow M' \in \mathcal{T}$).

Thus $\mathcal{T}$ is a Serre subcategory. As before we have $\mathrm{Hom}_{\mathcal{A}/\mathcal{T}}(M, U) = \varinjlim \mathrm{Hom}(M', U) = \mathrm{Hom}(M, U)$

showing that $\mathcal{U} \to \mathcal{A}/\mathcal{T}$ is fully faithful, then essentially surjective by 2).
Also defining τM by

$$0 \to \tau M \to M \xrightarrow{\varepsilon_M} \tilde{M}$$

we have $\mathrm{Hom}(T, \tau M) \xrightarrow{\sim} \mathrm{Hom}(T, M) \quad \forall T$
so τ is right adjoint to the inclusion $\mathcal{T} \subset \mathcal{A}$.

Next suppose $\mathcal{A}$ Grothendieck (generator + AB5). Then $M \in \mathcal{T} \iff \mathrm{Hom}(M, U) = 0 \ \forall U$ implies $\mathcal{T}$ is closed under direct sums. Then one knows that $\mathcal{A}/\mathcal{T}$ is Grothendieck, hence also $\mathcal{U}$ and that injectives in $\mathcal{U}$ are the same as $\mathcal{T}$-free injectives in $\mathcal{A}$.

August 26, 1994

Thick subcategory [redacted] $\mathcal{V}$ of a triangulated category $\mathcal{D}$ according to Verdier (SGA $4\frac{1}{2}$) is a full subcategory closed under translation and cones, such that for any map $X \to Y$ in $\mathcal{D}$, if the cone is in [redacted] $\mathcal{V}$ and the map factors through an object of $\mathcal{V}$ then both X, Y are in $\mathcal{V}$.

Check that if $\mathcal{V}, \mathcal{W} \subset \mathcal{D}$ satisfy

1) $\mathrm{Hom}(V, W) = 0 \quad \forall V \in \mathcal{V},\ W \in \mathcal{W}$

2) $\forall X$ in $\mathcal{D}$ $\exists \Delta \quad V \to X \to W \to$

then both $\mathcal{V}$ and $\mathcal{W}$ are thick.

First note that these conditions imply for $V \to X \to W \to$ as in 2) that

$$\mathrm{Hom}(V', V) \xrightarrow{\sim} \mathrm{Hom}(V', X) \qquad \forall V' \in \mathcal{V}$$

$$\mathrm{Hom}(W, W') \xrightarrow{\sim} \mathrm{Hom}(X, W') \qquad \forall W' \in \mathcal{W}$$

which means that $\mathcal{V} \subset \mathcal{D}$ has right adjoint $X \mapsto V$ and $\mathcal{W} \subset \mathcal{D}$ has left adjoint $X \mapsto W$.

Then
$$\mathrm{Hom}(V', X) = 0 \quad \forall V' \implies X \xrightarrow{\sim} W \in \mathcal{W}$$
$$\mathrm{Hom}(X, W') = 0 \quad \forall W' \implies V \xrightarrow{\sim} X \in \mathcal{V}$$

Now suppose given $f : X \to Y$ in $\mathcal{D}$. If the Cone on f is in $\mathcal{V}$ we have

$$\mathrm{Hom}(Y, W') \xrightarrow{\sim} \mathrm{Hom}(X, W') \qquad \forall W'$$

and if f factors [redacted] $X \to V'' \to Y$ then this isom. factors through $\mathrm{Hom}(V, W') = 0$. $\therefore X, Y \in \mathcal{V}$.

Similarly $\mathcal{W}$ is thick.

The significance of the factorization condition is not clear to me. It must somehow be used in constructing the Δated structure on $\mathcal{D}/\mathcal{V}$. All you need to define this category is the class of maps to be inverted, and these are the ones whose cones are in $\mathcal{V}$. Such $\mathcal{V}$ isoms. are closed under composition by the octahedral axiom.

Let's go back to $\mathcal{A}$ Grothendieck, $\mathcal{T}$ a Serre subcategory (closed under $\oplus$s), $\mathcal{A}/\mathcal{T}$ the quotient Grothendieck category. We have adjoint functors

$$\mathcal{T} \underset{\iota^!}{\overset{\iota_*}{\rightleftarrows}} \mathcal{A} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathcal{A}/\mathcal{T}$$

and an equivalence $\mathcal{T}^\perp \xrightarrow{\sim} \mathcal{A}/\mathcal{T}$.

Consider next the derived category situation: There are functors

$$\underbrace{D^+(\mathcal{A})_{\mathcal{T}}}_{\text{means homology groups in } \mathcal{T}} \underset{R\iota^!}{\overset{\iota_*}{\rightleftarrows}} D^+(\mathcal{A}) \underset{Rj_*}{\overset{j^*}{\rightleftarrows}} D^+(\mathcal{A}/\mathcal{T})$$

Because $\mathcal{A}$, $\mathcal{A}/\mathcal{T}$ are Grothendieck, they have sufficiently many injectives, hence $R\iota^!$, Rj_* $\exists$ and are calculated by injective resolutions.

We know that $D^+(\mathcal{A}) \simeq K^+(\mathrm{Inj}(\mathcal{A}))$

$\boxed{\mathrm{Hom}(T,Q)=0 \ \forall T\in\mathcal{T}}$

and $D^+(\mathcal{A}/\mathcal{T}) \simeq K^+(\mathrm{Inj}(\mathcal{A}/\mathcal{T}))$. Moreover j_* induces an equivalence $\mathrm{Inj}(\mathcal{A}/\mathcal{T}) \xrightarrow{\sim} \{Q \in \mathrm{Inj}(\mathcal{A}) \mid \ \}$

I want to check that fits into the $\mathcal{V}, \mathcal{W} \subset \mathcal{D}$ discussion. Here $\mathcal{D} = D^+(\mathcal{A})$, $\mathcal{V} = D^+(\mathcal{A})_{\mathcal{T}}$, and $\mathcal{W}$ is the full subcategory of $\mathcal{D}$ consisting of complexes W satisfying $\mathrm{Hom}^*_{\mathcal{D}}(T, W) = 0$ for all $T \in \mathcal{T}$, T being considered as a complex supported in degree zero.

First check that $\mathrm{Hom}^*_{\mathcal{D}}(V, W) = 0$ for $V \in \mathcal{V}$, $W \in \mathcal{W}$. This follows from the Postnikov filtration of V, which increases, adding one homology group at a time.

Next given $X \in \mathcal{D}$ we want to construct a triangle $V \to X \to W \to$. Here W will be $Rj_*(j^*X)$. More precisely, we choose an injective resolution of j^*X, which is a complex in $\mathrm{Inj}(\mathcal{A}/\mathcal{T}) \xrightarrow[\sim]{j_*} \left\{ Q \;\middle|\; \begin{matrix} Q \text{ injective in } \mathcal{A} \\ \mathrm{Hom}(T, Q) = 0 \ \forall T \in \mathcal{T} \end{matrix} \right\}$. Thus we have a complex Q of $\mathcal{T}$-free injectives in $\mathcal{A}$ and a quis $j^*X \to j^*Q$, equivalently a map $X \to j_*(j^*Q) = Q$ whose cone has homology gps in $\mathcal{T}$. Thus the desired triangle $V \to X \to W \to$ is given by $W = Q$ and $V = \mathrm{Cone}(X \to W)$.

The rest now should fall in place: $Ri^!$, Rj_* are right adjoints of i_*, j^*. Equivalence of $\mathcal{W} = \{ W \in \mathcal{D} \mid \mathrm{Hom}^*_{\mathcal{D}}(T, W) = 0 \ \forall T \in \mathcal{T} \}$ with $D^+(\mathcal{A}/\mathcal{T})$.

Let's try to make things clear[redacted]er in the derived category situation. Suppose I replace $D^+(\mathcal{A})$ with the equivalent category $K^+(\mathrm{Inj}(\mathcal{A}))$. Similarly replace $D^+(\mathcal{A}/\mathcal{T})$ by $K^+(\mathcal{T}\text{-free }\mathrm{Inj}(\mathcal{A}))$. [redacted] Notation:

$$\mathcal{D} = K^+(\mathrm{Inj}(\mathcal{A})) \qquad \mathcal{W} = K^+(\mathcal{T}\text{free }\mathrm{Inj}(\mathcal{A}))$$

$$\mathcal{V} = K^+(\mathrm{Inj}(\mathcal{A}))_{\mathcal{T}}$$ complexes of injectives with homology in $\mathcal{T}$.

Note $\mathcal{V}$, [redacted] $\mathcal{W}$ are full subcats. of [redacted] $\mathcal{D}$. Then the conditions $\mathrm{Hom}_{\mathcal{D}}(V,W)=0$, $\forall M \ \exists \Delta$ $V \to M \to W \to$ are satisfied, so one [redacted] knows that [redacted] there are adjoints

$$\begin{tikzcd} \mathcal{V} \arrow[r, hook, shift left] & \mathcal{D} \arrow[l, shift left] \arrow[r, shift left] & \mathcal{W} \arrow[l, hook, shift left] \end{tikzcd}$$

such that both

$$\mathcal{V} \longrightarrow \mathcal{D} \longrightarrow \mathcal{W}$$
$$\mathcal{W} \longrightarrow \mathcal{D} \longrightarrow \mathcal{V}$$

are 'exact', [redacted] equivalently: $\mathcal{W} \to \mathcal{D}/\mathcal{V}$ and $\mathcal{V} \to \mathcal{D}/\mathcal{W}$ are equivalences of Δated categories.

~~So~~ the only issue remaining here is the relation of $D^+(\mathcal{A})_{\mathcal{T}}$ with $D^+(\mathcal{T})$. Restricting to injective complexes, I want to know whether a complex of injectives V with homology in $\mathcal{T}$ is quasi-isomorphic to a complex in $\mathcal{T}$.

Let's consider $\mathcal{A} = mod(R)$, $\mathcal{T} = tors(R, I)$, $\mathcal{A}/\mathcal{T} = \mathcal{M}(R, I)$. We want conditions sufficient that $D^+(R)_{tors} \xleftarrow{\sim} D^+(\mathcal{T})$. In other words given an injective (bdd to the left) complex of R-mods with torsion homology, we would like it to be quasi-isomorphic to a complex of torsion modules.

1st condition is stability (i.e. tors is a stable torsion theory). This means E injective R-module $\Longrightarrow \tau E$ is injective R-module.

This condition implies

2nd condition: E injective R-module $\Longrightarrow E/\tau E$ is an injective R-module.

Suppose the 2nd condition holds. Then if E is an injective complex with torsion homology we have an exact sequence of complexes

$$0 \longrightarrow \tau E \longrightarrow E \longrightarrow E/\tau E \longrightarrow 0$$

where $E/\tau E$ is torsion-free + injective. But we know that $RHom_R(E, E/\tau E) \simeq 0$ because E has torsion homology and $E/\tau E$ is solid injective. Thus $E \longrightarrow E/\tau E$ is homotopic to zero, whence $\exists$ $E \longrightarrow \tau E$ such that $E \longrightarrow \tau E \longrightarrow E$ is homotopic to the identity.

It's simpler to look at homology where we have short exact sequences

$$0 \longrightarrow H^{n-1}(E/\tau E) \longrightarrow H^n(\tau E) \longrightarrow H^n(E) \longrightarrow 0$$

Because $E/\tau E$ solid injective its first non-zero homology group is solid hence torsion-free.

This means $E/\tau E$ must be acyclic, and so $\tau E \to E$ is a quis

Observe that the 2nd condition holds if proj. dim. $(R) \leq 1$, since then any ~~quotient~~ of an injective is injective.

In the case $I = I^2$ we know the first condition (stability) is equivalent to R/I being a flat R^{op} module.

Similarly as $E/\tau E = \mathrm{Hom}_R(I, E)$ for E injective and

$$\mathrm{Hom}_R(M, \mathrm{Hom}_R(I, E)) = \mathrm{Hom}_R(I \otimes_R M, E)$$

we see that $E/\tau E$ injective for all $E \iff I \otimes_R -$ exact $\iff I$ is R^{op}-flat.

In the case $I = I^2$ we have a necessary and sufficient condition for $D^+(\mathrm{tors}) \xrightarrow{\sim} D^+(R)_{\mathrm{tors}}$, namely $I \overset{L}{\otimes}_R R/I \simeq 0$, and we know this condition depends only on the ring I.

In general we have the criterion that proj. dim. $(R) \leq 1$, also stability holds when R is commutative noetherian.

It's possible that Joachim's approximate h-unitality is relevant here. One can pose the question of whether any $X \in D^b(R)_{\mathrm{null}}$ is quis a complex in $D^b(R/I^n)$ for some n. Idea of using the bar construction to obtain the derived category of modules in some adic sense - e.g. pro-nilpotent completion of π_1

August 30, 1994

Return to the question of whether

$$\mathcal{M}(R,I) \longrightarrow \mathrm{add}(\mathrm{firm}(R^{op},I^{op}), \mathcal{Ab})$$
$$M \longmapsto -\otimes_R M$$

is fully faithful, hopefully the essential image consists of the right continuous functors.

The idea I have is to show that I can recover $\tilde{M}$ $(= j_*j^*M)$ from the family of abelian groups $F\otimes_R M$ with F a firm flat right module. In fact we would like to use F of the form $F(\alpha)$, α a sequence in I.

Some evidence that this might be possible. First we know that the torsion elements $m \in M$ are the T-nilpotent elements: $\forall$ sequence $a_1, \ldots$ in I $\exists n$ $a_n \cdots a_1 m = 0$. ~~This~~ This means that ${}_IM$ is the intersection of the kernels of the canonical maps $M \longrightarrow F(\alpha)\otimes_R M$, where α runs over all sequences in I. (A related result is that there are enough solid injectives of the form $\mathrm{Hom}_{\mathbb{Z}}(F(\alpha), \mathbb{Q}/\mathbb{Z})$. Why? If $m \notin {}_IM$, then $\exists \alpha$ such that m does not go to zero under $M \to F(\alpha)\otimes_R M$. Then there is a character χ such that m is not killed by $M \longrightarrow F(\alpha)\otimes_R M \xrightarrow{\chi} \mathbb{Q}/\mathbb{Z}$. By

$$\mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(F(\alpha), \mathbb{Q}/\mathbb{Z})) = \mathrm{Hom}_{\mathbb{Z}}(F(\alpha)\otimes_R M, \mathbb{Q}/\mathbb{Z})$$

we get a map $M \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(F(\alpha), \mathbb{Q}/\mathbb{Z})$, such that the composition with $\mathrm{Hom}_{\mathbb{Z}}(F(\alpha), \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ does not kill M.)

Further evidence comes from the commutative R finitely generated I case. If $I = \sum_{i=1}^{n} Rf_i$, then we know that

$$\tilde{M} = \Gamma\Big(\overbrace{\mathrm{Spec}(R) - \mathrm{Spec}(R/I)}^{\bigcup \mathrm{Spec}(R_{f_i})},\ \text{sheaf assoc. to } M\Big)$$

$$= \mathrm{Ker}\Big\{\prod_i M_{f_i} \longrightarrow \prod_{i,j} M_{f_i f_j}\Big\}$$

where $M_f = F(f,f,\ldots) \otimes_R M$.

Let's look at the case $I = I^2$. Here we know the final result holds: namely equivalence

$$\begin{tikzcd}
\mathcal{M}(R,I) \arrow[r] & \text{rt cont add}(\mathrm{firm}(R^{op},I^{op}), ab) \\
\mathrm{firm}(R,I) \arrow[r] \arrow[u, "\simeq"] & \text{rt cont add}(\mathcal{M}(R^{op},I^{op}), ab) \arrow[u, "\simeq"']
\end{tikzcd}$$

where $\Longrightarrow$ $I = I^2$ is used

because is an equivalence in general.

But in any case we can try to see if our approach works, namely to express

$$\tilde{M} = \mathrm{Hom}_R(I^{(2)}, M)$$

somehow in terms of $F(\alpha) \otimes_R M$.

Example. $I = ReR$ where $e^2 = e$. Then we know $ReR = Re \otimes_S eR$ where $S = eRe$ so $\tilde{M} = \mathrm{Hom}_R(Re \otimes_S eR, M) = \mathrm{Hom}_S(eR, \mathrm{Hom}_R(Re, M))$

$$\therefore \quad \tilde{M} = \mathrm{Hom}_S(eR, eM)$$

If we choose a presentation

$$S^{(\Lambda_1)} \longrightarrow S^{(\Lambda_0)} \longrightarrow eR \longrightarrow 0$$

then we get

$$0 \longrightarrow \tilde{M} \longrightarrow (eM)^{\Lambda_0} \longrightarrow (eM)^{\Lambda_1}$$

where $\quad eM = F(e, e, \ldots) \otimes_R M$.

If we choose a presentation of eR ~~by~~ free S-R bimodules $S \otimes_{\mathbb{Z}} R$, then $\tilde{M}$ is a kernel of a map between products of the R-module

$$\mathrm{Hom}_S(S \otimes_{\mathbb{Z}} R, eM) = \mathrm{Hom}_{\mathbb{Z}}(R, eM).$$

Another example: suppose I is a firm flat right module. Then we have

$$\tilde{M} = \mathrm{Hom}_R(I, I \otimes_R M)$$

since $\mathrm{Hom}_R(I, -) \Rightarrow \mathrm{Hom}_R(I^{(2)}, -)$ inverts the null-isom $I \otimes_R M \longrightarrow M$. Choosing a presentation

$$R^{(\Lambda_1)} \longrightarrow R^{(\Lambda_0)} \longrightarrow I \longrightarrow 0$$

of left modules, we get

$$0 \longrightarrow \tilde{M} \longrightarrow (I \otimes_R M)^{\Lambda_0} \longrightarrow (I \otimes_R M)^{\Lambda_1}$$

where $I \otimes_R M$ has the form $F \otimes_R M$ at least.

Q: Is it possible to reduce to this case by Morita equivalence?

September 1, 1994

Consider the derived category Δ business in more generality. Start with a Grothendieck category $\mathcal{A}$ and localizing Serre subcategory $\mathcal{T}$:

$$\mathcal{T} \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathcal{A} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathcal{A}/\mathcal{T}$$

$$i^! i_* = 1 \qquad j^* j_* = 1$$

For any $M \in \mathcal{A}$ we have an exact sequence

$$0 \longrightarrow i_* i^! M \longrightarrow M \longrightarrow j_* j^* M$$

where the last map is surjective when M is an injective object. NOT CLEAR!

Let's check this last point. It suffices to take any map $N \xrightarrow{f} j_* j^* M$, then show $\exists$

$$\begin{tikzcd} N_1 \arrow[r] \arrow[d] & M \arrow[d] \\ N \arrow[r, "f"] & j_* j^* M \end{tikzcd}$$

for some $N_1 \twoheadrightarrow N$ surjective. The arrow f is equivalent to a map in

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{T}}(j^*N, j^*M) = \varinjlim_{N' \to N \ \mathcal{T}\text{ isom.}} \mathrm{Hom}_{\mathcal{A}}(N', M)$$

So f can be represented by a correspondence

$$\begin{tikzcd} & N' \arrow[dl] \arrow[dr] & \\ N & & M \end{tikzcd}$$

Let us factor $N' \to N$ into
an injection followed by a surjection
$$N' \hookrightarrow N_1 \twoheadrightarrow N$$
say via Grothendieck's graph $N' \subset N \times U \to N$
Then we have

$$\begin{tikzcd} N' \arrow[r] \arrow[d, hook] & M \\ N_1 \arrow[ur, dashed, "\exists \text{ if } M \text{ is injective}"'] \arrow[d] & \\ N & \end{tikzcd}$$

I need to arrange that $N \leftarrow N_1 \to M$ also represents f, and it's clear then that I want $N_1 \twoheadrightarrow N$ to have kernel in $\mathcal{T}$. So I have to be able to factor a $\mathcal{T}$-isomorphism into an injection followed by surjection where these are $\mathcal{T}$-isos. ?

This doesn't seem to work, but perhaps it is not needed.

September 2, 1994

Start with $\mathcal{A}$ a Grothendieck category $\mathcal{T}$ a Serre subcategory closed under $\oplus$'s, whence we have adjoint functors

$$\mathcal{T} \underset{\iota^!}{\overset{\iota_*}{\rightleftarrows}} \mathcal{A} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathcal{A}/\mathcal{T}$$

such that ι_*, j^* are exact

$j^*\iota_* = 0$

$$\iota^!\iota_* = 1 \qquad j^*j_* = 1 \qquad \iota^! j_* = 0$$

$$0 \to \iota_*\iota^! M \to M \longrightarrow j_*j^*M \qquad \text{exact } \forall M$$

Now consider

$$D^+(\mathcal{T}) \underset{R\iota^!}{\overset{\iota_*}{\rightleftarrows}} D^+(\mathcal{A}) \underset{Rj_*}{\overset{j^*}{\rightleftarrows}} D^+(\mathcal{A}/\mathcal{T})$$

Check these are adjoint functors as indicated. Let X, Y, Z be left bounded complexes of injective objects in $\mathcal{A}, \mathcal{T}, \mathcal{A}/\mathcal{T}$ resp.

$$\begin{aligned}
R\,\mathrm{Hom}_{\mathcal{A}}(X, Rj_*(Z)) &= R\,\mathrm{Hom}_{\mathcal{A}}(X, j_*Z) && Z \text{ inj} \\
&= \mathrm{Hom}_{\mathcal{A}}(X, j_*Z) && j_*Z \text{ inj since } j_* \text{ has exact left adjoint} \\
&= \mathrm{Hom}_{\mathcal{A}/\mathcal{T}}(j^*X, Z) \\
&= R\,\mathrm{Hom}_{\mathcal{A}/\mathcal{T}}(j^*X, Z) && Z \text{ inj}
\end{aligned}$$

$$\begin{aligned}
R\,\mathrm{Hom}_{\mathcal{T}}(Y, R\iota^!(X)) &= R\,\mathrm{Hom}_{\mathcal{T}}(Y, \iota^!X) && X \text{ inj} \\
&= \mathrm{Hom}_{\mathcal{T}}(Y, \iota^!X) && \iota^!X \text{ inj as } \iota^! \text{ has exact left adj.}
\end{aligned}$$

$$= \mathrm{Hom}_{a}(\iota_* Y, X)$$
$$= R\mathrm{Hom}_{a}(\iota_* Y, X)$$

(The above works obviously for a pair (f^*, f_*) with f^* exact; the fact that one or the other is injective is irrelevant.)

Now we know $\boxed{j^* Rj_* = 1}$

since $j^* Rj_*(Z) = j^* j_* Z = Z$. Thus

$$R\mathrm{Hom}_{a}(Rj_*(Z_1), Rj_*(Z_2)) = R\mathrm{Hom}_{a/\mathcal{I}}(j^* Rj_*(Z_1), Z_2)$$
$$= R\mathrm{Hom}_{a/\mathcal{I}}(Z_1, Z_2)$$

showing Rj_* <u>is fully faithful</u>.

But $(R\iota^!)\iota_*$ need not be the identity since ι_* need not preserve injectives.

$\boxed{\text{Assume } (R\iota^!)\iota_* = 1.}$

Then

$$R\mathrm{Hom}_{a}(\iota_* Y_1, \iota_* Y_2) = R\mathrm{Hom}_{\mathcal{I}}(Y_1, R\iota^!(\iota_* Y_2))$$
$$= R\mathrm{Hom}_{\mathcal{I}}(Y_1, Y_2)$$

so that i_* <u>is fully faithful</u>.

I should have noted earlier before making the above assumption the following

$\boxed{j^* \iota_* = 0}$ and $\boxed{R\iota^! Rj_* = 0}$ also follows by adjunction from $j^*\iota_* = 0$

$$R\iota^! Rj_*(Z) \underset{\substack{\uparrow \\ Z \text{ inj}}}{=} Ri^! j_* Z \underset{\substack{\uparrow \\ j_* Z \text{ inj}}}{=} \iota^! j_* Z = 0.$$

Then we have the orthogonality

$$R\,Hom_{\mathcal{A}}(\iota_* Y,\ R_{j_*}(Z)) = R\,Hom_{\mathcal{A}/\mathcal{J}}(j^*\iota_* Y, Z) = 0$$

Finally we want the canonical Δ

$$\iota_* R\iota^!(X) \longrightarrow X \longrightarrow R_{j_*}(j^*X) \longrightarrow$$

Let's define $\mathcal{U}$ to be the cofibre of the adjunction map $\iota_* R\iota^!(X) \longrightarrow X$, so that we have the triangle

$$\iota_* R\iota^!(X) \xrightarrow{\alpha} X \longrightarrow \mathcal{U} \longrightarrow$$

apply $R\iota^!$ to get the triangle

$$\begin{tikzcd}
R\iota^!\iota_* R\iota^!(X) \arrow[r, "R\iota^!\cdot\alpha"] & R\iota^!(X) \arrow[r] & R\iota^!(\mathcal{U}) \arrow[r] & {} \\
R\iota^!(X) \arrow[u, "\beta . R\iota^!"] \arrow[ur, "1"'] & & &
\end{tikzcd}$$

and recall that β is an isomorphism by assumption. Thus we find $R\iota^!(\mathcal{U}) = 0$.

Up to quis we can assume $\mathcal{U}$ is a minimal injective complex (left-bounded). Then $R\iota^!(\mathcal{U}) = 0$ means $\iota^!\mathcal{U}$ is acyclic. Assuming $\iota^!\mathcal{U}^{n'} = 0$ for $n' < n$ we have

$$\begin{array}{ccccccc}
 & & 0 & \longrightarrow & \iota^!\mathcal{U}^n & \longrightarrow & \iota^!\mathcal{U}^{n+1} \\
 & & & & \cap & & \cap \\
0 & \longrightarrow & Z^n & \longrightarrow & \mathcal{U}^n & \longrightarrow & \mathcal{U}^{n+1}
\end{array}$$

exact as $H^n(\iota^!\mathcal{U}) = 0$

so $Z^n \cap \iota^!\mathcal{U}^n = 0$. But for $\mathcal{U}$ to be minimal means that $Z^n \subset U^n$ is an essential extension, so we find $i^! U^n = 0$. Thus $i^! U = 0$ which means that U is a complex of $\mathcal{T}$-free injectives. Thus $U = j_*(j^* U)$ where $j^* U$ is a complex of injectives in $\mathcal{A}/\mathcal{T}$. $\therefore$ $U = Rj_*(j^* U)$. Finally apply j^* to the triangle $i_* R i^!(X) \to X \to U \to$ given $j^* X \xrightarrow{\sim} j^* U$, so we are done.

<u>September 3, 1994</u>

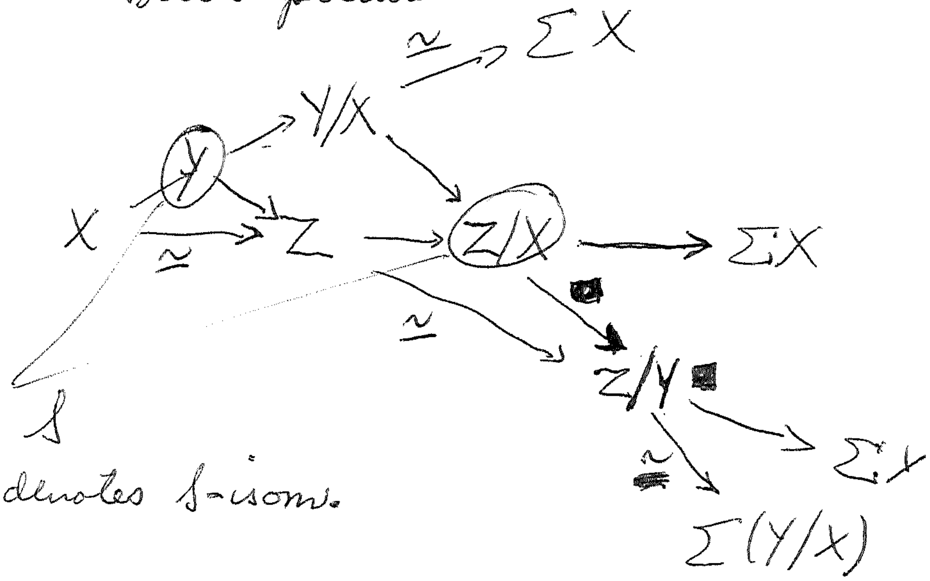
Recall [redacted] my misunderstanding of a thick subcategory of a triangulated category. Start with motivation for the actual definitions. If $F: \mathcal{X} \longrightarrow \mathcal{X}'$ is an exact functor between triangulated categories, then the 'kernel' $\mathcal{K}$ of F is thick: For any map $X \to Y$ having cone in the kernel $\mathcal{K}$ and factoring through an object in $\mathcal{K}$ has both X, Y in $\mathcal{K}$.

But my idea was to invert maps in $\mathcal{X}$ whose cone lies in a given full subcategory $\mathcal{S}$. I think that maps in the localization of $\mathcal{X}$ wrt $\mathcal{S}$-isomorphisms can be calculated by left or right fractions, provided only that $\mathcal{S}$ is closed under cones. (Assume $\mathcal{S}$ contains 0.)

Now suppose given $\mathcal{S}$ closed under cones but not thick. [redacted] Look at $\mathcal{X}[(\mathcal{S}\text{-isos})^{-1}]$. Since $\mathcal{S}$ is not thick there is a map $X \longrightarrow Z$ whose cone is in $\mathcal{S}$, which factors $X \longrightarrow Y \longrightarrow Z$ where Y is in $\mathcal{S}$. Think of these maps as inclusions and look at octahedron

$$\begin{tikzcd}
 & & & & \Sigma X \\
 & & & Y/X \arrow[ur, "\simeq"'] \arrow[dd] & \\
 & Y \arrow[urr] \arrow[dr] & & & \\
X \arrow[ur] \arrow[rr, "\simeq"'] & & Z \arrow[r] \arrow[dr, "\simeq"'] & Z/X \arrow[r] \arrow[d] & \Sigma X \\
 & & & Z/Y \arrow[d, "\simeq"'] \arrow[dr] & \\
 & & & {} & {}
\end{tikzcd}$$

Better picture



in $\mathcal{S}$

$\simeq$ denotes $\mathcal{S}$-isoms.

Then we have in $\mathcal{X}[(\mathcal{S}\text{-isos})^{-1}]$

$$\boxed{X \xrightarrow{\simeq} Z \xrightarrow{\simeq} Z/Y \xrightarrow{\simeq} \Sigma(Y/X) \xrightarrow{\simeq} \Sigma^2 X}$$

Lets check what we need to compute the maps in the localization by fractions. Any map in $\mathcal{X}[(\mathcal{S}\text{-isos})^{-1}]$ is a product of maps f_i and inverses s_i^{-1}. So to be able to represent these by fs^{-1} (right fractions) we need to be able to convert $s^{-1}f$ to this form.

$$\begin{tikzcd} W \arrow[r, dashed, "f'"] \arrow[d, dashed, "s'"'] & Z \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

This can always be done by one of the TR axioms I think. In any case for the examples I know one can assume up to isomorphism that Z is the h-fibre of $Y \to S$ where $S \in \mathcal{S}$, then define W to be $f^*Z = X \times_Y Z$, and $W =$ h-fibre of $X \to S$, etc.

Similarly we can represent maps in the localization category in the form fs^{-1}.

Next to have a calculus of right fractions we need $sf = 0 \Rightarrow \exists s'$ such that $fs' = 0$. Thus given

$$X \xrightarrow{f} Y \xrightarrow{s} Z \qquad sf = 0$$

we know f factors: $X \to \underbrace{h\text{-fibre}(Y \to Z)}_{S} \to Y$ this h-fibre being an [redacted] object S of $\mathcal{S}$. Then take $W = h\text{-fibre}(X \to S)$, whence [redacted] $W \xrightarrow{s'} X$ is an $\mathcal{S}$-iso such that $W \xrightarrow{s'} X \longrightarrow S$ is zero, hence $fs' = 0$.

I've left out the requirement that $s_1^{-1} s_2^{-1} = (s_2 s_1)^{-1}$ with $s_2 s_1$ an $\mathcal{S}$-isom. Thus I need $\mathcal{S}$-isos. to be closed under composition, which means $\mathcal{S}$ closed under cones.

The problem I'm working on is to show when $I = I^2$ in R one has an equivalence of Δated categories.

$$\text{firm}D(R,I) \xrightarrow{\sim} D^-(\mathcal{M}(R,I))$$

Here $\text{firm}D(R,I)$ is the full subcat of $D^-(R)$ consisting of complexes M such that

$$Li^*(M) = R/I \overset{L}{\otimes}_R M \quad \text{is} \simeq 0.$$

We know such an M is quis a firm flat complex (right bdd). Moreover we have an equivalence

$$\text{flat firm}(R,I) \xrightarrow{\sim} \text{flat}\mathcal{M}(R,I)$$

given by $(j_!, j^*)$. Also we know $\mathcal{M}(R,I)$ has sufficiently many flat objects, so that for any right-bdd complex N in $\mathcal{M}(R,I)$ $\exists$ a quis $F \to N$ with F a complex in flat $\mathcal{M}(R,I)$.

This missing ingredient is two equivalences

$$D^-(\text{flat } \mathcal{M}) \simeq D^-(\mathcal{M})$$

$$D^-(\text{flat firm}) \simeq \text{firm} D$$

where the derived categories are suitably defined, something like the quotient category by the acyclic complex subcategories.

September 5, 1994

Problem: I have two candidates for a firm derived category as follows. First is the full subcategory of $D^-(R)$ consisting of complexes U satisfying $R/I \overset{L}{\otimes}_R U = 0$; up to equivalence this is the same as the full subcategory of $D^-(R)$ consisting of firm flat complexes. On the other hand one can consider the homotopy category of firm flat complexes $K^-(\text{firmflat})$, and the thick subcategory of acyclic complexes, and form the quotient

$$K^-(\text{firmflat})/\text{acyc}$$

Let's denote the former by $\text{firm}D(R,I)$, so that we have $\text{firm}D \subset D^-(R)$. Denote the latter by $D^-(\text{firmflat})$. Note that the derived category makes sense for an exact category in my sense (full subcategory of an abelian category closed under extensions).

There is an obvious functor

$$* \qquad \mathbf{D}^-(\text{firmflat}) \longrightarrow \text{firm}D$$

and the question is whether it's an equivalence of categories. I know it's essentially surjective. In effect given U such that $R/I \overset{L}{\otimes}_R U = 0$, choose a quis $P \to U$ with P projective, then P/IP is acyclic, and one can deform the identity operator on P to $f: P \to IP \subset P$, then form $P[f^{-1}]$ to

obtain
$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "q"] & \underbrace{P[f^{-1}]}_{\text{firm flat}} \\ M & \end{tikzcd}$$

The question is then whether $*$ is fully faithful. Now maps ~~between~~ $F_1 \to F_2$ in firm D (i.e. in $D^-(R)$), where F_1, F_2 are firm flat, are represented by

$$\begin{tikzcd} & X \arrow[dl, "q"'] \arrow[dr] & \\ F_1 & & F_2 \end{tikzcd}$$

It follows that $X \in$ firm D. Our problem is to replace X by a firm flat F. We have quis $X \xleftarrow{q} P \xrightarrow{q} F$ as above, can suppose $X = P$. Then we reach the situation

$$\begin{tikzcd} & P \arrow[dl, "q"'] \arrow[d, "q"] \arrow[dr] & \\ F_1 & F & F_2 \end{tikzcd}$$

and it's not clear what to do.

September 9, 1994

Recall for a Morita context $\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$ and ideals $I \subset R$, $J \subset S$ the condition

$$* \left(PIQ \supset J^k, \quad QJP \supset I^k \quad \text{for some } k \geq 0. \right)$$

I claim this condition means that the topology on S defined by the ideals PI^nQ, $n \geq 0$ is the same as the J-adic topology, as well as the topology on R defined by QJ^nP, $n \geq 0$ is the same as the I-adic topology.

Let's check the condition $*$ implies this. We have

$$PI^nQ \supset (PIQ)^n \qquad \left(QP \subset R \text{ and } IR = I \right)$$

Also $* \Rightarrow QP \supset I^k$ so

$$(PIQ)^n = P(IQP)^{n-1}IQ \supset P(I^{1+k})^{n-1}IQ = PI^{nk+n-k}Q$$

Thus $\{PI^nQ\}$ gives the PIQ-adic topology on S.

Next we have $PIQ \supset J^k$ and

$$PI^{kn}Q \subset P(QJP)^nQ = PQ(JPQ)^n \subset J^n.$$

Simpler is $(PIQ)^k \subset PI^kQ \subset PQJPQ \subset J$.

So we see that PIQ-adic and J-adic topologies on S coincide. Thus $\{PI^nQ\}$ gives the J-adic topology on S. Similarly $\{QJ^nP\}$ gives the I-adic topology on R.

Conversely these imply $PIQ \supset J^k$ and $QIP \supset J^k$ for some k.

Assume again $QJP \supset I^k$, $PIQ \supset J^k$ some $k \geq 0$

Let $P_1 = PI$, $Q_1 = Q$. Then

$P_1 Q_1 = PIQ$, which has the same adic top as J:

$$J^k \subset PIQ$$

$$(PIQ)^k \subset PI^kQ \subset PQJPQ \subset J$$

and $Q_1 P_1 = QPI$, which has the same adic topology as I:

$$I^{k+1} \subset QJPI \subset QPI$$

$$QPI \subset I.$$

Thus we can suppose $QP = I$, $PQ = J$ by adjusting the ideals without changing the null modules.

September 11, 1994

Consider again a Morita context $\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$ and ideals $I \subset R$, $J \subset S$. Write $I \sim I'$ when the ideals I, I' give the same adic topology, i.e. $I \supset I'^k$, $I' \supset I^k$ for some k.

Consider the conditions

1) $QP \sim I$ and $PQ \sim J$.
2) $QJP \supset I^k$ and $PIQ \supset J^k$ ~~for~~ some k.
3) $QJP \sim I$ and $PIQ \sim J$.
4) $Q(JP) \sim I$ and $(JP)Q \sim J$

Then one has the implications

Why? Assume 1) i.e. $QP \supset I^k$ $\quad I \supset (QP)^k$ $\quad PQ \supset J^k$ $\quad J \supset (PQ)^k$ for some k.

Then $QJP \supset Q(PQ)^kP = (QP)^{k+1} \supset I^{k(k+1)}$ and similarly $PIQ \supset J^{k(k+1)}$, proving 2).

3) $\Rightarrow$ 2) is obvious since $QJP \sim I \Rightarrow QJP \supset I^k$ some k

2) $\Rightarrow$ 3) Assume 2). Then $PIQ \supset J^k$ is given and

$$(PIQ)^k \subset PI^kQ \subset PQJPQ \subset J$$

so $PIQ \sim J$. Similarly (really by symmetry) $QJP \sim I$, so 3) holds.

2) $\Rightarrow$ 4). Assume 2). Then

$$J \supset JPQ \supset JPIQ \supset J^{1+k} \quad \Rightarrow \quad J \sim JPQ.$$

$QJP \supset I^k$ is given and

$$(QJP)^k \subset QJ^kP \subset QPIQP \subset I \quad \Rightarrow \quad I \sim QJP$$

whence 4)

Significance of the above

2) $\Leftrightarrow$ 3) says that my hypothesis for Morita equivalence means $QJP \sim I$ and $PIQ \sim J$.

3) $\Rightarrow$ 4) says that if I replace the given Morita context with $\begin{pmatrix} R & Q \\ P_1 & S \end{pmatrix}$, where P_1 is JP (also PI works), then I have the situation 1).

When 1) holds I can assume $QP = I$ and $PQ = J$, which was my operating assumption for a long time.

Why $\begin{pmatrix} R & I \\ R & \tilde{I} \end{pmatrix}$ is a Morita context.

Consider $M_2(\tilde{R}) = \begin{pmatrix} \tilde{R} & \tilde{R} \\ \tilde{R} & \tilde{R} \end{pmatrix}$ and use e for the identity of R. Then $\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$ is an idempotent in $M_2(R)$ and so

$$\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \tilde{R} & \tilde{R} \\ \tilde{R} & \tilde{R} \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e\tilde{R}e & e\tilde{R} \\ \tilde{R}e & \tilde{R} \end{pmatrix} = \begin{pmatrix} R & R \\ R & \tilde{R} \end{pmatrix}$$

is a ring with the identity $\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$. Clearly $\begin{pmatrix} R & I \\ R & \tilde{I} \end{pmatrix}$ is a subring of $\begin{pmatrix} R & R \\ R & \tilde{R} \end{pmatrix}$.