Review yesterday. Assuming R is a flat algebra over a commutative ring k, we can construct a chain complex $P.$ of R-bimodules, which is good flat on the right and which is a resolution of R modulo bimodules null on the right. One then has for any complex M of modules

$$Lj_!(j^*M) = P \otimes_R M$$

In effect, we have $Lj_!(j^*M) = F$, where F is a complex of good flat modules which is a resolution of M modulo null modules. Then

$$F = R \otimes_R F \xleftarrow[\text{quis}]{} P \otimes_R F \xrightarrow[\text{quis}]{} P \otimes_R M$$

When $I \overset{L}{\otimes}_R I \simeq I$ (equivalently $I \overset{L}{\otimes}_R R/I = 0$), P is a bimodule resolution of I by bimodules which are good flat on the right. Thus

$$Lj_!(j^*M) = I \overset{L}{\otimes}_R M$$

and we get the Δ

$$Lj_!(j^*M) \longrightarrow M \longrightarrow \iota_* L\iota^*(M) = R/I \overset{L}{\otimes} M$$

with an arrow from $\iota_* L\iota^*(M)$ back to $Lj_!(j^*M)$.

Dually, let $M \longrightarrow Q$ be an injective resolution. Then with P as above

$$M \xrightarrow[\text{quis}]{} Q = \mathrm{Hom}_R(R, Q) \xrightarrow[\text{quis mod null}]{} \underbrace{\mathrm{Hom}_R(P, Q)}_{\text{good' injective}}$$

whence $\mathrm{Hom}_R(P, Q)$ [redacted] is a good' injective resolution of M modulo null modules. Thus

$$Rj_*(j^*M) = \mathrm{Hom}_R(P,Q) = R\mathrm{Hom}_R(P,M)$$

When $I \overset{L}{\otimes}_R I \xrightarrow{\sim} I$, P is quis to I hence

$$Rj_*(j^*M) = R\mathrm{Hom}_R(I,M)$$

and we get the Δ

$$\iota_* R\iota^!(M) \longrightarrow M \longrightarrow Rj_*(j^*M) \overset{\circledast}{\longrightarrow} \iota_* R\iota^!(M)$$

$$\iota_* R\iota^!(M) = R\mathrm{Hom}_R(R/I, M)$$

Another view point. Suppose we consider the full subcategory of $D(R\text{-mod})$ consisting of complexes such that the homology is I-null. This is a triangulated subcategory: If two objects in a triangle belong to the subcategory, then the third does also by the long exact homology sequence. Let's denote this subcategory by $D(R\text{-mod})_{I\text{-null}}$. Now I think there's a quotient triangulated category $D(R\text{-mod})/D(R\text{-mod})_{I\text{-null}}$ defined, which is constructed as a category of fractions.

There are various question like whether the obvious map

1) $$D(R\text{-mod})/D(R\text{-mod})_{I\text{-null}} \xrightarrow{j^*} D(R\text{-mod}/I\text{-null})$$

is an equivalence of triangulated categories. This I feel should be OK. We can also ask whether the obvious map

2) $$D(R/I\text{-mod}) \xrightarrow{\iota_*} D(R\text{-mod})_{I\text{-null}}$$

is an equivalence.

The map 1) is induced by j^* which is exact from R-mod to R-mod/I-null. It carries a complex in R-mod/I-null into an acyclic complex, which is quis to 0. There's I think a functor in the inverse direction which lifts the complex inductively over the skeleton, cutting down to make the differentials have square zero. Check this later.

Look at 2). If we have the triangle

$$Lj_!(j^*M) \longrightarrow M \longrightarrow \iota_* L\iota^*(M)$$

then $M \in D(R\text{-mod})_{I\text{-null}}$ (equivalently $j^*M = 0$) implies $M \xrightarrow{\sim} \iota_* L\iota^*(M) = R/I \overset{L}{\otimes}_R M$, so M is in the image of ι_*.

Dually given the triangle

$$\iota_* R\iota^!(M) \longrightarrow M \longrightarrow Rj_*(j^*M)$$

we have $M \in D(R\text{-mod})_{I\text{-null}} \Longrightarrow j^* M = 0 \Longrightarrow$

$$M \xleftarrow{\sim} \iota_* R\iota^!(M) = R\mathrm{Hom}_R(R/I, M).$$

It seems that we get then

$$R\mathrm{Hom}_R(R/I, M) \xrightarrow{\sim} M \xrightarrow{\sim} R/I \overset{L}{\otimes}_R M$$

for any complex of R/I-modules.

Let's check some of this directly. Let M be a complex of R-modules such that $I H_*(M) = 0$. Up to quasi-isomorphism we can suppose M is flat. Then we have an exact sequence of complexes

$$0 \longrightarrow I \otimes_R M \longrightarrow M \longrightarrow M/IM \longrightarrow 0$$

Suppose to begin with that I is right flat. Then we have

$$H_q(I \otimes_R M) = I \otimes_R H_q(M) = 0$$

so $I \otimes_R M$ is acyclic and M is quis the complex of R/I-modules M/IM.

More generally suppose $I \overset{L}{\otimes}_R I \xrightarrow[\text{quis}]{} I$. Then we know (under the assumption that R is a flat algebra over a commutative ring) that there is a bimodule resolution $P.$ of I consisting of good flat right modules. Then

$$I \otimes_R M \underset{\substack{\text{quis} \\ \text{as } M \\ \text{injective}}}{\longleftarrow} P \otimes_R M \quad \text{acyclic since each } P_p \otimes_R M \text{ is}$$

Thus again we have a quis $M \longrightarrow M/IM$

However notice that under the assumption $I \overset{L}{\otimes}_R R/I = 0$ we can always construct a resolution P of I by good flat right modules, so in any case $I \otimes_R M$ is quis $P \otimes_R M$ which is acyclic and we have M quis M/IM.

Dually we can suppose up to quis that M is injective. Then we have an exact sequence of complexes

$$0 \longrightarrow \mathrm{Hom}_R(R/I, M) \longrightarrow M \longrightarrow \mathrm{Hom}_R(I, M) \longrightarrow 0$$

Assuming M has I-null homology it follows that $\mathrm{Hom}_R(I, M)$ has I-null homology.

Suppose I is right flat. Then $\mathrm{Hom}_R(I, M)$ is a complex of good' injectives. Let's check that this together with the fact that its homology is I-null implies $\mathrm{Hom}_R(I, M)$ is acyclic.

Let E° be a complex of good' injectives whose homology is I-null, and suppose it's bdd below, say $E^n = 0$ for $n < 0$. Then we have

$$0 \rightarrow H^0(E) \rightarrow E^0 \rightarrow E^1$$

so $H^0(E)$ satisfies both $I\,H^0(E) = 0$, ${}_I H^0(E) = 0$ and thus $H^0(E) = 0$. Then $E^0 \hookrightarrow E^1$ and E^0 injective means $E^1 = E^0 \oplus E^{1'}$. It's clear that E° is acyclic.

In greater generality, assuming only $I \overset{L}{\underset{R}{\otimes}} R/I = 0$ we get P a [redacted] resolution of I by bimodules which are good flat on the right. Then $\mathrm{Hom}_R(I, M) \xrightarrow[\text{quis}]{} \mathrm{Hom}_R(P, M)$, where $\mathrm{Hom}_R(P, M)$ is good' injective, so again $\mathrm{Hom}_R(P, M)$ is acyclic. Thus $\mathrm{Hom}_R(I, M)$ is acyclic whence

$$\mathrm{Hom}_R(R/I, M) \longrightarrow M$$

is a quis.

June 19, 1994

Some additional comments arising from the past 2 days work:

First let's check in the case $I \overset{L}{\otimes}_R I \xrightarrow{\sim} I$ that $D(R/I\text{-mod})$ is a full subcategory of $D(R\text{-mod})$ i.e. if M_1, M_2 are cxs of R/I-modules then

$$R\,Hom_{R/I}(M_1, M_2) \xrightarrow[\text{qis}]{} R\,Hom_R(M_1, M_2)$$

(I think once you have that $D(R/I\text{-mod}) \to D(R\text{-mod})$ is fully faithful, then the essential image is $D(R\text{-mod})_{I\text{-null}}$, because any object in this last category can be built up a la Postnikov, the point being that fully faithful implies the k-invariants always lie in the essential image of $D(R/I\text{-mod})$).

Let $M_2 \xrightarrow[\text{qis}]{} Q$ with Q an injective R-mod cx. Then we showed as a consequence of $I \overset{L}{\otimes}_R I \xrightarrow{\sim} I$ that

$$Hom_R(R/I, Q) \xrightarrow[\text{qis}]{} Q$$

Now $Hom_R(R/I, Q)$ is an injective R/I-mod cx., so

$$\begin{array}{ccc} R\,Hom_{R/I}(M_1, M_2) & \longrightarrow & R\,Hom_R(M_1, M_2) \\ \big| \text{qis} & & \big| \text{qis} \\ Hom_{R/I}\big(M_1, Hom_R(R/I, Q)\big) & \xrightarrow{\sim} & Hom_R(M_1, Q) \end{array}$$

(the lower arrow: adjunction isom.)

whence the assertion.

Second idempotent functors and reflections, $I \overset{L}{\otimes}_R I \simeq I$, etc. Defer this.

Third, note that when I is right flat, then $j_!$ is exact (because $j_! j^* M = I \otimes_R M$ is exact). Consequently if Q is an injective module then

$$\mathrm{Hom}_R(j_! U, Q) = \mathrm{Hom}_{R\text{-mod}/I\text{-null}}(U, j^* Q)$$

is exact in U showing that <u>j^* respects injectives when I is right flat.</u>

Similarly if I is left projective then j_* is exact. In the $I = I^2$ situation this is clear from $j_*(j^* M) = \mathrm{Hom}_R(I^g, M)$ and $I^g = I$. But it holds in general because the 'good' modules:

$$M \xrightarrow{\sim} \mathrm{Hom}_R(I, M)$$

evidently form an abelian category with exact forgetful functor to modules. If P is a projective module, then

$$\mathrm{Hom}_R(P, j_* U) = \mathrm{Hom}_{R\text{-mod}/I\text{-null}}(j^* P, U)$$

is an exact functor of U, showing that <u>j_* respects projectives when I is left projective.</u>

It seems that the **condition** $I \overset{L}{\otimes}_R I \xrightarrow[\text{quis}]{} I$ depends only on the nonunital ring I. In effect we know this condition is equivalent to the existence of a good flat resolution P of I. First of all we know that good modules depend only on I. On the other good flat modules are those modules M

such that $M = IM$ which satisfy the Cartan-Eilenberg linear equations criterion where the coefficients are in I.

June 20, 1994

Prop. M a complex of R modules (bdd. below for lower indexing). Then $R/I \overset{L}{\otimes}_R M = 0$ $\iff$ M quis to a complex P of good flat modules (bdd below).

Proof. ($\Leftarrow$) $R/I \overset{L}{\otimes}_R M = R/I \overset{L}{\otimes}_R P = R/I \otimes_R P = 0$.

($\Rightarrow$) We can suppose M is a complex of projective modules.

Since M is flat $R/I \overset{L}{\otimes}_R M$ is quis to M/IM, so M/IM is acyclic. Since M consists of projective modules, the canon map $M \to M/IM$ is null-homotopic. Choosing a null-homotopy and lifting it to a degree 1 operator $h : M \to M$, we obtain a map $f = 1 - [d,h] : M \to M$ compatible with d which is homotopic to the identity and whose image is contained in IM. Let

$$P = \varinjlim \{ M \xrightarrow{f} M \xrightarrow{f} M \longrightarrow \cdots \}$$

Then P_n, being a filtered inductive limit of free modules, is flat. Also $f(M) \subset IM \Rightarrow IP = P$. Finally since homology commutes with filtered $\varinjlim$'s, we have $H_*(P) = \varinjlim \{ H_*(M) \xrightarrow{id} H_*(M) \to \cdots \}$ so the obvious map $M \to P$ is a quis.

Here's a step toward Morita invariance in general. Let A be a left ideal in a unital algebra R. Let M be a good A-module: $A \otimes_A M \xrightarrow{\sim} M$. Then M has a unique R-module structure extending the A-module structure: $r(am) = (ra)m$, and this R-module structure is unital. The composition

$$A \otimes_A M \twoheadrightarrow AR \otimes_R M \longrightarrow M$$

is an isom., the first map is surjective, hence both maps are isos., showing that M is an AR-good R-module.

Conversely let N be an AR-good R-module: $AR \otimes_R N \xrightarrow{\sim} N$. One has an exact sequence

$$0 \longrightarrow K \longrightarrow A \otimes_A R \longrightarrow AR \longrightarrow 0$$

where $KA^2 = 0$: Given $\sum a_i \otimes_A r_i \in K$, i.e. $\sum a_i r_i = 0$, then $\left(\sum a_i \otimes_A r_i\right) a a' = \sum a_i r_i a \otimes_A a' = 0$. Then

$$\begin{array}{ccccccc} K \otimes_R N & \longrightarrow & A \otimes_A R \otimes_R N & \longrightarrow & AR \otimes_R N & \longrightarrow & 0 \\ & & \| & & \downarrow \cong & & \\ & & A \otimes_A N & \longrightarrow & N & & \end{array}$$

and $K \otimes_R N = K \otimes_R AN = K \otimes_R A^2 N = KA^2 \otimes_R N = 0$, showing that N is a good A-module.

June 21, 1994

Again: I ideal in R unital. Let X be a right R-module which is I-good:

$$X \otimes_R I \xrightarrow{\sim} X.$$

Consider the Serre subcategory $\mathcal{S}$ of $\mathcal{C} =$ R-mod consisting of M such that $I^n M = 0$ for some n. I claim that $X \otimes_R -$ inverts $\mathcal{S}$-isomorphisms.

To prove this consider an $\mathcal{S}$-iso $M_1 \to M_2$, so the kernel and cokernel are killed by some power of I. To show $X \otimes_R M_1 \xrightarrow{\sim} X \otimes_R M_2$ we can factor the map into a surjection followed by an injection, so it suffices to consider these cases.

If $M_1 \hookrightarrow M_2$ with $I^n(M_2/M_1) = 0$, then we have a diagram with exact rows

($I^{(n)} = I^{\otimes_R n}$)

$$\begin{array}{ccccccccc} & & I^{(n)} \otimes_R M_1 & \longrightarrow & I^{(n)} \otimes_R M_2 & \longrightarrow & I^{(n)} \otimes_R (M_2/M_1) & \longrightarrow & 0 \\ & & \downarrow & \swarrow & \downarrow & & \downarrow 0 & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_2/M_1 & \longrightarrow & 0 \end{array}$$

One sees easily that there exists a unique dotted arrow such the two triangles including it are commutative. Tensoring with X yields

$$\begin{array}{ccc} X \otimes_R I^{(n)} \otimes_R M_1 & \longrightarrow & X \otimes_R I^{(n)} \otimes_R M_2 \\ \cong \downarrow & \swarrow & \downarrow \cong \\ X \otimes_R M_1 & \longrightarrow & X \otimes_R M_2 \end{array}$$

where the vertical arrows are isomorphisms, hence $X \otimes_R M_1 \xrightarrow{\sim} X \otimes_R M_2$.

On the other hand if $M_1 \to M_2$ is surjective and its kernel K satisfies $I^n K = 0$, then one has

$$\begin{array}{ccccccc} & & I^{(n)} \otimes_R K & \longrightarrow & I^{(n)} \otimes_R M_1 & \longrightarrow & I^{(n)} \otimes_R M_2 & \longrightarrow 0 \\ & & {\scriptstyle 0}\downarrow & & \downarrow & \swarrow & \downarrow & \\ 0 & \longrightarrow & K & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow 0 \end{array}$$

where the dotted arrow is unique such that the two triangles involving it commute. Again we conclude $X \otimes_R M_1 \xrightarrow{\sim} X \otimes_R M_2$.

Here's a more powerful proof. [redacted]

[redacted] For a flat right R module P which is I-good ($PI = P$), the functor $P \otimes_R -$ is <u>exact</u> and it kills M such that $IM = 0$. Thus its obvious that $P \otimes_R -$ inverts $\mathcal{S}$-isomorphisms. But we know that any right I-good module X has a presentation $P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$, where P_0, P_1 are right flat & I-good. So if $M_1 \to M_2$ is an $\mathcal{S}$-isom we have

$$\begin{array}{ccccccc} P_1 \otimes_R M_1 & \longrightarrow & P_0 \otimes_R M_1 & \longrightarrow & X \otimes_R M_1 & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow & & \\ P_1 \otimes_R M_2 & \longrightarrow & P_0 \otimes_R M_2 & \longrightarrow & X \otimes_R M_2 & \longrightarrow & 0 \end{array}$$

and [redacted] it's clear.

This proof is more powerful because it shows that $X \otimes_R -$ inverts a larger class of maps. Specifically let us consider three Serre subcategories $\mathcal{S}_i$ $i = 0, 1, 2$ of R-mod defined as follows.

$\mathcal{S}_0$ is the category $\mathcal{S}$ above consisting of $M \ni I^n M = 0$

for some n,

$\mathcal{S}_1$ is the category of I-torsion modules considered previously. Thus $\mathcal{S}_1$ consists of M such that $\mathrm{Hom}_R(M, E) = 0$ for all injective modules E such that ${}_I E = 0$. Alternatively $M \in \mathcal{S}_1$ when the transfinitely defined filtration

$$F^{\alpha+1} M = \{ m \in M \mid Im \in F^{\alpha} M \}$$

$$F^{\alpha} M = \bigcup_{\alpha' < \alpha} F^{\alpha'} M \qquad \alpha \text{ limit ordinal}$$

exhausts M. Alternatively, for all submodules $N < M$, $\{ m \mid I \cdot m \subset N \} > N$.

$\mathcal{S}_2$ is the category consisting of M such that $P \otimes_R M = 0$ for all [redacted] flat I-good right modules P. It suffices to take P to be a generating flat I-good module.

We have the following inclusions

$$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \mathcal{S}_2$$

Check: Let $M \in \mathcal{S}_1$, let N be the largest submodule of M such that $P \otimes_R N = 0$. If $N < M$ then $N' = \{ m \in M \mid Im \subset N \}$ is $> N$ and

$$\underset{\substack{\| \\ 0}}{P \otimes_R N} \longrightarrow P \otimes_R N' \longrightarrow \underset{\substack{\| \\ 0}}{P \otimes_R (N'/N)} \to 0$$

so we have a contradiction showing $P \otimes_R M = 0$, and $M \in \mathcal{S}_2$.

Notice that $\mathcal{S}_1$, $\mathcal{S}_2$ are closed under direct sums, hence they are localizing [redacted] subcategories, i.e. torsion theories.

Let $\mathcal{A} = R\text{-mod}$, $\mathcal{S}_0 =$ Serre subcategory of $M \ni \exists n, I^n M = 0$. Let's calculate the maps in $\mathcal{A}/\mathcal{S}_0$.

Quite generally one has

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{S}_0}(M_1, M_2) = \varinjlim_{\left\{\begin{smallmatrix} N_1 \subset M_1 & M_1/N_1 \in \mathcal{S} \\ N_2 \subset M_2 & N_2 \in \mathcal{S}\end{smallmatrix}\right\}} \mathrm{Hom}_{\mathcal{A}}(N_1, M_2/N_2)$$

$$= \varinjlim_{\{\text{cat of } \mathcal{S}_0 \text{ isos } M' \to M_1\}} \mathrm{Hom}_{\mathcal{A}}(M', M_2)$$

$$= \varinjlim_{\{\text{cat of } \mathcal{S}_0 \text{ isos. } M_2 \to M''\}} \mathrm{Hom}_{\mathcal{A}}(M_1, M'')$$

where the first formula has the advantage that the category over which the limit is taken is small (a directed set in fact).

I claim

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{S}_0}(M_1, M_2) = \varinjlim_n \mathrm{Hom}_R(I^{(n)} \otimes_R M_1, M_2)$$

$$= \varinjlim_n \mathrm{Hom}_R(M_1, \mathrm{Hom}_R(I^{(n)}, M_2))$$

We just have to check that the objects $\{I^{(n)} \otimes_R M \longrightarrow M\}$, $n \geq 0$ are cofinal in the filtering category of $\mathcal{S}_0$-isos $M' \longrightarrow M$. The dual assertion results by adjointness.

So given an $\mathcal{S}_0$-iso $M' \longrightarrow M$ factor it into surjection followed by injection. On pp 640-641

We've seen there are dotted arrows as follows

$$\begin{array}{ccccc} & & & & I^{(n_2+n_1)}\otimes_R M \\ & & & \swarrow & \downarrow \\ & & I^{(n_2)}\otimes_R M'' & & I^{(n_1)}\otimes_R M \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ M' & \twoheadrightarrow & M'' & \hookrightarrow & M \end{array}$$

Filling in the top by applying $I^{(n_2)}\otimes_R -$ and naturality we win.

Next let's check that the functors $I\otimes_R -$ and $\mathrm{Hom}_R(I,-)$ on $\mathcal{A}$ descend to $\mathcal{A}/\mathcal{S}_0$ and are inverse. If $M_1 \to M_2$ is an $\mathcal{S}_0$ isom. then

$$\begin{array}{ccc} I\otimes_R M_1 & \longrightarrow & I\otimes_R M_2 \\ \mathcal{S}_0\text{ isom}\downarrow & & \downarrow \mathcal{S}_0\text{-isom} \\ M_1 & \xrightarrow{\mathcal{S}_0\text{-isom}} & M_2 \end{array}$$

shows that $I\otimes_R M_1 \to I\otimes_R M_2$ is an $\mathcal{S}_0$-isom. Thus $I\otimes_R -$ descends to $\mathcal{A}/\mathcal{S}_0$ and similarly for $\mathrm{Hom}_R(I,-)$.

Next we have

$$M \xrightarrow{\mathcal{S}_0\text{ isom}} \mathrm{Hom}_R(I,M)$$

$$\begin{array}{ccc} I\otimes_R M & \xrightarrow{\mathcal{S}_0\text{-isom}} & I\otimes_R \mathrm{Hom}_R(I,M) \\ & \searrow^{\mathcal{S}_0\text{-isom}} & \downarrow \\ & & M \end{array} \qquad \begin{array}{ccc} x\otimes m & \mapsto & x\otimes(y\mapsto ym) \\ & \searrow & \downarrow \\ & & xm \end{array}$$

and

$$I\otimes_R M \xrightarrow{\mathcal{S}_0\text{ isom}} M$$

$$\begin{array}{ccc} \mathrm{Hom}_R(I, I\otimes_R M) & \xrightarrow{\mathcal{S}_0\text{ isom}} & \mathrm{Hom}_R(I,M) \\ \uparrow & \nearrow_{\mathcal{S}_0\text{-isom}} & \\ M & & \end{array} \qquad \begin{array}{ccc} (x\mapsto x\otimes m) & \longmapsto & (x\mapsto xm) \\ \uparrow & \nearrow & \\ m & & \end{array}$$

Thus we have canonical $\mathcal{S}_0$-isoms

$$I \otimes_R \mathrm{Hom}_R(I, M) \longrightarrow M$$

$$M \longrightarrow \mathrm{Hom}_R(I, I \otimes_R M)$$

showing the functors $I \otimes_R -$ and $\mathrm{Hom}_R(I, -)$ are inverse on $\mathcal{A}/\mathcal{S}_0$.

June 22, 1994 (54 years old)

For the proof of Morita equivalence we need to replace the ideal I by certain bimodules, something like $I \otimes_R I$, which need not be an ideal in R.

Consider pairs (L, ∂) where L is an R-bimodule equipped with a bimodule map $\partial: L \to R$ satisfying $\partial(l_1) l_2 = l_1 \partial(l_2)$. We note that the image ∂L is an ideal in R, and the kernel $\operatorname{Ker} \partial$ is an R-bimodule killed on both sides by the ideal ∂L:

$$\partial(l_1) \in \partial L, \quad l_2 \in \operatorname{Ker}\partial \implies \partial(l_1) l_2 = l_1 \partial(l_2) = 0.$$

Given two pairs (L, ∂), (L', ∂) of this sort their tensor product is $(L \otimes_R L', \partial)$, where $\partial(l \otimes l') = \partial(l)\partial(l')$. This map $\partial: L \otimes_R L' \to R$ is a well-defined R-bimodule map:

$$\partial(lr)\partial(l') = \partial(l) r \partial(l') = \partial(l)\partial(rl')$$

$$\partial(rl \otimes l') = \partial(rl)\partial(l') = r\,\partial(l)\partial(l') = r\,\partial(l \otimes l')$$

and similarly for right mult. Finally

$$\begin{aligned}
\partial(l_1 \otimes l_1') l_2 \otimes l_2' &= \partial(l_1)\partial(l_1') l_2 \otimes l_2' \\
&= l_1 \partial(\partial(l_1') l_2) \otimes l_2' \\
&= l_1 \partial(l_1')\partial(l_2) \otimes l_2' \\
&= l_1 \otimes \partial(l_1')\partial(l_2) l_2' \\
&= l_1 \otimes l_1' \partial(\partial(l_2) l_2') \\
&= l_1 \otimes l_1' \partial(l_2)\partial(l_2') \\
&= l_1 \otimes l_2 \; \partial(l_2 \otimes l_2')
\end{aligned}$$

Given (L, ∂) we can form an inverse system of bimodules

$$\longrightarrow L\otimes_R L\otimes_R L \longrightarrow L\otimes_R L \longrightarrow L \longrightarrow R$$

as follows: Note that the condition $\partial(l_1)l_2 = l_1\partial(l_2)$ means that the possible face operators

$$\partial_i : L^{\otimes_R n} \longrightarrow L^{\otimes_R n-1} \qquad (l_1,\dots,l_n) \longmapsto (l_1,\dots,l_{i-1}\partial(l_i), l_{i+1},\dots,l_n)$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$$
$$\qquad\qquad\qquad\qquad\qquad\qquad (l_1,\dots,l_{i-1}, \partial(l_i)l_{i+1},\dots,l_n)$$

coincide:

$$\begin{aligned}\partial_i(l_1,\dots,l_n) &= (l_1,\dots,l_{i-1}, \partial(l_i)l_{i+1},\dots,l_n)\\ &= (l_1,\dots,l_{i-1}, l_i\partial(l_{i+1}),\dots,l_n)\\ &= \partial_{i+1}(l_1,\dots,l_n)\end{aligned}$$

We want to check now that if (L,∂) is a pair as above such that the ideal [redacted] $\partial(L)$ defines the same adic topology as I, i.e. $\partial(L)^n \subset I$, $I^n \subset \partial(L)$ for some n, then maps in the category $R\text{-mod}/\underbrace{\{M \mid \exists n, I^n M = 0\}}_{\text{call this } I\text{-null}}$ are given by

$$\varinjlim_n \operatorname{Hom}_R(L^{\otimes_R n}\otimes_R M_1, M_2)$$

$$= \varinjlim_n \operatorname{Hom}_R(M_1, \operatorname{Hom}_R(L^{\otimes_R n}, M_2))$$

It suffices to check that the maps $L^{\otimes_R n}\otimes_R M \to M$ are cofinal in the category of [redacted] I-null isomorphisms $M' \to M$ with target M. First [redacted] check that for any module M the map $L\otimes_R M \to M$ (given by ∂) is an I-null isom. (I should have earlier mentioned that we can assume $\partial L = I$). [redacted] We have exact sequences

$$0 \longrightarrow K \longrightarrow L \longrightarrow I \longrightarrow 0$$

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

where K [redacted] is killed by I. Then

$$K\otimes_R M \longrightarrow L\otimes_R M \longrightarrow I\otimes_R M \longrightarrow 0$$

$$0 \to K' \longrightarrow I\otimes_R M \longrightarrow M \longrightarrow M/IM \longrightarrow 0$$

where $I\cdot K' = 0$ and $I(M/IM) = 0$. This shows that $L\otimes_R M \longrightarrow M$ is an isomorphism mod I-null.

Next suppose $M' \longrightarrow M$ is an I-null isom. and factor it $M' \twoheadrightarrow M'' \hookrightarrow M$; let K_1 be the kernel of $M' \twoheadrightarrow M''$. For n large I^n kills K_1 and M/M'' hence dotted arrows exist in

$$\begin{array}{ccccccc}
L^{\otimes_R n}\otimes_R K_1 & \longrightarrow & L^{\otimes_R n}\otimes_R M' & \longrightarrow & L^{\otimes_R n}\otimes_R M'' & \longrightarrow & 0 \\
\downarrow 0 & & \downarrow & & \downarrow & & \\
0 \longrightarrow K_1 & \longrightarrow & M' & \longrightarrow & M'' & \longrightarrow & 0
\end{array}$$

$$g_1 : L^{\otimes_R n}\otimes_R M'' \dashrightarrow M'$$

$$\begin{array}{ccccccc}
L^{\otimes_R n}\otimes_R M'' & \longrightarrow & L^{\otimes_R n}\otimes_R M & \longrightarrow & L^{\otimes_R n}\otimes_R M/M'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow 0 & & \\
0 \longrightarrow M'' & \longrightarrow & M & \longrightarrow & M/M'' & \longrightarrow & 0
\end{array}$$

$$g_2 : L^{\otimes_R n}\otimes_R M \dashrightarrow M''$$

keeping these diagrams commutative. Thus

$$\begin{array}{ccccc}
L^{\otimes 2n}\otimes M' & \longrightarrow & L^{\otimes 2n}\otimes M'' & \longrightarrow & L^{\otimes 2n}\otimes M \\
\downarrow & & \downarrow & & \downarrow \\
L^{\otimes n}\otimes M' & \longrightarrow & L^{\otimes n}\otimes M'' & \longrightarrow & L^{\otimes n}\otimes M \\
\downarrow & & \downarrow & & \downarrow \\
M' & \longrightarrow & M'' & \longrightarrow & M
\end{array}$$

$$L^{\otimes n}\otimes g_2 : L^{\otimes 2n}\otimes M \dashrightarrow L^{\otimes n}\otimes M'', \qquad g_1 : L^{\otimes n}\otimes M'' \dashrightarrow M'$$

So it works.

June 23, 1994

$$R = T(V) = \bigoplus_{n\geq 0} V^{\otimes n}, \quad I = \bigoplus_{n>0} V^{\otimes n} = T^{>0}(V)$$

An R-module is the same as a vector space M equipped with a linear map $V \otimes M \to M$.

M is I-solid $\iff$ $V \otimes M \xrightarrow{\sim} M$
M is I-cosolid $\iff$ $M \xrightarrow{\sim} \mathrm{Hom}(V, M)$

Suppose V finite-dimensional $\neq 0$, let x_i be a basis for V, y_i the dual basis for V^*.
A solid M is the same as a module over

$$\mathcal{O}_V = T(V \oplus V^*) \Big/ \begin{array}{l} yx = \text{[redacted]} \langle y|x\rangle \\ \sum x_i y_i = 1 \end{array}$$

A cosolid M, i.e. $M \simeq V^* \otimes M$ is the same as a module over

$$\mathcal{O}_{V^*} = T(V \oplus V^*) \Big/ \begin{array}{l} xy = \langle y|x\rangle \\ \sum y_i x_i = 1 \end{array}$$

Note that
solid $T^{>0}(V)$ modules = cosolid $T^{>0}(V^*)$ modules
cosolid ———————— = solid ————————

Recall what we learned about the Cuntz-Krieger algebra $\mathcal{O}_E$. Here A is a unital algebra (ring), E a unital bimodule over A which is a finitely generated projective generator for [redacted] A-mod.
Then
$$\mathcal{O}_E = T_A(E \oplus E^*) \Big/ \begin{array}{l} yx = \langle y|x\rangle \\ \sum x_i y_i = 1 \end{array}$$

where $E^* = \mathrm{Hom}_{A^{op}}(E, A)$, $\langle y|x\rangle$ is the canonical map $E^*\otimes_A E \longrightarrow A$ and

$$\sum_i x_i\otimes y_i \in E\otimes_A E^* \xrightarrow{\sim} \mathrm{Hom}_{A^{op}}(E,E)$$

gives the identity operator on E. The ring $\mathcal{O}_E$ is $\mathbb{Z}$-graded $\mathcal{O}_E = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_E^n$ [redacted] and $\mathcal{O}_E^1\mathcal{O}_E^{-1} = \mathcal{O}_E^{-1}\mathcal{O}_E^1 = \mathcal{O}_E^0$, so that $\mathcal{O}_E^1 = \mathcal{O}_E^0\otimes_A E$ is an invertible bimodule over $\mathcal{O}_E^0$ with inverse $\mathcal{O}_E^{-1} = E^*\otimes_A\mathcal{O}_E^0$. Better to say that $\mathcal{O}_E$ is the $\mathbb{Z}$-graded tensor algebra on the invertible bimodule $\mathcal{O}_E^1$ over $\mathcal{O}_E^0$.

Go back to $R = T(V)$, $I = T^{>0}(V)$, $\dim V < \infty$. We know in this case that the **solid** and cosolid modules [redacted] four abelian categories which may be identified with $\mathcal{O}_V$-mod and $\mathcal{O}_{V^*}$-mod respectively. Also the ~~forgetful~~ inclusion functors to R-modules are exact. These ~~forgetful~~ inclusion functors are restriction of scalars associated to canonical homomorphisms

$$T(V) \longrightarrow \mathcal{O}_V \qquad T(V) \longrightarrow \mathcal{O}_{V^*}$$

[redacted] Hence the inclusion functors

$$I\text{-solid} \subset R\text{-mod} \qquad I\text{-cosolid} \subset R\text{-mod}$$

should both have left and right adjoints.

Let's make some general observations that should apply at least in the case $R = T(V)$, $I = T^{>0}(V)$.

Recall M is I-cosolid when $M \xrightarrow{\sim} \mathrm{Hom}_R(I, M)$.

Given any R-module M we have an inductive system $M \longrightarrow \mathrm{Hom}_R(I, M) \longrightarrow \mathrm{Hom}_R(I^{\otimes_R 2}, M) \longrightarrow \cdots$ and we can take the inductive limit. When I is finitely presented as R-module we know $\mathrm{Hom}_R(I, -)$ commutes with filtered $\varinjlim$'s and so

$$\mathrm{Hom}_R\left(I, \varinjlim_n \mathrm{Hom}_R(I^{\otimes_R n}, M)\right)$$

$$= \varinjlim_n \mathrm{Hom}_R\left(I, \mathrm{Hom}_R(I^{\otimes_R n}, M)\right)$$

$$= \varinjlim_n \mathrm{Hom}_R(I^{\otimes_R n+1}, M)$$

showing that $\varinjlim_n \mathrm{Hom}_R(I^{\otimes_R n}, M)$ is I-cosolid. Moreover the canonical map from M to this limit is an isomorphism modulo I-torsion modules. Thus the localization functor when I is finitely presented as left R-module is

$$M \longmapsto \varinjlim_n \mathrm{Hom}_R(I^{\otimes_R n}, M)$$

Now suppose I is finitely generated projective as left R-module. Then the same is true for

$$I^{\otimes_R n} = I \otimes_R I \otimes_R \cdots \otimes_R I$$

and

$$\mathrm{Hom}_R(I^{\otimes_R n}, M) = \mathrm{Hom}_R(I^{\otimes_R n}, R) \otimes_R M$$

so

$$\varinjlim_n \mathrm{Hom}_R(I^{\otimes_R n}, M) = \left(\varinjlim_n \mathrm{Hom}_R(I^{\otimes_R n}, R)\right) \otimes_R M$$

Take now $R = T(V)$, $I = T^{>0}(V)$.

Then
$$I^{\otimes_R n} = R\otimes V \otimes_R (R\otimes V)\otimes_R \cdots \otimes_R (R\otimes V)$$
$$= R \otimes V^{\otimes n}$$

as left R-module, so
$$\mathrm{Hom}_R(I^{\otimes_R n}, M) = \mathrm{Hom}(V^{\otimes n}, M) = V^{*\otimes n}\otimes M$$

and the localization functor is
$$M \longmapsto \varinjlim_n \left(M \longrightarrow V^*\otimes M \longrightarrow V^*\otimes V^*\otimes M \longrightarrow \cdots\right)$$
$$= \underbrace{\varinjlim_n \left(T(V) \longrightarrow V^*\otimes T(V) \longrightarrow V^{*\otimes 2}\otimes T(V)\longrightarrow \cdots\right)}_{\text{this should be } \mathcal{O}_{V^*}} \otimes_R M$$

Next consider the solid case. Given any R-module M one has an inverse system
$$M \longleftarrow I\otimes_R M \longleftarrow I^{\otimes_R 2}\otimes_R M \longleftarrow \cdots$$
and one can take the inverse limit. When I is finitely generated projective as a right R-module $I\otimes_R -$ commutes with $\varprojlim$'s, so the functor

(*) $$M \longmapsto \varprojlim_n I^{\otimes_R n}\otimes_R M$$

should be right adjoint to the inclusion of solid modules in R-modules.

It seems like there is some kind of Cuntz-Krieger algebra here in the case of an ideal $I \subset R$ which is finitely generated projective as right R-module. Call this algebra $\mathcal{O}_I$. Its desired property is that there's a homomorphism $R \to \mathcal{O}_I$ such $\mathcal{O}_I$-modules are equivalent to I-solid R-modules via restriction of scalars.

Assuming I fin. gen. projective as right R-module we have

$$I \otimes_R M = \mathrm{Hom}_R(I^*, M)$$

where $I^* = \mathrm{Hom}_{R^{op}}(I, R)$ is the right dual of I. Then

$$\begin{aligned} I \otimes_R I \otimes_R M &= \mathrm{Hom}_R(I^*, I \otimes_R M) \\ &= \mathrm{Hom}_R(I^*, \mathrm{Hom}_R(I^*, M)) \\ &= \mathrm{Hom}_R(I^* \otimes_R I^*, M) \end{aligned}$$

We know the right adjoint functor to the inclusion I-solid $\hookrightarrow$ R-mod is

$$\begin{aligned} M \longmapsto & \varprojlim_n I^{\otimes_R n} \otimes_R M \\ &= \varprojlim_n \mathrm{Hom}_R(I^{* \otimes_R n}, M) \\ &= \mathrm{Hom}_R(\varinjlim_n I^{* \otimes_R n}, M) \end{aligned}$$

Now this should be $\mathrm{Hom}_R(\mathcal{O}_I, M)$ if the inclusion I-solid $\subset$ R-mod is restriction of scalars associated to a homom. $R \longrightarrow \mathcal{O}_I$. Thus we should have

$$\mathcal{O}_I = \varinjlim_n I^{* \otimes_R n} = \varinjlim_n \mathrm{Hom}_{R^{op}}(I^{\otimes_R n}, R)$$

In the case $R = T(V)$, $I = T^{>0}(V)$, then

$$\mathrm{Hom}_{R^{op}}(I^{\otimes_R n}, R) = \mathrm{Hom}_{R^{op}}(V^{\otimes n} \otimes R, R) = R \otimes V^{* \otimes n}$$

so that $\mathcal{O}_I = \varinjlim_n T(V) \otimes V^{* \otimes n}$. This should be the Cuntz $\mathcal{O}_V$, while the algebra encountered with cosolid modules is $\varinjlim_n \mathrm{Hom}_R(I^{\otimes_R n}, R)$. So the difference between these is whether we use left or right duals. [redacted]

Here's some comments [redacted] to make the preceding a bit clearer.

Assuming I finitely generated projective as right module an R-module M is solid: $I \otimes_R M \xrightarrow{\sim} M$ iff it is a module over $\varinjlim \mathrm{Hom}_{R^{op}}(I^{\otimes_R n}, R)$.

Assuming I fin. gen. proj as left module an R-module M is cosolid: $M \xrightarrow{\sim} \mathrm{Hom}_R(I, M)$ iff it is a module over $\varinjlim_n \mathrm{Hom}_R(I^{\otimes_R n}, R)$.

Put $I_r^* = \mathrm{Hom}_{R^{op}}(I, R)$, $I_\ell^* = \mathrm{Hom}_R(I, R)$. Then M solid means one has both R and I_r^* mapping into $\mathrm{Hom}_{\mathbb{Z}}(M, M)$. Also M cosolid means [redacted] $M \xrightarrow{\sim} I_\ell^* \otimes_R M$ so that both R and I_ℓ^* map into $\mathrm{Hom}_{\mathbb{Z}}(M, M)$.

Q: Is it possible to construct other R-algebras [redacted] $R \longrightarrow \mathcal{O}$ by combining the [redacted] natural transformation $I \otimes_R M \longrightarrow M$ and $M \longrightarrow \mathrm{Hom}_R(I, M)$ [redacted] to get something which is inverted exactly in $\mathcal{O}$?

One could hope for enough $\mathcal{O}$ to form the analogue of an open affine covering of a projective scheme.

June 25, 1994

Recall

$$\mathrm{Hom}_{R\text{-mod}/I\text{-nilp}}(M_1, M_2) = \varinjlim_n \mathrm{Hom}_R(I^{\otimes_R n} \otimes_R M_1, M_2)$$

$$\simeq \varinjlim_n \mathrm{Hom}_R(M_1, \mathrm{Hom}_R(I^{\otimes_R n}, M_2))$$

Of particular interest is the ring

$$\mathcal{O} = \mathrm{Hom}_{R\text{-mod}/I\text{-nilp}}(R, R) = \mathrm{Hom}_{R\text{-mod}/I\text{-nilp}}(I, I)$$

$$= \varinjlim_n \mathrm{Hom}_R(I^{\otimes_R n}, R) = \varinjlim_n \mathrm{Hom}_R(I^{\otimes_R n}, I)$$

of endomorphisms of the canonical generator $R \simeq I$ for the category R-mod/I-nilp. [redacted] Recall that this ring $\mathcal{O}$ depends only upon the [redacted] nonunital ring I.

Consider the case $I = I^2$. Then

$$\mathcal{O} = \mathrm{Hom}_R(I \otimes_R I, I) \simeq \mathrm{Hom}_R(I \otimes_R I, I \otimes_R I)$$

At ~~this~~ point it would probably be best to adopt the nonunital ring viewpoint, prerings in the terminology suggested by Husemoller.

[redacted]

Let I be a prering such that $I^2 = I$, let $A = I \otimes_I I$ be its canonical solid extension, so that $I = A/K$, where K is [redacted] a null ideal in A. Better: Recall that a prering I such that $I^2 = I$ can be canonically written $I = A/K$ where A is a solid ring: $A \otimes_A A \xrightarrow{\sim} A$ and K is a <u>null ideal</u> in A.

Before continuing with the analysis preceding, it is worthwhile discussing examples of Morita equivalences. The basic result is the following:

Consider a Morita context

$$\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$$

that is, a unital ring with a 2x2 block matrix decomposition (equivalently a unital ring together with an idempotent element).

Let $I = QP \subset R$, $J = PQ \subset S$. These are ideals of R and S respectively. Then there are equivalences of categories

1) $\dfrac{R\text{-mod}}{I\text{-nilp}} \rightleftarrows \dfrac{S\text{-mod}}{J\text{-nilp}}$ $\qquad M \longmapsto P \otimes_R M$, $\quad Q \otimes_S N \longleftarrow N$

2) $(R,I)\text{-solid} \rightleftarrows (S,J)\text{-solid}$ $\qquad$ ———

3) $(R,I)\text{-cosolid} \rightleftarrows (S,J)\text{-cosolid}$ $\qquad M \longmapsto \operatorname{Hom}_R(Q, M)$, $\quad \operatorname{Hom}_S(P, N) \longleftarrow N$

Examples.

1. Suppose $S \subset R$ is an inclusion of unital rings, let J be an <u>ideal in S which is also a left ideal</u> in R: $\boxed{RJ = J}$, and let $\boxed{I = JR}$ be the ideal in R generated by J. Then we have a Morita context

$$\begin{pmatrix} R & Q \\ P & S \end{pmatrix} \overset{\text{defn}}{=} \begin{pmatrix} R & J \\ R & S \end{pmatrix} \underset{\text{subring}}{\subset} \begin{pmatrix} R & R \\ R & R \end{pmatrix}$$

$$\begin{pmatrix} R & J \\ R & S \end{pmatrix}\begin{pmatrix} R & J \\ R & S \end{pmatrix} = \begin{pmatrix} R^2+JR & RJ+JS \\ R^2+SR & RJ+S^2 \end{pmatrix} = \begin{pmatrix} R & J \\ R & S \end{pmatrix}$$

and $QP = JR =$ the ideal I in R
$PQ = RJ =$ the ideal J in S.

Thus we have the equivalence of categories

$$M \longmapsto R\otimes_R M = M$$ restriction of scalars from R to S

$$J\otimes_S N \longleftarrow N$$ which puts an R-module structure on any J-solid N: $J\otimes_S N \xrightarrow{\sim} N$.

Special case: If J is an ideal in S, then $I = JR = J$, so we find (at least in the case [redacted] $S \subset R$) that the [redacted] three categories assoc. to (B, J) and (R, J) are the same.

2. Suppose $S \subset R$ as above, J an ideal in S which is a <u>right ideal</u> in R ($JR = J$); let $\boxed{I = RJ.}$ Then we have a Morita context

$$\begin{pmatrix} R & Q \\ P & S \end{pmatrix} \overset{\text{defn}}{=\!=} \begin{pmatrix} R & R \\ J & S \end{pmatrix} \subset M_2(R)$$

Check: $$\begin{pmatrix} R & R \\ J & S \end{pmatrix}\begin{pmatrix} R & R \\ J & S \end{pmatrix} \subset \begin{pmatrix} R^2+RJ & R^2+RS \\ JR+SJ & JR+S^2 \end{pmatrix} = \begin{pmatrix} R & R \\ J & S \end{pmatrix}$$

$$QP = RJ = I$$

$$PQ = JR = J$$ so this time the

equivalence is given by

$$M \longmapsto J\otimes_R M$$

$$R\otimes_S N \longleftarrow N$$ base extension from S to R

3. Suppose $R/K = S$, the ideal I in R is such that $\boxed{KI = 0}$, and J is the image of I in S. Then we have an ideal

$$\begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix} \text{[redacted]} \subset \begin{pmatrix} R & R \\ I & R \end{pmatrix}$$

Check:

$$\begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}\begin{pmatrix} R & R \\ I & R \end{pmatrix} = \begin{pmatrix} KI & KR \\ KI & KR \end{pmatrix} = \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}$$

$$\begin{pmatrix} R & R \\ I & R \end{pmatrix}\begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix} = \begin{pmatrix} 0 & RK \\ 0 & IK+RK \end{pmatrix} = \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}$$

hence we obtain a Morita context:

$$\begin{pmatrix} R & Q \\ P & S \end{pmatrix} \stackrel{\text{def}}{=\!=} \begin{pmatrix} R & S \\ I & S \end{pmatrix} = \begin{pmatrix} R & R \\ I & R \end{pmatrix} \Big/ \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}$$

Then $QP = SI = (R/K)\cdot I = RI = I$

$PQ = IS = (\text{Image of } I \text{ in } S)S = JS = J$

so the equivalence is given by

$M \longmapsto I \otimes_R M$ — which means for M solid that M is killed by K hence M is an S-module

$N = S \otimes_S N \longleftarrow N$ — restriction of scalars from S to R.

Special case: If $I \cap K = 0$ so that $I \xrightarrow{\sim} J$, then we get for a surjection $R \twoheadrightarrow S$ the independence of the good categories on the embedding as an ideal in a unital algebra.

4. Suppose $R/K = S$, I is an ideal in R such that $\boxed{IK = 0}$, let J be the image of I in S. Then we have an ideal:

$$\begin{pmatrix} 0 & 0 \\ K & K \end{pmatrix} \subset \begin{pmatrix} R & I \\ R & R \end{pmatrix}$$

Check:

$$\begin{pmatrix} 0 & 0 \\ K & K \end{pmatrix}\begin{pmatrix} R & I \\ R & R \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ KR & KI+KR \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ K & K \end{pmatrix}$$

$$\begin{pmatrix} R & I \\ R & R \end{pmatrix}\begin{pmatrix} 0 & 0 \\ K & K \end{pmatrix} = \begin{pmatrix} IK & IK \\ RK & RK \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ K & K \end{pmatrix}$$

hence we obtain a Morita context

$$\begin{pmatrix} R & Q \\ P & S \end{pmatrix} \overset{\text{def}}{=\!=} \begin{pmatrix} R & I \\ S & S \end{pmatrix} = \begin{pmatrix} R & I \\ R & R \end{pmatrix} \Big/ \begin{pmatrix} 0 & 0 \\ K & K \end{pmatrix}$$

Then

$$QP = I \cdot S = I \cdot (R/K) = I$$

$$PQ = SI = S(\text{Image of } I \text{ in } S) = SJ = J$$

so the equivalence is given by

$$M \longmapsto S \otimes_R M \qquad \text{base extn from } R \text{ to } S$$

$$I \otimes_S N \longleftarrow N$$

June 26, 1999

Let $A = A \otimes_A A$ be a solid ring, let $K \subset A$ be an ideal and let $B = A/K$. Let's try to relate the right multiplier [redacted] rings $\mathrm{Hom}_A(A, A)$ and $\mathrm{Hom}_B(B, B)$.

First we have $\mathrm{Hom}_B(B,B) = \mathrm{Hom}_A(B, B)$ since A maps onto B.

Next from the exact sequence

$$0 \longrightarrow K \longrightarrow A \longrightarrow B \longrightarrow 0$$

one gets

$$0 \longrightarrow \mathrm{Hom}_A(B,B) \longrightarrow \mathrm{Hom}_A(A,B) \longrightarrow \mathrm{Hom}_A(K,B)$$

Because A is solid as A-module we have

$$\mathrm{Hom}_A(A, B) \xleftarrow{\sim} \mathrm{Hom}_A(A, \underbrace{A \otimes_A B}_{\text{solidification of } B})$$

One has also the exact sequence

$$\begin{array}{ccccccc} A \otimes_A K & \longrightarrow & A \otimes_A A & \longrightarrow & A \otimes_A B & \longrightarrow & 0 \\ & & \wr\!\downarrow & & & & \\ & & A & & & & \end{array}$$

so that $\quad A/AK \xleftarrow{\sim} A \otimes_A B$

Now take $K = \mathrm{ann}(A_r) = \{a \mid Aa = 0\} =$ largest ideal such that $AK = 0$. Then any A-module map $f : K \to B$ has image in $\{b \in B \mid Ab = 0\}$. Let $\pi : A \to B$ be the canonical surj, suppose $A\pi(a) = 0$, i.e. $\pi(Aa) = 0$, or $Aa \subset K$. Then $Aa = A^2 a \subset AK = 0$, so $a \in K$ and we conclude $\mathrm{Hom}_A(K, B) = 0$.

Thus we have

$$
\begin{aligned}
\mathrm{Hom}_B(B,B) &= \mathrm{Hom}_A(B,B) \\
&\xrightarrow{\sim} \mathrm{Hom}_A(A,B) \\
&\xleftarrow{\sim} \mathrm{Hom}_A(A, A\otimes_A B) \\
&\xrightarrow{\sim} \mathrm{Hom}_A(A,A)
\end{aligned}
$$

Something slightly more efficient is to note that since $AK=0$, the surjection $A \to B$ is an isomorphism modulo A-null, hence as A is solid $\mathrm{Hom}_A(A,A) \xrightarrow{\sim} \mathrm{Hom}_A(A,B)$.

Here's a check on the above calculation: Since $AK=0$ we know the equivalence between A-solid and B-solid is $M \longmapsto B\otimes_A M = M/KM$, so that the [redacted] A/KA is the B-solid module corresponding to the A-solid module A. Then the surjection $A/KA \longrightarrow B$ is an isom modulo B-null. But because $B\cdot B = B$ and ${}_B B = 0$ we know $\mathrm{Hom}_{B\text{-mod}/B\text{-null}}(B,B) = \mathrm{Hom}_B(B,B)$.

In fact this argument can be carried out for an arbitrary A such that $A=A^2$. Namely let $K = {}_A A = \{a\in A \mid Aa=0\}$, so that $AK=0$. Then $A \to A/K$ is an isomorphism mod A-null so

$$
\begin{aligned}
\mathrm{Hom}_{A\text{-mod}/A\text{-null}}(A,A) &= \mathrm{Hom}_{A\text{-mod}/A\text{-null}}(A/K, A/K) \\
&= \mathrm{Hom}_A(A/K, A/K) \quad \left\{ \begin{array}{l} \text{since } A\cdot A/K = A/K \\ \text{and } {}_A(A/K) = 0 \end{array} \right. \\
&= \mathrm{Hom}_{A/K}(A/K, A/K)
\end{aligned}
$$

Return to $A = A \otimes_A A$ and let J be any ideal in A such that $AJ = 0$. We want to compare $\mathrm{Hom}_A(A,A)$ with $\mathrm{Hom}_{A/J}(A/J, A/J) = \mathrm{Hom}_A(A/J, A/J)$. We have the exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(A/J, A/J) \longrightarrow \mathrm{Hom}_A(A, A/J) \longrightarrow \mathrm{Hom}_A(J, A/J)$$

$$\uparrow \simeq$$

$$\mathrm{Hom}_A(A, A)$$

(Since A solid and $A \to A/J$ is iso mod A-null)

Thus $\mathrm{Hom}_A(A/J, A/J) \xrightarrow{\sim} \{\theta \in \mathrm{Hom}_A(A,A) \mid \theta(J) \subset J\}$

We would now like an example of a solid ring A and J such that $AJ = 0$ where [redacted] $\mathrm{Hom}_A(A,A)$ [redacted] does not [redacted] preserve J. Start with any solid ring B such that $K = \sqrt[B]{B} \neq 0$ and put $A = B \oplus B$. Take $J = \Delta K \subset B \oplus B$, where $\Delta K = \{(a,a) \mid a \in K\}$. Take $\theta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ on $B \oplus B$.

Further discussion of quotients $A/K = B$ of a ring A (such that $A = A^2$ if necessary)

First let's give the prering formulation of the two Morita equivalence cases:

$\underline{KA = 0}$:

$$M \longmapsto A \otimes_A M \qquad \begin{pmatrix} A & {}_A\tilde{B}_B \\ {}_BA_A & B \end{pmatrix} \qquad QP = \tilde{B}A = A$$

$$N = \tilde{B} \otimes_B N \longleftarrow N \qquad\qquad PQ = A\tilde{B} = B$$

$\underline{AK = 0}$:

$$M \longmapsto \tilde{B} \otimes_A M = M/KM \qquad \begin{pmatrix} A & A \\ \tilde{B} & B \end{pmatrix} \qquad QP = A\tilde{B} = A$$

$$A \otimes_B N \longleftarrow N \qquad\qquad PQ = \tilde{B}A = B$$

Combining these we have the case $\boxed{AKA=0}$:
Check that $\begin{pmatrix} 0 & AK \\ KA & K \end{pmatrix} \subset \begin{pmatrix} A & A \\ A & A \end{pmatrix}$ is an ideal:

$$\begin{pmatrix} 0 & AK \\ KA & K \end{pmatrix}\begin{pmatrix} \tilde{A} & A \\ A & \tilde{A} \end{pmatrix} = \begin{pmatrix} 0+AKA & AK\tilde{A} \\ KA\tilde{A}+KA & KA^2+K\tilde{A} \end{pmatrix} \subset \begin{pmatrix} 0 & AK \\ KA & K \end{pmatrix}$$

$$\begin{pmatrix} \tilde{A} & A \\ A & \tilde{A} \end{pmatrix}\begin{pmatrix} 0 & AK \\ KA & K \end{pmatrix} = \begin{pmatrix} AKA & \tilde{A}AK+AK \\ \tilde{A}KA & A^2K+\tilde{A}K \end{pmatrix} = \begin{pmatrix} 0 & AK \\ KA & K \end{pmatrix}$$

Thus we get a Morita context

$$\begin{pmatrix} \tilde{A} & A/AK \\ A/KA & \tilde{A}/K \end{pmatrix} = \begin{pmatrix} \tilde{A} & A/AK \\ A/KA & \widetilde{A/K} \end{pmatrix} \qquad \begin{matrix} QP = (A/AK)(A/KA) = A^2 \\ PQ = (A/KA)(A/AK) = A^2+K/K \end{matrix}$$

Note the following: Given A let $\operatorname{ann}(A_\ell) = \{a \in A \mid aA = 0\}$, $\operatorname{ann}_\ell(A) = \{a \in A \mid Aa = 0\}$. Let $\pi : A \longrightarrow A/\operatorname{ann}_\ell(A)$ be the canonical surjection.

Suppose $a \in A$ such that $\pi(a) \in \operatorname{ann}_r(A/\operatorname{ann}_\ell(A))$, that is, $(A/\operatorname{ann}_\ell(A))\pi(a) = 0$, equivalently $Aa \subset \operatorname{ann}_\ell(A)$, which is the same as $AaA = 0$. Thus

$$\pi^{-1}\left(\operatorname{ann}_r\left(A/\operatorname{ann}_\ell(A)\right)\right) = \{a \mid AaA = 0\}$$

which means that it's the same as the inverse image in A of $\operatorname{ann}_\ell\left(A/\operatorname{ann}_r(A)\right)$. One has a square of ideals

$$\begin{matrix} \operatorname{ann}_\ell(A) \cap \operatorname{ann}_r(A) & \subset & \operatorname{ann}_r(A) \\ \cap & & \cap \\ \operatorname{ann}_\ell(A) & \subset & \{a \mid AaA = 0\} \end{matrix}$$

whose quotients are Morita equivalent to A.

Consider next the subring situation $A \subset B$.

1. A is a <u>left ideal</u> in B: $BA \subset A$. Then one has a Morita context

$$\begin{pmatrix} A & \tilde{B} \\ A & B \end{pmatrix} \qquad \begin{matrix} M \mapsto A \otimes_A M \\ N = \tilde{B} \otimes_B N \leftarrow\!\shortmid N \end{matrix} \qquad \begin{matrix} QP = \tilde{B}A = A \\ PQ = A\tilde{B} \end{matrix}$$

hence a Morita equivalence between B and $A\tilde{B}$ = the ideal generated by the left ideal A. Another context giving the same Morita equivalence is

$$\begin{pmatrix} A & A\tilde{B} \\ A & B \end{pmatrix} \qquad \begin{matrix} QP = \overbrace{A\tilde{B}A}^{A} = A^2 \sim A \quad \text{commensurable} \\ PQ = A^2\tilde{B} = A\tilde{B}A\tilde{B} = (A\tilde{B})^2 \sim A\tilde{B} \end{matrix}$$

2. A is a <u>right ideal</u> in B: $AB \subset B$. Then one has the Morita contexts

$$\begin{pmatrix} A & A \\ \tilde{B} & B \end{pmatrix} \qquad \begin{matrix} QP = A\tilde{B} = A \\ PQ = \tilde{B}A = \text{the ideal in } B \text{ gen. by } A \end{matrix}$$

$$\begin{pmatrix} A & A \\ \tilde{B}A & B \end{pmatrix} \qquad \begin{matrix} QP = A\tilde{B}A = A^2 \\ PQ = \tilde{B}A^2 = \tilde{B}A\tilde{B}A = (\tilde{B}A)^2 \end{matrix}$$

giving the same Morita equivalence between A and $\tilde{B}A$.

3. Assume $ABA \subset A$. Then one has the Morita context

$$\begin{pmatrix} A & A\tilde{B} \\ \tilde{B}A & B \end{pmatrix} \qquad \begin{matrix} QP = A\tilde{B}A \quad \text{note } A^2 \subset A\tilde{B}A \subset A \\ PQ = \tilde{B}A^2B \end{matrix}$$

Unfortunately $\tilde{B}A^2\tilde{B}$ does not seem to contain any power of $\tilde{B}AB$ in general, so we don't get a Morita equivalence between A and $\tilde{B}A\tilde{B}$, only one between A and $\tilde{B}A^2\tilde{B}$.

However if one makes the stronger hypothesis $ABA \subset A^2$ (whence $A\tilde{B}A = A^2$) then

$$(\tilde{B}A\tilde{B})^2 = \tilde{B}A\tilde{B}A\tilde{B} = \tilde{B}A^2\tilde{B},$$

so indeed A and $\tilde{B}A\tilde{B}$ are Morita equivalent. The nice Morita context in this case is

$$\begin{pmatrix} A & A\tilde{B} \\ \tilde{B}A & B \end{pmatrix} \qquad \begin{aligned} QP &= A\tilde{B}A = A^2 \\ PQ &= \tilde{B}A^2\tilde{B} = (\tilde{B}A\tilde{B})^2 \end{aligned}$$

Note that the hypothesis $A\tilde{B}A = A^2$ is satisfied when A is either a left or right ideal in B.

Check: Given $A \subset B$ such that $ABA \subset A$, then A is a left ideal in $A\tilde{B}$ since $(A\tilde{B})A \subset A$. Thus one has a Morita equivalence between A and the ideal in $A\tilde{B}$ generated by A, namely $A^2\tilde{B}$. Then $A^2\tilde{B}$ is a right ideal in B, so one has a Morita equiv. of $A^2\tilde{B}$ with the ideal in B it generates namely $\tilde{B}A^2\tilde{B}$. Again one gets a Morita equiv. between A and $\tilde{B}A^2\tilde{B}$.

June 27, 1994

Let M be a chain complex of R-bimodules such that $M \overset{L}{\otimes}_R R/I \simeq 0$. Up to quasi-isomorphism we can suppose M consists of free [redacted] R-bimodules. [redacted] A free R-bimodule is a direct sum [redacted] $\bigoplus_\Lambda R \otimes_k R$ for some set Λ. Let's assume that $R \otimes_k R$ is a flat right R-module, for example this is true when R is flat over the ground ring k. Then M will be flat as right R-module and so

$$M \overset{L}{\otimes}_R R/I \simeq M \otimes_R R/I = M/MI$$

is acyclic, equivalently the inclusion $MI \hookrightarrow M$ is a qis of R-bimodules. Since M is free as R bimodule, there is a bimodule map $f: M \to MI \subset M$ which is homotopic to the identity. Let

$$P = \varinjlim \{ M \xrightarrow{f} M \xrightarrow{f} M \xrightarrow{f} \cdots \}$$

Then P is a flat R-bimodule complex such that $P = PI$. Because P is a filtered direct limit of [redacted] direct sums of copies of $R \otimes_k R$, which is flat as [redacted] right R-module, we know P is flat as right R-module. Thus P is an I-solid right R-module. We also have an evident qis $M \to P$.

Note the direction of the arrow, which is unlike the resolution $P \to I$ constructed [redacted] when $I \overset{L}{\otimes}_R R/I = 0$.

June 28, 1994

In the sheaf theory situation

$$\mathrm{Sh}_Y \underset{\iota^!}{\overset{\iota^*}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\ \iota_*\ }}}}} \mathrm{Sh}_X \underset{j_*}{\overset{j_!}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\ j^*\ }}}}} \mathrm{Sh}_U$$

one has that ι^* and $j_!$ are exact functors and one has ~~an~~ short exact sequence

$$0 \longrightarrow j_! j^* F \longrightarrow F \longrightarrow \iota_* \iota^* F \longrightarrow 0$$

Further a sheaf F on X is equivalent to the triple $(\iota^* F, j^* F, \varphi : \iota^* F \longrightarrow \iota^* j_* (j^* F))$. This follows from

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \iota_* \iota^! F & = & \iota_* \iota^* \iota_* \iota^! F & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & j_! j^* F & \longrightarrow & F & \longrightarrow & \iota_* \iota^* F & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & j_! j^* j_* j^* F & \longrightarrow & j_* j^* F & \longrightarrow & \iota_* \iota^* j_* j^* F & \longrightarrow & 0
\end{array}$$

which shows that the square is cartesian, hence given $(F_Y, F_U, \varphi : F_Y \longrightarrow \iota^* j_* F_U)$ [redacted] this triple corresponds to the sheaf F defined by the fibre product:

$$\begin{array}{ccc}
F & \longrightarrow & \iota_* F_X \\
\downarrow & & \downarrow {\scriptstyle \iota_*(\varphi)} \\
j_* F_U & \longrightarrow & \iota_* \iota^* j_* F_U
\end{array}$$

adjunction arrow $1 \longrightarrow \iota_* \iota^*$

Let's now consider the module situation with R, I with $I = I^2$:

$$(R/I)\text{-mod} \;\underset{\iota^!}{\overset{\iota^*}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\;\iota_*\;}}}}}\; R\text{-mod} \;\underset{j_*}{\overset{j_!}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\;j^*\;}}}}}\; \mathcal{M}(R,I)$$

$$\iota^*(M) = M/IM \qquad j_!(j^*M) = I^S \otimes_R M$$

$$\iota^!(M) = \operatorname{Hom}_R(R/I, M) \qquad j_*(j^*M) = \operatorname{Hom}_R(I^S, M)$$

$\iota^*(M) = R/I \otimes_R M$ is exact iff R/I is flat as a right R-module and this we know is equivalent to $\forall x_1, \dots, x_n \in I \;\; \exists x \in I \ni (1-x)x_i = 0 \;\; \forall i$, also equivalent to $\forall x_1 \in I \;\; \exists x \in I \ni (1-x)x_1 = 0$.

Assume R/I is right flat. Then

$$\begin{array}{ccccc} 0 \longrightarrow & R/I \otimes_R I & \longrightarrow & R/I \otimes_R R & \\ & \| & & \| & \qquad \text{so } I = I^2. \\ & I/I^2 & \xrightarrow{\;0\;} & R/I & \end{array}$$

and the [redacted] exact sequence $0 \to I \to R \to R/I \to 0$ shows that I is also right flat. Thus $I^S = I \otimes_R I \xrightarrow{\sim} I$. It follows then that $j_!(j^*M) = I \otimes_R M$ [redacted] is exact in M, hence $j_!$ is an exact functor.

We also have the exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & I \otimes_R M & \longrightarrow & M & \longrightarrow & R/I \otimes_R M & \longrightarrow 0 \\ & \| & & \| & & \| & \\ 0 \longrightarrow & j_!(j^*M) & \longrightarrow & M & \longrightarrow & \iota_* \iota^* M & \longrightarrow 0 \end{array}$$

and the diagram

$$\begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ & & & \operatorname{Hom}_R(R/I, M) & = & R/I \otimes_R \operatorname{Hom}_R(R/I, M) & \\ & & & \downarrow & & \downarrow & \\ 0 \longrightarrow & I \otimes_R M & \longrightarrow & M & \longrightarrow & R/I \otimes_R M & \longrightarrow 0 \\ & \downarrow \cong & & \downarrow & & \downarrow & \\ 0 \longrightarrow & I \otimes_R \operatorname{Hom}_R(I, M) & \longrightarrow & \operatorname{Hom}_R(I, M) & \longrightarrow & R/I \otimes_R \operatorname{Hom}_R(I, M) & \longrightarrow 0 \end{array}$$

because I solid $\Rightarrow I \otimes_R -$ inverts null isos.

$\leftarrow$ exact as R/I flat

from which one gets the cartesian square

$$\begin{array}{ccc} M & \longrightarrow & R/I \otimes_R M \\ \downarrow & & \downarrow \\ \mathrm{Hom}_R(I,M) & \longrightarrow & R/I \otimes_R \mathrm{Hom}_R(I,M) \end{array}$$

This should imply that an R module is equivalent to a triple (N, Q, φ) [redacted] where N is an R/I-module, [redacted] Q is a I-cosolid R-module, and $\varphi: N \longrightarrow R/I \otimes_R Q$ is a map of R/I-modules.

Example: $I = \bigoplus_\alpha ke_\alpha$, $R = k \oplus I$. A cosolid I module Q has the form $\prod_\alpha V_\alpha$ where the V_α are vector spaces, [redacted] and $Q/IQ = \bigoplus V_\alpha$. Thus a triple (N, Q, φ) amounts to a vector space N, a family of vector spaces $\{V_\alpha\}$, and a map $\varphi: N \to \prod V_\alpha / \bigoplus V_\alpha$. The corresponding R-module M (unital) is given by pull-back

$$\begin{array}{ccccccccc} 0 & \to & \bigoplus V_\alpha & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \bigoplus V_\alpha & \longrightarrow & \prod V_\alpha & \longrightarrow & \prod V_\alpha / \bigoplus V_\alpha & \longrightarrow & 0 \end{array}$$

and M amounts to a factorization $\bigoplus V_\alpha \to M \to \prod V_\alpha$ of the canonical injection $\bigoplus V_\alpha \hookrightarrow \prod V_\alpha$.

Notice that $ke_\alpha = Re_\alpha$ is a projective R-module so $I = \bigoplus ke_\alpha$ is projective, so R/I has projective dimension 1. $j_* j^* M = \mathrm{Hom}_R(I,M)$ is exact in M, hence j_* is exact; this is also clear from the fact that $\mathcal{M}(R,I)$ is a product category of $\mathrm{Mod}(k)$ for each α, so every object is both injective and projective.

The only local cohomology is in degrees 0, 1:

$$
\begin{array}{ccccccccc}
0 \to & i_* i^! M & \longrightarrow & M & \longrightarrow & j_* j^* M & \longrightarrow & i_* R^1 i^!(M) & \to 0 \\
 & & & \| & & \| & & & \\
 & & & M & \longrightarrow & \prod V_\alpha & & &
\end{array}
$$

Next I want to look at the derived category situation. In the sheaf situation we should have the 3x3 diagram

$$
\begin{array}{ccccc}
j_! j^* i_* i^! F \;(= 0) & \longrightarrow & i_* i^! F & \xrightarrow{\sim} & i_* i^* i_* i^! F \\
\downarrow & & \downarrow & & \downarrow \\
j_! j^* F & \longrightarrow & F & \longrightarrow & i_* i^* F \\
\downarrow \wr & & \downarrow & & \downarrow \\
j_! j^* j_* j^* F & \longrightarrow & j_* j^* F & \longrightarrow & i_* i^* j_* j^* F
\end{array}
$$

where the rows and columns are triangles, and here I should have derived functors $Ri^!$, Rj_*. It appears from this diagram that $F \in D(X)$ is equivalent to a triple consisting of $F_Y \in D(Y)$, $F_U \in D(U)$ and a map $\varphi : F_Y \longrightarrow i^* Rj_*(F_U)$. In the module case, assuming the h-unitality: $I \overset{L}{\otimes}_R R/I \simeq 0$, we should have the diagram

$$
\begin{array}{ccccc}
I \overset{L}{\otimes}_R R\mathrm{Hom}_R(R/I, M) \;(= 0 \text{ by h-unitality}) & \longrightarrow & R\mathrm{Hom}_R(R/I, M) & \xrightarrow{\sim} & R/I \overset{L}{\otimes}_R R\mathrm{Hom}_R(R/I, M) \\
\downarrow & & \downarrow & & \downarrow \\
I \overset{L}{\otimes}_R M & \longrightarrow & M & \longrightarrow & R/I \overset{L}{\otimes}_R M \\
\downarrow \wr & & \downarrow & & \downarrow \\
I \overset{L}{\otimes}_R R\mathrm{Hom}_R(I, M) & \longrightarrow & R\mathrm{Hom}_R(I, M) & \longrightarrow & R/I \overset{L}{\otimes}_R R\mathrm{Hom}_R(I, M)
\end{array}
$$

so that again M is a suitable homotopy fibre product.

June 29, 1994

Let $X: \quad \to X^{-1} \to X^0 \to X^1 \to \cdots$

be a complex consisting of injective modules. We want to split off a contractible complex in order to obtain a minimal complex. I recall that a complex K is contractible iff it has a special contraction $[d,h]=1$, $h^2=0$, and that a special contraction is equivalent to a choice of complement:

$K^n = C^n \oplus Z^n$ for each $n \in \mathbb{Z}$.

Fix n and consider the extension $Z^n \subset X^n$. Choose $Y^n \subset X^n$ maximal such that $Z^n \cap Y^n = 0$. One knows, because X^n is injective, that Y^n is injective and that $Z^n \hookrightarrow X^n/Y^n$ is an injective hull for Z^n. Notice that $Y^n \cap Z^n = 0$ $\Longrightarrow$ $Y^n \subset X^n \xrightarrow{d} X^{n+1}$ is monic. Thus one has an injection of complexes

$$\begin{array}{ccccccccc} \to & 0 & \to & Y^n & \xrightarrow{1} & Y^n & \to & 0 & \to \cdots \\ & & & \downarrow & & \downarrow & & & \\ & & \to & X^n & \xrightarrow{d} & X^{n+1} & \to & X^{n+2} & \to \cdots \end{array}$$

The top complex is $\mathrm{Cone}\{Y_n[0] \to Y_n[0]\}[-n-1]$; denote it $C(Y_n, n)$.

Choosing Y_n in this way $\forall n$ we get an injection of $\bigoplus_n C(Y_n,n) = \prod_n C(Y_n,n)$ into X. But for any complex K one has

$$\underbrace{Z^0\mathrm{Hom}(K, C(Y_n,n))}_{\text{maps in category of complexes}} = \mathrm{Hom}(K_{n+1}, Y_n)$$

so that by choosing a retraction of X^{n+1} onto dY^n for each n we obtain a retraction of X onto $\prod C(Y_n, n)$.

In this way we can split off a contractible complex from X and obtain a homotopy equivalent complex X_{min} having the property that $\forall n$ $Z^n \hookrightarrow X^n_{min}$ is an injective hull of Z^n.

Consider now $R \supset I$ ideal and let X be a complex bounded below (upper indexing) consisting of injective modules. Then we know

$$R\,Hom_R(R/I, X) \simeq Hom_R(R/I, X) = {}_IX$$

(Here bdd below is required, example of complete resolutions used in Tate cohomology).

Suppose that X is minimal as above and that $R\,Hom_R(R/I, X) \simeq 0$, i.e. that ${}_IX$ is acyclic. Look at the lowest degree n such that ${}_IX^n \neq 0$; we can suppose $n=0$. Then $H^0({}_IX) = Z^0({}_IX) = 0$, i.e. ${}_IX^0 \cap Z^0 = 0$. By minimality X^0 is an injective hull of Z^0, so we conclude ${}_IX^0 = 0$ a contradiction. Therefore X is a complex of I-cofirm injectives. We've almost proved:

<u>Prop.</u> <u>Let</u> $M \in D^+(R\text{-mod})$. <u>Then</u> $R\,Hom_R(R/I, M) \simeq 0$ $\iff$ $M \underset{quis}{\simeq}$ <u>a complex X bdd below</u> of I-<u>cofirm</u> <u>injectives.</u>

The direction $\Leftarrow$ is trivial because

$$R\,Hom_R(R/I, M) = Hom_R(R/I, X) = 0$$

Conversely given M satisfying $R\mathrm{Hom}_R(R/I, M) \cong 0$ we know that M is quasi-isomorphic to a complex X bdd below of injectives which is minimal as above. Then we have seen that X is I-cofirm.

We have already proved

Prop. <u>Let</u> $M \in D_+(R\text{-mod})$. <u>Then</u> $R/I \otimes^L_R M = 0$ $\iff$ $M \underset{\text{quis}}{\simeq}$ <u>a complex (bdd below for lower indexing) consisting of</u> I-<u>firm</u> <u>flat modules.</u>

June 30, 1994

<u>Prop.</u> [redacted] If $M/AM = 0$ and ${}_A N = 0$, then

(This also holds for M firm or N cofirm)

$$\mathrm{Hom}_A(M,N) \xrightarrow{\sim} \mathrm{Hom}_{\bar{m}(A)}(M,N)$$

Proof. $0 \to K \to A \otimes_A M \to M \to 0$

$$0 \to \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(A\otimes_A M, N) \to \mathrm{Hom}_A(K,N)$$

and $\mathrm{Hom}_A(K,N) = 0$ because $AK = 0$ and ${}_A N = 0$. Thus

$$\mathrm{Hom}_A(M,N) \xrightarrow{\sim} \mathrm{Hom}_A(A\otimes_A M, N) \xrightarrow{\sim} \mathrm{Hom}_A(A^{\otimes 2}\otimes_A M, N) \xrightarrow{\sim}$$

since $A\otimes_A M \twoheadrightarrow M \implies A^{\otimes 2}\otimes_A M \twoheadrightarrow A\otimes_A M$ etc. Thus

$$\mathrm{Hom}_A(M,N) \xrightarrow{\sim} \varinjlim_n \mathrm{Hom}_A(A^{\otimes n}\otimes_A M, N) = \mathrm{Hom}_{m(A)}(M,N)$$

Similarly for "good" tensor product defined by

$$m(A^{op}) \times m(A) \xrightarrow{\overset{g}{\otimes}_A} \mathrm{Pro}\,Ab$$

$$X, M \longmapsto \{X\otimes_A A^{\otimes n}\otimes_A M\}$$

<u>Prop.</u> If $XA = X$ and $AM = M$, then

$$X \overset{g}{\otimes}_A M \xrightarrow{\sim} X\otimes_A M$$

(This also holds for either X or M firm)

In effect $0 \to K \to A\otimes_A M \to M \to 0$

yields $X\otimes_A K \to X\otimes_A A \otimes_A M \to X\otimes_A M \to 0$

$$X\otimes_A K = XA\otimes_A K = X\otimes_A AK = 0.$$

so

$$X\otimes_A A^{\otimes 2}\otimes_A M \xrightarrow{\sim} X\otimes_A A\otimes_A M \to X\otimes_A M \to 0$$

July 2, 1994

Suppose M firm: $I \otimes_R M \xrightarrow{\sim} M$, equivalently $- \otimes_R M$ inverts isos mod I-nilp of right R-modules. Now

$$R \longrightarrow \mathrm{Hom}_{R^{op}}(I, R) \qquad r \mapsto (a \mapsto ra)$$

has its kernel + cokernel killed by I on the right. (the kernel is $\{r / rI = 0\}$) Thus we have

$$\boxed{M \xrightarrow{\sim} \mathrm{Hom}_{R^{op}}(I, R) \otimes_R M}$$

Iterating we have

$$M \xrightarrow{\sim} \underrightarrow{\lim}\, \mathrm{Hom}_{R^{op}}(I^{\otimes_R n}, R) \otimes_R M$$

the O_r for right modules $\longrightarrow \mathrm{Hom}_{\mathcal{M}(R^{op}, I^{op})}(R, R)$

Put another way the functor $- \otimes_R M$ from R^{op}-mod $\longrightarrow$ Ab descends to $\mathcal{M}(R^{op}, I^{op})$ so its value on R, namely $R \otimes_R M = M$, is natural acted on by the endomorphisms of R in the category $\mathcal{M}(R^{op}, I^{op})$.

Recall that the canonical map

$$\mathrm{Hom}_R(I, R) \otimes_R M \longrightarrow \mathrm{Hom}_R(I, M)$$

is an isomorphism when I is finitely gen projective as left R-module, or when I is finitely presented as left R-module and M is flat. Notice that this uses a <u>different dual</u> of I. Notation:

<u>left dual</u> $I_\ell^* = \mathrm{Hom}_R(I, R)$ <u>right dual</u> $I_r^* = \mathrm{Hom}_{R^{op}}(I, R)$

Consider the diagram

$$\begin{array}{ccc} R\otimes_R M & \longrightarrow & I_\ell^* \otimes_R M \\ \| & & \downarrow \\ M & \longrightarrow & \mathrm{Hom}_R(I,M) \end{array}$$

Then we have

<u>Prop</u>: Assume $R \longrightarrow I_\ell^* = \mathrm{Hom}_R(I,R)$ has kernel + cokernel killed by I^n on the <u>right</u> for some n.

Assume I is f.g. proj R-module, or a fin. pres. R-module and that M is flat.

Then M firm $\Longrightarrow$ M cofirm.

Thus <u>in the comm. noetherian case we have firm + flat $\Longrightarrow$ cofirm.</u>

Consider next the square

$$\begin{array}{ccc} \mathrm{Hom}_R(I_r^*, M) & \longrightarrow & \mathrm{Hom}_R(R,M) \\ \uparrow & & \| \\ I\otimes_R M & \longrightarrow & M \end{array}$$

Then we have

<u>Prop</u>: Assume $R \longrightarrow I_r^* = \mathrm{Hom}_{R^{op}}(I,R)$ has kernel + cokernel killed by I^n on the <u>left</u> for some n

Assume I is fin gen proj as right R-module

Then M cofirm $\Longrightarrow$ M firm

<u>Example</u>: $R = T(V)$, $I = T^{>0}(V)$ (dim $V < \infty$). Here the (assuming dim $V \geq 2$) hypotheses that $R \longrightarrow I_\ell^*$ be a right mod nilp_I isom. and that $R \longrightarrow I_r^*$ be a left mod nilp_I isom. fail.

Why? $R \longrightarrow Hom_R(I,I) \subset Hom_R(I,R)$ is found to be

$$\begin{array}{ccccc} R & \longrightarrow & Hom_R(I,I) & \subset & Hom_R(I,R) \\ \| & & \| & & \| \\ R & \longrightarrow & V^* \otimes V \otimes R & \subset & V^* \otimes R \\ 1 & \longmapsto & v_i^* \otimes v_i \otimes 1 & & \end{array}$$

so $Hom_R(I,I)/I$ is a free right R-module $\neq 0$, hence not killed by any I^n.

July 5, 1994

I ideal $\subset R$, recall there [redacted] is a torsion theory τ_I on R-modules whose torsion-free modules are the M such that ${}_IM = 0$, and whose torsion modules are the M such that $\mathrm{Hom}_R(M,E)$ for every torsion-free injective. [redacted] Torsion theories of the form τ_I for some ideal are called regular torsion theories in Golan's book (Ch. 30). One has the following description of the torsion modules.

Prop. M is τ_I-torsion $\iff$ $\forall m \in M$, $\forall$ sequence $a_1, a_2, \ldots \in I$, [redacted] $\exists n$ s.t. $a_n a_{n-1} \cdots a_1 m = 0$.

Proof: ($\Longrightarrow$) M is torsion $\iff$ $\forall N < M$ one has [redacted] ${}_I(M/N) \neq 0$. One can then define a filtration by transfinite induction: $F^0(M) = 0$

$$F^{\alpha+1}M / F^{\alpha}M = {}_I(M/F^{\alpha}M)$$

$$F^{\alpha}M = \bigcup_{\beta<\alpha} F^{\beta}M \qquad \alpha \text{ limit ordinal}$$

Suppose given $m \in M$ and (a_n) in I, for each n let α_n be [redacted] least s.t. $a_n a_{n-1} \cdots a_1 m \in F^{\alpha_n}M$. [redacted]

If $a_n a_{n-1} \cdots a_1 m \neq 0$ then $\alpha_n > 0$ and α_n is not a limit ordinal, so $a_{n+1} \cdots a_1 m \in F^{\alpha_n - 1}M$, whence $\alpha_{n+1} < \alpha_n$. But one can't have an infinite decreasing sequence $\alpha_0 > \alpha_1 > \cdots$ of ordinals, so $a_n \cdots a_1 m = 0$ for some n.

$\Longleftarrow$ If M is not torsion, $\exists N < M$ such that

${}_I(M/N) = 0.$

Let $m \in M - N$. We will construct (a_n) in I such that $\forall n$ $a_n \cdots a_1 m \notin N$. By replacing M by M/N we can suppose ${}_I M = 0$. Thus $Im \neq 0$, so $\exists a_1 \ni a_1 m \neq 0$. Then $I a_1 m \neq 0 \Rightarrow \exists a_2 \ni a_2 a_1 m \neq 0$ etc.

That's the proof I saw yesterday. Here's a more efficient one.

Let (a_n) be a sequence in I and let $m \in M$. The condition

⊛ $\exists n$ s.t. $a_n \cdots a_1 m = 0$ means that M is in the kernel of the canon map

$$M \longrightarrow \varinjlim_n \left(M \xrightarrow{a_1} M \xrightarrow{a_2} \cdots\cdots \right)$$

$$\| $$

$$\underbrace{\left(\varinjlim R \xrightarrow{a_1 \cdot} R \xrightarrow{a_2 \cdot} \cdots\cdots \right)}_{\text{denote this } F((a_n))} \otimes_R M$$

Note that $F((a_n))$ is firm flat right R-module.

Consider the full sub. category of all M such that $F((a_n)) \otimes_R M = 0$ for all sequences (a_n) in I. Because $F((a_n))$ is flat this category is closed under submodules. It is a Serre subcategory closed under $\oplus$'s. It clearly contains I-null modules, hence all I-torsion modules. This gives the implication $(\Rightarrow)$ in the above proposition.

Prop: (TFAE) 1) M is I-torsion [redacted]

2) $X \otimes_R M = 0$ for all right modules X s.t. $X = XI$.

3) $F((a_n)) \otimes_R M = 0$ for all sequences $(a_n) \in I$.

It remains to check $1) \Rightarrow 2)$. Because $X \otimes_R -$ respects $\varinjlim$'s there is a largest submodule $N \subset M$ such that $X \otimes_R N = 0$. If $N < M$, then, as M is assumed torsion, $\exists\ N'$ s.t. $N < N' \subset M$ and $I(N'/N) = 0$. Then we have

$$X \otimes_R N \longrightarrow X \otimes_R N' \longrightarrow X \otimes_R (N'/N) \longrightarrow 0$$
$$X \otimes_R (N'/N) = 0$$

so $N = M$ and $X \otimes_R M = 0$.

Here's an attempt to show that firm modules form an abelian category at least in the case where R is noetherian commutative.

Recall that in general an (additive) functor $R\text{-mod} \xrightarrow{F} Ab$ respecting $\varinjlim$'s (i.e. respecting $\oplus$'s and right exact) has the form $X \otimes_R -$, where the right module X is $F(R)$. [redacted] Next $X \otimes_R -$ descends to $R\text{-mod}/I\text{-tors} \iff X$ is firm. Thus firm right modules [redacted] are equivalent to functors $R\text{-mod}/I\text{-tors} \longrightarrow Ab$ respecting $\varinjlim$'s. (I think here one [redacted] needs that the canonical functor $R\text{-mod} \longrightarrow R\text{-mod}/I\text{-tors}$ respects $\varinjlim$'s because it has a right adjoint.)

Now suppose R left noetherian. Then the finitely generated R-modules form a noetherian abelian category whose associated ind-object category (also the associated locally noetherian category) is R-mod. This should also be true for the quotient categories: R-mod/I-tors is the locally noetherian cat. assoc. to the noetherian category fg R-mod/fg I-tors. I think then that right continuous (resp. lim's) functors R-mod/I-tors ⟶ Ab are equivalent to functors fg R-mod/fg I-tors ⟶ Ab which are right exact.

If all this holds, then we have an equivalence between firm modules X and right ~~exact~~ functors fg R-mod/fg I-tors ⟶ Ab. In fact this should be clear directly.

So now the idea was that right exact functors $\mathcal{A} \longrightarrow$ Ab, where $\mathcal{A}$ is a small abelian category, should form an abelian category. I think Gabriel proves the result for left exact functors.

I think there's a problem with right exact functors because the left-derived functor: $L_0 F \to F$ is constructed using inverse limits, as opposed to the right-derived functor: $F \longrightarrow R^0 F$, which is constructed using direct limits.

July 7, 1994

Given I an ideal $\subset R$ we have a canonical functor

1) $$I^{op}\text{-firm} \longrightarrow \varinjlim \text{ cont Fun}\left(R\text{-mod}/I\text{-torsion}, Ab\right)$$
$$X \longmapsto (M \longmapsto X \otimes_R M)$$

Why? The functor $X\otimes_R - : R\text{-mod} \longrightarrow Ab$, where X is an R^{op}-module, is $\varinjlim$ continuous as it has the right adjoint $N \mapsto \text{Hom}_{\mathbb{Z}}(X, N)$. On the other hand we have adjoint functors

$$I\text{-tors} \underset{i^!}{\overset{i_*}{\rightleftarrows}} R\text{-mod} \underset{j_*}{\overset{j^*}{\rightleftarrows}} R\text{-mod}/I\text{-tors}$$

$R\text{-mod}$ —localization→ I-coferm, I-coferm —inclusion→ $R\text{-mod}$; $R\text{-mod}/I\text{-tors} \simeq I\text{-coferm}$

from the theory of torsion theories. In particular the canonical map j^* to the quotient category is $\varinjlim$ continuous as it has a right adjoint.

When X is I^{op}-firm we know that $X\otimes_R -$ inverts isoms modulo I-tors. Thus this functor descends to a functor $R\text{-mod}/I\text{-tors} \longrightarrow Ab$.

To see it is $\varinjlim$ cont suppose given a functor $C \longrightarrow R\text{-mod}/I\text{-tors}$. Then using j_* it can be lifted to a functor $i \mapsto M_i$, $C \to R\text{-mod}$, and we have $j^*(\varinjlim M_i) = \varinjlim j^* M_i$. Then

$$X\otimes_R \varinjlim(j^*M_i) = X\otimes_R j^*(\varinjlim M_i) = X\otimes_R \varinjlim M_i = \varinjlim X\otimes_R j^* M_i.$$

This seems OKAY.

Next we want to show that 1) is an equivalences of categories. So start with Φ : R-mod/I-tors $\longrightarrow$ Ab $\varinjlim$ cont., then Φj^* : R-mod $\longrightarrow$ Ab is $\varinjlim$ continuous.

Now there is a canonical map

$$\Phi j^*(R) \otimes_R M \longrightarrow \Phi j^*(M)$$

(for an arbitrary functor this is true) which is an isomorphism iff the functor is $\varinjlim$ cont. Thus we have $\Phi j^*(M) = X \otimes_R M$ with $X = \Phi j^*(R)$.

Since Φ descends it inverts mod I-tors isoms., hence X is I^{op} firm.

<u>Thus we have an equivalence of categories</u>

$$I^{op}\text{-firm} \xrightarrow{\sim} \varinjlim \text{ cont Fun}(R\text{-mod}/I\text{-tors}, Ab)$$

<u>Prop.</u> TFAE

1) I is (left) T-nilpotent ($\forall$ sequence (a_n) in I $\exists n \ni a_n a_{n-1} \cdots a_1 = 0$).

2) I-cofirm = 0

3) I^{op}-firm = 0

Proof. Since I-cofirm $\simeq$ R-mod/I-tors, I-cofirm = 0 means every R-module is in I-tors, equivalently R is I-torsion.

1) $\Rightarrow$ 2). If I-cofirm $\neq 0$, then there exists a torsion-free module N, i.e. ${}_I N = 0$. Pick $n \in N$, $n \neq 0$. Then $In \neq 0$ so can pick $a_1 \in I$ s.t. $a_1 n \neq 0$, then $I a_1 n \neq 0$, so

can pick $a_2 \in I$ s.t. $a_2 a_1 n \neq 0$, etc. showing I is not T-nilpotent.

3) $\Rightarrow$ [redacted]) If I is not T-nilpotent, there is a sequence a_n in I such that $\forall n \quad a_n \cdots a_1 \neq 0$. This means that [redacted]

$$F = \varinjlim \{ R \xrightarrow{a_1 \cdot} R \xrightarrow{a_2 \cdot} \cdots \} \neq 0$$

We know that F is a flat and firm [redacted] right module, so I^{op}-firm $\neq 0$.

2) $\Rightarrow$ 3) [redacted] If I-cofirm $= 0$, then R-mod$/I$-tors $= 0$ so I^{op}-firm which is the cat right exact functors from [redacted] R-mod$/I$-tors to Ab is zero.

As a check, suppose there exists a nonzero flat I-firm right module F. Then consider the class of modules M such that $F \otimes_R M = 0$. This is clearly a Serre subcat of R-mod closed under $\oplus$'s. It contains any I-null modules, hence all I-torsion modules. But it doesn't contain R so we see I-tors $<$ R-mod. Thus [redacted] I^{op}-firm $\neq 0 \Rightarrow I$-cofirm $\neq 0$, so 2) $\Rightarrow$ 3).

Here seems to be the standard example of an I which is (left) T-nilpotent but not right T-nilpotent. Consider infinite strictly upper triangular matrices with finite support and entries say in a [redacted] field k. Given a sequence $a_1, \cdots \in I$, note that because a_1 has finite support a_1 is contained in a left ideal which is finite dimensional, namely matrices supported in columns $1 \leq j \leq m$ for some m.

This left ideal is killed by I^{m-1} so it's clear that $a_m \cdots a_1 = 0$.

On the other hand denoting by e_{ij}, $i<j$, the basis matrix with 1 in the (i,j) th position we have

$$e_{12} e_{23} \cdots e_{n-1\, n} = e_{1n} \neq 0$$

for all n so I is not right T-nilpotent.

Then for such a ring I we have

I^{op}-firm $= 0$, I-cofirm $= 0$

I-firm $\neq 0$, I^{op}-cofirm $\neq 0$.

Let's calculate these categories. Take R to be the path algebra of the quiver

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet} \longrightarrow \underset{4}{\bullet} \longrightarrow$$

and I the ideal of paths of length ≥ 1. This R is non unital. It is a tensor algebra

$$R = S \oplus \overbrace{B \oplus B \otimes_S B \oplus \cdots}^{I}$$

where the summands can be visualized as the matrices supported in the various diagonals starting the main diagonal (which gives $S = \bigoplus_{i=1}^{\infty} k e_{ii}$)

An I-firm R-module should be the same as a firm S module M equipped with an isom.

$$(*) \qquad B \otimes_S M \xrightarrow{\sim} M$$

of S modules. So M we know is $\bigoplus_{i=1}^{\infty} M_i$ where

$M_i = e_{ii} M$. Then $(*)$ gives

$$e_{i\,i+1} \otimes M_{i+1} \xrightarrow{\sim} M_i$$

So a firm module in this case is a representation of the quiver

$$\underset{\bullet}{M_1} \xrightarrow{\sim} \underset{\bullet}{M_2} \xrightarrow{\sim} \underset{\bullet}{M_3} \xrightarrow{\sim}$$

such that the arrows are isomorphisms. I should be more careful:

$$e_{12} \otimes M_2 \xrightarrow{\sim} M_1$$
$$e_{23} \otimes M_3 \xrightarrow{\sim} M_2$$
$$\ldots$$

Consider now an I^{op} firm R^{op} module, which should be the same as an S^{op}-module M together with an isomorphism of right S-mods

$$M \otimes_S B \xrightarrow{\sim} M$$

This time we have

$$\bigoplus_{i=1}^{\infty} M_i \otimes e_{i\,i+1} \xrightarrow{\sim} \bigoplus_{i=0}^{\infty} M_{i+1}$$

which means

$$0 \xrightarrow{\sim} M_1$$
$$M_1 \otimes e_{12} \xrightarrow{\sim} M_2$$
$$M_2 \otimes e_{23} \xrightarrow{\sim} M_3$$
$$\ldots$$

hence the only I^{op}-firm module is zero.

Note this is consistent with I T-nilpotent $\Longleftrightarrow$ I^{op}-firm $= 0$.

Notice the following consequence of
I^{op}-firm $\simeq$ $\varinjlim$ cont. Fun(I-cofirm, Ab).

Namely if I-cofirm $\simeq$ $\mathcal{O}$-mod for some ring $\mathcal{O}$, then I^{op}-firm $\simeq$ $\mathcal{O}^{op}$-mod.

Let's consider the case where I is a finitely generated proj R-module. Then

$$Hom_R(I, M) = \underbrace{Hom_R(I,R)}_{I^*} \otimes_R M$$

so an I-cofirm module is an R-module such that $M \xrightarrow{\sim} I^* \otimes_R M$. This means that besides the operators $T_r : m \mapsto rm$ for $r \in R$ we also have operators $T^*_\varphi : M \longrightarrow I^* \otimes_R M \simeq M$
for $\varphi \in I^* = Hom_R(I,R)$ $\quad m \mapsto \varphi \otimes m \mapsto T^*_\varphi m$

An I^{op}-firm module is an R^{op}-module X such that $X \otimes_R I \xrightarrow{\sim} X$. This means in addition to the operators $T_r : x \longrightarrow xr$ we have operators
$T^*_\varphi : X \xrightarrow{\sim} X \otimes_R I \xrightarrow{1 \otimes \varphi} X$.

An interesting point here is that supposedly these types of modules depend only on I and not R. How do you see this? It is obviously meaningless to expect I to be finitely generated projective over $\tilde{I}$.

July 8, 1994

About pure exact sequences + pure injective ($=$ algebraically compact) modules. References:
Books: Jensen + Lessing – Model theoretic algebra.
Prest – Model theory + modules.

Prop. For an exact sequence $0 \to M' \to M \to M'' \to 0$ of R-modules TFAE

1) For any fin pres. R^{op} module X (resp. any R^{op}-module X), the functor $X \otimes_R -$ applied to this sequence is exact.

2) For any fin pres R-module N, the functor $\mathrm{Hom}_R(N, -)$ applied to the sequence is exact.

3) The sequence of R^{op} mods

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(M'', \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(M', \mathbb{Q}/\mathbb{Z}) \to 0$$

is split exact.

4) The given exact sequence is a filtered inductive limit of split exact sequences.

Proof. First discuss the equivalence of 1) and 2). Consider

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}_R(N,M') & \to & \mathrm{Hom}_R(N,M) & \to & \mathrm{Hom}_R(N,M'') & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M'^P & \to & M^P & \to & M''^P & \to & 0 \\
 & & \downarrow r^t & & \downarrow r^t & & \downarrow r^t & & \\
0 & \to & M'^Q & \to & M^Q & \to & M''^Q & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & X \otimes_R M' & \to & X \otimes_R M & \to & X \otimes_R M'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Here starting with $R^q \xrightarrow{\cdot r} R^p \longrightarrow M \longrightarrow 0$
we define X by $R^p \xrightarrow{r^t\cdot} R^q \longrightarrow X \longrightarrow 0$.

Assuming 1) we have $X \otimes_R M' \hookrightarrow X \otimes_R M$, whence by the serpent lemma $\mathrm{Hom}_R(N, M) \twoheadrightarrow \mathrm{Hom}_R(N, M'')$. Thus 1) $\Rightarrow$ 2), and the other direction is similar.

[illegible]

Check that the two cases of 1) are equivalent. The point is that <u>any module is a filtered inductive limit of fin. presented modules</u>. Choose a presentation

$$R^{(\Lambda')} \xrightarrow{\varphi} R^{(\Lambda)} \longrightarrow M \longrightarrow 0$$

Then consider the poset of finite subsets pair (S', S) such that $S' \subset \Lambda'$, $S \subset \Lambda$ and $\varphi(R^{S'}) \subset R^S$. This poset is directed and $M = \varinjlim_{(S',S)} \mathrm{Coker}\left(R^{S'} \xrightarrow{\varphi} R^S\right)$.

This result I think <u>identifies</u> <u>the ind-category of fin pres. modules with the category of R-modules</u>. Actually the important point is that for N finitely presented $\mathrm{Hom}_R(N, -)$ commutes with filtered $\varinjlim$'s.

Next write M'' as a direct limit of finitely presented modules N_i and consider the pull back sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M \times_{M''} N_i & \longrightarrow & N_i & \longrightarrow & 0 \\ & & \| & & \downarrow & \swarrow & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

Assuming 2) the upper sequence splits, so we see 2) $\Rightarrow$ 4). The converse is obvious.

Finally we have

$$\mathrm{Hom}_{R^{op}}(X, \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})) = \mathrm{Hom}_{\mathbb{Z}}(X \otimes_R -, \mathbb{Q}/\mathbb{Z})$$

for any R^{op} module X. Assuming 1) the right side applied to our given exact sequence is exact for any X, hence so is the left side and this implies 3). Conversely 3) implies the right side applied to the given sequence is exact, and then because $\mathbb{Q}/\mathbb{Z}$ is a faithful injective, 1) follows.

The conditions of the prop above define the notion of pure short exact sequences. Then this leads to the notions of pure projective and pure injective module.

Notice that any f.p. module is pure-proj by 2), hence any summand of a direct sum of f.p. modules is pure-proj. Given any module M, because fin. pres. modules form a small category (essentially) we can manufacture an epimorphism

$$0 \longrightarrow K \longrightarrow \bigoplus_i N_i \longrightarrow M \longrightarrow 0$$

where the N_i are f.p., such that every map from a f.p. module to M lifts. Thus the above exact sequence is pure exact, and M is a pure quotient of a pure projective module. If M is pure projective this sequence splits, so pure projective is equivalent to summand of a direct sum of fin. pres. modules.

It's clear one can construct pure exact pure projective resolutions unique up to homotopy.

Next let's examine pure-injectives.

From

$$\mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(X, E)) = \mathrm{Hom}_{\mathbb{Z}}(X \otimes_R M, E)$$

it is clear that $\forall$ R^{op}-module X and injective $\mathbb{Z}$ module E then $\mathrm{Hom}_{\mathbb{Z}}(X, E)$ is a pure-injective. (Note that X flat $\Rightarrow$ $\mathrm{Hom}_{\mathbb{Z}}(X, E)$ is injective)

Again we can manufacture for any module M an injection

$$0 \longrightarrow M \longrightarrow \prod_i \mathrm{Hom}_{\mathbb{Z}}(X_i, \mathbb{Q}/\mathbb{Z}) \longrightarrow C \longrightarrow 0$$

such that for any f.p. right module X, the induced map

$$X \otimes_R M \longrightarrow X \otimes_R \prod_i \mathrm{Hom}_{\mathbb{Z}}(X_i, \mathbb{Q}/\mathbb{Z})$$

is injective. Thus the exact sequence above is pure exact, so any module M is a pure submodule of a pure injective. Then we can construct [redacted] pure injective resolution unique up to homotopy and do "pure" homological algebra.

The next stage is to consider the functor categories of covariant + contravariant functors from fp mod(R) [redacted] to Ab. [redacted] We have the two embeddings

$$\Phi : \mathrm{mod}(R) \longrightarrow \mathrm{Fun}(\mathrm{fp\,mod}(R)^{op}, \mathrm{Ab})$$
$$M \longmapsto h_M = \mathrm{Hom}_R(-, M)$$

$$\Psi : \mathrm{mod}(R^{op}) \longrightarrow \mathrm{Fun}(\mathrm{fp\,mod}(R), \mathrm{Ab})$$
$$X \longmapsto X \otimes_R -$$

where the images are the left exact and right exact functors resp. I recall that there are canonical maps for any F:

$$\text{contrav.} \qquad F(M) \longrightarrow \mathrm{Hom}_R(M, F(R))$$

$$\text{cov.} \qquad F(R) \otimes_R M \longrightarrow F(M)$$

which are isomorphisms for M finitely presented iff F is left exact (resp. right exact).

However in this "pure" game one improves this characterization as follows.

Φ induces equivalences

$$\mathrm{fg\,mod}(R) \xrightarrow{\sim} \text{f.g. proj functors} : \mathrm{fp\,mod}(R)^{op} \to Ab$$

$$\mathrm{pure\,proj}(R) \xrightarrow{\sim} \text{proj functors}$$

$$\mathrm{mod}(R) \xrightarrow{\sim} \text{flat functors}$$

A fin. gen. functor is a quotient of a representable one, a fin. pres. functor thus has a presentation $h_M \to h_{M'} \to E \to 0$, a flat functor F is such that any map $E \to F$ with E fin pres factors through a representable functor. A flat functor is a filtered inductive limit of representable functors, so it's clear that flat functors come from modules.

Ψ induces equivalences

$$\mathrm{mod}(R) \xrightarrow{\sim} \text{fp-injective functors} : \mathrm{fp\,mod}(R) \to Ab$$

$$\text{pure-inj}(R) \xrightarrow{\sim} \text{injective functors.}$$

A finite-presentation-injective functor Q is such that any map $U \longrightarrow Q$ where U is a f.g. subfunctor of a representable functor $h^E = \mathrm{Hom}_R(E, -)$, E finitely presented, can be extended to a map $h^E \longrightarrow Q$.

Question: Assume I neither left T-nilpotent nor right T-nilpotent. Does there exist a nonzero module M such that $M/IM = {}_I M = 0$?

For example if we take I to be matrices (a_{ij}), with $(i,j) \in \mathbb{Z}\times\mathbb{Z}$ of finite support and upper triangular: $a_{ij} = 0$ for $i \geq j$, then firm I-modules should be representations of the quiver

$$\xrightarrow{\sim} M_{-1} \xrightarrow{\sim} M_0 \xrightarrow{\sim} M_1 \xrightarrow{\sim}$$

This needs checking at some point.

Let's study

$$\Phi : \mathrm{mod}(R) \longrightarrow \mathrm{Fun}(\mathrm{fp\,mod}(R^{op}), \mathcal{Ab})$$

$$M \longmapsto (X \longmapsto X \otimes_R M)$$

We wish to understand injective functors. Given an $X \in \mathrm{fp\,mod}(R^{op})$ and injective abelian group E, we have an exact contravariant functor $\mathrm{Fun} \longrightarrow \mathcal{Ab}$

$$F \longmapsto \mathrm{Hom}_{\mathbb{Z}}(F(X), E)$$

This is $\underrightarrow{\lim}$ continuous, so it should be representable:

$$\mathrm{Hom}_{\mathbb{Z}}(F(X), E) = \mathrm{Hom}_{\mathrm{Fun}}(F, G)$$

for some $G \in \text{Fun}$. Taking $F = h^Y = \text{Hom}_R(Y, -)$ we then have

$$G(Y) = \text{Hom}_{\text{Fun}}(h^Y, G)$$
$$= \text{Hom}_{\mathbb{Z}}(h^Y(X), E)$$
$$= \text{Hom}_{\mathbb{Z}}(\text{Hom}_R(Y, X), E)$$

Now Y is finitely presented, so we have exact sequences

$$R^p \xrightarrow{r\cdot} R^q \longrightarrow Y \longrightarrow 0$$

$$X^p \xleftarrow{\cdot r} X^q \longleftarrow \text{Hom}_R(Y, X) \longleftarrow 0$$

(E inj) $$\text{Hom}_{\mathbb{Z}}(X,E)^p \longrightarrow \text{Hom}_{\mathbb{Z}}(X,E)^q \longrightarrow \text{Hom}_{\mathbb{Z}}(\text{Hom}_R(Y,X), E) \longrightarrow 0$$

$$\| \qquad\qquad\qquad \|$$

$$R^p \otimes_R \text{Hom}_{\mathbb{Z}}(X,E) \longrightarrow R^q \otimes_R \text{Hom}_{\mathbb{Z}}(X,E) \longrightarrow Y \otimes_R \text{Hom}_{\mathbb{Z}}(X,E) \longrightarrow 0$$

Thus $G(-) = - \otimes_R \text{Hom}_{\mathbb{Z}}(X, E)$.

Now we know (p 691) for any right R-mod X and injective $\mathbb{Z}$-module E that $\text{Hom}_{\mathbb{Z}}(X, E)$ is a pure injective module. Moreover any pure injective is a summand of a product of $\text{Hom}_{\mathbb{Z}}(X_i, \mathbb{Q}/\mathbb{Z})$ for X_i fin pres. Thus we have shown that $- \otimes_R Q$ is an injective functor for Q pure injective.

July 9, 1994

Yesterday I wrote a proof that an injective functor in $\mathrm{Fun}(\mathrm{fpmod}(R^{op}), \mathcal{Ab})$ is of the form $X \mapsto X \otimes_R Q$ with Q purely injective and conversely. I would like to have a direct proof that $- \otimes_R Q$ with Q pure injective is injective in the functor category, but the proof is indirect in the sense that Q pure injective $\iff$ Q summand of $\prod_i \mathrm{Hom}_{\mathbb{Z}}(X_i, \mathbb{Q}/\mathbb{Z})$ for some family X_i in $\mathrm{fpmod}(R^{op})$, and so one reduces to the case $Q = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})$. In this case one has a funny double dual argument.

The proof in Jensen + Lenzing is different and proceeds by first characterizing functors of the form $- \otimes_R M$ as fp-injectives in the functor category. Clearly injective $\Longrightarrow$ fp injective, so an injective functor has the form $- \otimes_R Q$. Now $0 \to M' \to M \to M'' \to 0$ pure exact $\iff$ $0 \to (- \otimes_R M') \to (- \otimes_R M) \to (- \otimes_R M'') \to 0$ exact $\Longrightarrow$ $0 \to \mathrm{Hom}_R(M'', Q) \to \mathrm{Hom}_R(M, Q) \to \mathrm{Hom}_R(M', Q) \to 0$ exact (since the embedding $M \mapsto - \otimes_R M$ is fully faithful; M can be recovered from the value of this functor on R). Thus $- \otimes_R Q$ injective $\Longrightarrow$ Q is pure injective.

I still need a proof that Q pure-inj $\Longrightarrow$ $- \otimes_R Q$ is an injective functor.

But first let's examine fp-injective functors.

A functor $G \in \mathrm{Fun} = \mathrm{Fun}(\mathrm{fp\,mod}(R^{op}), \mathcal{Ab})$ is fp-injective when $\mathrm{Ext}^1(F,G)$ for any fin pres functor F. For $\overset{\mathrm{Fun}}{F}$ to be finitely presented means it is cokernel

$$h^{X_1} \longrightarrow h^{X_0} \longrightarrow F \longrightarrow 0$$

of representable functors. To calculate $\mathrm{Ext}^1_{\mathrm{Fun}}(F,G)$ choose an epim. $h^X \twoheadrightarrow F$ and one finds that G is fp-injective iff any diagram

$$\begin{array}{ccc} U & \hookrightarrow & h^X \\ \downarrow & \swarrow \exists & \\ G & & \end{array}$$

can be completed where U is a fin gen functor (quotient of a representable one).

I claim G fp-inj $\Rightarrow$ G right exact. In effect given $X_1 \to X_0 \to X \to 0$ exact in $\mathrm{fp\,mod}(R^{op})$ we wish to show

$$G(X_1) \longrightarrow G(X_0) \longrightarrow G(X) \longrightarrow 0$$

is exact. But

$$0 \longrightarrow h^X \longrightarrow h^{X_0} \longrightarrow h^{X_1}$$

is exact. So

$$\begin{array}{ccc} h^X & \hookrightarrow & h^{X_0} \\ \downarrow & \swarrow \exists & \\ G & & \end{array} \quad \Longrightarrow \quad G(X_0) \twoheadrightarrow G(X)$$

an

$$\begin{array}{ccc} h^{X_0}/h^X & \hookrightarrow & h^{X_1} \\ \downarrow & \swarrow \exists & \\ G & & \end{array} \quad \Longrightarrow \quad G(X_1) \twoheadrightarrow \mathrm{Ker}\big(G(X) \to G(X_0)\big)$$

Finally F right exact $\Longrightarrow$ canon map X [redacted] $\otimes_R F(R) \longrightarrow F(X)$ is an isomorphism for $X \in \text{fp mod}(R^{op})$.

The missing argument that Q pure-injective $\Longrightarrow$ $- \otimes_R Q$ is an injective functor goes as follows. Use the fact that the functor category Fun has enough injectives, to embed $- \otimes_R Q$ into an injective functor E. Then injective $\Rightarrow$ fp injective so we know $E = - \otimes_R M$ for [redacted]. $M = E(R)$. Then $- \otimes_R Q \hookrightarrow - \otimes_R M$ means that Q is a pure submodule of M, so because Q is assumed [redacted] pure injective, we know Q is a summand of M, hence $- \otimes_R Q$ is [redacted] a summand of $- \otimes_R M = E$, so $- \otimes_R Q$ is injective.

The argument I gave yesterday in effect constructs enough pure injectives of the form $\text{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})$ and shows explicitly by the double dual argument:

$$Y \otimes_R \text{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \text{Hom}_{\mathbb{Z}}(\text{Hom}_{R^{op}}(Y, X), \mathbb{Q}/\mathbb{Z})$$

(because isom. for $Y = R$ and both sides right exact), also

$$\text{Hom}_{\text{Fun}}(F, \text{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})) = \text{Hom}_{\mathbb{Z}}(F(X), \mathbb{Q}/\mathbb{Z})$$

that the corresponding functor $- \otimes_R \text{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})$ is [redacted] injective in Fun.

I have left out the implication that $- \otimes_R M$ is always fp-injective. Suppose $U \subset h^{X_1}$ is a finitely generated subfunctor of a representable functor h^{X_1}. Then we have an epim. $h^{X_0} \twoheadrightarrow U$. Then if $X = \mathrm{Coker}(X_0 \to X_1)$ we have

$$(*) \qquad 0 \longrightarrow h^{X} \longrightarrow h^{X_0} \longrightarrow h^{X_1}$$

whence $U = h^{X_0}/h^{X}$. Then given $U \xrightarrow{\varphi} - \otimes_R M$, φ is equivalent to an element of the kernel of $X_0 \otimes_R M \longrightarrow X \otimes_R M$, which by exactness of

$$X_1 \otimes_R M \longrightarrow X_0 \otimes_R M \longrightarrow X \otimes_R M \longrightarrow 0$$

comes from an elt. of $X_1 \otimes_R M$, i.e. a map $h^{X_1} \longrightarrow - \otimes_R M$ extending φ.

The construction of $(*)$ shows that any finitely generated functor in $\mathrm{Fun}(\mathrm{fp\,mod}(R^{op}), \mathcal{Ab})$ has a projective resolution of length ≤ 2.

I want now to examine again the case $R = k[x,y]$, $I = (x,y)$, keeping in mind also the graded module situation. Recall that $R\text{-mod}/I\text{-tors}$ $\simeq$ quasi-coherent sheaves on the affine plane with origin removed. In the graded situation we get the category of quasi-coherent sheaves on $\mathbb{P}^1$.

Recall we have

$$I\text{-firin} \;\simeq\; \text{lim cont } \mathrm{Fun}(R\text{-mod}/I\text{-tors}, \mathcal{Ab})$$
$$M \longmapsto (M \mapsto X \otimes_R M)$$

On the other hand because R is noetherian comm. it should be true that ~~[crossed out]~~ a lim cont.

functor $R\text{-mod}/I\text{-tors} \to \mathcal{Ab}$ should be equivalent to a right exact functor

$$\text{fg mod}(R)/\text{fg } I\text{-tors} \longrightarrow \mathcal{Ab}$$

I want to look at the other functor in this situation:

$$\text{Mod}(R)/I\text{-tors} \longrightarrow \text{Fun}(I\text{-firm}, \mathcal{Ab})$$

I recall that there is a canonical isom.

$$\text{Tor}_n^R(k, M) = \text{Ext}_R^{2-n}(k, M)$$

so that flat $\Longrightarrow$ I-cofirm

inj $\Longrightarrow$ I-firm.

~~Consider the image~~

July 10, 1994

Here's an improvement concerning flat firm resolutions.

Prop. Let M be an R-module. Then $\mathrm{Tor}_j^R(R/I, M) = 0$ for $0 \le j \le n$ iff $\exists$ a resolution [redacted]

$$\cdots \longrightarrow E_n \to E_{n-1} \to \cdots \to E_0 \to M \to 0$$

where E_j is firm flat $0 \le j \le n$.

Proof: ($\Leftarrow$) One has because $E_0, .., E_n$ are flat

$$\mathrm{Tor}_j^R(R/I, M) = H_j(E/IE)$$

and $E_j/IE_j = 0$ in degrees $\le n$ because they are firm.

($\Rightarrow$) Let $\to P_1 \to P_0 \to M \to 0$ be a projective resolution of M. Then $\mathrm{Tor}_j^R(R/I, M) = H_j(P/IP) = 0$ for $j \le n$. Let $\pi : P \to P/IP$ be the canon surjection, let $P(\le n)$ be the n-skeleton of P. Then because $P(\le n)$ consists of proj modules and $H_j(P/IP) = 0$ for $j \le n$, the restriction $P(\le n) \longrightarrow P/IP$ of π is null homotopic. Choose[redacted] a null homotopy and lift[redacted] it to an [redacted] operator h of degree one on P s.t. $h(P_j) = 0$, $j > n$, and let $f = 1 - [d, h]$. Then $\pi(f) = \pi - [d, \pi(h)] = 0$ on P_j for $j \le n$, so $f(P_j) \subset IP_j$ for $j \le n$. Let

$$E = \varinjlim \left(P \xrightarrow{f} P \xrightarrow{f} \cdots \right)$$

E is flat, firm in degrees $\le n$, and a resolution of

M since $H_*(E) = \varinjlim (H_*(P) \xrightarrow{1} H_*(P) \to \ldots)$

In the case R noetherian commutative we can try to show that
$$\text{mod}(R)/I\text{-tors} \longrightarrow \text{Fun}(I\text{-firm}, Ab)$$
is fully faithful as follows. Let $I = \sum_{j=1}^{n} Ra_i$. Then we have the finite open affine covering $Sp(R) - Sp(R/I) = \bigcup Sp(R_{a_j})$ which leads to Čech formula
$$0 \to j_* j^* M \to \prod_j \Gamma(Sp(R_{a_i}), j^* M) \to \prod_{j,k} \Gamma(Sp(R_{a_j a_k}), j^* M)$$
or
$$0 \to j_* j^* M \to \underbrace{\prod_j R_{a_j}}_{X_0} \otimes_R M \to \underbrace{\prod_{j,k} R_{a_j a_k}}_{X_1} \otimes_R M$$

Thus if $\Phi : - \otimes_R M \to - \otimes_R N$ is a map of functors, where say M and N are cofirm, then
$$\begin{array}{ccccccc} 0 \to & M & \to & X_0 \otimes_R M & \to & X_1 \otimes_R M \\ & & & \downarrow & & \downarrow \\ 0 \to & N & \to & X_0 \otimes_R N & \to & X_1 \otimes_R N \end{array}$$
yields a map $f : M \to N$. We now want to show that $\Phi = 1 \otimes f$. Subtracting, we can assume $f = 0$, i.e. $\Phi : - \otimes_R M \to - \otimes_R N$ is a map which vanishes for $- = R_{a_j}$

At this point we need to know something more about flat firm modules.

<u>July 11, 1994</u>

Suppose I finitely generated as right R-module: $I = \sum_{i=1}^{n} a_i R$. Consider an R-module M such that $IM = M$. To fix the ideas suppose R is a k-algebra, k a field; let V be a [redacted] vector space with basis $v_1, \ldots, v_n$ and let $v_1^*, \ldots, v_n^*$ be the dual basis for V^*.

The idea is to construct a flat firm R-module mapping onto M, but always using the fact that $M = IM = \sum a_i M$, so that any [redacted] $m \in M$ can be written $m = \sum a_i m_i$ for some choice of m_i, $1 \leq i \leq n$.

Let φ be the map of R-modules

$$R \longrightarrow R \otimes V \otimes V^* \longrightarrow R \otimes V^*$$
$$r \longmapsto \sum_i r \otimes v_i \otimes v_i^* \longmapsto \sum_i r a_i \otimes v_i^*$$

We construct module maps

$$\begin{array}{ccccccc} R & \xrightarrow{\varphi} & R \otimes V^* & \xrightarrow{\varphi \otimes 1} & R \otimes V^* \otimes V^* & \longrightarrow & \\ \downarrow u_0 & & \downarrow u_1 & & \downarrow u_2 & & \\ M & = & M & = & M & = & \end{array}$$

[redacted] Choose $m \in M$ and let $u_0(r) = rm$. Choose $m_i \in M$ such that $m = \sum a_i m_i$ and let $u_1(\sum r_i \otimes v_i^*) = \sum r_i m_i$. Then

$$u_1(\varphi(r)) = u_1\left(\sum r a_i \otimes v_i^*\right) = \sum r a_i m_i = rm = u_0(r).$$

Choose $m_{ji} \in M$ such that $m_i = \sum_j a_j m_{ji}$ $\forall i$, and put $u_2(r_{ji} \otimes v_j^* \otimes v_i^*) = \sum r_{ji} m_{ji}$

Then $u_2 (\varphi \otimes 1)(1 \otimes v_i^*) = u_2(\sum_j a_j \otimes v_j^* \otimes v_i^*)$

$$= \sum_j a_j m_{ji} = m_i = u_1(1 \otimes v_i^*)$$

Thus $u_2 \circ (\varphi \otimes 1) = u_1$. It's clear the construction continues with choosing $m_{ji} = \sum a_k m_{kji}$ etc.

This construction shows that

$$\varinjlim \left(R \xrightarrow{\varphi} R \otimes V^* \xrightarrow{\varphi \otimes 1} R \otimes V^* \otimes V^* \longrightarrow \cdots \right)$$

is a generators for the category of firm flat modules. In the case $R \rightleftarrows T(V)$, $a_i \leftrightarrow v_i$ the above limit should be the Cuntz algebra $\mathcal{O}_V$ with generators T_v, T_λ^* for $v \in V$, $\lambda \in V^*$ subject to the relations $T_\lambda^* T_v = \langle \lambda, v \rangle$, $\sum_i T_{v_i} T_{v_i^*} = 1$.

The reason for this is the fact that a firm module is equivalent to a vector space M equipped with an isomorphism $V \otimes M \xrightarrow{\sim} M$.

It seems that the above inductive limit is just the base extension of $\mathcal{O}_V$:

$$R \otimes_{T(V)} \mathcal{O}_V$$

and moreover it should be describable also as $R \otimes T(V^*) / \sum a_i v_i^* = 1$.

Given M such that $M = IM$ as above, so that $V \otimes M \longrightarrow M$ is
$v_i \otimes m \longmapsto a_i m$

surjective, if we choose a lifting then we obtain operators T_λ $\lambda \in V^*$

$$M \xrightarrow{\text{lifting}} V \otimes M \xrightarrow{\lambda \otimes 1} M$$

such that $\sum a_i T_{v_i^*} = 1$. (Note the lifting is $m \mapsto v_i \otimes T_{v_i^*} m$.) Thus M becomes a module over $R \otimes_{T(V)} \mathcal{O}_V$.

In general M is a quotient of a direct sum of copies of the R-module $R \otimes_{T(V)} \mathcal{O}_V$. The choice of the lifting gives a systematic choice of solutions for $m = \sum_i a_i m_i$, $m_i = \sum a_j m_{ji}$, etc.

Consider now the case $R = S(V)$ the polynomial ring. I would like to understand the firm modules in this case, in particular whether they form an abelian category. It would be nice if there were a smaller generator than $R \otimes_{T(V)} \mathcal{O}_V = R \otimes T(V^*)/\sum v_i v_i^* = 1$

Notice that there is commutative version of this, namely $S(V) \otimes S(V^*)/\sum_i v_i v_i^* = 1$.

There are various questions to ask. For example one might restrict to graded modules and consider $\mathbb{Z}$-graded $R = S(V)$ commutative algebras $S = \bigoplus_{n \in \mathbb{Z}} S_n$ such that $VS_{-1} = S_0$. In this case it seems that $S_1 = S_0 V^{-1}$ is an invertible S_0 module with inverse S_{-1}. So we are looking at rings over $S(V)$ which invert the ideal $S(V)V$.

Examples: $S(V)_v = S(V)[\lambda]/(\lambda v = 1)$

or $S(V) \otimes S(V^*) / \sum v_i v_i^* = 1$.

Notice that $S(V) \otimes S(V^*) / \sum v_i v_i^* = 1$ has the associated variety consisting of pairs $\lambda \in V^*$, $v \in V$ such that $\lambda(v) = 1$. This is an affine variety which fibres with affine space fibres over both $V^* - 0$ and $V - 0$.

There's something reminiscent of Morita equivalence here.

July 12, 1994

Preadditive category = additive category without the existence of 0 and $\oplus$.

A small pre-additive category $\mathcal{a}$ is the same as a non-unital ring A with a matrix decomposition

$$A = \bigoplus A_{\alpha\beta} \qquad \alpha, \beta \in \operatorname{Ob} \mathcal{a}$$

$$A_{\alpha\beta} A_{\gamma\delta} \subset \begin{cases} 0 & \beta \neq \gamma \\ A_{\alpha\delta} & \beta = \gamma \end{cases}$$

such that $\exists\, e_\alpha \in A_{\alpha\alpha}$ such that

$$e_\alpha f = f \quad \text{for } f \in A_{\alpha\beta}$$

$$f e_\alpha = f \quad \text{for } f \in A_{\beta\alpha}$$

Thus if $\operatorname{Ob} \mathcal{a}$ is finite $\mathcal{a}$ is the same as a unital ring with a matrix decomposition. In particular a Morita context

$$\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$$

is the same as a pre-additive category with two objects. Some other examples:

$$\begin{pmatrix} R & Q & R & R \\ P & S & P & P \\ R & Q & R & R \\ R & Q & R & R \end{pmatrix} \qquad \begin{pmatrix} \ddots & & \vdots & & \\ & R_0 & R_1 & R_2 & \\ \cdots & R_{-1} & R_0 & R_1 & \cdots \\ & R_{-2} & R_{-1} & R_0 & \\ & & \vdots & & \ddots \end{pmatrix}$$

"Hankel" matrix ring where $R = \bigoplus_{n \in \mathbb{Z}} R_n$ is $\mathbb{Z}$-graded

Suppose then $A = \bigoplus_{\alpha,\beta} A_{\alpha\beta}$ corresponds to a preadditive category. One has

$$Ae_\beta = \bigoplus_\alpha A_{\alpha\beta}$$

so as left A-module one has

$$A = \bigoplus_\beta Ae_\beta$$

which means that A is a projective $\tilde{A}$-module.

Recall the picture

$$\text{null} \rightleftarrows \text{mod}(\tilde{A}) \rightleftarrows \mathcal{M}(A)$$

In this situation $\tilde{A}/A$ is a flat A^{op} module so $i^*(M) = \tilde{A}/A \otimes_{\tilde{A}} M = M/AM$ is exact. Because A is a projective A^{op} module, $j_*(j^*M) = \text{Hom}_{\tilde{A}}(A, M)$ is exact, so j_* is exact.

We should have an identification

$$\mathcal{M}(A) = \text{Additive functors}(\mathcal{A}, \mathcal{Ab})$$

with $j_!(F) = \bigoplus_\alpha F(\alpha)$ $\quad j_*(F) = \prod_\alpha F(\alpha)$. Also

$$0 \to i_* i^!(M) \longrightarrow M \longrightarrow j_* j^* M \longrightarrow i_* R^1 i^!(M) \longrightarrow 0$$

$$M = M, \quad j_*j^*M = \prod_\alpha e_\alpha M, \quad M \longrightarrow \prod_\alpha e_\alpha M$$

and $R^q j_*(j^*M) \xrightarrow{\sim} i_* R^{q+1} i^!(M) = 0$ for $q \geq 1$, because j_* is exact.

How to calculate $Lj_!(j^*M)$.

First remark is that the construction of a firm flat resolution-modulo-null-modules of M makes sense for a complex bdd below. Say $M_\cdot$ is a chain complex of R-modules, and pick $\forall n$ a surjection $F_n' \twoheadrightarrow I^g \otimes_R M_n$. Then

$$\begin{array}{ccccccc} \cdots \longrightarrow & F_2' \oplus F_1' & \longrightarrow & F_1' \oplus F_0' & \longrightarrow & 0 & \longrightarrow \\ & \downarrow & & \downarrow & & & \\ \cdots \longrightarrow & I^g \otimes_R M_1 & \longrightarrow & I^g \otimes_R M_0 & \longrightarrow & 0 & \longrightarrow \end{array}$$

gives a surjection of [redacted] complexes $F_{0\cdot} \twoheadrightarrow I^g \otimes_R M_\cdot$ such that F is firm flat. Now proceed as before

$$0 \to K_{1\cdot} \longrightarrow F_{0\cdot} \longrightarrow I^g \otimes_R M_\cdot \longrightarrow 0$$

$$0 \longrightarrow K_{2\cdot} \longrightarrow F_{1\cdot} \longrightarrow I \otimes_R K_{1\cdot} \longrightarrow 0$$

and we obtain a double complex $F_{\cdot\cdot}$ of firm flat modules together with an augmentation $F_{0\cdot} \longrightarrow M_\cdot$ which "horizontally resolves $M_\cdot$ modulo null modules.

Now if $E \longrightarrow R$ is a right R-module resolution-modulo-null-modules with E firm flat, then one has qis

$$E \otimes_R M \longleftarrow E \otimes_R F \longrightarrow F$$

and any of these complexes represents $Lj_!(j^*M)$.

<u>Remark</u> that F is a complex of R-modules, but $E \otimes_R M$ is only a complex of abelian groups.

There's a problem it seems in constructing E as a bimodule complex.

Here's another way to obtain $L_{j!}j^*M$:

$$L_{j!}(j^*M) = \varprojlim \left\{ \cdots \longrightarrow I \overset{L}{\underset{R}{\otimes}} I \overset{L}{\underset{R}{\otimes}} M \longrightarrow I \overset{L}{\underset{R}{\otimes}} M \longrightarrow M \right\}$$

where the inverse system is essentially constant in the sense that the maps become more and more connected. To prove this formula let $F \longrightarrow M$ be as above and let C be the cone on this map. Then F is a complex of flat modules, so

$$I \overset{L}{\underset{R}{\otimes}} F \longrightarrow IF = F$$

is a quis, so the inverse system in the case of F is constant up to quis. Next the triangle

$$I \overset{L}{\underset{R}{\otimes}} C \longrightarrow R \overset{L}{\underset{R}{\otimes}} C \longrightarrow R/I \overset{L}{\underset{R}{\otimes}} C \longrightarrow$$
$$R \overset{L}{\underset{R}{\otimes}} C = C$$

shows that the homology group of $I \overset{L}{\underset{R}{\otimes}} C$ are null, since C (and obviously $R/I \overset{L}{\underset{R}{\otimes}} C$) have this property. But if the lowest nonzero homology group of C is in degree n, then $H_j(I \overset{L}{\underset{R}{\otimes}} C) = 0$ for $j < n$ and $H_n(I \overset{L}{\otimes}_R C) = I \otimes_R H_n(C) = 0$ since $I = I^2$ and $I H_n(C) = 0$. Thus $I \overset{L}{\otimes}_R C$ is at least 1 more connected than C. Thus $\{ I^{\overset{L}{\otimes}_R n} \overset{L}{\otimes}_R C \}$ is essentially zero, and $F = \{ I^{\overset{L}{\otimes}_R n} \overset{L}{\otimes}_R F \} \sim \{ I^{\overset{L}{\otimes}_R n} \otimes_R M \}$ establishing the formula.

July 13, 1994

We continue with the derived category situation. In $D_+(\text{mod}(R))$ we have a canonical distinguished triangle

1)
$$\begin{array}{ccccccc} I \overset{L}{\otimes}_R M & \longrightarrow & R \overset{L}{\otimes}_R M & \longrightarrow & R/I \overset{L}{\otimes}_R M & \longrightarrow \\ & & \wr\!\!\mid & & \wr\!\!\mid & \\ & & M & \longrightarrow & i_* Li^*(M) & \end{array}$$

Thus to produce a canonical distinguished triangle

2) $$Lj_! j^*(M) \longrightarrow M \longrightarrow i_* Li^*(M) \longrightarrow$$

we must produce a canonical isomorphism

3) $$Lj_! j^*(M) \simeq I \overset{L}{\otimes}_R M$$

In particular we must have

4) $$0 \simeq I \overset{L}{\otimes}_R R/I$$

[redacted] Assuming 4) we now construct 2) + 3).

$Lj_! j^* M$ can be calculated using any [redacted] complex in $C_+(M)$ which is quis $j^* M$ and which consists of flat objects. [redacted] Better: $Lj_! j^* M \simeq F$ where F is any complex (bdd below) of firm flat modules equipped with a quis $F \to M$ modulo null modules. Construction of such an F, using the fact that any complex N^0 such that $IN = N$ is a quotient $F \twoheadrightarrow N$ of a flat firm complex:

$$0 \longrightarrow K_1 \longrightarrow F_0 \longrightarrow I^g \otimes_R M \longrightarrow 0$$
$$0 \longrightarrow K_2 \longrightarrow F_1 \longrightarrow I \otimes_R M \longrightarrow 0$$

This yields a double complex F of firm flat mods together with a 'horizontal' augmentation $F \to M$ which is a quis mod null.

Suppose now that M is a complex of projective modules. The condition $I \overset{L}{\otimes}_R R/I = 0$ means that I has a resolution E by firm flat right modules. Consider the diagram

$$\begin{array}{ccccc} E \otimes_R F & \xrightarrow{\alpha} & I \otimes_R F & = & F \\ \downarrow \beta & & \downarrow \gamma & & \\ E \otimes_R M & \xrightarrow{\delta} & I \otimes_R M & & \end{array}$$

The map α is a quis because $E \to I$ is a quis and F is flat; similarly δ is a quis because we are assuming M projective.
The map β is a quis because E is firm flat and $H_*(F) \to H_*(M)$ is an isom mod null modules.

Thus γ is a quis. But γ is a map of R-module complexes (unlike β, α, δ). Thus we have

$$Lj_! j^* M \simeq F \simeq I \otimes_R M = I \overset{L}{\otimes}_R M$$

Next consider $Rj_*(j^* M)$. In $D^+(\mathrm{mod}(R))$ we have a canonical dist Δ

$$\begin{array}{ccccc} RHom_R(R/I, M) & \longrightarrow & RHom_R(R, M) & \longrightarrow & RHom_R(I, M) \longrightarrow \\ \downarrow \simeq & & \downarrow \simeq & & \\ \iota_* R\iota^!(M) & \longrightarrow & M & & \end{array}$$

so that to have a canonical distinguished Δ

$$\Gamma_X R i^!(M) \to M \to Rj_* j^*(M) \to$$

we need a canonical isom.

$$R\mathrm{Hom}_R(I, M) \simeq Rj_* j^*(M)$$

Up to isom. in $D^+(\mathrm{mod}(R))$ we can suppose M injective. Recall that $Rj_* j^* M \simeq Q$ where Q is a complex of cofirm injective modules equipped with a quis $M \to Q$ modulo null modules. Consider the diagram of complexes (of abelian groups)

$$\begin{array}{ccc} \mathrm{Hom}_R(I, M) & \xrightarrow{\alpha} & \mathrm{Hom}_R(F, M) \\ \downarrow \beta & & \downarrow \gamma \\ Q = \mathrm{Hom}_R(I, Q) & \xrightarrow{\delta} & \mathrm{Hom}_R(F, Q) \end{array}$$

where F is a firm flat left module resolution of I. Because M, Q are injective and $F \to I$ is a quis, the maps α, δ are quis. To see γ is a quis it's equivalent to show that $\mathrm{Hom}_R(F, C)$ is quis 0, where C is the cone on $M \to Q$. Thus C is a complex of injectives whose homology is null.

So we reach the problem of showing $R\mathrm{Hom}_R(F, N) = 0$ where F is a flat firm complex and N is complex with null homology. But it's clear that I shouldn't have introduced F. ?

Let's start again with the whole derived category business.

We wish to construct a canonical functorial distinguished triangle in $D_+(\mathrm{mod}(R))$

1) $$L j_! j^*(M) \longrightarrow M \longrightarrow i_* L i^*(M) \longrightarrow$$

Since one has the distinguished Δ

$$\begin{array}{ccccc} I \overset{L}{\otimes}_R M & \longrightarrow & R \overset{L}{\otimes}_R M & \longrightarrow & R/I \overset{L}{\otimes} M \longrightarrow \\ & & \downarrow\wr & & \downarrow\wr \\ & & M & \longrightarrow & i_* L i^*(M) \end{array}$$

our task amounts to constructing a canon isom

2) $$L j_! j^*(M) \xrightarrow{\sim} I \overset{L}{\otimes}_R M$$

Now 2) implies that if M is a complex (bdd below) with null homology, then $I \overset{L}{\otimes}_R M = 0$. In particular $2) \Longrightarrow I \overset{L}{\otimes}_R R/I = 0$.

Conversely assume $I \overset{L}{\otimes}_R R/I = 0$, i.e. $\mathrm{Tor}^R_*(I, R/I) = 0$. I claim then that $\mathrm{Tor}^R_*(I, N) = 0$ for all null modules N. In effect assume $\mathrm{Tor}^R_i(I, N) = 0$ for all $i < n$ and null modules N. Choose an exact sequence $0 \to N_1 \to (R/I)^{\wedge} \to N \to 0$. Then $\mathrm{Tor}^R_n(I, N) \xrightarrow{\sim} \mathrm{Tor}^R_{n-1}(I, N_1) = 0$.

It follows then from the spectral sequence

$$E^2_{pq} = \mathrm{Tor}^R_p(I, H_q(M)) \Longrightarrow H_n(I \overset{L}{\otimes}_R M)$$

that $I \overset{L}{\otimes}_R M \simeq 0$ for any complex M with null homology.

Finally $Lj_! j^*M$ can be calculated using a flat resolution of j^*M in $\mathcal{M}$. This amounts to a firm flat complex F together with a quis $F \to M$ modulo null modules. The cone C for this map then has null homology so $I \overset{L}{\otimes}_R C \simeq 0$, whence $I \overset{L}{\otimes}_R F \longrightarrow I \overset{L}{\otimes}_R M$ is a quis. (I forgot to mention that $Lj_! j^*M = F$ above). Then we have

$$Lj_! j^*M = F \xleftarrow[\text{quis}]{} I \overset{L}{\otimes}_R F \xrightarrow{\text{quis}} I \overset{L}{\otimes}_R M$$

yielding 2).

Next we wish to obtain a canonical distinguished Δ in $D^+(\mathrm{mod}(R))$

3) $$\iota_* R\iota^!(M) \longrightarrow M \longrightarrow Rj_* j^*M \longrightarrow$$

and since one has the dist. Δ

$$\begin{array}{ccccc} R\mathrm{Hom}_R(R/I, M) & \longrightarrow & R\mathrm{Hom}_R(R, M) & \longrightarrow & R\mathrm{Hom}_R(I, M) \longrightarrow \\ \| & & \| & & \\ \iota_* R\iota^*(M) & \longrightarrow & M & & \end{array}$$

we need a canonical isom

4) $$Rj_* j^*M \simeq R\mathrm{Hom}_R(I, M)$$

Note that 4) $\Rightarrow$ $R\mathrm{Hom}_R(I, M)$ when $j^*M = 0$, i.e. M has null homology. In particular $\mathrm{Ext}^*_R(I, N) = 0$ for all null modules N.

Let $P \to I$ be a projective resolution. Take $N = \mathrm{Hom}_{\mathbb{Z}}(R/I, \mathbb{Q}/\mathbb{Z})$, where left mult. of R on N comes from the right mult on R/I. Then

$$\mathrm{Ext}^*_R(I,N) = H^* \mathrm{Hom}_R(P, \mathrm{Hom}_{\mathbb{Z}}(R/I, \mathbb{Q}/\mathbb{Z}))$$
$$= H^* \mathrm{Hom}_{\mathbb{Z}}(R/I \otimes_R P, \mathbb{Q}/\mathbb{Z})$$
$$= \mathrm{Hom}_{\mathbb{Z}}(\underbrace{H_*(R/I\otimes_R P)}_{\mathrm{Tor}^R_*(R/I,I)}, \mathbb{Q}/\mathbb{Z})$$

Thus 4) $\Longrightarrow$

5) $\quad \mathrm{Tor}^R_*(R/I, I) = 0 \quad$ i.e. $\quad R/I \overset{L}{\otimes}_R I \simeq 0$.

Conversely assume 5). Then we claim $\mathrm{Ext}^*_R(I,N) = 0$ for all null modules N. By the preceding Ext calculation we know this holds for N of the form $\mathrm{Hom}_{\mathbb{Z}}(R/I, Q)$ with Q any injective $\mathbb{Z}$ module and any null N embeds in such a module. So again we can argue that if $\mathrm{Ext}^i_R(I,N)$ for all $i<n$ and null modules N, then upon embedding N into some $\mathrm{Hom}_{\mathbb{Z}}(R/I, Q)$ and letting N_1 be the cokernel, we have $0 = \mathrm{Ext}^{n-1}_R(I, N_1) \xrightarrow{\sim} \mathrm{Ext}^n_R(I,N)$, proving the claim.

From the spectral sequence
$$E_2^{pq} = \mathrm{Ext}^p_R(I, H^q(M)) \Longrightarrow H^n(R\mathrm{Hom}_R(I,M))$$
we conclude that $R\mathrm{Hom}_R(I,M) \simeq 0$ if M has null homology.

Finally $Rj_* j^* M$ can be calculated using an injective resolution of j^*M in $\mathcal{M}$, which amounts to an complex Q of cofirm injectives together with a quis $M \longrightarrow Q$ modulo null modules. One has $Rj_* j^* M = Q$.

Then

$$Rj_*j^*M = Q \longrightarrow RHom_R(I,Q) \longleftarrow RHom_R(I,M)$$

The first map is a quis because Q is cofirm injective the second because the cone C on $M \to Q$ has null homology, hence $RHom_R(I,C) \cong 0$. Thus we get the desired isom.

$$Rj_*j^*M \simeq RHom_R(I,M)$$

Observe we can replace I by a complex U. in the above arguments. Two cases:

1) If U is a complex of R^{op} modules (bdd below) then $U \overset{L}{\otimes}_R R/I = 0 \iff U \overset{L}{\otimes}_R -$ kills complexes with null homology (complexes in $D_+(mod(R))$).

2) If U is a complex of R modules (bdd below) then $R/I \overset{L}{\otimes}_R U = 0 \iff RHom_R(U,-)$ kills complexes with null homology (complexes in $D^+(mod(R))$).

Proof of 1). Enough to prove $\Rightarrow$, Given M bdd below with null homology, the Postnikov system of M reduces to the case where M is a null module N. Writing N as a quotient of $(R/I)^{(\Lambda)}$ with kernel N_1, then repeating to obtain N_2, etc. we have

$$U \overset{L}{\otimes}_R N \xrightarrow{\sim} U \overset{L}{\otimes}_R N_1[1] \xrightarrow{\sim} U \overset{L}{\otimes}_R N_2[2] \xrightarrow{\sim}$$

But these are getting more and more connected, so all homology groups of $U \overset{L}{\otimes}_R N$ are zero.

Proof of 2) Let P be a proj. resolution of U, let Q be any injective $\mathbb{Z}$-module. Then

$$
\begin{aligned}
&R\mathrm{Hom}_R(U, \mathrm{Hom}_{\mathbb{Z}}(R/I, Q)) \\
&\simeq \mathrm{Hom}_R(P, \mathrm{Hom}_{\mathbb{Z}}(R/I, Q)) \\
&= \mathrm{Hom}_{\mathbb{Z}}(R/I \otimes_R P, Q) \\
&\simeq \mathrm{Hom}_{\mathbb{Z}}(R/I \overset{L}{\otimes}_R U, Q)
\end{aligned}
$$

If $R\mathrm{Hom}_R(U, -)$ kills complexes with null homology, then this calculation shows $R/I \overset{L}{\otimes}_R U = 0$, so we obtain the direction $\Leftarrow$.

Conversely assume $R/I \overset{L}{\otimes}_R U = 0$. To prove $R\mathrm{Hom}_R(U, M) = 0$ for any complex in $D^+(\mathrm{mod}(R))$ with null homology, we can reduce to the case where N is a null module via the Postnikov system of N. The above calculation gives $R\mathrm{Hom}_R(U, N) = 0$ for N of the form $\mathrm{Hom}_{\mathbb{Z}}(R/I, Q)$, with Q any injective $\mathbb{Z}$-module. We can embed N in some $\mathrm{Hom}_{\mathbb{Z}}(R/I, Q)$, then if N^1 is the cokernel, embed N^1 similarly to obtain N^2, etc. Then

$$R\mathrm{Hom}_R(U, N) \simeq R\mathrm{Hom}_R(U, N^1)[-1] \simeq R\mathrm{Hom}_R(U, N^2)[-2] \simeq$$

so all the homology groups of $R\mathrm{Hom}_R(U, N)$ are zero. This proves $\Rightarrow$.

Notice that the above proofs hold without assuming $I = I^2$, provided we use the fact that any null module: $I^n N = 0$ is an extension of modules killed by I.

July 15, 1994

Morita invariance examples.

Suppose $A \subset B$ (subring), $B = \tilde{B}A\tilde{B}$ the ideal generated by A. Factor

$$A \subset A\tilde{B} \subset \tilde{B}A\tilde{B} = B$$

The second inclusion is such that $A\tilde{B}$ is a right ideal in B such that B is the ideal $\tilde{B}AB$ generated by $A\tilde{B}$.

Assume now that A is a left ideal in $A\tilde{B}$: $A\tilde{B}A \subset A$ and that $A\tilde{B}$ is the ideal in $A\tilde{B}$ generated by A: $\quad A\tilde{B} = \widetilde{AA\tilde{B}} = A + A^2\tilde{B}$.

Then $A\tilde{B}A = A^2 + \underbrace{A^2\tilde{B}A}$ $\quad A(A\tilde{B}A) \subset A^2$, so $\quad A\tilde{B}A = A^2$

Thus we get the conditions $\boxed{\tilde{B}A\tilde{B} = B, \ A\tilde{B}A = A^2.}$

Recall the cases:

A left ideal in B generating B as ideal $\quad M \longmapsto A \otimes_A M$, $\quad \tilde{B} \otimes_B N \longleftarrow N$ $\quad \begin{pmatrix} A & \tilde{B} \\ A & B \end{pmatrix}$ $\quad \tilde{B}A = A$, $A\tilde{B} = B$

A right ideal in B generating B as ideal $\quad X \longmapsto X \otimes_A A$, $\quad X \otimes_B \tilde{B} \longleftarrow Y$ $\quad \begin{pmatrix} A & A \\ \tilde{B} & B \end{pmatrix}$ $\quad A\tilde{B} = A$, $\tilde{B}A = B$

Combine in the above situation

$$\begin{pmatrix} A & A\tilde{B} & A\tilde{B} \\ A & A\tilde{B} & A\tilde{B} \\ \tilde{B}A & \tilde{B} & B \end{pmatrix}$$

$$\tilde{B} \otimes_{A\tilde{B}} A \longrightarrow\!\!\!\!\!\rightarrow \tilde{B}A$$

$$\widetilde{A\tilde{B}} \otimes_{A\tilde{B}} A\tilde{B} = A\tilde{B}$$

giving the composite Morita equivalence

$$\begin{pmatrix} A & A\tilde{B} \\ \tilde{B}A & B \end{pmatrix}$$

$$QP = A\tilde{B}\tilde{B}A = A\tilde{B}A = A^2 \subset A$$

$$PQ = \tilde{B}A^2\tilde{B} = \tilde{B}A\tilde{B}A\tilde{B} = (\tilde{B}A\tilde{B})^2 = B^2 \subset B.$$

Suppose $A/K = B$ where $AKA = 0$

Recall the two cases

$K'A = 0$ $\quad M \longmapsto A \otimes_A M$, $\quad \tilde{B} \otimes_B N \longleftarrow\!\shortmid N$

$$\begin{pmatrix} A & \tilde{B} \\ A & B \end{pmatrix} = \begin{pmatrix} A & \tilde{A} \\ A & A \end{pmatrix} \Big/ \begin{pmatrix} 0 & K' \\ 0 & K' \end{pmatrix}$$

$AK'' = 0$ $\quad X \longmapsto X \otimes_A A$, $\quad X \otimes_B \tilde{B} \longleftarrow\!\shortmid Y$

$$\begin{pmatrix} A & A \\ \tilde{B} & B \end{pmatrix} = \begin{pmatrix} A & A \\ \tilde{A} & A \end{pmatrix} \Big/ \begin{pmatrix} 0 & 0 \\ K'' & K'' \end{pmatrix}$$

Factor $A \longrightarrow A/K' \longrightarrow A/K$ want $AK \subset K'$

$$\begin{pmatrix} A & \tilde{A}/K' & A/K' \\ A & A/K' & A/K' \\ A/KA & \tilde{A}/K & A/K \end{pmatrix}$$

$\tilde{A}/K' \otimes_{A/K'} A/K' = A/K'$

$\tilde{A}/K \otimes_{A/K'} A = A/KA$

yielding the composite Morita equivalence

$$\begin{pmatrix} A & A/K' \\ A/K'' & A/K \end{pmatrix} = \begin{pmatrix} A & A \\ A & A \end{pmatrix} \Big/ \begin{pmatrix} 0 & K' \\ K'' & K \end{pmatrix}$$

(here I have replaced KA by K''. The conditions for [↑] to be a Morita context are

$$\begin{pmatrix} 0 & K' \\ K'' & K \end{pmatrix}\begin{pmatrix} A & A \\ A & A \end{pmatrix} = \begin{pmatrix} K'A & K'A \\ K''A + KA & K''A + KA \end{pmatrix} \subset \begin{pmatrix} 0 & K' \\ K'' & K \end{pmatrix}$$

so $\boxed{K'A = 0 \quad | \quad KA \subset K''}$ (assuming $K', K'' \subset K$ + they are ideals in A)

Also

$$\begin{pmatrix} A & A \\ A & A \end{pmatrix}\begin{pmatrix} 0 & K' \\ K'' & K \end{pmatrix} = \begin{pmatrix} AK'' & AK' + AK \\ AK'' & AK' + AK \end{pmatrix} \subset \begin{pmatrix} 0 & K' \\ K'' & K \end{pmatrix}$$

so $\boxed{AK'' = 0 \quad | \quad AK \subset K'}$

Starting from $AKA = 0$ we can take $K'' = KA$, $K' = AK$

this is a smallest possibility, which leads to the largest P, Q. We also have

$$K' = \{k \in K \mid kA = 0\} \qquad K'' = \{k \in K \mid Ak = 0\}$$

which leads to the smallest P, Q.

Given a Morita context $\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$ and ideas $I \subset R$, $J \subset S$. Assume

$$(*) \quad QJP \subset I \subset QP \qquad PIQ \subset J \subset PQ$$

Then

$$I^3 \subset QPIQP \subset QJP \subset I$$

$$J^3 \subset PQJPQ \subset PIQ \subset J$$

shows that $I \sim QJP$ and $J \sim PIQ$.

Moreover given $I \subset QP$ if we set $J = PIQ$, then $(*)$ holds: $PIQ = J \subset PQ$.

$$QJP = Q(PIQ)P \subset RIR = I \subset QP$$

Let's now go over the details of Morita equivalence when $\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$, $QJP \subset I \subset QP$, $PIQ \subset J \subset PQ$.

We first need to see

$$\mathcal{M}(R, I) \longrightarrow \mathcal{M}(S, J)$$

$$M \longmapsto P \otimes_R M$$

is well-defined. It suffices to show the functor $P \otimes_R -$: mod$(R) \to$ mod(S) carries I-null isomorphisms into J-null isomorphisms.

Suppose $M \xrightarrow{\varepsilon} N$ is a map of R-modules whose kernel and cokernel are killed by I. Better to do the kernel and cokernel seperately.

I claim that $\boxed{I \cdot \mathrm{Coker}(\varepsilon) = 0 \implies}$ $\boxed{PIQ \cdot \mathrm{Coker}(1\otimes\varepsilon) = 0}$, where $1\otimes\varepsilon : P\otimes_R M \to P\otimes_R N$.

Take $paq \in PIQ$ and $p_1 \otimes_R n \in P\otimes_R N$. Then

$$paq(p_1 \otimes_R n) = pa(qp_1)\otimes_R n = p\otimes_R a(qp_1)n.$$ But

$a\cdot \mathrm{Coker}(\varepsilon) = 0$ means $a(qp_1)n = \varepsilon(m)$ for some m.

Then $paq(p_1\otimes n) = p\otimes_R \varepsilon(m) = (1\otimes\varepsilon)(p\otimes_R m)$.

Next I show $\boxed{I \cdot \mathrm{Ker}(\varepsilon) = 0 \implies PIQ \cdot \mathrm{Ker}(1\otimes\varepsilon) = 0.}$

Take $paq \in PIQ$ and $\sum p_i \otimes_R m_i \in \mathrm{Ker}(1\otimes\varepsilon)$, i.e. $\sum p_i \otimes_R \varepsilon(m_i) = 0$. ~~Note that~~ We have a well-defined map

$$P\otimes_R N \longrightarrow N \qquad p'\otimes m \longmapsto (qp')m$$

hence $0 = \sum (qp_i)\varepsilon(m_i) = \varepsilon \sum (qp_i)m_i$. Then

$$paq \sum p_i \otimes m_i = \sum p\, a(qp_i)\otimes m_i = p\otimes a\sum (qp_i)m_i = 0$$

using the fact that $a\cdot \mathrm{Ker}(\varepsilon) = 0$. Thus $PIQ\cdot \mathrm{Ker}(1\otimes\varepsilon) = 0$.

Alternative approach: First I show for any R-module M that the canon map

$$P\otimes_R M \xrightarrow{\varphi} \mathrm{Hom}_R(Q,M) \qquad \varphi(p\otimes m) : q \longmapsto (qp)m$$

has its cokernel + kernel killed by PQ, hence by J.

Let $\sum p_i \otimes m_i \in \mathrm{Ker}(\varphi)$ i.e. $\sum (qp_i)m_i = 0 \quad \forall q$

Then $pq \sum p_i \otimes m_i = \sum p\otimes (qp_i)m_i = 0$ showing $PQ\,\mathrm{Ker}(\varphi) = 0$.

Let $f \in \mathrm{Hom}_R(Q, M)$. Then

$$((pq)f)(q') = f(q'\varphi pq) = f(q'p'q)$$
$$= q'p\, f(q) = \varphi(p \otimes f(q))(q')$$

Thus $(pq)f = \varphi(p \otimes f(q))$ showing $PQ\,\mathrm{Coker}(\varphi) = 0$.

Now let $\varepsilon : M \to N$ be a map of R modules, let $K = \mathrm{Ker}(\varepsilon)$, $C = \mathrm{Coker}(\varepsilon)$, and consider the diagram

$$\begin{array}{ccccccccc}
 & & P \otimes_R M & \xrightarrow{1 \otimes \varepsilon} & P \otimes_R N & \longrightarrow & P \otimes_R C & \longrightarrow & 0 \\
 & & \varphi \downarrow & & \downarrow \varphi & & & & \\
0 \longrightarrow \mathrm{Hom}_R(Q, K) & \longrightarrow & \mathrm{Hom}_R(Q, M) & \xrightarrow{\varepsilon_*} & \mathrm{Hom}_R(Q, N) & & & &
\end{array}$$

Take $paq \in PIQ$, $p' \otimes c \in P \otimes_R C$. Assuming $I C = 0$, then

$$paq(p' \otimes c) = p \otimes a(qp')c = 0$$

showing $P \otimes_R C$ is killed by PIQ.

Let $f \in \mathrm{Hom}_R(Q, K)$. Assuming $I \cdot K = 0$ we have $((paq)f)(q') = f(q'paq) = (q'p)\,a f(q) = 0$ so PIQ kills $\mathrm{Hom}_R(Q, K)$.

Now $PIQ \supset J^3$, and we have shown that φ maps are J-null isomorphisms. Thus we can conclude that ε I-null epic (resp. monic) $\Longrightarrow$ $1 \otimes \varepsilon$ and ε_* are J-null epic (resp. monic).