May 6, 1994

Return to [redacted] Pimsner's construction.

Consider a Morita equivalence situation

$$R = \begin{pmatrix} A & E^* \\ E & B \end{pmatrix} \qquad \begin{matrix} A = A^2 = E^*E & E^* = AE^* = BE^* \\ E = EA = BE & B = B^2 = EE^* \end{matrix}$$

where $B$ is unital. Then we know that $E$ is a f.g. projective $\tilde{A}_r$ module which is $A_r$-good (since $E \otimes_A A = B \otimes_B E = E$) and that $E^*$ is the dual f.g. proj $\tilde{A}$-module $\mathrm{Hom}_{A_r}(E, \tilde{A}) = \mathrm{Hom}_{A_r}(E, A)$.

Also $B = E \otimes_A E^* = \mathrm{Hom}_{A_r}(E, E)$.

[redacted]

Now suppose given a homomorphism $\phi : A \to B$, whence $E$, $E^*$ become $A$-bimodules. We can then form $\bigoplus_{n \geq 0} E^{\otimes_A n} = A \oplus E \oplus E \otimes_A E \oplus \cdots$, on which [redacted] we have operators $T_x$ for $x \in E$, $T_y^*$ for $y \in E^*$ satisfying $A$-bilinearity $T_{a_1 x a_2} = a_1 T_x a_2$, $T^*_{a_1 y a_2} = a_1 T^*_y a_2$ and [redacted] $T_y^* T_x = \langle y, x \rangle$. Let

$$\mathcal{T}_E = \bigoplus_{p,q \geq 0} E^{\otimes_A p} \otimes_A E^{*\otimes_A q}$$

be the Toeplitz algebra generated over $A$ by elements $T_x$, $T_y^*$ satisfying the above relations. Recall

$$\mathrm{Hom}_{A_r}(M \otimes_A E, N) = \mathrm{Hom}_{A_r}(M, \mathrm{Hom}_{A_r}(E, N))$$
$$= \mathrm{Hom}_{A_r}(M, N \otimes_A E^*)$$

so that $\mathrm{Hom}_{A_r}(E^{\otimes_A q}, E^{\otimes_A p}) = E^{\otimes_A p} \otimes_A E^{*\otimes_A q}$.

[redacted] Let $(x_i, y_i) \in E \otimes_A E^* = \mathrm{Hom}_{A_r}(E,E)$ correspond to the identity operator, so that

$$x = x_i \langle y_i, x\rangle \qquad y = \langle y, x_i\rangle y_i$$

Then $T_{x_i} T^*_{y_i}$ is an idempotent in $\mathcal{T}_E$. [redacted] We define $\mathcal{O}_E$, the Cuntz-Krieger algebra to be the quotient of $\mathcal{T}_E$ by the ideal $\{ T_1 T_2 - T_1 (T_{x_i} T^*_{y_i}) T_2 \mid T_1, T_2 \in \mathcal{T}_E \}$. Observe that

$$T^*_y \, T_{x_i} T^*_{y_i} = T^*_{\langle y, x_i\rangle y_i} = T^*_y$$

$$T_{x_i} T^*_{y_i} \, T_x = T_{x_i \langle y_i, x\rangle} = T_x$$

so that the ideal is spanned [redacted] by $T_1 (1 - T_{x_i} T^{+}_{y_i}) T_2$ of the form

$$T_{\xi_1} \cdots T_{\xi_k} (1 - T_{x_i} T^*_{y_i}) T^*_{\eta_1} \cdots T^*_{\eta_\ell} \qquad \begin{matrix} \xi_i \in E \\ \eta_\ell \in E^* \end{matrix}$$

This should means that

$$\mathcal{O}^{(n)}_E = \varinjlim_{p-q=n} E^{\otimes_A p} \otimes_A E^{* \otimes q}$$

By construction $\mathcal{O}_E$ is unital with $1 = T_{x_i} T^*_{y_i}$, also $\mathcal{O}^{(0)}_E = \mathcal{O}^{(1)}_E \mathcal{O}^{(-1)}_E$. The question is whether [redacted] $\mathcal{O}^{(-1)}_E \mathcal{O}^{(1)}_E = \mathcal{O}^{(0)}_E$. Recall picture

$$\begin{tikzcd}
\mathcal{O}^{(-1)}_E & \mathcal{O}^{(0)}_E & \mathcal{O}^{(1)}_E \\
E^* \arrow[d] & A \arrow[d] & E \arrow[d] \\
E \otimes_A E^* \otimes_A E^* \text{[redacted]} \arrow[d] & E \otimes_A E^* \arrow[d] & E \otimes_A E \otimes_A E^* \arrow[d] \\
{} & {} & {}
\end{tikzcd}$$

$$T_y^* T_x = \langle y, x\rangle \in \langle E^*, E\rangle = A$$

$$T_{y_1}^*\left(T_{x_1}T_{x_2}T_{y_2}^*\right) = \langle y_1, x_1\rangle T_{x_2}T_{y_2}^* \in AE\otimes_A E^*$$

$$\left(T_{x_1}T_{y_1}^* T_{y_2}^*\right)T_{x_2} = T_{x_1}T_{y_1}^*\langle y_2, x_2\rangle \in E\otimes_A E^*A$$

$$\left(T_{x_1}T_{x_2}T_{y_1}^*T_{y_2}^*T_{y_3}^*\right)T_{x_3} = T_{x_1}T_{x_2}T_{y_1}^*T_{y_2}^*\langle y_3, x_3\rangle \in E^{\otimes_A 2}\otimes_A E^{*\otimes_A 2}A$$

$$\left(T_{x_1}T_{y_1}^*T_{y_2}^*\right)\left(T_{x_2}T_{x_3}T_{y_3}^*\right) = T_{x_1}T_{y_1}^*\langle y_2, x_2\rangle T_{x_3}T_{y_3}^*$$

$$= T_{x_1}\langle y_1, \langle y_2, x_2\rangle x_3\rangle T_{y_3}^*$$

$$\in E\langle E^*, AE\rangle \otimes_A E^*$$

The point to note here is that $AE$ is really $\phi(A)E$, so it's not clear that $AE = E$; and similarly $E^*A$ does not have to be $E^*$; note that $EA = E$ and $AE^* = E^*$.

What's happening is this it seems: We know that $E$ is a fg projective $\tilde{A}_r$ module, hence so is $E\otimes_A E$. In effect $E$ is a summand of $\tilde{A}_r^n$ so $E\otimes_A E$ is a summand of $\tilde{A}^n\otimes_A E = E^{\oplus n}$. Corresponding to this projective module its an idempotent ideal of $A$ namely its trace which is spanned by

$$T_{y_1}^*T_{y_2}^*T_{x_2}T_{x_1} = T^*_{y_1\langle y_2, x_2\rangle}T_{x_1} = \langle y_1\langle y_2, x_2\rangle, x_1\rangle$$

Thus the ideal is $\langle E^*A, E\rangle = \langle E^*, AE\rangle$

Note that $E\langle E^*, AE\rangle$ contains $x_i\langle y_i, AE\rangle = AE$, so that $E\langle E^*, AE\rangle\otimes_A E^* \supset AE\otimes_A E^*$; also it contains $E\otimes_A E^*A$

Anyway it seems that one must make an assumption to get the invertibility. The simplest seems to be to assume that $\phi$ is such that $\phi(A)E = E$ or $E^*\phi(A) = E^*$. For example if $A$ is unital and $\phi$ is a unital homomorphism.

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Status of Pimsner Understanding.

I have analyzed the case where $E \otimes_A E^* = \mathrm{Hom}_{A^*}(E, E)$ - this is the case $K(E) = L(E)$ in Pimsner's article. The invertibility is unclear in general [illegible] and depends on $\phi$ - this is what Pimsner means by ~~[illegible] not being~~ the Hilbert bimodule $E_\infty$ over $\mathcal{F}_E$ not satisfying the condition that $\mathcal{F}_E$ is generated by scalar products.

Still to understand:

1) case where $\phi : A \longrightarrow L(E)$ has image outside $K(E)$, in particular $\mathcal{F}_E = \mathcal{O}_E$ when $\phi^{-1} K(E) = 0$.

2) examples

3) KK computations

<u>May 9, 1994</u>

<u>Excision.</u> Suppose $A \subset R$ is an ideal such that $A^2 = A$. Then one has an equivalence of categories

$$\begin{tikzcd} \frac{R\text{-mod}}{R/A\text{-mod}} \arrow[r, shift left, "M \mapsto M"] & \frac{A\text{-mod}}{A/A\text{-mod}} \arrow[l, shift left, "A^g \otimes_A N \leftarrow\!\shortmid N"] \end{tikzcd}$$

Check: We have an exact functor $R\text{-mod} \to A\text{-mod}$, given by restriction of scalars. It carries $R$-modules killed by $A$ into $A$-modules killed by $A$, so it descends ~~to~~ to yield an exact functor on the quotient categories.

Given an $A$-module $N$, $A^g \otimes_A N = A \otimes_A A \otimes_A N$ is naturally an $R$-module with $R$ acting by left multiplication on the first $A$. Moreover we know that $A^g \otimes_A -$ inverts null isoms., so we get a well-defined functor $\frac{A\text{-mod}}{A/A\text{-mod}} \longrightarrow R\text{-mod}$, $N \mapsto A^g \otimes_A N$, which we can follow by the projection to $\frac{R\text{-mod}}{R/A\text{-mod}}$.

Compute the composites

$$M \longmapsto M \longmapsto A^g \otimes_A M$$

$$N \longmapsto A^g \otimes_A N \longmapsto A^g \otimes_A N$$

There are canonical maps of functors

$$A^g \otimes_A M \longrightarrow M$$

$$A^g \otimes_A N \longrightarrow N$$

whose kernels + cokernels are killed by $A$, so the composites are canonically isomorphic to the identity.

There's a slight technical point that the maps $A^g \otimes_A M \to M$, as $M$ runs over $R$-mod, are compatible with maps in $R$-mod, but not every map in R-mod/R/A-mod comes from a map in R-mod. However every map in the quotient category is a composition of maps in R-mod and inverses of isomorphisms modulo the Serre subcategory R/A-mod, so $A^{\delta}\otimes_A M \longrightarrow M$ is a map of functors from R-mod/R/A-mod to itself.

A more concrete proof uses the equivalences

$$\frac{R\text{-mod}}{R/A\text{-mod}} \simeq \left\{ M\in R\text{-mod} \;\middle|\; A\otimes_R M \xrightarrow{\sim} M \right\}$$

$$\frac{A\text{-mod}}{A/A\text{-mod}} \simeq A\text{-gmod}$$

and the equivalence resulting from the following two observations

1) $A\otimes_A M \xrightarrow{\sim} A\otimes_R M$

2) If $A\otimes_A N \simeq N$, then there is a unique R-module on N extending the A-module structure: $r(an)=(ra)n$.

---

Let's try to prove Wodzicki's result in the case of an ideal A such that $A^2=A$, namely that a $R^{op}$ module X is $R^{op}$ flat iff it is $A^{op}$ flat. The first point is the formula

$$X\otimes_A M \xrightarrow{\sim} X\otimes_R M$$

for $M\in R$-mod. This shows that X $A^{op}$-flat $\Longrightarrow$ X is $R^{op}$ flat.

Assume now $X$ is $R^{op}$ flat. Then

$$0 \longrightarrow A \longrightarrow \tilde{R} \longrightarrow \tilde{R}/A \longrightarrow 0 \quad \text{exact}$$

implies

$$0 \longrightarrow X \otimes_R A \longrightarrow X \longrightarrow \underset{\overset{\shortparallel}{0}}{X/XA} \longrightarrow 0$$

$$\text{so } \overbrace{X \otimes_R A}^{X \otimes_A A \xrightarrow{\sim}} \xrightarrow{\sim} X \quad \text{showing } X \text{ is } A\text{-good.}$$

Then we know $M \longmapsto X \otimes_R M$ is an exact functor from $R$-mod to Ab which descends to $R$-mod / $R/A$-mod. Now use the equivalence

$$\frac{A\text{-mod}}{A/A\text{-mod}} \quad \underline{\sim} \quad \frac{R\text{-mod}}{R/A\text{-mod}}$$

$$N \longmapsto A^g \otimes_A N$$

Composing with $X \otimes_R -$ gives

$$N \longmapsto X \otimes_R A^g \otimes_A N = X \otimes_A A^g \otimes_A N = X \otimes_A N$$

We know this is an exact functor of $N \in A$-mod, so $X$ is $A$-flat.

The above is kind of hard and not as general as the proof via the Cartan-Eilenberg criterion (see p. 565)

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Suppose $A \subset R$ is a left ideal such that $A^2 = A$. Then we have a Morita equiv.

$$\begin{pmatrix} A & AR \\ A & AR \end{pmatrix}$$

excision above

$$\begin{tikzcd} A\text{-gmod} \arrow[r, shift left, "N \mapsto A \otimes_A N = N"] & AR\text{-gmod} \arrow[l, shift left, "M = AR \otimes_{AR} M \longleftarrow\!\shortmid M"] \end{tikzcd} = \{ M \in R\text{-mod} \mid AR \otimes_R M \xrightarrow{\sim} M \}$$

which means we have the following generalization of excision to a left ideal $A \subset R \ni A = A^2$ :

$$\begin{tikzcd} A\text{-gmod} & \{M \in R\text{-mod} \mid AR\otimes_R M \xrightarrow{\sim} M\} \\ N \arrow[r, mapsto] & A\otimes_A N = N \\ M & M \arrow[l, mapsto] \end{tikzcd}$$

May 12, 1994

Let $A$ be a nonunital ring such that $(1+A)^\times$ is a group. I claim that any simple $A$-module is null.

To see this suppose $M$ is a simple $A$-module such that $AM \neq 0$. Choose $m$ such that $Am \neq 0$, whence $Am = M$ by simplicity. There is then an element $a \in M$ such that $am = m$. Let $1-a'$ be the inverse of $1-a$ in the group $(1+A)^\times$, i.e. $a + a' - a'a = a + a' - aa = 0$. Then

$$m = am = (a'a - a')m = a'(am - m) = 0$$

contradiction.

Conversely suppose that $(1+A)^\times$ is not a group. One knows that there is an element $1-a$ which does not have a left inverse, i.e. $\forall a' \in A$ we have $(1-a')(1-a) \neq 1$, or equivalently $a + a' - a'a \neq 0$, or equivalently $a \notin A(1-a) = \{a'(1-a) \mid a' \in A\}$. By Zorn $\exists$ a left ideal $J < A$ such that $A(1-a) \subset J$ and such that $J$ is maximal such that $a \notin J$. Let's show that $J$ is a maximal left ideal. Let $b \in A - J$. Then $Ab + J > A$, so $a \in Ab + J$. Then for any $c \in A$ we have $ca \in Ab + J$ and $c(1-a) \in A(1-a) \subset J$, whence $A = Ab + J$. Thus $A/J$ is a simple $A$-module (as it is $\neq 0$ because of $a \notin J$). Also $Aa + J = A$ shows that $A(A/J) \neq 0$, so $A/J$ is not null. Thus we obtain

Prop. For any nonunital ring we have $(1+A)^\times$ is group $\iff$ every simple $A$-module is null.

Call such rings local (nonunital) rings. (radical ring better?)

May 12, 1994 (cont.)

Let $\mathbf{C}$ be a small category. Let $A$ be ring $\mathbb{Z}[\mathrm{Ar}\,\mathbf{C}]$ with basis the arrows in $\mathbf{C}$ with multiplication $f\cdot g = \begin{cases} fg & \text{when this defined} \\ 0 & \text{otherwise} \end{cases}$.

Notice that $A$ has "local units" i.e. for any finite subset $a_1, \dots, a_n \in A$ there exists an idempotent $e$ such that $ea_i = a_ie = a_i$. In effect the $a_i$ [redacted] are supported in a full subcategory with finitely many objects, so we [redacted] reduce to the case where $\mathbf{C}$ have finitely many objects. In this case $A$ is unital, the identity element of $A$ being $\sum_X \mathrm{id}_X$, where $X$ ranges over the objects of $\mathbf{C}$.

Before proceeding, consider the question of when an idempotent ring $A$ is such that its good modules are those modules $M$ such that $AM = M$. We know this is the case if $A$ is unital.

[redacted] Now

$$0 \to A \to \tilde{A} \to \mathbb{Z} \to 0$$

yields
$$0 \to \mathrm{Tor}_1^A(\mathbb{Z}, M) \to A \otimes_A M \to M \to M/AM \to 0.$$

Thus $(AM = M \implies M \text{ good})$ iff $(AM = M \implies \mathrm{Tor}_1^A(\mathbb{Z}, M) = 0)$.

In particular this holds when $\mathrm{Tor}_1^A(\mathbb{Z}, M) = 0$ for all $A$-modules $M$, i.e. when $\mathbb{Z}$ is flat as a right $A$-module.

Now $\mathbb{Z}$ is a projective $A^{op}$ module iff

$$0 \longrightarrow A \longrightarrow \tilde{A} \longrightarrow \mathbb{Z} \longrightarrow 0$$

has an $A^{op}$ linear splitting. Such a splitting is given by $\tilde{A} \longrightarrow A$, $x \longmapsto ex$ where $e \in A$ is such that $ea = a$, $\forall a \in A$. Thus $\mathbb{Z}$ is a projective $A^{op}$ module iff $A$ has a left identity.

Consequently if $A = \varinjlim A_i$ is a filtered inductive limit of rings $A_i$ having left identities, ~~then~~

$$\mathrm{Tor}_1^A(\mathbb{Z}, M) = \varinjlim_i \mathrm{Tor}_1^{A_i}(\mathbb{Z}, M) = 0$$

for all $M \in A$-mod, and ~~so~~ $\mathbb{Z}$ is $A^{op}$-flat.

In particular, a ring with local identities as above is a filtered union of unital subrings, so $\mathbb{Z}$ is flat as both $A$ and $A^{op}$ module. This means that good left (resp. right) modules are those satisfying $AM = M$ (resp. $MA = M$).

Let's consider an example. Let $I$ be an infinite set and $A$ the ideal $\bigoplus_I \mathbb{C} \subset \prod_I \mathbb{C}$. An $A$-module is the same as a vector space $V$ together with subspaces $V_i$, $i \in I$, which are independent; i.e. $\bigoplus_i V_i \longrightarrow V$ is injective. NO see below One has $V_i = e_i V$, where $e_i = 1 \in \mathbb{C} \xrightarrow{m_i} \bigoplus_I \mathbb{C}$. One has $AV = V \iff \bigoplus V_i = V$.

It should now be clear that for a small category $C$ and $A = \mathbb{Z}[\mathrm{Ar}\, C]$, that the good

A-modules are exactly the functors $\mathcal{C} \longrightarrow \mathcal{A}b$.

$\ast$ I have to be more careful about $A = \bigoplus_I \mathbb{Z} \subset \prod \mathbb{Z}$. A module $M$ over $A$ is [redacted] an abelian group together with operators $e_i$, $i \in I$, such that $e_i e_j = \delta_{ij} e_j$. Setting $M_i = e_i M$ one has canon. maps

$$\bigoplus_I M_i \xrightarrow{(m_i)} M \xrightarrow{(e_i)} \prod_I [redacted] M_i$$

whose composition is the inclusion of the direct sum into the direct product. Conversely, given a family $(M_i, i \in I)$ of abelian groups and a factorization of the canonical [redacted] inclusion from the direct sum to the direct product

$$\bigoplus M_i \longrightarrow M \longrightarrow \prod M_i$$

we get orthogonal projections $e_i$ on $M$ such that $M_i = e_i M$. This gives an equivalence of categories between A-modules and such data.

Clearly $AM = M \iff M \xleftarrow{\sim} \bigoplus M_i$. One has also [redacted]

$$\mathrm{Hom}_A(A, M) = \prod M_i$$

so that $\mathrm{ann}_A M = 0 \iff M \hookrightarrow \prod M_i$

and $M$ is good' $\iff M \xrightarrow{\sim} \prod M_i$

Let $R$ be a unital ring, let $A$ be a left ideal in $R$.

First observation: $R/A$ is a projective $R$-module $\Longleftrightarrow$ $A = Re$ where $e^2 = e$ $\Longleftrightarrow$ $A$ has a right identity.

We want now to recall the result (Wodzicki, but also I think in Faith's book) that $R/A$ is a flat $R$-module $\Longleftrightarrow$ $A$ has local right identities: given $a_1, \dots, a_n \in A$, $\exists$ $a \in A$ such that $a_i a = a_i$ $\forall i$.

($\Longleftarrow$) Given $a_1, \dots, a_n \in A$ we have the finitely presented module $R/\sum Ra_i$. Since $R/A$ is flat we have a factorization:

$$\begin{tikzcd} R/\sum Ra_i \arrow[r, "\cdot(x_j)"] \arrow[d, "\text{canon surj}"'] & R^n \arrow[d, "\cdot(m_j)"] \\ R/A \arrow[r, equal] & R/A \end{tikzcd}$$

i.e. there are elements $x_j \in R$ such that $(\sum Ra_i)x_j = 0$, and $r_j + A \in R/A$, such that $1 \equiv x_j r_j$ mod $A$. Thus $a = 1 - x_j r_j \in A$ satisfies $a_i a = a_i(1 - x_j r_j) = a_i$.

($\Longrightarrow$) Given a map $N \longrightarrow R/A$, where $N$ is finitely presented we know it lifts to $R/\sum_{i=1}^{m} Ra_i$ for some finitely generated left ideal $\sum Ra_i$. Choose $a \in A$ such that $a_i(1-a) = 0$, $\forall i$. Then we have

$$\begin{tikzcd} N \arrow[r] \arrow[d] & R/\sum Ra_i \arrow[dl] & R \\ R/A & & \end{tikzcd}$$

$$\begin{tikzcd} N \arrow[r] \arrow[d] & R/\sum Ra_i \arrow[r, "\cdot(1-a)"] \arrow[d] & R \arrow[d] \\ R/A \arrow[r, phantom, "="] & R/A \arrow[r, phantom, "="] & R/A \end{tikzcd}$$

Thus we have factored $N \to R/A$ into $N \longrightarrow R \longrightarrow R/A$, which by the Cartan-Eilenberg criterion implies $R/A$ is flat.

---

At this point we have the following equivalent conditions for a nonunital ring:

1) $\mathbb{Z}$ is a flat $\tilde{A}$-module.

2) there exists an embedding of $A$ as a left ideal of a unital ring $R$ such that $R/A$ is a flat $R$-module.

3) for any embedding of $A$ as a right ideal of a unital algebra $R$, one has that $R/A$ is a flat $R$-module.

4) $A$ has local right identities.

---

I recall the implication $\exists$ local right identities $\Rightarrow$ $R/A$ flat can be proved as follows. Consider the category with the single object given by the canon. map $R \to R/A$, $r \mapsto r+A$, and morphisms given by the monoid $(1-A)^{\times}$. This category is filtering, namely $1-a_1$, $1-a_2$ are equalized by $1-a$ where $(a_1-a_2)(1-a)=0$. The inductive limit of the functor sending the unique object to $R$ is $R/A$, so $R/A$ being a filtered inductive limit of free modules is flat. Note that we haven't used the full strength of local right identity, only that $\forall a_1 \in A$, $\exists a \in A$ such that $a_1 a = a_1$. It's slightly amazing how this works: Suppose given $a_1, a_2 \in A$. Consider $1-0$, $1-a_1$, $1-a_2$. Then we can find $a'$ such that

$$(1-0)(1-a') = (1-a_1)(1-a')$$

and [illegible] afterword find $a''$ such that

$$((1-a_1)(1-a'))(1-a'') = ((1-a_2)(1-a'))(1-a'')$$

setting $1-a \doteq (1-a')(1-a'')$, we then have

$$(1-0)(1-a) = (1-a_1)(1-a) = (1-a_2)(1-a)$$

which implies $\quad a_1(1-a_1) = a_2(1-a) = 0$.

[illegible] Actually I am being very inefficient, and I should proceed as follows. Given $a_1, a_2$ you choose $a'$ so that $a_1(1-a') = 0$, then choose $a''$ so that $a_2(1-a')(1-a'') = 0$. Then if $1-a = (1-a')(1-a'')$ we have $a_1(1-a) = a_2(1-a) = 0$.

May 25, 1994

Let $A$ be a nonunital ring, let $B$ be a left ideal in $A$. Consider the assertions

1) $\forall$ simple $A$-module $M$ one has $BM = 0$.

2) $1+B$ is a group under multiplication.

2) $\Rightarrow$ 1) Assume $\exists$ a simple $A$-module $M$ such that $BM \neq 0$. Choose $m \in M$ such that $Bm \neq 0$. $Bm$ is an $A$-submodule of $M$, and as $M$ is simple, one has $Bm = M$. Choose $b \in B$ such that $bm = m$. Then $(1-b)m = 0$ and $m \neq 0$, which shows $1-b$ does not have a left inverse in $\tilde{A}$.

1) $\Rightarrow$ 2). It suffices to show for any $b \in B$ that $\tilde{A}(1-b) = \tilde{A}$, i.e. that $\exists \alpha \in \tilde{A}$ such that $\alpha(1-b) = 1$. In effect one has $\alpha = 1 + \alpha b \in 1 + B$ (since $B$ is a left ideal). Thus every element of $1+B$ has a left inverse, which implies $1+B$ is a group.

Suppose that that $b \in B$ is such that $\tilde{A}(1-b) < \tilde{A}$, and choose a maximal left ideal $L$ of $\tilde{A}$ containing $\tilde{A}(1-b)$. Then $\tilde{A}/L$ is a simple $A$-module. Assuming 1) one has $B(\tilde{A}/L) = 0 \Rightarrow B\tilde{A} \subset L \Rightarrow b \in L$. Then $b, 1-b \in L \Rightarrow 1 \in L$, contradiction.

---

The above equivalence is the special case of the following when $R = \tilde{A}$:

Prop: Let $B$ be a left ideal in a unital algebra $R$. TFAE:

1) For every simple (unital) $R$-module $M$ one has $BM = 0$.

1') For every maximal left ideal $L$ of $R$ one has $BR \subset L$

2) For every maximal left ideal $L$ of $R$ one has $B \subset L$.

3) $1+B$ is a group under multiplication.

Proof. 1), 1') are equivalent since simple $R$-modules are of the form $R/L$ with $L$ a max. left ideal.

1') $\Rightarrow$ 2) obvious

2) $\Rightarrow$ 3). Suffices to show $\forall b \in B$ that $R(1-b) = R$, i.e. $\exists r \in R$ with $r(1-b) = 1$. In this case $r = 1 + rb \in 1+B$, [redacted] so every elt of $1+B$ has a left inverse, which implies $1+B$ is a group.

If $b \in B$ such that $R(1-b) < R$, choose $L$ max left ideal containing $R(1-b)$. [redacted] Then [redacted] $B \not\subset L$, otherwise $b, 1-b \in L \Rightarrow 1 \in L$. This contradicts 2).

3) $\Rightarrow$ 1). If $M$ simple $R$-module $\ni BM \neq 0$, choose $m \ni Bm \neq 0$, then $Bm = M$ as $M$ is simple, choose $b \ni bm = m$. Then $(1-b)m = 0$, $m \neq 0$ implies $1-b$ not invertible. □

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Defn. The Jacobson radical [redacted] $\mathrm{Jac}(R)$ of $R$ is the largest left ideal having the above equivalent conditions. By 2) it is the intersection of all maximal left ideals of $R$. By 1) it is the annihilator of all simple $R$-modules, hence it is an ideal. Hence ~~by~~ 3) it is the largest ideal in $R$ such that $1+B$ is a group. This condition is left right symmetric, hence $\mathrm{Jac}(R)$ can also be described as the intersection of all maximal right ideals, or as the annihilator of all simple right $R$-modules.

Recall also that $\mathrm{Jac}(R)$ is what occurs in Nakyama's lemma: $M$ fin. gen $R$-module, then $\mathrm{Jac}(R)\cdot M = M \implies M = 0$.

Question: Included in the above prop is the implication that if $B$ is a left ideal in $R$ unital such that $1+B$ is a group, then $1+BR$ is also a group. Can this be seen directly, i.e. without invoking Zorn's lemma?

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Let $A$ be nonunital. Then we can define $\mathrm{Jac}(A)$ to be the largest left ideal in $A$ satisfying the equivalent conditions

1) For every simple $A$-module $M$ one has $BM=0$.

2) $1+B$ is a group under multiplication (or equivalently, $B$ is a group under the operation $b_1+b_2-b_1b_2$.)

In fact we have $\mathrm{Jac}(A) = \mathrm{Jac}(\tilde{A})$. In effect simple $A$-modules are the same as simple unital $\hat{A}$-modules. Divide these into null modules: $AM=0$ which are the same as simple abelian groups, and those

those which are non null : $AM \neq 0$.

Since $\mathrm{Jac}(\mathbb{Z}) = \bigcap_{p\ \text{prime}} p\mathbb{Z} = 0$, it follows that $A$ is the annihilator all all the simple null $A$-modules. Moreover $\mathrm{Jac}(\tilde{A})$ is then the annihilator in $A$ of all the simple nonnull $A$ modules.

Note that if $M$ is a simple $A$ module, then ${}_A M = \mathrm{ann}_A(M) = \mathrm{Hom}_A(\mathbb{Z}, M)$ is a submodule of $M$, so ${}_A M = 0$ or $M$. Thus $M$ non-null $\iff {}_A M = 0$. In this case $\forall m \in M$, $m \neq 0$, one has $Am = M$, hence $M = A/l$, where $l = \mathrm{ann}_A(m)$ is a maximal left ideal of $A$.

Conversely let $l$ be a maximal left ideal of $A$ whence $A/l$ is a simple $A$-module. As $A(A/l) = (A^2 + l)/l$, we see $A/l$ is non null $\iff A^2 \not\subset l \iff A^2 + l = A$. So if $A^2 = A$, then $A/l$ with $l$ a maximal left ideal is always a non-null simple module. On the other hand in the case of the maximal ideal $\mathfrak{m}$ of a noetherian local (comm. say) ring, one has $J(\mathfrak{m}) = \mathfrak{m}$ since $1 + \mathfrak{m}$ is a group, but there are maximal left ideals for every one dimensional quotient of $\mathfrak{m}/\mathfrak{m}^2$.

Thus in the case $A = A^2$ we have

$$J(A) = \bigcap l \qquad l \text{ maximal ideal in } A$$

but not in general.

---

Suppose now that $A$ is an ideal in a unital algebra $R$. We wish to prove

$$\mathrm{Jac}(A) = A \cap \mathrm{Jac}(R).$$

Recall $\mathrm{Jac}(A) = \{a \mid aM = 0 \text{ for all simple } A\text{-mods } M\}$

$= \{a \mid aM = 0 \ \forall \text{ simple } A\text{-mods } M \ni AM \neq 0\}$.

$A \cap \mathrm{Jac}(R) = \{a \in A \mid aN = 0 \ \forall \text{ simple (unital) } R\text{-modules } N\}$

$= \{a \in A \mid aN = 0 \ \forall \text{ simple } R\text{-modules } N \ni AN \neq 0\}$

We are going to show simple $A$-modules $M \ni AM \neq 0$ are the same as simple $R$-modules $N$ such that $AN \neq 0$.

Consider such an $R$-module $N$. Since $A$ is an ideal in $R$, ${}_A N = \{n \mid An = 0\}$ is an $R$-submodule of $N$, which is $< O$ as $AN \neq 0$, hence ${}_A N = 0$. Then for $0 \neq n \in N$ we have $An$ is a nonzero $R$-submodule of $N$, hence $An = N$. This shows $N$ is simple as an $A$-module, and clearly $AN \neq 0$. Notice that because $AN = N$, one has $r(an) = (ra)n$, hence the $R$-module structure on $N$ is determined by the $A$-module structure, i.e. the multiplication homom. $A \longrightarrow \mathrm{End}_{\mathbb{Z}}(N)$ extends uniquely to a unital homom. $R \longrightarrow \mathrm{End}_{\mathbb{Z}}(N)$.

Conversely let $M$ be a simple $A$-module such that $AM \neq 0$. Then we have $AM = M$, hence an exact sequence

$$0 \longrightarrow K(M) \longrightarrow A \otimes_A M \longrightarrow M \longrightarrow 0$$

of $A$-modules where $A \cdot K(M) = 0$. Also ${}_A M$ is an $A$-submodule of $M$ which is $< M$, so ${}_A M = 0$. By applying the left exact functor ${}_A - = \mathrm{Hom}_A(\mathbb{Z}, -)$ to the above exact sequence, we get

$$\begin{tikzcd} 0 \arrow[r] & K(M) \arrow[r] & {}_A(A\otimes_A M) \arrow[r] & {}_A M \arrow[d, equal] \\ & & & 0 \end{tikzcd}$$

whence $K(M) \xrightarrow{\sim} {}_A(A\otimes_A M)$. Now $A\otimes_A M$ has an $R$-module structure: $r(a\otimes m) = ra\otimes m$ extending the $A$-module structure. As $A$ is an ideal in $R$ we know $K(M) = {}_A(A\otimes_A M)$ is an $R$-submodule of $A\otimes_A M$. Thus $M$ has an $R$-module structure extending the $A$-module structure which is unique as we've seen. Clearly $M$ is a simple $R$-module.

Thus we obtain

Prop. Let $A$ be any ideal in a unital algebra $R$. If $N$ is a simple unitary $R$-module such that $AN \neq 0$, then $N$ is a simple nonnull $A$-module. If $M$ is a simple non-null $A$-module, then there is a unique $R$-module structure on $M$ compatible with the $A$-module structure. In this way simple unitary $R$-modules $N \ni AN \neq 0$ may be identified with simple nonnull $A$-modules.

Cor. $\mathrm{Jac}(A) = A \cap \mathrm{Jac}(R)$.

May 27, 1994

Example. Let $A = q\mathbb{Z}$, $q>1$.

An A-module $M$ is the same as an abelian group equipped with an operator $T$ such that $T^2 = qT$. (In other words $\tilde{A} = \mathbb{Z}[T]/(T^2-qT)$. [redacted] Note that $A = \mathbb{Z}T = \{nT / n\in\mathbb{Z}\}$ with multiplication given by $T^2=qT$, so that

$$0 \to A \to \tilde{A} \to \mathbb{Z} \to 0$$

is

$$0 \to \mathbb{Z}[T]/(T-q) \xrightarrow{T} \mathbb{Z}[T]/(T^2-qT) \to \mathbb{Z}[T]/(T) \to 0$$

There is also another exact sequence

$$0 \to \mathbb{Z}[T]/(T) \xrightarrow{T-q} \mathbb{Z}[T]/(T^2-qT) \to \mathbb{Z}[T]/(T-q) \to 0$$

so that $\tilde{A} = \tilde{B}$ where $B = \mathbb{Z}(T-q)$. As

$$(q-T)^2 = q^2 - 2qT + T^2 = q^2 - 2qT + qT = q^2 - qT = q(q-T)$$

$B$ and $A$ are isomorphic via $T \mapsto q-T$. In other words given an abelian group $M$ with operator $T$ such that $T^2 = qT$, we get another operator $q-T$ satisfying the same relation.)

This digression finished, let's find the good A-modules.

Suppose $M$ is an A-module. Then

$$AM = \Big\{ \underbrace{\sum_i n_i T m_i}_{=T\sum n_i m_i} \in M \;\Big|\; \begin{matrix} m_i \in M \\ n_i \in \mathbb{Z} \end{matrix} \Big\}$$

[redacted] is contained in $TM$ and conversely so

$$AM = TM$$

Thus [redacted] $AM = M \iff TM = M$

in which case one has $0 = (T-q)TM = (T-q)M$, so that $T$ = multiplication by $q$ on $M$. Consequently $A$-modules such that $AM = M$ are the same as $q$-divisible abelian groups.

Next $A \otimes_A M = \mathbb{Z}T \otimes_{\mathbb{Z}T} M$ is the quotient of $\mathbb{Z}T \otimes_{\mathbb{Z}} M$ by the relations $aT \otimes_{\mathbb{Z}} m = a \otimes_{\mathbb{Z}} Tm$. Now one has the canonical isom:

$$\mathbb{Z}T \otimes_{\mathbb{Z}} M \overset{T \otimes_{\mathbb{Z}} m \,\mapsfrom\, m}{\xleftarrow{\ \sim\ }} M$$

[redacted] Any $a \in A$ has the form $a = nT$, [redacted] and $aT = nT^2 = nqT = Tnq$, so $aT \otimes_{\mathbb{Z}} m = Tnq \otimes_{\mathbb{Z}} m = T \otimes_{\mathbb{Z}} nqm$. On the other hand, $a \otimes_{\mathbb{Z}} Tm = nT \otimes_{\mathbb{Z}} Tm = T \otimes_{\mathbb{Z}} nTm$. Thus $A \otimes_A M$ can be identified with the quotient of $M$ by the relations $nTm = nqm$, $\forall n, m$. Thus

$$\begin{array}{ccc} A \otimes_A M & \simeq & M/(T-q)M \\ T \otimes m & \longleftrightarrow & m \end{array}$$

Thus in general we have the identification

$$\begin{tikzcd} A \otimes_A M \arrow[r, "\mu_M"] \arrow[d, equal] & M \arrow[d, equal] \\ M/(T-q)M \arrow[r, "T"] & M \end{tikzcd}$$

Suppose now that $M$ is good. Then $AM = M$ and we've seen that this implies $T = q$ on $M$ and that $q$ is surjective. Thus $T = q$ is actually an isomorphism and we've proved:

<u>Prop:</u> Let $A = q\mathbb{Z}$, $q > 1$. Then

1) $A$ modules $M$ satisfying $AM = M$ are the same as $q$-divisible abelian groups

2) good $A$ modules are the same as $q$-torsion free $q$-divisible abelian groups, i.e. unital $\mathbb{Z}[\frac{1}{q}]$-modules

May 28, 1994

Let $A$ be an ideal in a unital ring $R$. Consider the full subcategories

$$A\text{-gmod} = \{ M \in A\text{-mod} \mid A \otimes_A M \xrightarrow{\sim} M \} \quad \subset \quad A\text{-mod}$$

$$R\text{-g}_A\text{mod} = \{ N \in R\text{-umod} \mid A \otimes_R N \xrightarrow{\sim} N \} \quad \subset \quad R\text{-umod}$$

I claim these categories are canonically equivalent (even isomorphic).

Lemma: Let $X, N$ be right and left unitary $R$-modules respectively. If either $XA = X$ or $AN = N$, then the canonical surjection

$$X \otimes_A N \longrightarrow X \otimes_R N$$

is an isomorphism.

Proof: It suffices to show the universal $A$-bilinear map

$$X \times N \longrightarrow X \otimes_A N \qquad (x,n) \longmapsto x \otimes_A n$$

is $R$-bilinear: $xr \otimes_A n = x \otimes_A rn$. In the case $AN = N$ we can suppose $n$ of the form $an'$, whence $xr \otimes_A an' = xra \otimes_A n' = x \otimes_A ran'$. The other case is similar. ▮

Improvement: Let $X$ be a right $R$-module, $N$ a left $R$-module, where $R$ is a nonunital ring.

1) If $A$ is a left ideal in $R$ and $AN = N$ then $X \otimes_A N \xrightarrow{\sim} X \otimes_R N$.

2) If $A$ is a right ideal in $R$ and $XA = X$ then $X \otimes_A N \xrightarrow{\sim} X \otimes_R N$.

Suppose now that $A$ is an ideal in $R$ nonunital, and let

$$R\text{-}g_A\text{mod} = \{N \in R\text{-mod} \mid A \otimes_R N \xrightarrow{\sim} N\} \subset R\text{-mod}$$

~~Note that if $R$ is unital, then $AN = N \Rightarrow$ $N$ is a unitary $A$-module. Also if $RA = A$ then we have an exact sequence~~

~~$$0 \longrightarrow K \longrightarrow R \otimes_R A \longrightarrow A \longrightarrow 0$$~~

The condition $A \otimes_R N \xrightarrow{\sim} N \implies AN = N$ hence the lemma yields $A \otimes_A N \xrightarrow{\sim} A \otimes_R N$, and so $N$ is good as an $A$-module. Thus we have a functor

$$\textcircled{*} \qquad R\text{-}g_A\text{mod} \longrightarrow A\text{-gmod}$$

given by restricting scalars.

When $N$ is an $R$-module such that $AN = N$, the formula $r(an) = (ra)n$ shows that the action of $R$ on $N$ is determined by the action of $A$ on $N$. In particular an additive map $f : N \longrightarrow N'$ where $N'$ is another $R$-module is $R$ linear iff it is $A$-linear. ($\Rightarrow$ obvious and $\Leftarrow$: $f(r(an)) = f((ra)n) = raf(n) = rf(an)$.)

Thus the functor $\textcircled{*}$ is fully faithful.

Next let $M$ be a good $A$-module: $A \otimes_A M \xrightarrow{\sim} M$. Note that $A \otimes_A M$ is a $R$-module with action $r(a \otimes m) = ra \otimes m$. Thus using the isom. $A \otimes_A M \xrightarrow{\sim} M$ we obtain a $R$-module structure in $M$ such that

$r(am) = (ra)m$. We have

[redacted]

$$\begin{tikzcd} A\otimes_A M \arrow[r, "\sim"] \arrow[rr, bend right, "\cong"'] & A\otimes_R M \arrow[r] & M \end{tikzcd}$$

so $M$ with this $R$-module structure is an object of $R$-$g_A$mod. This shows the functor ⊛ is essentially surjective. [redacted]

It's clear that the following two structures on an abelian group [redacted] $M$:

i) $A$-module ∋ $A\otimes_A M \xrightarrow{\sim} M$

ii) $R$-module ∋ $A\otimes_R M \xrightarrow{\sim} M$

are equivalent. Hence the two categories are isomorphic.

Summarizing

Prop. Let $A$ be an ideal in the nonunital ring $R$. Then restriction of scalars

$$R\text{-}g_A\text{mod} \longrightarrow A\text{-gmod}$$

is an equivalence of category (isomorphism in fact).

Comments:

1. If $R$ is unital, then any $R$-module $N$ such that $AN = N$ is a unitary $R$-module. [redacted] In particular $R$-$g_A$mod $\subset$ $R$-umod.

2. Let's ask whether $R$-$g_A$mod $\subset$ $R$-gmod? Assume $RA = A$. Then we have an exact sequence of $R$-bimods

$$0 \longrightarrow K \longrightarrow R\otimes_R A \longrightarrow A \longrightarrow 0$$

such that $KA = 0$: if $\sum r_i \otimes a_i \longmapsto \sum r_i a_i = 0$, then

$$\left(\sum r_i\otimes a_i\right) a = \sum r_i \otimes a_i a = \left(\sum r_i a_i\right)\otimes a = 0.$$

Thus when $N$ is an $R$-module such that $AN = N$ we have

$$\begin{tikzcd} K\otimes_R N \arrow[r] \arrow[d, phantom, "\parallel"] & R\otimes_R A\otimes_R N \arrow[r] & A\otimes_R N \arrow[r] & 0 \\ K\otimes_R AN = KA\otimes_R N = 0. \end{tikzcd}$$

showing that $A\otimes_R N$ is $R$-good. Thus $A\otimes_R N \xrightarrow{\sim} N \implies N$ is $R$-good, so we find that if $RA = A$ <u>then</u> $R\text{-}g_A\text{mod} \subset R\text{-gmod}$.

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How I propose to use this: To calculate $A$-gmod I will use an embedding of $A$ as an ideal in a unital ring $R$ and then calculate $R\text{-}g_A\text{mod}$.

Example: Take $A = q\mathbb{Z} \subset R = \mathbb{Z}$.

$$A\otimes_R M = q\mathbb{Z}\otimes_{\mathbb{Z}} M \xleftarrow{\sim} M$$
$$q\otimes m \longleftarrow\!\shortmid\ m$$

so $A\otimes_R M \xrightarrow{\sim} M$ means $M \xrightarrow{q} M$ is an isomorphism, i.e. $M$ is a uniquely $q$-divisible abelian group.

Example: $R = \bigoplus_{n\geq 0} V^{\otimes n} = \mathbb{C}\oplus V \oplus V^{\otimes 2}\oplus\cdots$, $A = \bigoplus_{n\geq 1} V^{\otimes n}$. Then

$$A\otimes_R M = V\otimes R\otimes_R M = V\otimes M$$

and $\mu_M : A\otimes_R M \longrightarrow M$ is the map $V\otimes M \longrightarrow M$ given by the action of $V \subset R$ on $M$.

$$M \text{ is good}_A \iff V\otimes M \xrightarrow{\sim} M.$$

<u>Suppose $V$ finite dimensional</u>, and let $x_1,\dots,x_n$ be a basis. A unital $R$-module $M$ is a vector space [redacted] equipped with operators $x_1,\dots,x_n$. It is $A$-good [redacted] iff

$$M^n \longrightarrow M \quad , \quad (m_i) \longmapsto \sum_i x_i m_i$$

is an isomorphism. In this case there are operators $y_i$ on $M$, $1\le i\le n$ satisfying

$$y_i x_j = \delta_{ij} \quad , \quad \sum x_i y_i = 1$$

Thus $M$ is a unitary module over Cuntz's algebra $\mathcal{O}_n$ with generators $x_i, y_i$, $1\le i\le n$ satisfying the above relations. The converse evidently holds.

Thus for $\dim(V)=n$ good modules over the nonunital tensor algebra $\bigoplus_{n\ge1} V^{\otimes n}$ are equivalent to unitary $\mathcal{O}_n$ modules.

<u>Suppose next that $V$ is infinite dimensional.</u> Then a good $A$-module $A=\bigoplus_{n\ge1} V^{\otimes n}$ is a vector space $M$ equipped with an isomorphism $V\otimes M \xrightarrow{\sim} M$. I would like to describe [redacted] the category of good $A$-modules in a simpler way. Again [redacted] choose a basis $v_i$ for $V$, [redacted] where $i$ runs over an index set $\Lambda$. Thus $V=\mathbb{C}^{(\Lambda)}$. The map $V\otimes M \longrightarrow M$ is equivalent to operators $x_i$, $i\in A$, on $M$; an $A$-module, rather a unitary $R=\bigoplus_{n\ge0} V^{\otimes n}$ module, is just a vector space with these operators. The inverse map $M \longrightarrow V\otimes M = M^{(\Lambda)}$ is described by operators

$y_i$ on $M$, which satisfy the condition that for each $m \in M$ only finitely many $y_i(m)$ are nonzero.

The fact that we have an isomorphism $V \otimes M \xrightarrow{\sim} M$ implies that we have

$$y_i x_j = \delta_{ij} \qquad \text{and} \qquad \sum x_i y_i = 1$$

[redacted]

Consider the unital algebra with generators $x_i, y_i$, $i \in \Lambda$, satisfying the relations $y_i x_j = \delta_{ij}$. This should be the Toeplitz algebra $\bigoplus_{p,q \geq 0} V^{\otimes p} \otimes V^{* \otimes q}$ of Pimsner.

[redacted] Let's consider the nonunital algebra with same generators and relations, call this $B$. NO you need 1 for $y_i x_j = \delta_{ij}$ see $*$ below

$B$ has basis all non empty words

$$x_{i_1} \cdots x_{i_p} y_{j_1} \cdots y_{j_q}$$

In $B$ we have mutually annihilating idempotents $x_i y_i$, $i \in \Lambda$. If $S \subset \Lambda$ is a finite subset, then [redacted] $\sum_{i \in S} x_i y_i$ is an idempotent such that

$$\left( \sum_{i \in S} x_i y_i \right) x_j = \begin{cases} x_j & j \in S \\ 0 & j \notin S. \end{cases}$$

but $\left( \sum_{i \in S} x_i y_i \right) y_j$ is not $y_j$ at all. However

$$y_j \left( \sum_{i \in S} x_i y_i \right) = \begin{cases} y_j & j \in S \\ 0 & j \notin S. \end{cases}$$

[redacted] ~~In the case $\dim V = n$ this nonunital Toeplitz algebra $B$ is such that $B = BeB$ where $e = \sum_{i=1}^{n} x_i y_i$.~~ [redacted] ~~In effect take a basis~~ element

~~$\beta = x_{i_1} \cdots x_{i_p} y_{j_1} \cdots y_{j_q}$ where either $p$ or $q \geq 1$. If $p \geq 1$, then $e\beta = \beta$ so $\beta \in eB$, and if $q \geq 1$, then $\beta e = \beta$ so $\beta \in eB$.~~

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* To correct the above let $T$ be the Toeplitz algebra $\bigoplus_{p,q \geq 0} V^{\otimes p} \otimes V^{* \otimes q}$. Then $T$ has basis $x_\alpha y_\beta$ where $x_\alpha = x_{i_1} \cdots x_{i_p}$, $\alpha = (i_1, \dots, i_p)$. If $e = \sum_{i=1}^{n} x_i y_i$ assuming $x_1, \dots, x_n$ a basis for $V$, then

$$\begin{cases} e x_\alpha y_\beta = x_\alpha y_\beta & \text{if } |\alpha| > 0 \\ x_\alpha y_\beta e = x_\alpha y_\beta & \text{if } |\beta| > 0. \end{cases}$$

One has $e y_\beta = \sum x_i y_i y_\beta \in \operatorname{span}\{x_\alpha y_\beta \mid |\beta| > 0\}$.

Thus $Te = \bigoplus_{|p| \geq 0, |q| > 0} V^{\otimes p} \otimes V^{* \otimes q}$, $eT = \bigoplus_{|p| > 0, |q| \geq 0} V^{\otimes p} \otimes V^{* \otimes q}$

$eTe = \bigoplus_{|p|, |q| > 0} V^{\otimes p} \otimes V^{* \otimes q}$. Note that $TeT = T$ since it contains $y_1 e x_1 = y_1 x_1 = 1$.

Thus $T$ is Morita equivalent to $eTe$.

## May 29, 1994

Example: $R = k[x_0, x_1]$, $I = (x_0, x_1)$.

I would like to understood good $I$-modules, (equivalently unital $R$-modules which are $I$-good: $I \otimes_R M \xrightarrow{\sim} M$) in this situation. [Recall that I handled the noncommutative version yesterday and found good $I$-modules are equivalent to unital modules over Cuntz's algebra $\mathcal{O}_2$.)

Now $M$ is $I$-good iff [redacted] $\mathrm{Tor}_n^R(R/I, M) = 0$ for $n = 0, 1$. In the present example $R/I = k$ has a Koszul resolution

$$0 \longrightarrow R \xrightarrow{\binom{x_1}{-x_0}} R^2 \xrightarrow{(x_0, x_1)} R \longrightarrow k \longrightarrow 0$$

so $M$ is [redacted] $I$-good when

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\binom{x_1}{-x_0}"] & M^2 \arrow[r, "(x_0, x_1)"] & M \arrow[r] & 0 \\ & & & \phantom{x} \arrow[ul, bend left] \arrow[u, bend right] & \end{tikzcd}$$

is exact at the points $\vee$. The same Koszul resolution can be used to compute $\mathrm{Ext}_R^n(R/I, -)$ and leads to an isom.

$$\mathrm{Tor}_n^R(R/I, M) = \mathrm{Ext}_R^{2-n}(R/I, M)$$

so that $M$ is $I$-good $\Longleftrightarrow$ $\mathrm{Ext}_R^n(\overset{k}{\overset{\shortparallel}{R/I}}, M) = 0$ for $n = 1, 2$. Consequently any injective $R$ module is $I$ good in this example.

Some standard ideas holding more generally when $R$ (comm. unital) noetherian, $I$ ideal in $R$, $Z$ the corresponding closed subset of $\mathrm{Spec}(R)$: A (unital) $R$-module $M$ can be identified with a quasi-coherent sheaf on $\mathrm{Spec}(R)$. We have the following Serre subcategory and quotient category:

$$\begin{tikzcd} \begin{pmatrix}\text{I-torsion}\\ \text{R-mod}\end{pmatrix} \arrow[r, hook, "i_*"'] & (\text{R-mod}) \arrow[l, dashed, bend right, "i^*"'] \arrow[r, two heads, "j^*"] & \begin{pmatrix}\text{Quasi-coherent}\\ \text{Sheaves on}\\ \operatorname{Spec}(R)-Z\end{pmatrix} \arrow[l, hook, bend left, "j_*"] \end{tikzcd}$$

where I-torsion means each element is killed by a power of $I$, i.e. the associated sheaf is supported in $Z$, and $j : \operatorname{Spec}(R)-Z \hookrightarrow \operatorname{Spec}(R)$ is the obvious open immersion. We have the dual good notion [redacted]: $M \xrightarrow{\sim} j_*j^*M$, also the exact sequence

$$0 \longrightarrow \underline{H}^0_Z(M) \longrightarrow M \longrightarrow j_*j^*M \longrightarrow \underline{H}^1_Z(M) \longrightarrow 0$$

where

$$\underline{H}^i_Z(M) = \varinjlim_n \operatorname{Ext}^i_R(R/I^n, M)$$

from the theory of local cohomology.

[redacted] These very general ideas illustrate the more general picture arising for [redacted] torsion theories, but they do not help (it seems) to understand the example $R=k[x_0,x_1]$, $I=(x_0,x_1)$, e.g. to decide whether I-gmod is abelian.

In this example one might as well localize at the maximal ideal $I$. It might also help to restrict to graded $R$ modules. One has then

$$\begin{pmatrix}\text{graded}\\ \text{I-torsion}\\ \text{R-mods}\end{pmatrix} \hookrightarrow \begin{pmatrix}\text{graded}\\ \text{R-mod}\end{pmatrix} \begin{array}{c}\twoheadrightarrow\\ \hookleftarrow\end{array} \begin{array}{c}\text{Quasi-coherent}\\ \text{sheaves on } \mathbb{P}^1\end{array}$$

$$F \mapsto \Gamma(F(*)) = \bigoplus_{n\in\mathbb{Z}} \Gamma(\mathbb{P}^1, F(n)) .$$

Recall also the canonical exact sequence

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\binom{x_1}{-x_0}} \mathcal{O}(-1)^2 \xrightarrow{(x_0,x_1)} \mathcal{O} \longrightarrow 0$$

leads to

$$0 \longrightarrow \Gamma(F(*-2)) \longrightarrow \Gamma(F(*-1))^2 \longrightarrow \Gamma(F(*))$$

showing that the graded R-mod $M = \Gamma(F(*))$ satisfies $\mathrm{Ext}^i_R(R/I, M) = 0$ for $i = 0, 1$. This is a kind of dual good condition.

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Let's return to the example $R = \bigoplus_{n \geq 0} V^{\otimes n}$, $I = \bigoplus_{n > 0} V^{\otimes n}$. We have seen that I-good modules (same as R-modules satisfying $I \otimes_R M \xrightarrow{\sim} M$) are the same as vector spaces M equipped with an isomorphism $V \otimes M \xrightarrow{\sim} M$. When $\dim(V) = n < \infty$, we obtain, on choosing a basis of V, an identification of such an M with a unitary module over Cuntz's algebra $\mathcal{O}_n$ with generators $x_i, y_i$ for $1 \leq i \leq n$ and relations $y_i x_j = \delta_{ij}$, $\sum x_i y_i = 1$.

I now want to understand the case where V has infinite dimension, say countable. This seems to be hard. It might be impossible to describe $\{M$ equipped with $V \otimes M \xrightarrow{\sim} M\}$ as unitary modules over a unital ring, or even as good modules over a ring A such that $A = A^2$.

The infinite Cuntz algebra $\mathcal{O}_\infty$, i.e. $\mathcal{O}$ of Hilbert space is just the Toeplitz algebra, and I don't see how to get this from I-good R-modules.

---

Observation. Suppose I ideal in R unital such that I is right R-flat. Then I-good R-modules: $I \otimes_R M \xrightarrow{\sim} M$ form a full subcategory of R-mod which is closed under kernels and cokernels.

In effect given $M_1 \to M_2$ in $R$-$g_I$mod let $K, C$ be its kernel & cokernel in $R$-umod. We have

$$\begin{tikzcd}
0 \arrow[r] & I\otimes_R K \arrow[r] \arrow[d] & I\otimes_R M_1 \arrow[r] \arrow[d, "\cong"] & I\otimes_R M_2 \arrow[r] \arrow[d, "\cong"] & I\otimes_R C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

where the top row is exact by flatness of $I$. This diagram shows $K$ and $C$ are $I$-good.

It follows that $R$-$g_I$mod is an abelian category when $I$ is right $R$-flat.

June 5, 1994

Let $I$ be an ideal in $R$ unital. The problem is whether $R\text{-}g_I\text{mod}$ is an abelian category. Note that the inclusion (which is fully faithful)

$$(1) \qquad R\text{-}g_I\text{mod} \subset R\text{-mod}$$

is compatible with $\varinjlim$'s. Moreover, because $I \otimes_R -$ respects $\varinjlim$'s (i.e. this functor on $R$-mod commutes with arbitrary direct sums and it is right exact), given an functor $y \longmapsto M_y$ from a small cat $\mathcal{Y}$ to $R\text{-}g_I\text{mod}$, we know the inductive limit $\varinjlim M_y$ in $R$-mod [redacted] actually lies in $R\text{-}g_I\text{mod}$ and is the inductive limit in this subcategory.

We also know I think from the construction of flat good modules that $R\text{-}g_I\text{mod}$ has a generator. Thus it should follow from general considerations that the inclusion functor (1) has a right adjoint $N \longmapsto N^g$: For all good $M$ one has

$$\text{Hom}_R(M,N) \xleftarrow{\sim} \text{Hom}_R(M,N^g)$$

Let's consider some examples.

1) $I = q\mathbb{Z} \subset R = \mathbb{Z}$. In this case $R\text{-}g_I\text{mod}$ can be identified with $\mathbb{Z}[\frac{1}{q}]$-mod, and the inclusion function (1) is restriction of scalars from $\mathbb{Z}[\frac{1}{q}]$ to $\mathbb{Z}$. In this situation we have adjoint functors

$$\begin{tikzcd}
\mathbb{Z}[\frac{1}{q}]\text{-mod} \arrow[r] & \mathbb{Z}\text{-mod} \arrow[l, bend right, "\mathbb{Z}[\frac{1}{q}]\otimes_{\mathbb{Z}}"'] \arrow[l, bend left, "\text{Hom}_{\mathbb{Z}}(\mathbb{Z}[\frac{1}{q}],-)"]
\end{tikzcd}$$

The right adjoint to the inclusion of good modules is

$$N \longmapsto \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\tfrac{1}{q}], N) = \varprojlim \{ N \xleftarrow{\delta} N \xleftarrow{\delta} N \xleftarrow{\delta} \cdots \}$$

which is a kind of $q$-adic completion, but not the same in general. Note that because $\mathbb{Z}[\frac{1}{q}]$ is not projective as $\mathbb{Z}$-module (since projectives are free in the case of a PID), there exist $N$ for which $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}[\frac{1}{q}], N) \neq 0$. This means that $N \mapsto \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\frac{1}{q}], N)$ is not exact

I should have mentioned that this functor is $N \mapsto N^g$.

$$\boxed{N^g = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\tfrac{1}{q}], N)}$$

2) $I = I^2$. Then we have

$$\begin{tikzcd}
R\text{-}g_I\text{mod} \arrow[r, hook, shift left] \arrow[r, hook, shift right=4, "\mathrm{Hom}_R(I^g, -)"'] & R\text{-mod} \arrow[l, shift left, "\cancel{I^g \otimes_R -}"]
\end{tikzcd}$$

Hence $N^g = I^g \otimes_R N$ where $I^g = I \otimes_R I$, and $N \mapsto N^g$ is bicontinuous (respects both $\varinjlim$'s and $\varprojlim$'s). The inclusion of good modules is only right exact in general.

---

Let's continue with the problem of when the good modules form an abelian category. I propose to go over the arguments assuming the right adjoint $N \mapsto N^g$ to the inclusion $R\text{-}g_I\text{mod} \hookrightarrow R\text{-umod}$ exists. The existence of this right adjoint should follow from general arguments, as I explained.

Let's consider $u : M_1 \longrightarrow M_2$ a map of good modules, and look at the associated exact sequences in R-umod

$$0 \longrightarrow K \longrightarrow M_1 \longrightarrow L \longrightarrow 0$$
$$0 \longrightarrow L \longrightarrow M_2 \longrightarrow C \longrightarrow 0$$

Then $C$ is good and $L = IL$. We know $C$ is the cokernel of $u$ in $R\text{-}g_I\text{mod}$: $C = \mathrm{Cokerg}(u)$. We know also by left exactness of $N \mapsto N^g$ that $\mathrm{Kerg}(u) = K^g$. It's more or less clear that the canonical map $K^g \longrightarrow K$ has image $K^\# =$ largest submodule of $K$ such that $IK^\# = K^\#$.

Then we should have

$$\mathrm{Im\,g(ood)}\,(u) = \mathrm{Kerg}(M_2 \longrightarrow C) = L^g$$
$$\mathrm{Coimg}\ (u) = \mathrm{Cokerg}(K \longrightarrow M_1) = M_1/K^\#.$$

Thus for any submodule $L$ of a good module such that $IL = L$, if we have $L = M/K$ with $M$ good we want to have $L^g = M/K^\#$.

A stronger condition would be for any $L$ such that $IL = L$, if we have $L = M/K$ with $M$ good, then we have $L^g = M/K^\#$.

However the stronger condition is false in the case $g\mathbb{Z} < \mathbb{Z}$. First express the stronger condition: whenever $K$ is a submodule of $M$ good, then $(M/K)^g = M/K^\#$. Now take $M = \mathbb{Z}[\frac{1}{p}]$, $K = \mathbb{Z}$ where $g = p$ is prime.

Then
$$\mathbb{Z}[1/p]/\mathbb{Z}^\# = \mathbb{Z}[\tfrac{1}{p}]$$

$$\left(\mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z}\right)^g = \varprojlim \left\{ \to \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{p} \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{p} \mathbb{Q}_p/\mathbb{Z}_p \right\}$$
$$= \mathbb{Q}_p \blacksquare$$

Here we have used $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z} = \mathbb{Q}_p/\mathbb{Z}_p$ for $p$ prime.

---

At this point I am becoming convinced that good modules are interesting primarily in the case $I = I^2$.

Before leaving the general case, let's consider the dual good situation briefly. Suppose $M$ an $R$-mod such that
$$M \xrightarrow{\sim} \mathrm{Hom}_R(I, M)$$
In particular $\mathrm{ann}_I(M) = \{m \mid Im = 0\} = 0$.

I claim $\mathrm{ann}_I(M) = 0 \implies \mathrm{Hom}_R(I,M) \xrightarrow{\sim} \mathrm{Hom}_I(I,M)$. In effect this map is injective always. Let $f \in \underbrace{\qquad}$ and consider $f(ry) - rf(y) \in M$ where $r \in R, y \in M$. Then $\forall x \in I$, $x(f(ry) - rf(y)) = f(xry) - xrf(y) = 0$ since $f$ is $I$-linear. Then $\mathrm{ann}_I(M) = 0 \implies f(ry) = rf(y)$ showing $f$ is $R$-linear.

Thus if $M$ is a $I$-good' $R$-module, $M$ is a good' $I$-module. Conversely if $M$ is a **good'** $I$-mod:

[redacted] $M \xrightarrow{\sim} \mathrm{Hom}_I(I, M)$

then $M$ has an evident [redacted] $R$-module structure obtained from the right multiplication of $R$ on $I$: $y(rm) = (yr)m \quad \forall y \in I, \forall r \in R, \forall m \in M$. This is the unique $R$-module structure extending the $I$-module structure. Thus we have proved that [redacted] $I$-good' $R$-modules are equivalent to good' $I$-modules.

Examples:

1) $q\mathbb{Z} \subset \mathbb{Z}$. Then

$$\begin{array}{ccccc} M & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(q\mathbb{Z}, M) & \xrightarrow{\simeq} & M \\ m & \longmapsto & (nq \mapsto nqm) & & \\ & & f & \longmapsto & f(q) \end{array}$$

is again multiplication by $q$ so that good' modules are the same as good modules.

2) $R = T(V)$, $I = T(V) \otimes V$. Then $M$ good' when

$$M \xrightarrow{\sim} \mathrm{Hom}_{T(V)}(T(V) \otimes V, M) = \mathrm{Hom}(V, M)$$

This time choosing a basis $v_s$ for $V$ a good' module $M$ is the same as a vector space equipped with an [redacted] isomorphism

$$V \xrightarrow{\sim} \prod_S V$$

3) $R = k[x,y]$, $I = (x,y)$. Good' in general means $\mathrm{Ext}^i_R(R/I, M) = 0$ for $i = 0, 1$. In the present example, restricting to graded modules, we get exactly the graded modules corresponding to quasi-coherent sheaves on $\mathbb{P}^1$, it seems.

June 8, 1994

Suppose $A = \bigoplus_{\lambda\in\Lambda} A_\lambda$, $A_\lambda A_{\lambda'} \subset \begin{cases} 0 & \lambda\neq\lambda' \\ A_\lambda & \lambda=\lambda' \end{cases}$ is a direct sum of rings $A_\lambda$. Let's find the good and good' modules for $A$.

Let $M$ be a good $A$-module $A\otimes_A M \xrightarrow{\sim} M$. Then $\bigoplus_\lambda (A_\lambda \otimes_A M) = A\otimes_A M \xrightarrow{\sim} M$, so that $M = \bigoplus_\lambda M_\lambda$ where

$$M_\lambda = A_\lambda M \xleftarrow{\sim} A_\lambda \otimes_A M.$$

Note that $A_\lambda M_{\lambda'} = 0$ for $\lambda\neq\lambda'$, hence $A_\lambda \otimes_A M_{\lambda'} = A_\lambda \otimes_A A_{\lambda'} M \subset A_\lambda A_{\lambda'} \otimes_A M = 0$ for $\lambda=\lambda'$. Also $A_\lambda \otimes_A M_\lambda = A_\lambda \otimes_{A_\lambda} M_\lambda$ (either because $A_{\lambda'}$ acts trivially on both $A_\lambda$ and $M_\lambda$ for $\lambda'\neq\lambda$, or because $A_\lambda$ is an ideal in $A$ $\ni$ [redacted] $A_\lambda M_\lambda = M_\lambda$:

$$M_\lambda = A_\lambda M = A_\lambda\Big(M_\lambda \oplus \bigoplus_{\lambda'\neq\lambda} M_{\lambda'}\Big) = A_\lambda M_\lambda.\Big)$$

Thus $M_\lambda \xleftarrow{\sim} A_\lambda \otimes_A M = A_\lambda \otimes_A \Big(M_\lambda \oplus \cancel{\bigoplus_{\lambda'\neq\lambda} M_{\lambda'}}\Big) = A_\lambda \otimes_{A_\lambda} M$, $\forall \lambda\in\Lambda$. We conclude that <u>good $A$-modules are exactly those modules of the form $\bigoplus_{\lambda\in\Lambda} M_\lambda$, where $M_\lambda$ is a good $A_\lambda$-module</u>.

Take then $A = \bigoplus_{n=1}^{\infty} A_n$, where $A_n^{n+1}=0$, $A_n^n \neq 0$. <u>This gives a nonnilpotent ring $A$ such that the only good module is zero.</u>

Next return to $A = \bigoplus_{\lambda\in\Lambda} A_\lambda$ as above, and let's find the good' modules:

$$N \xrightarrow{\sim} \mathrm{Hom}_A(A, N) = \prod_\lambda \mathrm{Hom}_A(A_\lambda, N)$$

The kernel of the $\lambda$-th projection $N \to \mathrm{Hom}_A(A_\lambda, N)$ is $\mathrm{ann}_{A_\lambda}(N) = {}_{A_\lambda}N$. Put

$$N_\lambda = \mathrm{Ker}\left\{N \longrightarrow \prod_{\lambda' \neq \lambda} \mathrm{Hom}_A(A_{\lambda'}, N)\right\}$$

Then $\boxed{N = N_\lambda \oplus {}_{A_\lambda}N}$, so

$$\mathrm{Hom}_A(A_\lambda, N) = \mathrm{Hom}_A(A_\lambda, N_\lambda) \oplus \mathrm{Hom}_A(A_\lambda, {}_{A_\lambda}N)$$

The second summand is zero since given $f : A_\lambda \to {}_{A_\lambda}N$ the image of $f$ is killed by $A_{\lambda'}$ for $\lambda' \neq \lambda$ since $A_{\lambda'} f(A_\lambda) = f(A_{\lambda'} A_\lambda) = 0$, and also by $A_\lambda$. Thus

$$N_\lambda \xrightarrow{\sim} \mathrm{Hom}_A(A_\lambda, N) = \mathrm{Hom}_A(A_\lambda, N_\lambda) = \mathrm{Hom}_{A_\lambda}(A_\lambda, N_\lambda)$$

and so we conclude that *good'* $A = \oplus A_\lambda$ *modules are exactly those of the form* $\prod_\lambda N_\lambda$ *where* $N_\lambda$ *is a good'* $A_\lambda$ *module for each* $\lambda$.

June 9, 1940

Let $A$ be an ideal in $R$ unital, and restrict ~~attention~~ to unital $R$-modules. It seems that the 'A-good' theory fits into the general subject of torsion theories.

Recall an $R$-module $M$ is 'A-good' when

$$(1) \qquad M \xrightarrow{\sim} \mathrm{Hom}_R(A, M).$$

Note $(1) \Longrightarrow {}_A M = 0$, where ${}_A M = \mathrm{Hom}_R(R/A, M)$.

If $E$ is an injective $R$-module, then one has an exact sequence

$$0 \longrightarrow {}_A E \longrightarrow E \longrightarrow \mathrm{Hom}_R(A, E) \longrightarrow 0$$

hence ${}_A E = 0 \Longrightarrow E$ is 'A good'.

Thus we have a distinguished family of injective $R$-modules, namely those such that ${}_A E = 0$. This suggests there is a torsion theory $\tau$ on $R$-mod corresponding to these injectives. The $\tau$-torsion-free modules are those satisfying ${}_A M = 0$. Note that when ${}_A M = 0$ the injective hull $E(M)$ of $M$ satisfies ${}_A E(M) = 0$, (since ${}_A M = M \cap {}_A E(M)$ so as $M \subset E(M)$ is an essential extension if ${}_A E(M)$ were $\neq 0$, then the intersection ${}_A M$ would be $\neq 0$.)

The $\tau$-torsion modules are those $M$ such that $\mathrm{Hom}_R(M, E) = 0$ for all injectives $E$ such that ${}_A E = 0$. Notice that the class of such $M$ is a Serre subcategory closed under $\oplus$'s as it should be.

Let $M$ be an arbitrary $R$-module and define by transfinite induction an increasing family of submodules $T_\alpha(M)$ as follows:

$$T_{\alpha+1}(M) = \text{[redacted]} \{m \mid Am \subset T_\alpha(M)\}$$

$$T_\alpha(M) = \bigcup_{\alpha' < \alpha} T_{\alpha'}(M) \qquad \alpha \text{ limit ordinal}$$

Then $T_\alpha(M)$ is a $\tau$-torsion module for $\forall \alpha$, since

$$0 \longrightarrow T_\alpha(M) \longrightarrow T_{\alpha+1}(M) \longrightarrow \underbrace{T_{\alpha+1}(M)/T_\alpha(M)}_{\text{killed by } A} \longrightarrow 0$$

This increasing family becomes stationary for some $\alpha$, i.e. $\exists \alpha \ni \; T_\alpha(M) = T_{\alpha+1}(M)$, i.e.

$${}_A\big(M/T_\alpha(M)\big) = 0.$$

Put $T(M) = T_\alpha(M)$. Then $T(M)$ is a $\tau$-torsion submodule such that $M/T(M)$ is $\tau$-torsion-free. $T(M)$ is the largest $\tau$-torsion submodule of $M$.

Now at this point the rest of the picture should follow from basics about torsion theories. We have the localizing subcategory [redacted] $\tau$-tors and the quotients abelian category

$$\begin{tikzcd} \tau\text{-tors} \arrow[r, hook] & R\text{-mod} \arrow[r, two heads, shift left] & R\text{-mod}/\tau\text{-tors} \arrow[l, shift left] \end{tikzcd}$$

as well as a right-adjoint section functor for the canonical map to the quotient. One knows that injectives in $R$-mod/$\tau$-tors lift by the section functor to injectives $E$ satisfying ${}_A E = 0$. [redacted] As any object in the ~~quotient~~ category is the kernel of a map between injectives, the object lifts to the kernel of a map between A-good' injectives (as the section functor is left exact). Thus the image of the section functor lies in the full subcategory of A-good' modules.

The general theory gives the following recipe to construct the [redacted] section

$$M \longrightarrow M^{g'}$$

[redacted]. First make $M$ torsion-free $M \twoheadrightarrow M/T(M)$, then embed $M/T(M)$ into a torsion-free injective $E$, e.g.

$$M/T(M) \hookrightarrow E(M/T(M))$$

[redacted] Then $M^{g'}$ is the largest submodule of the injective $E$ [redacted] such that $M/T(M)$ is $\tau$-dense in $M^{g'}$, i.e. $M^{g'}/\mathrm{Im}(M)$ is $\tau$-torsion. In other words

[redacted]

$$\begin{tikzcd} M \arrow[r, two heads] & M/T(M) \arrow[r, phantom, "\subset"] & M^{g'} \arrow[r, phantom, "\subset"] & E \arrow[r] & \underbrace{E/M^{g'}}_{\text{torsion-free}} \end{tikzcd}$$

$\uparrow$ kernel torsion (under $M \twoheadrightarrow M/T(M)$); $\uparrow$ cokernel torsion (under $M/T(M) \subset M^{g'}$)

so we have an embedding $E/M^{g'}$ into a torsion-free injective $E'$ an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & M^{g'} \arrow[r] & E \arrow[r] & E' \\ & M \arrow[u, "\text{isom. mod torsion}"'] & & \end{tikzcd}$$

Then it's clear that $M^{g'}$ is good', as it should be.

June 12, 1994

Consider again the example $R = k[x,y]$, $I = (x,y)$, but think of $R$-modules as quasi-coherent sheaves on $Sp(R)$. In this case the $I$-torsion modules are those modules such that every element is killed by a power of $I$. Let's check this.

⟦ Given $0 \to M' \to M \to M'' \to 0$, an exact sequence of $R$-modules, it is clear that if every $m \in M$ is killed by some $I^n$ the same is true for $M'$ and $M''$. Conversely suppose $M'$ and $M''$ have this property and let $m \in M$. Then $\exists n_1$ s.t. $I^{n_1} m \in M'$. Now the module $I^{n_1} m$ is *finitely generated* so choosing a finite set of generators and $n_2$ large enough so that $I^{n_2}$ kills these generators, we have $I^{n_2} I^{n_1} m = 0$.

Example to show fin. generated is required. Take $A = \bigoplus_{n=1}^{\infty} A_n$ where $A_n$ is a nonunital ring such that $A_n^n \neq 0$, $A_n^{n+1} = 0$. Take $R = \tilde{A}$, $I = A$. Then every $a \in A$ belongs to $\bigoplus_{n=1}^{N} A_n$ for some $N$ and so $A^{N+1} a = 0$. Also $R/A$ is killed by $A$, so we see that $R$ is an extension of the modules $A$, $R/A$ having the property that any elt is killed by a power of $A$. But $R$ itself does not have this property.

(A different example cited in Golan's book namely $R = k[X_n]_{n \geq \mathbb{Z}}$, $I = (X_n)_{n \geq \mathbb{Z}}$, and the module $R / \sum R X_n^{n+1}$.) ⟧

Returning to our example we have R-mod = quasi-coh. sheaves on $X = Sp(R)$, I-torsion mods = such sheaves supported in the closed subset $Z$ corresponding to $I$. So one has the familiar picture

$$\begin{tikzcd} Z \arrow[r, hook, "\iota"] & X & U \arrow[l, hook', "j"'] \end{tikzcd}$$

and functors

$$\begin{tikzcd} I\text{-tors} \arrow[r, "\iota_*"] & R\text{-mod} \arrow[l, dashed, bend right, "\iota^*"'] \arrow[l, bend left, "\iota^!"] \arrow[r, "j^*"] & \mathcal{O}_U\text{-mod} \arrow[l, dashed, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

The dotted arrows $\iota^*$, $j_!$ are defined only in some proobject or derived category sense. Thus $\iota^*(M) = \{M/I^nM\}$ I think and $j_!$ is the thing introduced by Deligne.

There should be a description of a sheaf $F$ on $X$ as a triple consisting of sheaves $F_U$, $F_Z$ on $U, Z$ resp. together with a map

$$F_Z \longrightarrow \iota^* j_* F_U$$

In fact this holds [redacted] for sheaves of abelian groups: $F \longmapsto F_U = j^*F$, $F_Z = \iota^*F$, and the canonical map $\iota^*F \longrightarrow (\iota^*j_*)j^*F$ induced by the adjunction arrow $1 \longrightarrow j_*j^*$.

For quasi-coherent sheaves Husemoller reports that Deligne told him there are triangles in the derived category

a) $\iota_*\iota^! F \longrightarrow F \longrightarrow j_*j^*F$

b) $j_!j^*F \longrightarrow F \longrightarrow \iota_*\iota^*F$

In the present case I am interested in the first triangle. [redacted] For a single R-module $M$ we have

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(R/I^n, M) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \mathrm{Hom}_R(R, M) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \mathrm{Hom}_R(I^n, M) \arrow[r] & \mathrm{Ext}^1_R(R/I^n, M) \arrow[r] & 0 \\
& {}_{I^n}M & M & & &
\end{tikzcd}$$

which upon taking $\varinjlim_n$ yields

$$\begin{tikzcd}
0 \arrow[r] & \iota_* \iota^! M \arrow[r] \arrow[d, phantom, "\shortparallel"] & M \arrow[r] \arrow[d, phantom, "\shortparallel"] & j_* j^* M \arrow[r] \arrow[d, phantom, "\shortparallel"] & \iota_* R^1 \iota^! (M) \arrow[r] \arrow[d, phantom, "\shortparallel"] & 0 \\
0 \arrow[r] & H^0_Z(M) \arrow[r] & M \arrow[r] & H^0(U, M) \arrow[r] & H^1_Z(M) \arrow[r] & 0
\end{tikzcd}$$

This exact sequence

1) $$\boxed{0 \to H^0_Z M \to M \to j_* j^* M \to H^1_Z M \to 0}$$

we recognize as obtained from the module $M$ to its good' hull. This sequence defines an element of $\mathrm{Ext}^2_R(H^1_Z M, H^0_Z M)$. Now the injective hull of an $I$-torsion module should be an $I$-torsion module. This is certainly true in our example where the injective hull of $k$ is $\varinjlim \mathrm{Hom}_k(R/I^n, k)$. Thus this class in $\mathrm{Ext}^2_R(H^1_Z M, H^0_Z M)$ can be represented by an exact sequence

2) $$\boxed{0 \to H^0_Z M \to I^0 \to F \to H^1_Z M \to 0}$$

where $I^0, F$ are torsion. Specifically let $I^\circ$ be a minimal injective resolution of $H^0_Z M$ and pullback:

$$\begin{tikzcd}
0 \arrow[r] & H^0_Z M \arrow[r] \arrow[d, phantom, "\shortparallel"] & I^0 \arrow[r] \arrow[d, phantom, "\shortparallel"] & F \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & H^1_Z M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0_Z M \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots
\end{tikzcd}$$

Now we have a map from 1) to the resolution $M \to I$, which is unique up to homotopy, so we should have a canonical map [redacted] up to homotopy

$$\begin{tikzcd}
0 \arrow[r] & H^0_Z M \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & j_* j^* M \arrow[r] \arrow[d] & H^1_Z M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H^0_Z M \arrow[r] & I^0 \arrow[r] & F \arrow[r] & H^1_Z M \arrow[r] & 0
\end{tikzcd}$$

The preceding is a crude way of interpreting the fact that $M$ is the h-fibre of a map

$\mathbb{R} j_*$ [redacted] $j^*(M) \longrightarrow i_* Ri^!(M)[1].$

I guess the practical content is that an $R$-module $M$ is essentially equivalent to a good' module $N$ together with a map from $N$ to a length one complex of torsion modules:

$$\begin{tikzcd}
M \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\text{cart}"] & N \arrow[d] \\
I^\bullet \arrow[r] & F
\end{tikzcd}$$

It seems that this is no more significant than the exact sequence 1).

What I was hoping ~~to~~ do was to try to get good modules out of good' modules by some canonical modification, but it seems I just get a little better understand of lifting a good' module to a module.

Let's again consider the example $R = k[x,y]$, $I = (x,y)$. Recall $M$ good (resp. good') means

$$\mathrm{Tor}_n^R(k, M) = 0 \qquad (\text{resp. } \mathrm{Ext}_R^n(k, M) = 0)$$

for $n = 0, 1$. On the other hand, because of the standard Koszul complex resolution of $k$ one has a canonical isom:

$$\boxed{\mathrm{Ext}_R^n(k, M) = \mathrm{Tor}_{2-n}^R(k, M)}$$

Thus $M$ flat $\Longrightarrow$ $M$ good'

$M$ injective $\Longrightarrow$ $M$ good

Certain modules are both good and good', namely

injective modules $E$ such that $_I E = 0$.

flat modules $P$ such that $P = IP$.

On the other hand there are good' modules which do not seem to correspond to good modules. For example $R$ is good' (recall $\mathrm{Ext}_R^n(k, R) = \begin{cases} 0 & n \neq 2 \\ k & n = 2 \end{cases}$), but a map $M \longrightarrow R$ where $IM = M$ (e.g. if $M$ good) is necessarily zero.

Similarly the injective hull of $k$ is a good module which is torsion, so it doesn't seem to correspond to any good' module.

June 13, 1994

Consider $R = T(V)$, $I = \bigoplus_{n \geq 1} V^{\otimes n}$ where $V$ is 2 dimensional. [redacted] An $R$-module $M$ (unital understood) is the same as a vector space $M$ equipped with a map $V \otimes M \longrightarrow M$. The module is good (resp. good') when this map is an isom. (resp. when the transposed map $M \longrightarrow \underset{\substack{\parallel \\ V^* \otimes M}}{Hom(V, M)}$ is an isomorphism.) If we choose an isomorphism $V \simeq V^*$ then these categories become equivalent. I suspect there is not a canonical equivalence.

Let's try to calculate the composition:

$$\begin{tikzcd} (\text{good}) \arrow[r, hook] & (R\text{-mod}) \arrow[r] & (\text{good}') \end{tikzcd}$$

$\uparrow$
'localization' functor

The point is that the ideals $I^n$ are finitely generated as left $R$-modules, and I think this implies that a module is torsion iff [redacted] each element is killed by a power of $I$. Check this

$$0 \longrightarrow M' \longrightarrow \underset{\substack{\cup \\ m}}{M} \longrightarrow M'' \longrightarrow 0$$

$$I^{n_0} m \longrightarrow 0$$

Given $m \in M$, $\exists n_0$ such that $I^{n_0} m \longrightarrow 0$. Since $I^{n_0} = \sum R z_j$, one has $I^{n_0} m = \sum R z_j m$, and $\exists n_1$ such that $I^{n_1} z_j m = 0$, $\forall j$. Then $I^{n_1 + n_0} m = \sum I^{n_1} R z_j m$ $= \sum I^{n_1} z_j m = 0$.

So we know

$$\mathrm{Tors}(M) = \bigcup_n {}_{I^n}M = \varinjlim_n \mathrm{Hom}_R(R/I^n, M)$$

for any R-module M.

Suppose now that M is torsion free: ${}_IM = 0$. Then $\varinjlim_n \mathrm{Hom}_R(I^n, M)$ should be the localization of M. Let L(M) denote this localization. One knows that if [redacted] we embed M in a torsion-free injective E, e.g. the injective hull of M, then [redacted] L(M) may be identified with the submodule of E containing M such that

[redacted] $L(M)/M = \mathrm{Tors}(E/M)$.

$$\begin{tikzcd}
 & & & M \arrow[d, hook] & \\
 & & & \mathrm{Tors}(E/M) \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & E/M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varinjlim_n \mathrm{Hom}_R(I^n, M) \arrow[r] & \varinjlim_n \mathrm{Hom}_R(I^n, E) \arrow[r] & \varinjlim_n \mathrm{Hom}_R(I^n, E/M) &
\end{tikzcd}$$

From this diagram we see that

$$\varinjlim_n \mathrm{Hom}_R(I^n, M)/M \simeq \mathrm{Tors}(E/M)$$

so $L(M) = \varinjlim_n \mathrm{Hom}_R(I^n, M)$ at least when M is torsion-free.

June 14, 1994

Let $I$ be an ideal in $R$ unital, and suppose $I = \sum Rx_i$ is finitely generated as left module. Then

$$I^2 = \sum_i IRx_i = \sum_i Ix_i = \sum_i \Big(\sum_j Rx_j\Big)x_i$$

and similarly

$$I^n = \sum_{i_1,\dots,i_n} R x_{i_1} \cdots x_{i_n}$$

is finitely generated as left module.

Let $M$ be an $R$-module, and [redacted] let $M'$ be the submodule $\varinjlim_n \operatorname{Hom}_R(R/I^n, M) = \bigcup_{n=1}^{\infty} I^n M$

Suppose $m \in M$ such that $Im \in M'$. There exists a $k$ such that $x_i m \in {}_{I^k}M$ for all $i$, whence $I^{k+1}m = I^k \sum_i Rx_i m = \sum_i I^k x_i m = 0$, so $m \in M'$. Thus [redacted] $M/M'$ is $I$-torsion-free and as $M'$ is $I$-torsion obviously, we find that $M'$ is the $I$-torsion submodule of $M$.

Let's introduce the notation

$$H^0_I(M) = \varinjlim_n \operatorname{Hom}_R(R/I^n, M)$$

for the $I$ torsion submodule.

The question I am interested in is when is the localization of $M$, denote it [redacted] $j_* j^* M$ given by $\varinjlim_n \operatorname{Hom}_R(I^n, M)$. This is always true in the commutative noetherian case I believe.

Recall that besides the inverse system $\{I^n\}$ one also has the inverse system of tensor powers $\{I^{\otimes_R n}\}$:

$$\longrightarrow I\otimes_R I\otimes_R I \longrightarrow I\otimes_R I \longrightarrow I \longrightarrow R$$

The point here is that the different possible 'face' operator obtained by multiplying adjacent copies of $I$ all agree.

Now consider the canonical map

$$\underbrace{\varinjlim_n \operatorname{Hom}_R(I, \operatorname{Hom}_R(I^{\otimes_R n}, M))}_{= \operatorname{Hom}_R(I^{\otimes_R n+1}, M)} \to \operatorname{Hom}_R(I, \varinjlim \operatorname{Hom}_R(I^{\otimes_R n}, M))$$

This map is injective if $I$ is a finitely generated R-module, and bijective if $I$ is a finitely presented R-module.

So if $F(-) = \operatorname{Hom}_R(I,-)$, then $I$ fin. pres. $\Rightarrow$ $F$ respects filtered inductive limits, so if $F^\infty = \varinjlim_n F^n$ with respect to the canonical map $1 \to F \to F^2 \to \cdots\cdots$, then $F^\infty \xrightarrow{\sim} FF^\infty$.

Then we can conclude

$$j_* j^* M = F^\infty(M) = \varinjlim_n \operatorname{Hom}_R(I^{\otimes_R n}, M)$$

There's a remaining point of why we can replace $\{I^{\otimes_R n}\}$ with $\{I^n\}$ here in the commutative noetherian case.

~~To handle the~~

Let's show directly that the canonical map

$$\varinjlim_{n} \mathrm{Hom}_R(I^n, M) \longrightarrow j_*j^*M$$

is an isomorphism (in the comm. noetherian case). Both sides are left exact functors of $M$, so one reduces to the case where $M$ is injective.

When $M$ is injective have exact sequence

$$0 \longrightarrow \varinjlim_{n} \mathrm{Hom}_R(R/I^n, M) \longrightarrow M \longrightarrow \varinjlim_{n} \mathrm{Hom}_R(I^n, M) \longrightarrow 0$$

whence $\varinjlim_{n} \mathrm{Hom}_R(I^n, M) = M/H^0_I M$. On the other hand $j_*j^*M$ is obtained by embedding $M/H^0_I M$ in its injective hull $E$), then

$$j_*j^*M/(M/H^0_I M) = H^0_I(E/(M/H^0_I M))$$

It suffices to prove

Lemma: If $M$ is injective then $H^0_I M$ is injective.

In effect, then $H^0_I M$ is a summand of $M$, so $M/H^0_I M$ is injective, and so $j_*j^*M = M/H^0_I M$.

To prove the lemma, let $\mathfrak{a}$ be an ideal in $R$ (supposed comm. noeth.), and let $f : \mathfrak{a} \longrightarrow H^0_I M$ be a map. We have to show $f$ extends to a map $g : R \longrightarrow H^0_I M$. Because $\mathfrak{a}$ is fin. gen $f$ factors through $\mathfrak{a}/I^n\mathfrak{a}$ for some $n$. Consider

$$\begin{tikzcd}
\mathfrak{a} \arrow[r, phantom, "\subset"] \arrow[d] \arrow[dd, bend right=60, "f"'] & R \arrow[d] \arrow[dd, bend left=60, "g"] \\
\mathfrak{a}/I^n\mathfrak{a} \arrow[r, hook] \arrow[d] & R/I^n\mathfrak{a}+I^N \arrow[dl, dashed, "\exists \text{ as } M \text{ is injective}"] \\
M &
\end{tikzcd}$$

$\Longrightarrow \mathfrak{a} \cap (I^n\mathfrak{a} + I^N) = I^n\mathfrak{a} + \mathfrak{a} \cap I^N = I^n\mathfrak{a}$

When $R$ comm. noeth. We know $\exists N$ large enough so that $\underbrace{I^n\mathfrak{a} \supset I^N \cap \mathfrak{a}}$. Thus its clear.

---

Along the way we've managed to prove (in the comm. noeth case) that $\forall M$ the canonical map

$$\varinjlim_n \mathrm{Hom}_R(I^n, M) \longrightarrow \varinjlim \mathrm{Hom}_R(I^{\otimes_R n}, M)$$

is bijective. This amounts to $\forall n$, $\exists n' \geq n \ni \exists$ dotted arrow in

$\circledast$

$$\begin{tikzcd}
I^{\otimes_R n} & \\
I^{\otimes_R n'} \arrow[u] \arrow[r, two heads] & I^{n'} \arrow[ul, dashed]
\end{tikzcd}$$

[redacted]

It would be nice to show this directly, [redacted] say by replacing $\{I^{\otimes_R n}\}$ by $\{\mathrm{Sym}^R_n(I)\}$ which should be cofinal and using the surjective homom.

$$\mathrm{Sym}^R(I) \twoheadrightarrow \bigoplus_{n=0}^{\infty} I^n$$

---

Observation: Let $S$ be a ring, let $B$ be an $S$ bimodule, everything unital, let

$$R = T_S(B) = S \oplus B \oplus B\otimes_S B \oplus B\otimes_S B \otimes_S B \oplus \cdots$$

$$I = T_S^{>0}(B) = \quad B \oplus B\otimes_S B \oplus \cdots\cdots$$

Then $I = R\otimes_S B$, and if $M$ is an $R$-module [redacted]

$$\mathrm{Hom}_R(I, M) = \mathrm{Hom}_R(R\otimes_S B, M) = \mathrm{Hom}_S(B, M).$$

Thus $I$-good' [redacted] $R$-modules are the same as $S$-modules $M$ equipped with an $S$-module isomorphism $M \xrightarrow{\sim} \mathrm{Hom}_S(B, M)$. In particular such things form an abelian category.

On the other hand $I = B\otimes_S R$ as right $R$-module and $I\otimes_R M = B\otimes_S R \otimes_R M = B\otimes_S M$. Thus $I$-good $R$-modules are the same as $S$-modules $M$ equipped with an $S$-module isomorphism $B\otimes_S M \xrightarrow{\sim} M$. Do the $I$-good $R$-modules form an abelian cat? at the moment I know this only when $B$ is right $S$-flat.

June 17, 1994

Suppose $I = I^2 \subset R$ unital. Recall the adjoint functors

$$\begin{tikzcd}
R/I\text{-mod} \arrow[d, equal] \arrow[r, hook, "i_*"] & R\text{-mod} \arrow[d, equal] \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, two heads, "j^*"] & R\text{-mod}/R/I\text{-mod} \arrow[d, equal] \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \\
\mathcal{S} & \mathcal{A} & \mathcal{A}/\mathcal{S}
\end{tikzcd}$$

where $i^*(M) = M/IM$ $\qquad j_! j^* M = I^{g} \otimes_R M$

$i^!(M) = \mathrm{Hom}_R(R/I, M)$ $\qquad j_* j^* M = \mathrm{Hom}_R(I^{g}, M)$

We have exact sequences

$$\begin{tikzcd}
j_! j^* M \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, equal] & i_* i^* M \arrow[r] \arrow[d, equal] & 0 \\
I^{g} \otimes_R M \arrow[r] & M \arrow[r] & M/IM \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & i_* i^! M \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, equal] & j_* j^* M \arrow[d, equal] \\
0 \arrow[r] & {}_I M \arrow[r] & M \arrow[r] & \mathrm{Hom}_R(I^{g}, M)
\end{tikzcd}$$

I would like to complete these to triangles on the derived category level as in the theory of sheaves:

$$\mathbb{L}j_! j^* M \longrightarrow M \longrightarrow i_* \mathbb{L} i^* M \longrightarrow \Sigma(\mathbb{L}j_! j^* M)$$

$$i_* \mathbb{R} i^!(M) \longrightarrow M \longrightarrow \mathbb{R}j_* j^* M \longrightarrow \Sigma(i_* \mathbb{R} i^!(M))$$

---

Let's begin by recalling (p. $\underset{551-554}{534-538}$) the construction of $\mathbb{L}j_!(j^*M)$.

Identifying the quotient ab. cat. R-mod/R/I-mod with the good module category $R\text{-}g_I\text{mod}$, the functor $j_!$ becomes the inclusion and $j^*(M) = I^{g} \otimes_R M$. $\mathbb{L}j_!(j^*M)$ is calculated by choosing a complex of flat good modules $F_\cdot$ which is a resolution of $M$ modulo null-modules. Specifically:

$$0 \longrightarrow K_1 \longrightarrow F_0 \longrightarrow I^g \otimes_R M \longrightarrow 0$$

$$0 \longrightarrow K_2 \longrightarrow F_1 \longrightarrow I \otimes_R K_1 \longrightarrow 0$$

$$0 \longrightarrow K_3 \longrightarrow F_2 \longrightarrow I \otimes_R K_2 \longrightarrow 0$$

Then $L j_!(j^*M)$ is represented by the complex $F_\cdot$

Alternatively because

$$L_n j_!(j^*M) = \mathrm{Torg}^R_n(I, M)$$

we can resolve $I$ as right modules similarly:

$$0 \longrightarrow K_1 \longrightarrow P_0 \longrightarrow I^g \longrightarrow 0$$

$$0 \longrightarrow K_2' \longrightarrow P_1 \longrightarrow K_1 \otimes_R I \longrightarrow 0$$

$$0 \longrightarrow K_3' \longrightarrow P_2 \longrightarrow K_2' \otimes_R I \longrightarrow 0$$

Consider $P_\cdot \otimes_R F_\cdot$. One has

$$P_p = P_p \otimes_R I$$

$$H_q(P_p \otimes_R F_\cdot) \underset{\substack{\uparrow \\ P_p \text{ flat}}}{=} P_p \otimes_R H_q(F_\cdot) = \begin{cases} 0 & q > 0 \\ P_p \otimes_R I^g \otimes_R M & q = 0 \\ \quad \parallel & \\ P_p \otimes_R M & \end{cases}$$

$$H_q(P_\cdot \otimes_R F_p) = H_q(P_\cdot) \otimes_R F_p = \begin{cases} 0 & q > 0 \\ I^g \otimes_R F_p = F_p & q = 0 \end{cases}$$

Thus $\quad L j_!(j^*M) \sim F_\cdot \xleftarrow[\text{quis}]{} P_\cdot \otimes_R F_\cdot \xrightarrow[\text{quis}]{} P_\cdot \otimes_R M$

There's one problem with the preceding, namely, because $I$ is an $R$-bimodule, we want the 'resolution' $P_\cdot$ to [redacted] be a complex of $R$-bimodules which are flat+good as right modules.

If we are working over a field $k$, then in the construction of $P$, we choose first a flat good right module $Q_n$ mapping onto $K'_n \otimes_R I$ and then $P_n = P \otimes_k Q_n$ is a bimodule mapping onto $K'_n \otimes_R I$ which is flat ~~good~~ as right $R$-mod.

In general there do not seem to be enough bimodules which are right flat. If [redacted] so, then $R \otimes_{\mathbb{Z}} R$ would be right $R$-flat, i.e. for every fin. gen. left ideal $\mathfrak{a} \subset R$ ~~the~~ the map

$$\begin{tikzcd}
(R\otimes_{\mathbb{Z}} R)\otimes_R \mathfrak{a} \arrow[r] \arrow[d, equal] & (R\otimes_{\mathbb{Z}} R)\otimes_R R \arrow[d, equal] \\
R\otimes_{\mathbb{Z}} \mathfrak{a} \arrow[r] & R\otimes_{\mathbb{Z}} R
\end{tikzcd}$$

is injective. Consider $R =$ [redacted] $\tilde{A}$ where $A$ is an abelian group with zero multiplication. Suppose $A = A_1 \oplus A_2$ and $\mathfrak{a} \subset A_2$. Then we have

$$A_1 \otimes_{\mathbb{Z}} \mathfrak{a} \hookrightarrow A_1 \otimes_{\mathbb{Z}} A_2$$

which isn't always true.

Because of this problem, let's assume at least that we are working over a commutative ground ring $k$ such that $R$ is $k$-flat. Then if $Q$ is a right flat $R$-module, then $R \otimes_k Q$ is also right $R$-flat:

$$M \longmapsto (R\otimes_k Q)\otimes_R M = \underbrace{R\otimes_k (\underbrace{Q\otimes_R M}_{\text{exact in } M})}_{\therefore \text{ also exact.}}$$

Now return to our formula

$$Lj_!(j^*M) = P_\cdot \otimes_R M$$

and ask about the triangle (hoped for):

$$\begin{tikzcd}
Lj_!(j^*M) \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, equal] & \iota_* L\iota^*(M) \arrow[r] \arrow[d, equal] & \cdots \\
P_\cdot \otimes_R M & R \overset{L}{\otimes}_R M & R/I \overset{L}{\otimes}_R M &
\end{tikzcd}$$

Thus [redacted] we have this $\Delta \iff P_\cdot$ is actually a resolution of $I$.

Lemma: An $R$-module $M$ has a resolution by flat good modules iff $\mathrm{Tor}_*^R(R/I, M) = 0$

Proof. ($\Rightarrow$) If $M \sim P_\cdot$ where $P$ flat good, then

$$\mathrm{Tor}_n^R(R/I, M) = H_n(R/I \otimes_R P) = H_n(\underbrace{P/IP}_{=0})$$

($\Leftarrow$) As $\mathrm{Tor}_0^R(R/I, M) = M/IM = 0$, $\exists$ exact seq

$$0 \longrightarrow K_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with $P_0$ good flat. Then

$$\begin{tikzcd}
 & 0 \arrow[r] & \mathrm{Tor}_2^R(R/I, M) \arrow[dll] \\
\mathrm{Tor}_1^R(R/I, K_1) \arrow[r] & 0 \arrow[r] & \mathrm{Tor}_1^R(R/I, M) \arrow[dll] \\
K_1/IK_1 \arrow[r] & P_0/\text{[redacted]}IP_0 \arrow[r] \arrow[d, equal] & {} \\
 & 0 &
\end{tikzcd}$$

[redacted] better: $\mathrm{Tor}_{n+1}^R(R/I, M) \xrightarrow{\sim} \mathrm{Tor}_n^R(R/I, K_1)$ for all $n$, so we can choose $P_1 \twoheadrightarrow K_1$, etc. to construct the resolution.

---

So now apply this to the right module situation in the case of $I$. [redacted] Then we can find a bimodule resolution $P_\cdot$ of $I$ [redacted], where the $P_n$ are flat good as right modules, iff $\mathrm{Tor}_n^R(I, R/I) = 0 \quad \forall n$. This is equivalent to

$$\begin{cases} \mathrm{Tor}_n^R(I,I) = 0 & n>0 \\ I \otimes_R I \xrightarrow{\sim} I & \text{[redacted]} \end{cases}$$

and it's a relative version of h-unitality.

Next consider the dual [redacted] situation, i.e. with $Rj_*(j^*M)$. Recall [redacted] this is calculated by means of a suitable resolution modulo null modules consisting of good' injectives:

$$\begin{tikzcd} 0 \arrow[r] & \overset{\mathrm{Hom}_R(I^g,M)}{\overset{\shortparallel}{j_*j^*M}} \arrow[r] & E^0 \arrow[r] & C^1 \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_R(I,C^1) \arrow[r] & E^1 \arrow[r] & C^2 \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_R(I,C^2) \arrow[r] & E^2 \arrow[r] & C^3 \arrow[r] & 0 \end{tikzcd}$$

Then [redacted] $Rj_*(j^*M)$ is given by the complex $E^\circ$,
so

$$R^n j_*(j^*M) = \begin{cases} j_*j^*M & n=0 \\ \mathrm{Hom}_R(I,C^n)/\mathrm{Im}(C^n) & n>0 \end{cases}$$

As before we try to construct such a resolution functorial in $M$ by resolving the $I^g$ in $j_*j^*M = \mathrm{Hom}_R(I^g, M)$.

Suppose $I \overset{!}{\otimes}_R I \xrightarrow{\sim} I$ and let $P_\cdot$ be ~~the~~ the right good flat bimodule resolution of $I$ discussed before. Let $Q^\cdot$ be an injective R-module resolution of M. Consider the bicomplex $\mathrm{Hom}_R(P_\cdot, Q^\cdot)$.

This is a (bi)complex of injective R-modules:

$$\mathrm{Hom}_R(N, \mathrm{Hom}_R(P, Q)) = \underbrace{\mathrm{Hom}_R(\underbrace{P \otimes_R N}_{\text{exact in } N}, Q)}_{\text{exact in } N \text{ as } Q^\cdot \text{ inj.}} \quad \text{as } P \text{ flat}$$

which are also good' since

$$\mathrm{Hom}_R(R/I, \mathrm{Hom}_R(P, Q)) = \mathrm{Hom}_R(\underbrace{P \otimes_R R/I}_{=0}, Q)$$

~~So far we haven't used the fact that $P_\cdot$ is a resolution of $I = I^g$. This yields a quis~~ ~~$R\mathrm{Hom}_R(I, M)$~~

~~$\mathrm{Hom}_R(P_\cdot, Q_\cdot) \longleftarrow \mathrm{Hom}_R(I, Q_\cdot) =$~~

Suppose $P_\cdot$ such that $H_0(P_\cdot) = I^g$ and $H_n(P)$ null for $n \geq 1$. If Q is an injective R-module, then

$$H^n(\mathrm{Hom}_R(P_\cdot, Q)) = \begin{cases} \mathrm{Hom}_R(I^g, Q) = j_* j^* Q & n = 0 \\ \underbrace{\mathrm{Hom}_R(H_n(P), Q)}_{\text{null}} & n > 0 \end{cases}$$

Thus $\mathrm{Hom}_R(P_\cdot, Q)$ is a complex of good' injectives of the sort used to construct $Rj_*(j^*Q)$:

$$Rj_*(j^*Q) \underset{\text{quis}}{\simeq} \mathrm{Hom}_R(P_\cdot, Q)$$

So it seem clear that it ought to be possible to improve this quis to hold for Q a complex of injective R-modules. Certainly both sides are quis-invariant. Thus it should be true in general that if $P_\cdot$ is an appropriate good flat resolution of $I$, then $\mathrm{Hom}_R(P_\cdot, Q^\cdot)$ is an appropriate good' injective resolution of $Rj_*(j^*Q^\cdot)$.

Assuming this now suppose that we are in the h-unital case: $I \overset{L}{\otimes}_R I = I$. Then $P_\cdot$ is actually a resolution of $I$, so we have

$$Rj_*(j^*Q) \simeq \mathrm{Hom}_R(P_\cdot, Q^\cdot) \xleftarrow[\text{quis}]{} \mathrm{Hom}_R(I, Q^\cdot)$$

whence $Rj_*(j^*M) \underset{\text{quis}}{\simeq} R\mathrm{Hom}_R(I, M)$ for all (qcs.) $M$. This gives the desired triangle

$$\begin{tikzcd}
i_* Ri^!(M) \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, equal] & Rj_*(j^*M) \arrow[r] \arrow[d, equal] & {} \\
R\mathrm{Hom}_R(R/I, M) \arrow[r] & R\mathrm{Hom}_R(R, M) \arrow[r] & R\mathrm{Hom}_R(I, M) \arrow[r] & {}
\end{tikzcd}$$