February 1, 1994

Given $B \subset A \xrightarrow{\rho} B$, ρ a B-bimod map such that $\exists\, x_i, y_i \in A$ such that

$$1) \qquad x_i\rho(y_i a) = a \qquad\qquad \rho(a x_i) y_i = a$$

we have seen that $x_i \otimes y_i \in A \otimes_B A$ is central: $a x_i \otimes y_i = x_i \otimes y_i a$

Conversely assume $x_i \otimes y_i \in A \otimes_B A$ is central. Applying $\rho \otimes 1 : A \otimes_B A \longrightarrow B \otimes_B A = A$

$$a_1 \otimes a_2 \longmapsto \rho(a_1) a_2$$

we get

$$\rho(a x_i) y_i = \rho(x_i) y_i a$$

Similarly applying $1 \otimes \rho$ we get

$$x_i \rho(y_i a) = a x_i \rho(y_i)$$

Thus we find:

Prop: Given $B \subset A \xrightarrow{\rho} B$, ρ a B-bimodule map, and $x_i \otimes y_i \in A \otimes_B A$. Then 1) holds iff $x_i \otimes y_i$ is central in the bimodule $A \otimes_B A$ and $\rho(x_i) y_i = 1 = x_i \rho(y_i)$.

Suppose now that we take $B = \mathbb{C}$ and A to be a separable algebra. In this situation there is a canonical element $x_i \otimes y_i \in A \otimes A$ which satisfies

$$a x_i \otimes y_i = x_i \otimes y_i a \qquad\qquad x_i a \otimes y_i = x_i \otimes a y_i$$

$$x_i \otimes y_i = y_i \otimes x_i \qquad\qquad x_i y_i = y_i x_i = 1.$$

It is the unique symmetric separability element.

If $\tau(a) = tr(l_a)$ is the canonical trace coming from the regular representation, then $\{x_i\}$ $\{y_i\}$ are dual bases for A relative to the symmetric bilinear form $\tau(a_1 a_2)$:

$$\tau(x_i y_j) = \delta_{ij}$$

Let $1 = c_j y_j$ with c_j scalars. Then $\tau(x_i) = \tau(x_i c_j y_j) = c_i$, so we have $\tau(x_i)y_i = 1$, and similarly $x_i \tau(y_i) = \tau(y_i)x_i = 1$ by symmetry. Thus

Prop. If A is a separable algebra over $\mathbb{C}$, τ is the canonical trace, and [redacted] $x_i \otimes y_i \in A \otimes A$ is the canonical separability element, then we have $x_i \tau(y_i a) = a$, $\tau(a x_i) y_i = a$ for all $a \in A$.

Next I want to describe in the separable algebra case all the possible ρ. We know that each ρ determines a central element of $A \otimes A$. Recall that

$$\begin{tikzcd}
0 \arrow[r] & (\Omega^1 A)^\natural \arrow[r] \arrow[d, "\cong"] & (A \otimes A)^\natural \arrow[r] \arrow[d, "\cong"] & A^\natural \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & (\Omega^1 A)_\natural \arrow[r] \arrow[d, "\cong"] & (A \otimes A)_\natural \arrow[r] \arrow[d, equal] & A_\natural \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & {[A,A]} \arrow[r] & A \arrow[r] & A_\natural \arrow[r] & 0
\end{tikzcd}$$

so that there is a 1-1 correspondence between central elements of $A \otimes A$ and elements of A induced by the map $a_1 \otimes a_2 \longmapsto a_2 a_1$. This shows that the central elements are of the form $x_i w \otimes y_i = x_i \otimes w y_i$ [redacted] as w runs over [redacted] A.

Suppose that $\rho : A \to \mathbb{C}$ is a linear function of the type desired and let $x_i w \otimes y_i$ be the corresponding central element so that

$$\rho(x_i w) y_i = \rho(x_i) w y_i = 1$$

$$x_i w \rho(y_i) = x_i \rho(w y_i) = 1.$$

Now we know that $\{x_i w\}$ and $\{w y_i\}$ must be bases of A, hence w is invertible. Also ρ is uniquely determined by w since the $\rho(x_i)$ are the coordinates of 1 relative to the basis $w y_i$. In fact from $x_i \rho(w y_i) = 1$ we get $\tau(y_j) = \tau(y_j x_i \rho(w y_i))$ $= \delta_{ji} \rho(w y_i) = \rho(w y_j)$. Thus $\tau(a) = \rho(w a)$ for all $a \in A$ and so $\rho(a) = \tau(w^{-1} a) = \tau(a w^{-1})$.

Prop. Again for A separable, the linear functionals ρ for which $\exists\, x_i', y_i'$ satisfying $x_i' \rho(y_i' a) = \rho(a x_i') y_i' = a$ are of the form $\rho = \tau w^{-1}$ where w is an invertible element of A. The corresponding central element $x_i' \otimes y_i'$ is $x_i w \otimes y_i$.

The best case then to consider is where $\rho = \tau$ it seems. Note that in this case $\rho(1) = \dim A$, and the corresponding bimodule map $\mu : A \otimes_{\mathbb{C}} A \longrightarrow A$ sends the identity element $x_i \otimes y_i$ to $x_i y_i = 1$.

Our next project might be to look at the generalization of the above for B more generally.

Let review the program. It started with aim of constructing a lifting

$$\bar{C}^\lambda(A) \longrightarrow C^\lambda(A)/C^\lambda(\mathbb{C})$$

compatible with the differential. An easier problem seems to be to construct a lifting

$$1) \qquad \bar{C}^\lambda(A) \longrightarrow \bar{C}^\lambda(A \times \mathbb{C}[\varepsilon])$$

and further to improve it to an explicit homotopy equivalence. For insight I consider an analogous problem for principal bundles, namely to make the obvious injection

$$2) \qquad \Omega(P) \longrightarrow \Omega(P) \otimes W(\mathfrak{g})$$

into a homotopy equivalence having a suitable sort compatibility with the $\mathfrak{g}[\varepsilon]$ action. Now we get a retraction for 2) from a connection A on P, and analogously a retraction $\rho : A \to \mathbb{C}$ gives rise to a lifting 1). This we have worked out (at least for A finite diml), but it remains to handle the homotopy which should fit with the choice of 'connection'.

Recall $W = W(\mathfrak{g}) = \Lambda\mathfrak{g}^*_X \otimes S\mathfrak{g}^*_\varphi$, where

$$dX + X^2 = 0, \qquad \iota_X X = X, \qquad L_X X + [X, X] = 0, \quad \text{etc.}$$

We have a 1-parameter family of homoms.

$$u_t : \Omega \otimes W \longrightarrow \Omega \otimes W$$

$$u_t = \text{id on } \Omega = \Omega(P)$$

$$u_t(X) = X_t = tX + (1-t)A$$

$$u_t(\varphi) = dX_t + X_t^2 = t\varphi + (1-t)F + (t^2-t)(X-A)^2$$

We know that $u_t = t^D$ where D arises from the grading on $\Omega\otimes W$ obtained as follows. Put $\alpha = X-A$ so that $X = A+\alpha$ and

$$\varphi = d(A+\alpha) + (A+\alpha)^2$$
$$= d\alpha + \underbrace{dA + A^2 + [\alpha, A] + \alpha^2}_{\in\ \Omega\otimes\Lambda\mathfrak{g}^*_\alpha}$$

Thus $\Omega\otimes W = \Omega\otimes\Lambda\mathfrak{g}^*_\alpha\otimes S\mathfrak{g}^*_{d\alpha}$ and $u_t = t^D$ where D is the α degree, i.e. $D(\Omega)=0$ $D(\alpha)=\alpha$, $D(d\alpha)=d\alpha$. Clearly $[D,d]=0$

~~Actually, [illegible] better for our purposes than $d\alpha$ is $\nabla\alpha = d\alpha + [A,\alpha]$ which is horizontal. Recall that any of the operators $\nabla_t = d + \mathrm{ad}\, X_t$ preserves $(\Omega\otimes W)_{hor}$. $\nabla = \nabla_0$ has the advantage of commuting with D.~~

~~We have $\iota_X\alpha = 0$, $L_X\alpha + [X,\alpha] = 0$, and $(\Omega\otimes W)_{hor} = \Omega_{hor}\otimes\Lambda\mathfrak{g}^*_\alpha\otimes S\mathfrak{g}^*_{\nabla\alpha}$~~

Next as $\alpha = X-A$ we have

$$\iota_X\alpha = 0 \qquad L_X\alpha + [X,\alpha] = 0.$$

In particular $\alpha\in(\Omega\otimes W)_{hor}$. Recall that if we ~~take~~ choose a connection in $\Omega\otimes W$, then we get a derivation ∇ on $(\Omega\otimes W)_{hor}$. We have a choice of connection X_t leading to $\nabla_t = d - X_t^a L_a$, but the simplest is to take ∇ to be $\nabla_0 = d - A^a L_a$. Then we have $[D,\nabla]=0$. Also

$$\nabla\alpha = d\alpha - A^a(-[X_a,\alpha]) = d\alpha + [A,\alpha].$$

As a check note

$$\iota_X\left(d\alpha + [A, \alpha]\right) = L_X\alpha + [X, \alpha] = 0.$$

We then have

$$\Omega \otimes W = \Omega_{hor} \otimes \Lambda \mathfrak{g}_A^* \otimes \Lambda \mathfrak{g}_\alpha^* \otimes S\mathfrak{g}_{\nabla\alpha}^*$$

$$(\Omega \otimes W)_{hor} = \Omega_{hor} \otimes \Lambda \mathfrak{g}_\alpha^* \otimes S\mathfrak{g}_{\nabla\alpha}^*$$

$$\nabla = d - A^a L_a \quad \text{on } \Omega_{hor}$$

$$\nabla\alpha = d\alpha + [A, \alpha]$$

$$\nabla^2\alpha = \text{[illegible]}\left(-F^a L_a\right)(\alpha)$$

$$= F^a[X_a, \alpha] = [F, \alpha].$$

~~Note here that $\nabla = d + \mathrm{ad}\,A$ on $\Lambda\mathfrak{g}_\alpha^* \otimes S\mathfrak{g}_{\nabla\alpha}^*$ because both are derivations~~

February 4, 1994

Recall $\Omega = \Omega(P)$, [illegible] where P is a principal G bundle with connection A, whence $\Omega = \Omega_{hor} \otimes \Lambda \mathfrak{g}^*_A$. $W = W(\mathfrak{g}) = \Lambda \mathfrak{g}^*_X \otimes S\mathfrak{g}^*_\varphi$ where $\varphi = dX + X^2$. We have

$$\Omega \otimes W = \Omega \otimes \Lambda \mathfrak{g}^*_\alpha \otimes S\mathfrak{g}^*_{d\alpha} \qquad \alpha = X - A$$

and let D be the derivation corresponding to the degree in α, i.e. $D = 0$ on Ω, $D\alpha = \alpha$, $D(d\alpha) = d\alpha$.

We have $\iota_X \alpha = X$ $\quad L_X \alpha + [X, \alpha] = 0$.

Let h be the degree -1 derivation of $\Omega \otimes W$ such that $h(\Omega) = 0$, $h(\alpha) = 0$, $h(d\alpha) = \alpha$.

Then $[d,h](\Omega) = 0$, $[d,h](\alpha) = \cancel{dh\alpha} + hd\alpha = \alpha$,

$[d,h](d\alpha) = dh(d\alpha) = d\alpha$, so $\boxed{[d,h] = D}$. Also

$$[\iota_X, h](\Omega) = \cancel{\iota_X h(\Omega)} + h\,\iota_X(\Omega) \subset h(\Omega) = 0$$

$$[\iota_X, h](\alpha) = (\cancel{\iota_X} h + h\iota_X)\alpha = 0$$

$$[\iota_X, h](d\alpha) = \iota_X h d\alpha + h\iota_X d\alpha = \cancel{\iota_X \alpha} + hL_X\alpha$$

$$= -h[X,\alpha] = 0 \quad \text{since } [X_b, \alpha] = f^a_{bc}\alpha^c X_a$$

Thus $\boxed{[\iota_X, h] = 0}$. [illegible] Also

$[L_X, h](\Omega) = 0$, $[L_X, h]\alpha = \cancel{L_X h\alpha} - hL_X\alpha$

$= h[X,\alpha] = 0$, $[L_X, h]d\alpha = L_X h d\alpha - hL_X d\alpha$

$= L_X\alpha - hdL_X\alpha = -[X,\alpha] + \underbrace{hd[X,\alpha]} \ - [X, \underbrace{hd\alpha}_{\alpha}] = 0$

Thus $\boxed{[L_X, h] = 0}$, hence [illegible]

$$[\iota_X, D] = [\iota_X, [d,h]] = [L_X, h] - [d, [\iota_X, h]] = 0$$

$$[L_X, D] = [L_X, [d,h]] = 0.$$

Summarizing we find

$$[\iota_X, h] = 0 \qquad [\iota_X, D] = 0$$

$$[L_X, h] = 0 \qquad [L_X, D] = 0$$

$$[d, h] = D \qquad [d, D] = 0$$

In other words D is an infinitesimally symmetry of $\Omega \otimes W$ as DG algebra with $\mathfrak{g}[\varepsilon]$ action, and h is a null homotopy for D such h commutes with $\mathfrak{g}[\varepsilon]$ action.

Let's next compute the horizontal subalgebra.

Let $\nabla = d - A^a L_a$ (mainly restricted to $(\Omega \otimes W)_{hor}$)

Then $\nabla \alpha = d\alpha + A^a[X_a, \alpha] = d\alpha + [A, \alpha]$ is horizontal. We know $\nabla^2 = -F^a L_a$, so $\nabla(\nabla\alpha) = -F^a L_a \alpha = +F^a[X_a, \alpha] = [F, X]$. We have

$$\Omega \otimes W = \Omega_{hor} \otimes \Lambda \mathfrak{g}^*_A \otimes \Lambda \mathfrak{g}^*_\alpha \otimes S\mathfrak{g}^*_{\nabla\alpha}$$

$$(\Omega \otimes W)_{hor} = \Omega_{hor} \otimes \Lambda \mathfrak{g}^*_\alpha \otimes S\mathfrak{g}^*_{\nabla\alpha}$$

$$\nabla\alpha = d\alpha + [A, \alpha]$$

$$\nabla(\nabla\alpha) = [F, \alpha] = (d + \mathrm{ad}\, A)^2 \alpha$$

Look at h.

$$h(\Omega_{hor}) = 0$$

$$h(\alpha) = 0$$

$$h(\nabla\alpha) = \alpha$$

since $h[A, \alpha] = [\underset{0}{\underset{\shortparallel}{hA}}, \alpha] - [A, \underset{0}{\underset{\shortparallel}{h\alpha}}] = 0$

At this point we see h gives a really nice homotopy operator on $(\Omega \otimes W)_{bas}$.

February 5, 1994

Yesterday we analyzed

$$\Omega \otimes W = \Omega_{hor} \otimes \Lambda \mathfrak{g}^*_A \otimes \Lambda \mathfrak{g}^*_\chi \otimes S \mathfrak{g}^*_\varphi \qquad d\chi + \chi^2 = \varphi$$

$$= \Omega_{hor} \otimes \Lambda \mathfrak{g}^*_A \otimes \Lambda \mathfrak{g}^*_\alpha \otimes S \mathfrak{g}^*_{d\alpha} \qquad \alpha = \chi - A$$

and found the homotopy operator h, defined by $h(\Omega) = h(\alpha) = 0$, $h(d\alpha) = \alpha$, which satisfies $[\iota_X, h] = [L_X, h]$, $[d,h] = D$ the "degree in α" grading.

Now we want to go back to the analogy where Ω corresponds to $T(A^*)$ and $\Omega \otimes W$ to $T((A \times \mathbb{C}[\varepsilon])^*)$. Recall the isomorphism

$$\begin{array}{ccc} A \times \mathbb{C}[\varepsilon] & = & \tilde{A} \oplus \mathbb{C}\varepsilon \\ (e,1) & \longleftrightarrow & 1 \\ (e,0) & \longleftrightarrow & e \\ (0,\varepsilon) & \longleftrightarrow & \varepsilon \\ (X_i,0) & \longleftrightarrow & X_i \end{array}$$

Here e, X_i is a basis for A with dual basis ρ, θ^i; $e = 1_A$.

$$T(A^*) = \mathbb{C}\langle \rho, \theta^i \rangle \quad \text{with}$$

$$d\rho + \rho^2 = \underbrace{-f^0_{jk}\theta^j\theta^k}_{\omega} \qquad X_j X_k = f^0_{jk} e + f^i_{jk} X_i$$

$$d\theta^i + \rho\theta^i + \theta^i\rho + f^i_{jk}\theta_j\theta_k = 0$$

$T(\mathbb{C}[\varepsilon]^*) = \mathbb{C}\langle \chi, \varphi \rangle$, where χ, φ is dual to the basis $1, -\varepsilon$, and $d\chi + \chi^2 = \varphi$. Here $d = d' + d''$ where d' and d'' come from the product and diff'l ($d\varepsilon = 1$) in $\mathbb{C}[\varepsilon]$.

Recall that [redacted] d' and d'' are defined by $[d',\theta] + \theta^2 = 0$, $[d'',\theta] = 0$ where $\theta = \chi 1 - \varphi\varepsilon$: Thus

$$0 = [d',\theta] + \theta^2 = (d'\chi)1 - (d'\varphi)\varepsilon +$$ [redacted] $$\chi^2 1 - [\chi,\varphi]\varepsilon$$

$$0 = [d'',\theta] = (d''\chi)1 - (d''\varphi)\varepsilon - \varphi \underset{\displaystyle 1}{\underset{\shortparallel}{d\varepsilon}}$$

imply $d'\chi + \chi^2 = 0$, $d'\varphi + [\chi,\varphi] = 0$

$d''\chi = \varphi$, $d''\varphi = 0$.

Now use the projections [redacted]

$\tilde{A} \oplus \mathbb{C}$ [redacted] ε

$$= A \times \mathbb{C}[\varepsilon] \begin{array}{l} \longrightarrow A \\ \longrightarrow \mathbb{C}[\varepsilon] \end{array}$$

to get

$$\begin{array}{ccccc} T(A^*) & * & T(\mathbb{C}[\varepsilon]^*) & \xrightarrow{\sim} & T((\tilde{A} \oplus \mathbb{C}\varepsilon)^*) \\ \shortparallel & & \shortparallel & & \shortparallel \\ \mathbb{C}\langle \rho, \theta^i \rangle & * & \mathbb{C}\langle \chi, \varphi \rangle & & \mathbb{C}\langle \rho, \theta^i, \chi, \varphi \rangle . \end{array}$$

Then $\rho, \theta^i, \chi, \varphi$ is the dual basis to $e, X_i, e^\perp, -\varepsilon$. [redacted] For example e, X_i go to zero in $\mathbb{C}[\varepsilon]$ and $e^\perp, \varepsilon$ go to $1, \varepsilon$. Thus $\chi(e^\perp) = 1$ and $\chi(e) = \chi(X_i) = \chi(\varepsilon) = 0$. [redacted] Also $e^\perp, \varepsilon$ go to zero in A and e, X_i go to e, X_i so that $\rho(e) = 1$, $\rho(e^\perp) = \rho(\varepsilon) = \rho(X_i) = 0$. Notice that $\chi : \tilde{A} \oplus \mathbb{C}\varepsilon \to \mathbb{C}$ is the augmentation homomorphism

Interesting calculation $\left(\begin{smallmatrix}\text{already}\\ \text{on page 357}\end{smallmatrix}\right)$ 370

Let $u_t = t^D : R \circlearrowleft$, D a $\mathbb{N}$-grading.

Then $\mathring{u}_t = t^{D-1}D$ so $\iota(u_t, \mathring{u}_t)_* : \Omega^1 R \to R$ is given by $(u_t, \dot{u}_t)_*(x\,dy) = (t^D x)(t^{D-1}Dy)$ $= \frac{1}{t}(t^D x)(t^D Dy) = \frac{1}{t} t^D (x\,dy)$. Thus

$$\int_0^1 dt\; i(u_t, \dot{u}_t)_{\wedge}^{(xdy)} = \int_0^1 \frac{dt}{t} t^D (xDy) = \left[\frac{t^D}{D}\right]_0^1 (xDy) = \underbrace{\frac{1-P}{D}}_{\substack{\text{Green's}\\ \text{operator}}} (xDy)$$

where $P = \lim_{t\to 0} t^D =$ projection on nullspace

$\Omega^1_{W(g)} = W(g) \otimes g^*_{\mathring{X}} \oplus W(g) \otimes g^*_{\mathring{\varphi}}$ as $W(g)$-module, where the canonical derivation $W \to \Omega^1_W$ is $X, \varphi \xmapsto{\partial} \mathring{X}, \mathring{\varphi}$. d on Ω^1_W is induced by d on W. Thus from $\varphi = dX + X^2$ we get

$$\mathring{\varphi} = d\mathring{X} + [X, \mathring{X}] = (d + \operatorname{ad} X)\mathring{X}$$

and from $d\varphi + [X, \varphi] = 0$ we get

$$d\mathring{\varphi} + [X, \mathring{\varphi}] + [\mathring{X}, \varphi] = 0$$

i.e. $(d + \operatorname{ad} X)\mathring{\varphi} = [\varphi, \mathring{X}]$ which agrees with

$(d + \operatorname{ad} X)\mathring{\varphi} = (d + \operatorname{ad} X)^2 \mathring{X} = (\operatorname{ad} \varphi)\mathring{X}$.

Also we have a $g[\varepsilon]$ action on Ω^1_W, compatible with the W-module structure and $g[\varepsilon]$ action on W, such that

$$\iota_X \mathring{X} = 0 \qquad L_X \mathring{X} + [X, \mathring{X}] = 0$$

$$\iota_X \mathring{\varphi} = 0 \qquad L_X \mathring{\varphi} + [X, \mathring{\varphi}] = 0$$

Thus $\Omega^1_{W,hor} = S\mathfrak{g}^*_\varphi \otimes \mathfrak{g}^*_{\mathring{X}} \oplus S\mathfrak{g}^*_\varphi \otimes \mathfrak{g}^{-*}_{\mathring{\varphi}}$

$\Omega^1_{W,bas}$ has basis $tr(\varphi^n \mathring{X})$, $tr(\varphi^n \mathring{\varphi})$

where

$$d(tr(\varphi^n \mathring{X})) = tr((d + ad\, X)(\varphi^n \mathring{X}))$$
$$= tr(\varphi^n \mathring{\varphi}) = \partial\ tr\left(\frac{\varphi^{n+1}}{n+1}\right)$$

The reason for looking at Ω^1_W is in connection with general families $u_t : W \longrightarrow \Omega(P)$. In the case of interest at the moment where we have $u_t = t^D$ on $\Omega(P) \otimes W(\mathfrak{g})$, it suffices to invert D.

February 7, 1994

Let's return to the problem of constructing an explicit S operator on $\bar{C}^\lambda(A)$ starting from a retraction $\rho : A \longrightarrow \mathbb{C}$. Recall the idea that the construction should be functorial [redacted] [redacted] with respect to the pair (A, ρ), hence $S_\rho : \bar{C}^\lambda(A) \longrightarrow \bar{C}^\lambda(A)[2]$ should be determined by $S_{\tilde{\rho}} : \bar{C}^\lambda(\tilde{A}) \longrightarrow \bar{C}^\lambda(\tilde{A})[2]$, with $\tilde{\rho} : \tilde{A} \longrightarrow \mathbb{C}$ the pullback of ρ via the canonical surjection $\tilde{A} \longrightarrow A$. Thus [redacted] the construction [redacted] we seek should be determined by the case of an augmented algebra $A = \tilde{a}$, in which case the retraction $\tilde{a} \longrightarrow \mathbb{C}$ is equivalent to an arbitrary linear map $a \longrightarrow \mathbb{C}$.

Let's recall what we did for the bar construction, and [redacted] let's fix the notation. It's awkward to flip between A and a, so let A be a (possibly) nonunital algebra. Then we have the bar construction $(T(A), b')$ which is a DG coalgebra. Given any $\rho \in A^*$ we have a twisted bar construction $(T(A), b'_\rho)$, where $b'_\rho = b' - \iota_\rho(1-\lambda)$ is a coderivation but not necessarily of square zero. In the case where A is unital and $\rho(1) = 1$, then b'_ρ on $T(A)$ descends to $T(\bar{A})$.

Now we have seen that a good way to understand b'_ρ, b_ρ etc. is the following. We have the map of canonical exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A^{\otimes n} \arrow[r] \arrow[d] & \bar\Omega^n\tilde A \arrow[r] \arrow[d] \arrow[l, dashed, bend right, "r=(0\ 1)"'] & A^{\otimes n+1} \arrow[r] \arrow[d] \arrow[l, dashed, bend right, "\ell=\binom{1}{0}"'] & 0 \\
0 \arrow[r] & \bar A^{\otimes n} \arrow[r] & \bar\Omega^n A \arrow[r] & \bar A^{\otimes n+1} \arrow[r] & 0
\end{tikzcd}$$

The top sequence is defined in general for A nonunital, the bottom sequence and vertical arrows for A unital. The top sequence has a standard splitting which we have indicated relative to which $\check b = \begin{pmatrix} b & 1-\lambda \\ & b' \end{pmatrix}$, $\tilde B = \begin{pmatrix} 0 & 0 \\ N_\lambda & 0 \end{pmatrix}$. Now given $\rho \in A^*$ we can alter the standard splitting to

$$\begin{tikzcd}
0 \arrow[r] & A^{\otimes n} \arrow[r] & \bar\Omega^n\tilde A \arrow[r] \arrow[l, dashed, bend right, "r_\rho=(\iota_\rho\ 1)"'] & A^{\otimes n+1} \arrow[r] \arrow[l, dashed, bend right, "\ell_\rho=\binom{1}{-\iota_\rho}"'] & 0
\end{tikzcd}$$

Thus

$$\ell_\rho(a_0,\dots,a_n) = a_0da_1\cdots da_n - \rho(a_0)da_1\cdots da_n$$
$$= (a_0-\rho(a_0))da_1\cdots da_n$$

$$r_\rho(a_0da_1\cdots da_n) = \rho(a_0)(a_1,\dots,a_n)$$
$$r_\rho(da_1\cdots da_n) = (a_1,\dots,a_n)$$

Note that when A is unital and $\rho(1)=1$, ~~then~~ the r_ρ, ℓ_ρ splitting~~s~~ descends to a splitting of the bottom exact sequence

$$\begin{tikzcd}
0 \arrow[r] & d\Omega^{n-1}A \arrow[r] & \bar\Omega^n A \arrow[r] \arrow[l, dashed, bend right, "r_\rho"'] & d\Omega^n A \arrow[r] \arrow[l, dashed, bend right, "\ell_\rho"'] & 0
\end{tikzcd}$$

$$(a_0-\rho a_0)da_1\cdots da_n \qquad da_0\cdots da_n$$

$$\rho(a_0)da_1\cdots da_n \longleftarrow\!\shortmid\; a_0da_1\cdots da_n$$

In terms of the ρ splitting let's calculate $\tilde{b}$, $\tilde{B}$.

$$\begin{tikzcd}
 & & \begin{pmatrix} b & 1-\lambda \\ 0 & -b' \end{pmatrix} & & \\
0 \arrow[r] & A^{\otimes n} \arrow[r] \arrow[d, equal] & \bar{\Omega}^n \tilde{A} \arrow[r] \arrow[l, bend right, "(0\ 1)"'] \arrow[d, "\begin{pmatrix} 1 & 0 \\ -\iota_\rho & 1 \end{pmatrix}"] & A^{\otimes n+1} \arrow[r] \arrow[d, equal] \arrow[l, bend right, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"'] & 0 \\
0 \arrow[r] & A^n \arrow[r, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"'] & \bar{\Omega}^n \tilde{A} \arrow[r] & A^{\otimes n+1} \arrow[r] \arrow[l, bend right, "\begin{pmatrix} 1 \\ -\iota_\rho \end{pmatrix}"'] & 0
\end{tikzcd}$$

$$\tilde{b}\begin{pmatrix} 1 & 0 \\ -\iota_\rho & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\iota_\rho & 1 \end{pmatrix}\begin{pmatrix} ? \end{pmatrix}$$

$$\begin{pmatrix} ? \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \iota_\rho & 1 \end{pmatrix}\begin{pmatrix} b & 1-\lambda \\ & -b' \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\iota_\rho & 1 \end{pmatrix} = \begin{pmatrix} b_\rho & 1-\lambda \\ -\iota_\omega(1+\lambda) & -b'_\rho \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ \iota_\rho & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ N_\lambda & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\iota_\rho & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ N_\lambda & 0 \end{pmatrix}$$

So now let's calculate the S operator on $C^\lambda(A)$ which arises from the modified splitting

$$\begin{tikzcd}
0 \arrow[r] & A^{\otimes n} \arrow[r] & \bar{\Omega}^n \tilde{A} \arrow[r] & A^{\otimes n+1} \arrow[r] \arrow[l, bend right, "\begin{pmatrix} 1 \\ -\iota_\rho \end{pmatrix}"'] & 0
\end{tikzcd}$$

We proceed as in [redacted] CQ2.

$$0 \longrightarrow C^\lambda[1] \xrightarrow{i = \begin{pmatrix} 0 \\ N_\lambda \end{pmatrix}} P\tilde{\Omega}\tilde{A} \xrightarrow{j = (\pi\ 0)} C^\lambda \longrightarrow 0$$

Recall $C \xrightarrow{\pi} C^\lambda$ canonical surjection

Define $\bar{N}_\lambda, \bar{P}_\lambda : C^\lambda \longrightarrow C$ so that $N_\lambda = \bar{N}_\lambda \pi$ and $P_\lambda = \bar{P}_\lambda \pi$. Thus $\bar{P}_\lambda$ is the lifting of C^λ into C.

Define $l_\rho : C^\lambda \longrightarrow P\bar{\Omega}\tilde{A}$ by

$$l_\rho \pi = P\begin{pmatrix}1\\-\iota_\rho\end{pmatrix}P_\lambda = \begin{pmatrix}P_\lambda & 0\\ b'G_\lambda - G_\lambda b & P_\lambda\end{pmatrix}\begin{pmatrix}1\\-\iota_\rho\end{pmatrix}P_\lambda$$

$$= \begin{pmatrix}P_\lambda\\ -G_\lambda b P_\lambda - P_\lambda \iota_\rho P_\lambda\end{pmatrix} \quad \text{i.e. } l_\rho = \begin{pmatrix}\bar{P}_\lambda\\ -G_\lambda b \bar{P}_\lambda - P_\lambda \iota \bar{P}\end{pmatrix}$$

and define $S_\rho : C^\lambda \longrightarrow C^\lambda[2]$ by $-iS_\rho = \tilde{b} l_\rho - l_\rho b$

$$\tilde{b} l_\rho - l_\rho b = \tilde{b} P\begin{pmatrix}1\\-\iota_\rho\end{pmatrix}\bar{P}_\lambda - P\begin{pmatrix}1\\-\iota_\rho\end{pmatrix}\bar{P}_\lambda b$$

$$= P\left\{\begin{pmatrix}b & 1-\lambda\\ & -b'\end{pmatrix}\begin{pmatrix}1\\-\iota_\rho\end{pmatrix}\bar{P}_\lambda - \begin{pmatrix}1\\-\iota_\rho\end{pmatrix}\bar{P}_\lambda b\right\}$$

$$= P\left\{\begin{pmatrix}(b-(1-\lambda)\iota_\rho)\bar{P}_\lambda\\ b'\iota_\rho \bar{P}_\lambda\end{pmatrix} + \begin{pmatrix}-\bar{P}_\lambda b\\ \iota_\rho \bar{P}_\lambda b\end{pmatrix}\right\}$$

$$= \begin{pmatrix}P_\lambda & 0\\ b'G_\lambda - G_\lambda b & P_\lambda\end{pmatrix}\begin{pmatrix}[b,\bar{P}_\lambda] - (1-\lambda)\iota_\rho \bar{P}_\lambda\\ b'\iota_\rho \bar{P}_\lambda + \iota_\rho \bar{P}_\lambda b\end{pmatrix}$$

$$= \begin{pmatrix}P_\lambda[b,\bar{P}_\lambda]\\ (b'G_\lambda - G_\lambda b)\left([b,\bar{P}_\lambda] - (1-\lambda)\iota_\rho \bar{P}_\lambda\right) + P_\lambda\left(b'\iota_\rho \bar{P}_\lambda + \iota_\rho \bar{P}_\lambda b\right)\end{pmatrix}$$

$$-\underbrace{(b'G_\lambda - G_\lambda b)(1-\lambda)}_{\boxed{b'P_\lambda^\perp - P_\lambda^\perp b' = -b'P_\lambda + \cancel{P_\lambda b'}}}\iota_\rho \bar{P}_\lambda + \cancel{P_\lambda} b'\iota_\rho \bar{P}_\lambda + P_\lambda \iota_\rho \bar{P}_\lambda b$$

$$= b'\left(P_\lambda \iota_\rho \bar{P}_\lambda\right) + \left(P_\lambda \iota_\rho \bar{P}_\lambda\right)b$$

$$\therefore \underset{\substack{\shortparallel\\ -iS_\rho}}{\tilde{b} l_\rho - l_\rho b} = \underset{\substack{\shortparallel\\ -iS_0}}{\begin{pmatrix}0\\ (b'G_\lambda - G_\lambda b)[b,\bar{P}_\lambda]\end{pmatrix}} + \underset{-i\Delta S}{\begin{pmatrix}0\\ b'(P_\lambda \iota_\rho \bar{P}_\lambda) + (P_\lambda \iota_\rho \bar{P}_\lambda)b\end{pmatrix}}$$

Thus

$$-\bar{N}_\lambda \Delta S = b'(P_\lambda \iota_\rho \bar{P}_\lambda) + (P_\lambda \iota_\rho \bar{P}_\lambda) b$$

Consider this on elements $\xi \in C^\lambda_n$.

$$-\bar{N}_\lambda \Delta S = b' \frac{N_\lambda}{n} \iota_\rho \bar{P}_\lambda + \frac{N_\lambda}{n-1} \iota_\rho \bar{P}_\lambda b$$

$$= N_\lambda \left\{ b'\left(\frac{1}{n} \iota_\rho \bar{P}_\lambda\right) + \left(\frac{1}{n-1} \iota_\rho \bar{P}_\lambda\right) b \right\}$$

$$-\Delta S = \pi \left\{ \underline{\qquad\qquad\qquad\qquad} \right\}$$

$$= b\left(\frac{1}{n} \pi \iota_\rho \bar{P}_\lambda\right) + \left(\frac{1}{n-1} \pi \iota_\rho \bar{P}_\lambda\right) b$$

So

$$\boxed{S_\rho = S_0 - [b, h] \quad \text{where } h = \frac{1}{n} \pi \iota_\rho \bar{P}_\lambda \text{ on elements of } C^\lambda_n}$$

1

February 13, 1994

Let $C = \oplus C_n$, $C_n = \begin{cases} a^{\otimes n+1} & n \geq 0 \\ 0 & n < 0 \end{cases}$.

Exact sequence

$$0 \longrightarrow C[1] \longrightarrow \bar{\Omega}\tilde{a} \longrightarrow C \longrightarrow 0$$

with the standard splitting given by the lifting $(a_0, \dots, a_n) \longmapsto a_0 da_1 \dots da_n$ leads to

$$\tilde{b} = \begin{pmatrix} b & 1-\lambda \\ 0 & -b' \end{pmatrix} \qquad \tilde{B} = \begin{pmatrix} 0 & 0 \\ N_\lambda & 0 \end{pmatrix}$$

Given $\rho : a \to \mathbb{C}$ a linear functional, we can consider instead the splitting given by the lifting $(a_0, \dots, a_n) \longmapsto (a_0 - \rho(a_0)) da_1 \dots da_n$, which in terms of the standard isomorphism $\bar{\Omega}\tilde{a} = C \oplus C[1]$ is

$$\begin{pmatrix} 1 \\ -\iota_\rho \end{pmatrix} : C \longrightarrow \bar{\Omega}\tilde{a}.$$

Lets calculate the corresponding matrices for $\tilde{b}$ and $\tilde{B}$; denote these matrices by $\tilde{b}_\rho$, $\tilde{B}_\rho$ resp. One has

$$\begin{pmatrix} b & \overset{1-\lambda}{\text{[redacted]}} \\ 0 & -b' \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\iota_\rho & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\iota_\rho & 1 \end{pmatrix}\tilde{b}_\rho \qquad \text{i.e.}$$

$$\tilde{b}_\rho = \begin{pmatrix} 1 & 0 \\ \iota_\rho & 1 \end{pmatrix}\begin{pmatrix} b & 1-\lambda \\ 0 & -b' \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\iota_\rho & 1 \end{pmatrix} = \begin{pmatrix} b & 1-\lambda \\ \iota_\rho b & \iota_\rho(1-\lambda)-b' \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\iota_\rho & 1 \end{pmatrix}$$

$$= \begin{pmatrix} b - (1-\lambda)\iota_\rho & \text{[redacted]} \; 1-\lambda \\ \boxed{\iota_\rho b + b'\iota_\rho - \iota_\rho(1-\lambda)\iota_\rho} & -b' + \iota_\rho(1-\lambda) \end{pmatrix} = \begin{pmatrix} b_\rho & 1-\lambda \\ \iota_\omega(1+\lambda) & -b'_\rho \end{pmatrix}$$

I really only have to check that is $\iota_\omega(1+\lambda)$.

$$\begin{aligned}\iota_\rho b(a_0,\dots,a_n) &= \rho(a_0a_1)(a_2,\dots,a_n)\\ &\quad+\sum_{i=1}^{n-1}(-1)^i\rho(a_0)(a_1,\dots,a_ia_{i+1},\dots,a_n)\\ &\quad+(-1)^n\rho(a_na_0)(a_1,\dots,a_n)\end{aligned}$$

$$b'\iota_\rho(a_0,\dots,a_n) = \rho(a_0)\sum_{i=1}^{n-1}(-1)^{i-1}(a_1,\dots,a_ia_{i+1},\dots,a_n)$$

$$\therefore\ (\iota_\rho b+b'\iota_\rho)(a_0,\dots,a_n) = \rho(a_0a_1)(a_2,\dots,a_n) + (-1)^n\rho(a_na_0)(a_1,\dots,a_n)$$

Now $\iota_\rho(1-\lambda)\iota_\rho\ (a_0,\dots,a_n) = \rho(a_0)\,\iota_\rho\{(a_1,\dots,a_n)+(-1)^n(a_n,a_1,\dots,a_{n-1})\}$

$$= \rho(a_0)\rho(a_1)(a_2,\dots,a_n) + (-1)^n\rho(a_n)\rho(a_0)(a_1,\dots,a_{n-1})$$

Recall $\omega(a_0,a_1) = \rho(a_0a_1)-\rho(a_0)\rho(a_1)$. Thus

$$\boxed{\begin{aligned}&(\iota_\rho b+b'\iota_\rho-\iota_\rho(1-\lambda)\iota_\rho)(a_0,\dots,a_n)\\ &\quad= \omega(a_0,a_1)(a_2,\dots,a_n)+(-1)^n\omega(a_n,a_0)(a_1,\dots,a_{n-1})\\ &\quad= \iota_\omega(1+\lambda)(a_0,\dots,a_n)\end{aligned}}$$

Next recall that the standard lifting $\binom{1}{0}: C \longrightarrow \Omega\tilde{A}$ leads to the standard S_0 operator defined by

$$-iS_0 = \bar{b}\,l_0 - l_0 b \qquad l_0 = P\binom{1}{0}\bar{P}_\lambda$$

Here $\bar{P}_\lambda : C_\lambda \longleftrightarrow C$ is the lifting: $\pi\bar{P}_\lambda = 0$, $\bar{P}_\lambda\pi = P_\lambda$ where $\pi: C \longrightarrow C_\lambda$ is the canonical surjection, C_λ is the cyclic complex of a. We have calculated that

$$\bar N_\lambda S_0 = (-b'G_\lambda + G_\lambda b)[b, \bar P_\lambda]$$

$$S_0 = \tilde N_\lambda^{-1} P_\lambda (-b'G_\lambda + G_\lambda b)[b, \bar P_\lambda]$$

or

$$\boxed{\begin{aligned} S_0 &= \bar N_\lambda^{-1} P_\lambda (-b') G_\lambda b \bar P_\lambda \\ &= \tilde N_\lambda^{-1} P_\lambda c\, G_\lambda c \bar P_\lambda \end{aligned}}$$

Now we want to [illegible] calculate the S-operator S_ρ arising from the lifting $\begin{pmatrix} 1 \\ -\iota_\rho \end{pmatrix} : C \longrightarrow \tilde\Omega \tilde a$, i.e.

$$-iS_\rho = \tilde b\, l_\rho - l_\rho b \qquad l_\rho = P \begin{pmatrix} 1 \\ -\iota_\rho \end{pmatrix} \bar P_\lambda$$

Write $l_\rho = l_0 + \Delta l$, $S_\rho = S_0 + \Delta S$. Then

$$\Delta l = P \begin{pmatrix} 0 \\ -\iota_\rho \end{pmatrix} \bar P_\lambda = \begin{pmatrix} P_\lambda & 0 \\ b'G_\lambda - G_\lambda b & P_\lambda \end{pmatrix} \begin{pmatrix} 0 \\ -\iota_\rho \end{pmatrix} \bar P_\lambda = \begin{pmatrix} 0 \\ -P_\lambda \iota_\rho \bar P_\lambda \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \tilde N_\lambda \end{pmatrix} (-\tilde N_\lambda^{-1} P_\lambda \iota_\rho \bar P_\lambda) = \iota\, (\underbrace{-\tilde N_\lambda^{-1} P_\lambda \iota_\rho \bar P_\lambda}_{\text{set } Q = \text{this.}})$$

Then

$$\begin{aligned} -iS_\rho &= -i(S_0 + \Delta S) \\ &= \tilde b l_0 - l_0 b + \tilde b \Delta l - \Delta l\, b \\ &= -iS_0 + \tilde b \iota Q - iQb \\ &= -iS_0 - i(bQ + Qb) \end{aligned}$$

$$\therefore \quad \boxed{\Delta S = bQ + Qb \quad \text{where } Q = -\tilde N_\lambda^{-1} P_\lambda \iota_\rho \bar P_\lambda}$$

$$S_\rho = S_0 + \Delta S$$

$$= \bar{N}_\lambda^{-1} P_\lambda c G_\lambda c \bar{P}_\lambda \blacksquare b(\bar{N}_\lambda^{-1} P_\lambda \iota_\rho \bar{P}_\lambda) \blacksquare (\bar{N}_\lambda^{-1} P_\lambda \iota_\rho \bar{P}_\lambda) b$$

Now $b \bar{N}_\lambda^{-1} P_\lambda = \bar{N}_\lambda^{-1} b' P_\lambda$ since

$$\bar{N}_\lambda (b \bar{N}_\lambda^{-1} P_\lambda) = b' \bar{N}_\lambda \bar{N}_\lambda^{-1} P_\lambda = b' P_\lambda .$$

Thus $\boxed{S_\rho = \bar{N}_\lambda^{-1} \{ P_\lambda c G_\lambda c \bar{P}_\lambda \blacksquare b'(P_\lambda \iota_\rho \bar{P}_\lambda) \blacksquare (P_\lambda \iota_\rho \bar{P}_\lambda) b \}}$

One thing I would like to understand better is why this descends to $\bar{C}_\lambda(\mathcal{A})$ in the case $\mathcal{A}$ unital and $\rho(e) = 1$.

Let's calculate $P_\lambda(-b'_\rho) G_\lambda b_\rho P_\lambda$ to see if it gives rise to S_ρ the way $P_\lambda(-b') G_\lambda b P_\lambda$ gives rise to S_0. In some sense this [illegible] probably won't work because $P_\lambda(-b'_\rho) G_\lambda b_\rho P_\lambda$ is quadratee in ρ, whereas S_ρ is linear in ρ.

$$P_\lambda(-b'_\rho) G_\lambda b_\rho P_\lambda = P_\lambda\left(-b'_{\blacksquare} + \iota_\rho(1-\lambda)\right) G_\lambda \left(b - (1-\lambda)\iota_\rho\right) P_\lambda$$

$$= P_\lambda \left\{ -b' G_\lambda b + \iota_\rho P_\lambda^\perp b + b' P_\lambda^\perp \iota_\rho + \iota_\rho (1-\lambda) \iota_\rho \right\} P_\lambda$$

$$= P_\lambda(-b') G_\lambda b P_\lambda + P_\lambda \{ \iota_\rho b + b' \iota_\rho - \iota_\rho (1-\lambda) \iota_\rho \} P_\lambda$$

$$\blacksquare - P_\lambda \blacksquare \iota_\rho P_\lambda b P_\lambda - P_\lambda b' P_\lambda \iota_\rho P_\lambda$$

Now $P_\lambda b' P_\lambda = b' P_\lambda$ (since [illegible] $b'(P_\lambda C) \subset P_\lambda C$)

and $P_\lambda b P_\lambda = P_\lambda b$ (since $b(P_\lambda^\perp C) \subset P_\lambda^\perp C \Rightarrow P_\lambda b P_\lambda^\perp = 0$)

Thus we find

$$\boxed{\begin{aligned} &P_\lambda(-b'_\rho)G_\lambda b_\rho P_\lambda = \\ &P_\lambda(-b)G_\lambda b P_\lambda - (P_\lambda \iota_\rho P_\lambda)b - b'(P_\lambda \iota_\rho P_\lambda) \\ &\quad + P_\lambda \iota_\omega P_\lambda \end{aligned}}$$

This gives another waying of viewing S_ρ:

$$S_\rho = \bar{N}_\lambda^{-1} P_\lambda(-b'_\rho)G_\lambda b_\rho \bar{P}_\lambda - \bar{N}_\lambda^{-1} P_\lambda \iota_\omega \bar{P}_\lambda$$

which also shows that it descends to $\bar{C}_\lambda(a)$ when $\rho(e)=1$. One might be able to go from this to a Connes type formula for S_ρ.

We forget

$$\tilde{B}_\rho = \begin{pmatrix} 1 & 0 \\ \iota_\rho & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ N_\lambda & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\iota_\rho & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ N_\lambda & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\iota_\rho & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ N_\lambda & 0 \end{pmatrix} = \tilde{B}.$$

February 16, 1994

The situation: Given $\rho \in \mathcal{A}^*$ we have an associated operator S_ρ on $C_\lambda(\mathcal{A})$, and we have character forms $\frac{\omega^n}{n!}\bar{N}_\lambda \in C_\lambda^{2n-1}(\mathcal{A})$. The problem is to make explicit the fact that these character forms are related by. One would like to prove

$$(*) \qquad \left(\frac{\omega^n}{n!}\bar{N}_\lambda\right) S_\rho \overset{?}{=} \left(\frac{\omega^{n+1}}{(n+1)!}\bar{N}_\lambda\right)$$

but this seems a priori false because our S_ρ is of degree 1 in ρ and ω has degree 2, which means we can't have $\left(\omega \bar{N}_\lambda\right) S_\rho^n = \left(\frac{\omega^{n+1}}{(n+1)!}\bar{N}_\lambda\right)$.

To find the good version of $(*)$ wes may need a different S operator than S_ρ. It seems worthwhile to consider the possibilities. I recall that any S operator arises from a splitting of the canonical exact sequence

$$\begin{tikzcd} 0 \arrow[r] & C_\lambda(\mathcal{A})[1] \arrow[r, "\iota"] & \rho\bar{\Omega}\tilde{\mathcal{A}} \arrow[r, "j"] & C_\lambda(\mathcal{A}) \arrow[r] & 0. \end{tikzcd}$$

Thus I want [redacted] to go over how such ~~splittings~~ are constructed, say starting from a ρ

First consider

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A}^{\otimes n} \arrow[r, "{\binom{0}{1}}"'] & \bar{\Omega}^n\tilde{\mathcal{A}} \arrow[l, dashed, bend right, "{(\iota_\rho \ \ 1)}"'] \arrow[r, "{(1 \ \ 0)}"'] & \mathcal{A}^{\otimes n+1} \arrow[l, dashed, bend right, "{\binom{1}{-\iota_\rho}}"'] \arrow[r] & 0 \end{tikzcd}$$

where the splitting indicated descends to $\bar{\Omega}\mathcal{A}$ when $\mathcal{A}$ is unital and $\rho(e) = 1$.

This behaves nicely wrt the usual proof of the S-relations for the character forms which goes as follows.

$$\begin{tikzcd}
& & \Omega^{2n+1}\tilde{\mathcal{A}} \arrow[dl, "\tilde{b}"'] \arrow[r, two heads] \arrow[dd, "(\rho\omega^n\ \omega^n)\tilde{b} = (\omega^{n+1}(1+\lambda)\ 0)"'] & \mathcal{A}^{\otimes 2n+2} \arrow[dd, "\omega^{n+1}(1+\lambda)"] \\
\mathcal{A}^{\otimes 2n} \arrow[d, "\omega^n"] & \Omega^{2n}\tilde{\mathcal{A}} \arrow[l, "r=(\iota_\rho\ 1)"'] \arrow[dl, "(\rho\omega^n\ \omega^n)"] & & \\
\mathbb{C} \arrow[rr, phantom, "="] & & \mathbb{C} \arrow[r, phantom, "="] & \mathbb{C}
\end{tikzcd}$$

In words to find the S transform of $\omega^n\bar{N}_\lambda$ one extends this linear fnl on $C_\lambda(\mathcal{A})_{2n-1}$ to a linear functional f on $\Omega^{2n}\tilde{\mathcal{A}}$, such that $fbd=0$, then fb is a cyclic $2n+1$ cocycle representing the S transform. In the above diagram the extension is obtained by composing ω^n with the retraction $r=(\iota_\rho\ 1)$ to get $f=(\rho\omega^n\ \omega^n)$. As a check note that if we used the ρ-splitting of $\bar{\Omega}\tilde{\mathcal{A}}$, then $f=(0\ \omega^n)$ and

$$f\tilde{b} = (0\quad \omega^n)\begin{pmatrix} b_\rho & 1-\lambda \\ \iota_\omega(1+\lambda) & -b'_\rho \end{pmatrix} = (\omega^{n+1}(1+\lambda)\quad 0)$$

since $b'_\rho(\omega^n)=0$ (Bianchi) and $\omega^n\iota_\omega(1+\lambda) = \omega^{n+1}(1+\lambda)$.

So things seem to go rather smoothly before one tries to apply the harmonic projection.

I recall that Karoubi (essentially) has found other ways to contract $(\bar{\Omega}A, d)$ using a choice of ρ (see Jan 3, 94 p. 294). It seems worthwhile to explore the possible variations.

A contraction h for $(\bar{\Omega}A, d)$ determines a special contraction $hdh = h - h^2 d$ which in turn is equivalence to splittings of

$$(**) \qquad 0 \longrightarrow \bar{A}^{\otimes n} \longrightarrow \bar{\Omega}^n A \longrightarrow \bar{A}^{\otimes n+1} \longrightarrow 0$$

for all n (recall $\bar{A}^{\otimes 0} = 0$ by convention). We now want to go from a splitting of (**) to one for the harmonic sub exact sequence. Use the idea of averaging with respect to the cyclic group generated by $\tilde{\kappa}$, the finite order part of κ.

Consider then an exact sequence of G-modules

$$\begin{tikzcd} 0 \arrow[r] & W \arrow[r, "i"'] & V \arrow[l, bend right, "r"'] \arrow[r, "j"'] & V/W \arrow[l, bend right, "l"'] \arrow[r] & 0 \end{tikzcd}$$

A vector space splitting of this is equivalent to a retraction r for i, and also equivalent to a lifting l for j, and also equivalent to a pair (r, l) as above satisfying $ir + lj = 1$. (Note that $ir + lj = 1 \Longrightarrow$ $i = iri + \cancel{lji} \Longrightarrow ri = 1$ as i is injective. Similary $j = \cancel{jir} + jlj \Longrightarrow jl = 1$ as j is surjective.)

Given such a splitting we can average it to obtain an invariant splitting

$$1 = \frac{1}{|G|} \sum_g g(ir + lj)g^{-1}$$

$$= i\left(\frac{1}{|G|} \sum_g g r g^{-1}\right) + \left(\frac{1}{|G|} \sum_g g l g^{-1}\right) j$$

One gets the same splitting by averaging the retraction r or the lifting l, i.e. lifting the identity in either

$$0 \longrightarrow \mathrm{Hom}(V/W, W) \longrightarrow \mathrm{Hom}(V/W, V) \longrightarrow \mathrm{Hom}(V/W, V/W) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Hom}(V/W, W) \longrightarrow \mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}(W, W) \longrightarrow 0$$

Consider the spectral [redacted] decompositions

$$V = \bigoplus V_\alpha$$

$$W = \bigoplus W_\alpha \qquad W_\alpha = V_\alpha \cap W$$

where α runs over the inequivalent irred. repns.

$$\begin{tikzcd}
\mathrm{Hom}(V, W) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(W, W) \arrow[d, equal] \\
\bigoplus_{\alpha,\beta} \mathrm{Hom}(V_\alpha, W_\beta) \arrow[r] & \bigoplus_{\alpha,\beta} \mathrm{Hom}(W_\alpha, W_\beta)
\end{tikzcd}$$

There are no invariant elements in $\mathrm{Hom}(V_\alpha, W_\beta)$ for $\alpha \neq \beta$, and when G is abelian G acts by the character α on [redacted] V_α, W_α, hence G acts trivially on $\mathrm{Hom}(V_\alpha, W_\alpha)$. Thus the averaging process is

$$r \longmapsto \frac{1}{|G|} \sum g r g^{-1} = \sum_\alpha P_\alpha r P_\alpha$$

where $P_\alpha r P_\alpha$ really means the map $V_\alpha \to W_\alpha$ (in fact retraction) given by the composition

$$V_\alpha \subset V \xrightarrow{r} W \xrightarrow{P_\alpha} W_\alpha$$

February 18, 1994

Recall the contraction for $(\bar{\Omega}A, d)$ arising from Karoubi's paper, namely

$$(b' + \iota_\rho \lambda)(a_0, \dots, a_n) = \sum_{i=0}^{n-1} (-1)^{i-1} (a_0, \dots, a_i a_{i+1}, \dots, a_n)$$
$$+ (-1)^n (a_0, \dots, a_{n-1}) \rho(a_n)$$

or in differential form notation

$$(b' + \iota_\rho \lambda)\, a_0 da_1 \dots da_n = (-1)^{n-1} a_0 da_1 \dots da_{n-1} (a_n - \rho(a_n))$$

$$(b' + \iota_\rho \lambda)(\omega da) = (-1)^{|\omega|} \omega (a - \rho(a))$$

Check: $d(b' + \iota_\rho \lambda)(\omega da) = (-1)^{|\omega|} d(\omega(a - \rho(a)))$

$$= (-1)^{|\omega|} \left(d\omega (a - \rho(a)) + (-1)^{|\omega|} \omega da \right)$$

$$(b' + \iota_\rho \lambda) d(\omega da) = (-1)^{|\omega| + 1} d\omega (a - \rho(a))$$

$\therefore \ [d, b' + \iota_\rho \lambda] = 1$ in degrees ≥ 1.

In degree zero $[d, b' + \iota_\rho \lambda] \bar{a} = (b' + \iota_\rho \lambda) da = \overline{a - \rho(a)} = \bar{a}$

Thus $\boxed{[d, b' + \iota_\rho \lambda] = 1}$

Next

$$\omega da_1 da_2 \xmapsto{b' + \iota_\rho \lambda} (-1)^{|\omega|+1} \omega da_1 (a_2 - \rho(a_2))$$

[redacted] $= (-1)^{|\omega|+1} \left(\omega d(a_1 a_2) - \omega a_1 da_2 - \omega da_1 \rho(a_2) \right)$

$$\xmapsto{b' + \iota_\rho \lambda} \left(-\omega(a_1 a_2 - \rho(a_1 a_2)) + \omega a_1 (a_2 - \rho(a_2)) + \omega (a_1 - \rho(a_1)) \rho(a_2) \right)$$

$$= \omega \left(\rho(a_1 a_2) - \rho(a_1) \rho(a_2) \right).$$ Thus

$$\boxed{(b' + \iota_\rho \lambda)^2 = \iota_\omega \lambda^2}$$

Similarly $(b' \iota_\rho)^2 = -\iota_\omega$ which is consistent with our earlier formula

$$b_\rho'^{\,2} = (b' - \iota_\rho + \iota_\rho \lambda)^2 = -\iota_\omega(1-\lambda^2)$$

Now given any contraction h for $(\bar{\Omega} A, d)$ we can average it with respect to $\tilde{K}$. We have seen that because $\tilde{K}$ is locally cyclic this amounts to replacing h by its diagonal part relative to the eigenspace decomposition, i.e. $\sum P_\rho h P_\rho$. In particular we get the contraction PhP for $(P\bar{\Omega} A, d)$.

Let's calculate in the case $A = \tilde{a}$ using the standard isomorphism

$$\bar{\Omega}^n \tilde{a} = a^{\otimes n+1} \oplus a^{\otimes n}$$

We have $\tilde{a} \otimes a^{\otimes n} \longleftarrow \tilde{a} \otimes a^{\otimes n+1}$

$$(-1)^n (a_0 + c1)\, da_1 \cdots da_n (a_{n+1} - \rho a_{n+1}) \longleftarrow\!\shortmid (a_0 + c1)\, da_1 \cdots da_{n+1}$$

Thus $a_0\, da_1 \cdots da_{n+1} \longmapsto (-1)^n a_0\, da_1 \cdots da_n (a_n - \rho a_{n+1})$

$$\updownarrow$$

$$(b' + \iota_\rho \lambda)(a_0, \ldots, a_{n+1})$$

and $da_1 \cdots da_{n+1} \longmapsto (-1)^n da_1 \cdots da_n (a_{n+1} - \rho a_{n+1})$

$$= (-1)^n \Big(\sum_{i=1}^{n} (-1)^{n-i} da_1 \cdots d(a_i a_{i+1}) \cdots da_{n+1} - da_1 \cdots da_n \rho(a_{n+1}) + (-1)^n a_1\, da_2 \cdots da_{n+1} \Big)$$

$$\longleftrightarrow \begin{pmatrix} 1 \\ -(b' + \iota_\rho \lambda) \end{pmatrix} (a_1, \ldots, a_{n+1})$$

Thus we have

$$\begin{tikzcd} \tilde{a} \otimes a^{\otimes n} \arrow[r, shift right, "\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}"'] & \tilde{a} \otimes a^{\otimes n} \arrow[l, shift right, "\begin{pmatrix} b'+c_\rho\lambda & 1 \\ & -(b'+c_\rho\lambda) \end{pmatrix}"'] \end{tikzcd}$$

and the contraction property is clear.

Next let us consider the corresponding S operator. We take the lifting of $a^{\otimes n+1}$ into $\tilde{a} \otimes a^{\otimes n}$ that $b'+c_\rho\lambda$ gives, i.e.

$$\begin{pmatrix} 1 \\ -(b'+c_\rho\lambda) \end{pmatrix}$$

and then do the averaging which gives

$$P\begin{pmatrix} 1 \\ -(b'+c_\rho\lambda) \end{pmatrix}\bar{P}_\lambda = \underbrace{P\begin{pmatrix} 1 \\ -c_\rho \end{pmatrix}\bar{P}_\lambda}_{l_\rho} + \underbrace{P\begin{pmatrix} 0 \\ -b' \end{pmatrix}\bar{P}_\lambda}_{\begin{pmatrix} P_\lambda & 0 \\ b'G_\lambda - G_\lambda b & P_\lambda \end{pmatrix}\begin{pmatrix} 0 \\ -b'\bar{P}_\lambda \end{pmatrix}}$$

[redacted]

Thus we change the lifting l_ρ (which descends in the case a is unital and $\rho(e)=1$) [redacted] by $\begin{pmatrix} 0 \\ -P_\lambda b'\bar{P}_\lambda \end{pmatrix}$.

Notice that in the calculation on p375-6, also 389 nothing special about c_ρ is used. This calculation shows that if we vary the lifting $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ -h \end{pmatrix}$ then the operator S_0 changes by

$$\Delta S = bQ + Qb \qquad Q = -\bar{N}_\lambda^{-1} P_\lambda h \bar{P}_\lambda$$

Alternatively $-\bar{N}_\lambda \Delta S \pi = b'(P_\lambda h P_\lambda) + (P_\lambda h P_\lambda) b$.

Now take $h = b'$. Then

$$-\bar{N}_\lambda \Delta S_\pi = b' \underbrace{P_\lambda b' P_\lambda}_{\substack{b'P_\lambda \\ \text{as } \mathrm{Im}\, P_\lambda \text{ is} \\ \text{stable under } b'}} + P_\lambda \underbrace{b' P_\lambda}_{\substack{\text{proportional degreewise} \\ \text{to } b'N_\lambda b = b'b'N_\lambda = 0}} b$$

Thus we conclude that either lifting $\begin{pmatrix}1\\ -\iota_\rho\end{pmatrix}$ or $\begin{pmatrix}1\\ -b' - \iota_\rho\lambda\end{pmatrix}$ yields the same S operator.

This calculation supports the idea that there is a canonical S operator on $C_\lambda(\mathcal{A})$ associated to $\rho \in \mathcal{A}^*$ (or on $\bar{C}_\lambda(A)$ associated to $\rho \in A^*$ such that $\rho(1) = 1$). We still have the problem of proving the S relations explicitly for the Chern character forms $\frac{\omega^n}{n!}\bar{N}_\lambda$. This is theoretically possible.

February 19, 1994

Here's a slightly different problem concerning the explicit S operator. Suppose we take $\rho \in \mathcal{A}^*$ and the canonical S-operator on $C_\lambda(\mathcal{A})$ and try to understand the relations $\left(\frac{\omega^n}{n!}\bar{N}_\lambda\right)S \sim \frac{\omega^{n+1}}{(n+1)!}\bar{N}_\lambda$.* The point is that we already why this is true, and we ~~know~~ that S is given by diagram chasing, we can compare.

* Actually there is a sign missing which I shall ignore.

The usual proof proceeds by the following chase in the dual cyclic double complex

$$\begin{tikzcd}
C_\lambda & C \arrow[l, "\pi"'] \arrow[dd, "b"] & & \\
\frac{\omega^{n+1}}{(n+1)!}\bar{N}_\lambda & & & \\
& C & C \arrow[l, "1-\lambda"'] \arrow[dd, "-b'"] & \\
& \rho\frac{\omega^n}{n!} & & \\
& & C \arrow[r, leftrightarrow, "\bar{N}_\lambda"] & C_\lambda \\
& & \frac{\omega^n}{n!} & \frac{\omega^n}{n!}\bar{N}_\lambda
\end{tikzcd}$$

$\omega^n b' = \rho\omega^n(1-\lambda)$

$(\rho\omega^n)b = \omega^{n+1}(1+\lambda)$

$= \frac{\omega^{n+1}}{n+1}N_\lambda$

Now recall $S = \bar{N}_\lambda^{-1}P_\lambda(-b')G_\lambda b\bar{P}_\lambda$.

In general we can suppose given a cyclic cocycle $f_{n-1} \in C_\lambda^{n-1}(\mathcal{A})$, i.e. $f_{n-1} \in (\mathcal{A}^{\otimes n})^*$, $f_{n-1}b = f_{n-1}(1-\lambda) = 0$ and (ψ_n, φ_{n-1}) such that $\varphi_{n-1}N_\lambda = f_{n-1}$ and $\psi_n(1-\lambda) = \varphi_{n-1}b'$ whence $f_{n+1} = \psi_n b$ is a cyclic $n+1$ cocycle such that $f_{n-1}S \sim f_{n+1}$, i.e. the difference is a cyclic coboundary.

For simplicity suppose $(\psi_n \ \varphi_{n-1})$ is κ-invariant; since $(\psi_n \ \varphi_{n-1})\bar{b}d = 0$ this is equivalent to $\varphi_{n-1}(1-\lambda) = 0$. Then let's calculate $f_{n-1}S = f_{n-1}\bar{N}_\lambda^{-1}P_\lambda(-b')G_\lambda b\bar{P}_\lambda$. We then have $\varphi_{n-1} = f_{n-1}N_\lambda^{-1} = \frac{1}{n}f_{n-1}$. Diagram chasing:

$$\begin{tikzcd}
C_{\lambda,n+1} & C_{n+1} \arrow[l, "\pi"'] \arrow[d, "b"] & & \\
\frac{1}{n}f_{n-1}b'G_\lambda b & C_n & C_n \arrow[l, "1-\lambda"'] \arrow[d, "-b'"] & \\
 & \frac{1}{n}f_{n-1}b'G_\lambda & C_{n-1} & C_{\lambda,n-1} \arrow[l, "\bar{N}_\lambda"'] \\
 & & \varphi_{n-1} = \frac{1}{n}f_{n-1} & f_{n-1}
\end{tikzcd}$$

gives $-f_{n-1}S = \frac{1}{n}f_{n-1}b'G_\lambda b$.

We see that the real point is the difference between ψ_n and $\varphi_{n-1}b'G_\lambda$. Since $\psi_n(1-\lambda) = \varphi_{n-1}b'$ we know $\psi_n - \psi_n P_\lambda = \varphi_{n-1}b'G_\lambda$. Thus $\psi_n - \varphi_{n-1}b'G_\lambda = \psi_n P_\lambda$. This is a cyclic cochain whose coboundary is the difference $f_{n+1} - f_{n-1}S$.

In the example of interest where $(\psi_{2n} \ \varphi_{2n-1}) = (\rho\omega^n \ \omega^n)$, or better

$(\rho\omega^n \quad \omega^n)\left(\frac{1+K}{2}\right)$ what is $\psi_{2n} P_\lambda$?

Now $(\rho\omega)(a_0,a_1,a_2) = \rho(a_0)\big(\rho(a_1a_2) - \rho(a_1)\rho(a_2)\big)$ does not sum cyclically to zero it seems

$$(\rho\omega^n \quad \omega^n)\begin{pmatrix}\frac{1+\lambda}{2} & \\ -\frac{c}{2} & \frac{1+\lambda}{2}\end{pmatrix} = \left((\rho\omega^n)\left(\frac{1+\lambda}{2}\right) - \omega^n\frac{c}{2} \qquad \omega^n\frac{1+\lambda}{2}\right)$$

$$\left((\rho\omega)\left(\frac{1+\lambda}{2}\right) - \omega\frac{c}{2}\right)(a_0,a_1,a_2)$$

$$= \frac{1}{2}\rho(a_0)\omega(a_1,a_2) + \frac{1}{2}\rho(a_2)\omega(a_0,a_1)$$
$$-\frac{1}{2}\omega(a_2a_0,a_1)$$

$$= \frac{1}{2}\rho(a_0)\omega(a_1,a_2) - \frac{1}{2}\omega(a_2,a_0a_1) + \frac{1}{2}\omega(a_2,a_0)\rho(a_1)$$

There seems to be no reason for this to sum cyclically to zero.

Conclusion: This calculation suggests that the explicit S operator is not so useful.

February 20, 1994

Can we view HPT as a kind of Laplacian method? Recall the special deformation retraction data:

$$[d,h] = 1-e \qquad e^2=e \qquad h^2=eh=he=0.$$
$$[d,\theta] = \theta^2.$$

The basic calculation is

$$[d-\theta, \underbrace{h\frac{1}{1-\theta h}}_{\tilde{h}}] = 1 - \underbrace{\frac{1}{1-h\theta}\, e\, \frac{1}{1-\theta h}}_{\tilde{e}}$$

* see next page

Look at this as follows. One has

$$[d-\theta, h] = \underbrace{1-e}_{\text{idempotent}} - \underbrace{[h,\theta]}_{\text{(top) nilpotent}}$$

But from lifting idempotents we know there is a definite power series $f(x)$ such that when x^2-x is nilpotent, then $f(x)^2=f(x)$ and $f(x) \equiv x \mod (x^2-x)$

Thus applying f to $x=1-e-[h,\theta]$ gives an idempotent commuting with any operator commuting with x, e.g. $d-\theta$, h. Now observe that

$$1-e-[h,\theta] = (1-\theta h)(1-e)(1-h\theta)$$
$$\tilde{e}\,(1-e-[h,\theta]) = \frac{1}{1-h\theta}\, e\, \frac{1}{1-\theta h}\cdot(1-\theta h)(1-e)(1-h\theta) = 0$$
$$(1-e-[h,\theta])\,\tilde{e} = (1-\theta h)(1-e)(1-h\theta)\,\frac{1}{1-h\theta}\, e\, \frac{1}{1-\theta h} = 0$$

so $1-\tilde{e}$ commutes with $x=1-e-[h,\theta]$ and also $1-\tilde{e}$ is an idempotent congruent to x mod nilpotents. This suggests that $f(x) = 1-\tilde{e}$.

(* You have forgotten how much simpler it is to work with the conditions: $hdh = h$, $h^2 = 0$. These imply hd, dh are annihilating idempotents, hence $e = 1 - dh - hd$ is an idempotent.

Also $h(d-\theta)h = h - h\theta h = h(1-\theta h)$

so $h(d-\theta)h\frac{1}{1-\theta h} = h$, whence $\boxed{\tilde{h}(d-\theta)\tilde{h} = \tilde{h}}$

where $\tilde{h} = h\frac{1}{1-\theta h} = \frac{1}{1-h\theta}h$. The hard part is still $[d-\theta, \tilde{h}] = 1 - \tilde{e}$, and this is proved by

$$(1-h\theta)\left[(d-\theta)\tilde{h} + \tilde{h}(d-\theta)\right](1-\theta h)$$

$$= (1-h\theta)(d-\theta)h + h(d-\theta)(1-\theta h)$$

$$\begin{aligned} = \; & dh - \theta h + hd - h\theta \\ & -\underline{h}\theta dh + \underline{h}\theta^2 h - \underline{h}d\theta h + h\theta^2 h \end{aligned}$$

$$= 1 - e - \theta h - h\theta + h\theta^2 h = (1-h\theta)(1-\theta h) - e.)$$

The hope one gets from the above discussion is that anything one does with HPT might also be done by Laplacian methods. By these methods I mean introducing a homotopy operator h, then using spectral theory on $[d,h]$ (i.e. suitable functions of $[d,h]$) to obtain projection operators homotopic to zero. ▮

February 21, 1994

Return to problem of constructing a lifting $\bar{C}_\lambda(A) \longrightarrow C_\lambda(A)/C_\lambda(\mathbb{C})$. I propose to find a suitable homotopy operator on $C_\lambda(A)$ which yields a deformation retraction onto the fibre of the map $\bar{C}_\lambda(A) \longrightarrow C_\lambda(\mathbb{C})[1]$ given by the Chern character forms associated to a chosen ρ.

Recall we have the increasing filtration $F_p C_\lambda(A)$ associated to the filtration $F_{-1}A = 0$, $F_0 A = \mathbb{C}$, $F_1 A = A$ on A, and that

$$gr\, C_\lambda(A) = C_\lambda(\mathbb{C} \oplus \bar{A})$$

where $\mathbb{C} \oplus \bar{A} = (\bar{A})^\sim$, $\bar{A}$ having zero multiplication. Since ρ is chosen we have a vector space isomorphism $C_\lambda(A) = C_\lambda(\mathbb{C} \oplus \bar{A})$, and so it should be enough to find the homotopy in the case $A = \mathbb{C} \oplus \bar{A}$, $\bar{A}^2 = 0$.

Consider the dual picture where $\Sigma C_\lambda(A)^* = \bar{T}(A^*)_\natural$ and we have

$$T(A^*) = T(\bar{A}^*) * \mathbb{C}[\rho]$$

$$d\rho + \rho^2 = \omega \qquad (= -f^0_{jk}\,\theta^j\theta^k)$$

$$d\theta^i + \rho\theta^i + \theta^i\rho = -f^i_{jk}\,\theta^j\theta^k$$

We know that if we filter $T(A^*)$ adically using the ideal generated by $\bar{A}^*$ that we get $T(\bar{A}^*) * \mathbb{C}[\rho]$ with diffl: $d\rho + \rho^2 = 0$, $d\theta^i + [\rho, \theta^i] = 0$. This DG algebra should be the same as $T(A^*)$ when $A = \mathbb{C} \oplus \bar{A}$, $\bar{A}^2 = 0$.

Recall also the observation that this diffl: $d\rho + \rho^2 = 0$, $d\theta^i + [\rho, \theta^i] = 0$ on $T(\bar{A}^*) * \mathbb{C}[\rho]$ is the Alexander-Spanier diffl on the standard bimodule resolution $R * \mathbb{C}[h]$ in the case of the algebra $R = T(\bar{A}^*)$; here $h = -\rho$, (and there is an adjustment to make ~~because~~ R is graded).

This leads to the following: Find the homology of $(A * \mathbb{C}[h])_\natural$ with respect to the differential induced by the A-S diffl $da = [h, a]$, $dh = h^2$. Now

$$A * \mathbb{C}[h] = A \oplus A \otimes A \oplus A \otimes A \otimes A \oplus \cdots$$

$$d(a_0, \ldots, a_n) = \sum_{i=0}^{n+1} (-1)^i (\cdots, a_{i-1}, 1, a_i, \cdots)$$

$$(A * \mathbb{C}[h]) \otimes A = A_\natural \oplus A \oplus A \otimes A \oplus \cdots$$

$$(A * \mathbb{C}[h])_\natural = A_\natural \oplus A \oplus A_\lambda^{\otimes 2} \oplus A_\lambda^{\otimes 3} \oplus \cdots$$

This last also is a special case of

$$T_A(E)_\natural = A_\natural \oplus E \otimes_A \oplus [E \otimes_A]_\sigma^{(2)} \oplus \cdots$$

I now want to compute the diffl d on $(A * \mathbb{C}[h])_\natural$ obtained from the A-S diff d.

$$\begin{tikzcd} A^{\otimes n+1} \arrow[r, "d"] \arrow[d] & A^{\otimes n+2} \arrow[d, "\natural"] \\ A_\lambda^{\otimes n} \arrow[r, dashed] & A_\lambda^{\otimes n+1} \end{tikzcd}$$

Start with $(a_0, \ldots, a_n) = a_0 h \ldots h a_n \in A * \mathbb{C}[h]$.

$$d(a_0, \ldots, a_n) = (1, a_0, \ldots, a_n) + \sum_{i=1}^{n} (-1)^i (a_0, \ldots, a_{i-1}, 1, a_i, \ldots, a_n) + (-1)^{n+1} (a_0, \ldots, a_n, 1)$$

$$\begin{aligned}\natural\, d(a_0,\dots,a_n) &= (a_n,\cancel{a_0,\dots,a_{n-1})} + (-1)^{n+1}\cancel{(a_0,\dots,a_n)} \\ &\quad + \sum_{i=1}^{n}(-1)^i\,(a_na_0,\dots,a_{i-1},1,a_i,\dots,a_{n-1}) \\ &= \sum_{i=1}^{n}(-1)^i(-1)^{in}\quad(1,a_i,\dots,a_{n-1},a_na_0,a_1,\dots,a_{i-1})\end{aligned}$$

Indeed this depends on $\natural(a_0,\dots,a_n) = (a_na_0,a_1,\dots,a_{n-1})$

So we find that $d : A_\lambda^{\otimes n} \longrightarrow A_\lambda^{\otimes n+1}$ is given by

$$\begin{aligned}(a_0,\dots,a_{n-1}) \longrightarrow &\sum_{i=1}^{n}(-1)^{i(n-1)}(1,a_i,\dots,a_{n-1},a_0,\dots,a_{i-1}) \\ &= \sum_{i=0}^{n-1}(-1)^{i(n-1)}(1,a_{i+1},\dots,a_{n-1},a_0,\dots,a_i)\end{aligned}$$

We might as well denote this by B.

So we have the following picture of $(A*\mathbb{C}[h])_\natural$:

$$\begin{tikzcd} A_\natural \arrow[r, "0"] & A \arrow[r, "B"] & A_\lambda^{\otimes 2} \arrow[r, "B"] & A_\lambda^{\otimes 3} \arrow[r, "B"] & \cdots \end{tikzcd}$$

$$a \longmapsto (1,a)$$

$$(a_1,a_2) \longmapsto (1,a_1,a_2) - (1,a_2,a_1)$$

Observe that the positive degree part of this complex depends only on A as a vector space with element $1 \neq 0$. Also we have cycles $1 \in A$, $(1,1,1) \in A_\lambda^{\otimes 3}$, etc. The conjecture is that the homology of $(C_\lambda(A), B)$ is $C_\lambda(\mathbb{C})$.

February 22, 1994

Consider $A * \mathbb{C}[h]$ with grading given by $|a|=0$, $|h|=1$. There are various (super) derivations on $A * \mathbb{C}[h] = A \oplus AhA \oplus AhAhA \oplus \cdots$. One has b' of degree -1 defd by $b'(a)=0$, $b'(h)=1$. Also d, δ of degree $+1$ defined by

$$\delta(a) = 0 \qquad \delta(h) = h^2$$

$$d(a) = [h,a] \qquad d(h) = h^2$$

One has $b'^2 = \delta^2 = d^2 = 0$.

Also
$$(d+\delta)(a) = [h,a]$$
$$(d+\delta)(h) = 2h^2 = [h,h]$$

Thus $\boxed{d+\delta = ad(h)}$ and so

$$[d,\delta] = (d+\delta)^2 = ad(h)^2 = ad(h^2).$$

Further $[b',d] = [b',\delta] = 0$.

Recall that d is the Alexander-Spanier differential

$$d(a_0,\ldots,a_n) = \sum_{i=0}^{n+1} (-1)^i (\ldots, a_{i-1}, 1, a_i, \ldots)$$

while $\delta = -d + ad(h)$ is the negative of d with outside terms removed:

$$\delta(a_0,\ldots,a_n) = \sum_{i=1}^{n} (-1)^{i-1} (\ldots, a_{i-1}, 1, a_i, \ldots)$$

Now δ, b' are derivations relative to $A \subset A * \mathbb{C}[h]$ hence they induce differentials on the relative X supercomplex

$$X_A(A * \mathbb{C}[h]) \simeq X_A(T_A(E)) \qquad E = AhA \simeq A \otimes A$$

Recall that this consists of

$$T_A(E)\otimes_A = A_\natural \oplus [E\otimes_A] \oplus [E\otimes_A]^{(2)} \oplus \cdots$$

$$\Omega^1_A(T_A(E))_\natural = \left(T_A(E)\otimes_A E\otimes_A T_A(E)\right)_\natural = T_A(E)\otimes_A E\otimes_A$$

$$= [E\otimes_A] \oplus [E\otimes_A]^{(2)} \oplus \cdots$$

In general we have

$$\bar{X}_A(T_A(E)): \quad \bigoplus_{n\geq 1}[E\otimes_A]^{(n)} \underset{\partial = N_\sigma}{\overset{\beta = 1-\sigma}{\leftrightarrows}} \bigoplus_{n\geq 1}[E\otimes_A]^{(n)}$$

For $E = AhA$ with $|h| = 1$ we have

$$[E\otimes_A]^{(n)} \simeq Ah\cdots Ah \simeq A^{\otimes n}$$

where σ acts as λ on $A^{\otimes n}$ because h is of odd degrees.

To be more precise we will use the isomorphism

$$(A*\mathbb{C}[h])\otimes_A \xleftarrow{\sim} A_\natural \oplus Ah \oplus AhAh \oplus \cdots$$

i.e. identify the image of $a_0h\cdots a_nh$ in $(A*C[h])\otimes_A$ with the tensor $(a_0,\dots,a_n)\in A^{\otimes n+1}$. Then

$$\delta(a_0h\cdots a_nh) = \sum_{i=0}^{n}(-1)^i a_0\cdots a_i h^2 a_{i+1}\cdots$$

i.e. $$-\delta(a_0,\dots,a_n) = \sum_{i=1}^{n+1}(-1)^i(a_0,\dots,a_{i-1},1,a_i,\dots,a_n)$$

on $(A*\mathbb{C}[h])\otimes_A$.

We also use the isomorphism

$$\Omega^1_A(A*\mathbb{C}[h])_\natural \xleftarrow{\sim} A\hat{h} \oplus AhA\hat{h} \oplus AhAhA\hat{h} \oplus \cdots$$

where $\hat{h} = \partial h$, to identify the image of $a_0h\cdots ha_n\hat{h}$ in $\Omega^1_A(A*\mathbb{C}[h])_\natural$ with the tensor $(a_0,\dots,a_n)\in A^{\otimes n+1}$. Then

$$\delta(a_0 h a_1 \cdots h a_n \partial h)$$

$$= \sum_{i=1}^{n} (-1)^{i-1} a_0 \cdots a_{i-1} h^2 a_i \cdots a_n \partial h$$

$$+ (-1)^n a_0 h a_1 \cdots h a_n \underbrace{\delta(\partial h)}_{= \partial(\delta h) = \partial(h^2) = \partial h\, h + h \partial h}$$

$$(-1)^n a_0 h a_1 \cdots h a_n h \partial h + (-1)^n \underbrace{a_0 h a_1 \cdots h a_n \partial h\, h}_{(-1)^{n+1}}$$

Thus $$-\delta(a_0 h a_1 \cdots h a_n \partial h) = h a_0 \cdots h a_n \partial h$$

$$+ \sum_{i=1}^{n} (-1)^i a_0 \cdots h a_{i-1} h^2 a_i \cdots a_n \partial h$$

$$+ (-1)^{n+1} a_0 h a_1 \cdots h a_n h \partial h$$

i.e. $$-\delta(a_0, \ldots, a_n) = (1, a_0, \ldots, a_n)$$

$$+ \sum_{i=1}^{n} (-1)^i (a_0, \ldots, a_{i-1}, 1, a_i, \ldots, a_n)$$

$$+ (-1)^{n+1} (a_0, \ldots, a_n, 1)$$

Thus $-\delta$ on $\Omega^1_A(A * \mathbb{C}[h])_\natural$ is the Alexander-Spanier differential d, while $-\delta$ on $(A * \mathbb{C}[h])_\natural \otimes_A$ is $d - s$; more precisely, "is" means ~~[illegible]~~ "becomes" under the identification

$$(A * \mathbb{C}[h]/A)_\natural \otimes_A \xleftarrow{\sim} C(A)$$

$$\Omega^1_A(A * \mathbb{C}[h])_\natural \xleftarrow{\sim} C(A)$$

we have given. Next

$$\beta(a_0 h a_1 \cdots h a_n \hat{h}) = [a_0 h a_1 \cdots h a_n, h]$$

$$= a_0 h \cdots a_n h - (-1)^n \underbrace{h a_0 \cdots h a_n}$$

shows $\beta = 1-\lambda$ and

$$\partial(a_0h\cdots a_nh) = \sum_{i=0}^{n} a_0h\cdots a_i\hat{h}\,\underline{\cdots a_nh}$$

$$= \sum_{i=0}^{n}(-1)^{(n-i)n} a_{i+1}h\cdots a_nh\,a_0h\cdots a_{i-1}h\,a_i\hat{h}$$

$$= \sum_{j=0}^{n}(-1)^{jn} a_jh\cdots a_nh\,a_0h\cdots a_{j-1}\hat{h}$$

shows $\partial = N_\lambda$.

Lets check this further, by the analogue of Tsygan's proof of the identities $b(1-\lambda)=(1-\lambda)b'$ and $N_\lambda b = b'N_\lambda$. One has

$$(\lambda^i s\lambda^{-i})(a_0,\dots,a_n) = (-1)^{in}\lambda^i(1,a_i,\dots,a_n,a_0,\dots,a_{i-1})$$

$$= (-1)^i(a_0,\dots,a_{i-1},1,a_i,\dots,a_n)$$

whence $d = \sum_{i=0}^{n+1}\lambda^i s\lambda^{-i}$ on C_n

$d' = \sum_{i=1}^{n+1}\lambda^i s\lambda^{-i}$ where $d' = d-s$.

Then $\lambda d = \sum_{0}^{n+1}\lambda^{i+1}s\lambda^{-i} = \sum_{0}^{n}\lambda^{i+1}s\lambda^{-i-1}\lambda + \underbrace{\lambda^{n+2}s\lambda^{-n-1}}_{= s \text{ on } C_n}$

so $\lambda d = d'\lambda + s$; and $d = d'+s$

so $\boxed{(1-\lambda)d = d'(1-\lambda)}$ similarly

$$dN_\lambda = \sum_{i=0}^{n+1}\lambda^i s N_\lambda = N_\lambda s N_\lambda$$

$$N_\lambda d' = \sum_{i=1}^{n+1} N_\lambda s\lambda^{-i} = N_\lambda s N_\lambda$$

so $\boxed{dN_\lambda = N_\lambda d'}$

Remark that if instead of $-\delta$ on $A*\mathbb{C}[h]$ one takes the derivation b', then the induced differential on $(A*\mathbb{C}[h]/A)\otimes_A$ becomes b on $C(A)$, and the induced differential on $\Omega^1_A(A*\mathbb{C}[h])_\natural$ becomes b' on $C(A)$:

$$b'(a_0h\cdots a_nh) = \sum_{i=0}^{n-1}(-1)^i a_0h\cdots a_ia_{i+1}h\cdots a_nh$$
$$+(-1)^n \underbrace{{}^{\uparrow}a_0ha_1\cdots a_{n-1}ha_n}$$

$$b'(a_0h\cdots a_n\hat{h}) = \sum_{i=0}^{n-1}(-1)^i a_0h\cdots a_ia_{i+1}h\cdots a_n\hat{h}$$
$$+(-1)^n a_0ha_1\cdots ha_n\cdot\underbrace{b'(\hat{h})}_{0}$$

since $b'(h)=1$ and $b'(\hat{h}) = b'(\partial h) = \partial(b'(h)) = \partial(1) = 0$.

The reason we are interested in the effect of $-\delta$ on $(A*\mathbb{C}[h]/A)\otimes_A$ and $\Omega^1_A(A*\mathbb{C}[h])_\natural$ is in order to calculate the d homology of $(A*\mathbb{C}[h])_\natural = \{A_\natural \longrightarrow A \longrightarrow A^{\otimes 2}_\lambda \longrightarrow \cdots\}$. Note that $d+\delta = \mathrm{ad}(h)$ is trivial on $(A*\mathbb{C}[h])_\natural$, so $d=-\delta$ on this commutator quotient space.

Setting $R = A*\mathbb{C}[h]$, $\bar{R} = R/A$ we have the exact sequence of complexes

$$0 \longrightarrow \bar{R}_\natural \xrightarrow{\text{'}\partial\text{'}} \Omega^1_A R_\natural \xrightarrow{\beta} \bar{R}\otimes_A \xrightarrow{\partial} \Omega^1_A R_\natural \xrightarrow{\beta}$$

which we can write as a kind of cyclic bicomplex.

$$\begin{tikzcd}
 & {} & {} & {} & {} & \\
0 \arrow[r] & A^{\otimes 3}_{\lambda} \arrow[r, hook] \arrow[u, dash] & A^{\otimes 3} \arrow[r, "1-\lambda"] \arrow[u, dash] & A^{\otimes 3} \arrow[r, "N_\lambda"] \arrow[u, dash] & A^{\otimes 3} \arrow[r] \arrow[u, dash] & {} \\
0 \arrow[r] & A^{\otimes 2}_{\lambda} \arrow[r, hook] \arrow[u] & A^{\otimes 2} \arrow[r, "1-\lambda"] \arrow[u, "d"'] & A^{\otimes 2} \arrow[r, "N_\lambda"] \arrow[u, "-d'"'] & A^{\otimes 2} \arrow[r] \arrow[u, "d"'] & {} \\
0 \arrow[r] & A \arrow[r, hook] \arrow[u] & A \arrow[r, "1-\lambda"] \arrow[u, "d"'] & A \arrow[r, "N_\lambda"] \arrow[u, "-d'"'] & A \arrow[r] \arrow[u, "d"'] & {}
\end{tikzcd}$$

where the rows are exact. Now the d complex has homology zero except for $\mathbb{C}$ in degree 0; Karoubi gives an explicit contraction starting from a choice of $\rho : A \to \mathbb{C}$, $\rho(1) = 1$. The d' complex is contractible with $-b'$ as canonical contraction:

$$[-b', d'] = 1$$

This is because $0 = [b', d] = [b', s + d'] = 1 + [b', d']$.

Thus $$H_i\Big(\underset{0}{A} \to \underset{1}{A^{\otimes 2}_{\lambda}} \to \underset{2}{A^{\otimes 3}_{\lambda}} \to \cdots\Big) = \begin{cases} \mathbb{C} & i = 0, 2, 4, \ldots \\ 0 & \cancel{i}\ \text{otherwise} \end{cases}$$

I should also recall that the differential looks like B, [redacted] i.e. cyclic symmetrization followed by s.

The natural question is whether there is a nice way to go from ρ to a deformation retraction of the above complex onto its homology.

February 24, 1994

We have found the following transitions:

1) Start with $(C_\lambda(A), b)$ in the degenerate case $A = \mathbb{C} \oplus \bar{A}$, $\bar{A}^2 = 0$. The problem is to construct explicitly a deformation retraction of $C_\lambda(\mathbb{C} \oplus \bar{A})$ onto $C_\lambda(\mathbb{C}) \oplus C_\lambda(\bar{A})$, which is a subcomplex with $b = 0$.

We recall that $(C_\lambda(A), b)$ is essentially the cocommutator space of the bar construction $(T(A), b')$.

2) The dual of the preceding is

$$(C_\lambda(A), b)^* = (T(A^*), d)_\natural = (T(\bar{A}^*) * \mathbb{C}[h], d)_\natural$$

where $d(\xi) = [h, \xi]$, $d(h) = h^2$ is the AS differential. This is a graded version of [redacted]

$$(A * \mathbb{C}[h], d)_\natural \qquad da = [h, a] \qquad dh = h^2$$

where we have replaced $T(\bar{A}^*)$ above by an arbitrary alg A.

3) Now $(A * \mathbb{C}[h], d)_\natural$ is essentially the same as $(T(A), d)^\natural$, where $T(A)$ is the tensor coalgebra on $A[1]$ and d is the coderivation of degree [redacted] $+1$ which [redacted] lifts $T(A) \longrightarrow \mathbb{C}$ [redacted] $\hookrightarrow A$.

~~4) The dual of the preceding is~~

~~$T(A^*) = T(\bar{A}^*) \otimes$~~

Note that $(T(A), d)$ depends only on A as vector space equipped with distinguished element 1.

[redacted]

4) The dual of the preceding is $T(A^*)_\natural$ where $T(A^*)$ is equipped with the degree -1 [redacted] derivation which extends evaluation at 1: $A^* \longrightarrow \mathbb{C}$. [redacted] Let us choose a splitting $A = \mathbb{C} \oplus \bar{A}$, then

$$T(A^*) = T(\bar{A}^*) * \mathbb{C}[h]$$ [redacted]

equipped with the differential b' given by $b'(\xi) = 0$ for $\xi \in T(\bar{A}^*)$ and $b'(h) = 1$. This is a graded version [redacted] (because $\bar{A}^*$ has degree 1) of the standard resolution $(R * \mathbb{C}[h], b')$. Thus $T(A^*)_\natural = (T(\bar{A}^*) * \mathbb{C}[h], b')_\natural$ is (essentially) the cyclic complex of the free algebra $T(\bar{A}^*)$.

These transitions suggest that the deformation retractions I am looking for in the cases of $C_\lambda(\mathbb{C} \oplus \bar{A})$, or $(C_\lambda(A), d)$, or $(C_\lambda(R), b)$ with R-free are really the **same**, **or** at least closely related.

On HPT again: Suppose we consider

$$hdh = h \qquad [d, h^2] = 0$$

Then dh, hd are orthogonal idempotents, so $e = 1 - dh - hd$ is an idempotent. $[h, e] = -[h,[h,d]]$ $= -[h^2, d] = 0$.

Then $h(d-\theta)h = h - h\theta h$ so

$$\tilde{h}(d-\theta)\tilde{h} = \tilde{h} \qquad \text{where} \quad \tilde{h} = h\frac{1}{1-\theta h} = \frac{1}{1-h\theta}h$$

Also $(1-h\theta)[d-\theta, \tilde{h}](1-\theta h) = (1-h\theta)(d-\theta)h + h(d-\theta)(1-\theta h)$

$$= \begin{matrix} dh - \theta h \\ -h\theta dh + h\theta^2 h \end{matrix} \quad \begin{matrix} + hd - h\theta \\ - hd\theta h + h\theta^2 h \end{matrix} \quad \begin{matrix} = (1 - h\theta - \theta h + h\theta^2 h) - e \\ = (1-h\theta)(1-\theta h) - e \end{matrix}$$

so $[d-\theta, \tilde{h}] = 1-\tilde{e}$, $\tilde{e} = \frac{1}{1-h\theta} e \frac{1}{1-\theta h}$

Unfortunately $\tilde{e}$ seems not to be idempotent.

Return to $(C_\lambda(A), b)^* = (T(\bar{A}^*) * \mathbb{C}[h], d)_\natural$.

We have seen that $d+\delta = ad(h)$, where δ is the degree 1 derivation which kills $T(\bar{A}^*)$ and is such that $\delta(h) = h^2$. This means that $d = -\delta$ on the commutator quotient space. There should correspond to $-\delta$ a coderivation of degree -1 on $T(A)$ besides b' which corresponds to d. It should be b'_ρ where ρ is the augmentation on $A = \mathbb{C}\oplus\bar{A}$. We know this differential b'_ρ descends to $T(\bar{A})$. It corresponds to the nonunital algebra structure on A such that $\mathbb{C}\cdot\bar{A} = 0$, equivalently such that $A \longrightarrow \mathbb{C}\times\bar{A}$ is a homomorphism.

Thus $(C_\lambda(A), b)$ can be canonically identified with $(C_\lambda(\mathbb{C}\times\bar{A}), b)$, where $\mathbb{C}\times\bar{A}$ is the product of the scalars with the zero algebra $\bar{A}$, i.e. the semidirect product $\mathbb{C}\oplus\bar{A}$ where $\mathbb{C}\cdot\bar{A} = \bar{A}\cdot\bar{A} = 0$. Dual to

$$(T(\bar{A}^*) * \mathbb{C}[h])_\natural = \mathbb{C}[h]_\natural \oplus (\bar{A}^*\otimes\mathbb{C}[h]) \oplus (\bar{A}^*\otimes\mathbb{C}[h])^{\otimes 2}_\lambda \oplus \cdots$$

we have the GFT isomorphism

$$C_\lambda(\mathbb{C}\oplus\bar{A}) = C_\lambda(\mathbb{C}) \oplus (\bar{A}\otimes B) \oplus (\bar{A}\otimes B)^{\otimes 2}_\lambda \oplus \cdots$$

where B = Bar construction of $\mathbb{C}$.

There's a canonical special deformation retraction it seems of B

to $\mathbb{C}$, which is given by $\left|\begin{array}{l} h^{2n} \mapsto h^{2n-1} \\ h^{2n-1} \mapsto 0 \end{array}\right.$ dually on $\mathbb{C}[h]$.

Another way possibly to produce, better get control of, this homotopy is to use the additive isom

$$\mathbb{C}[h] = \Lambda(\mathbb{C}\varepsilon) \otimes S(\mathbb{C}u) \qquad |\varepsilon| = 1 \quad |u| = 2.$$

Then the derivation $\delta : h \mapsto h^2$ is the Koszul differential such that $\varepsilon \mapsto u$, while there is a homotopy $u^n \mapsto n u^{n-1}\varepsilon$ which is essentially the de Rham differential. I'm thinking in terms of the Euler vector field $x\partial_x$ and $L_{x\partial_x} = [d, \iota_{x\partial_x}]$

An interesting question is whether I can replace $T(\bar{A}^*) * \mathbb{C}[h]$ by $T(\bar{A}^*) * \mathbb{C}[\varepsilon, u]$ so I can use this Euler-Cartan homotopy formula.

February 25, 1994

Observation: $d' = d - s$ satisfies $d'^2 = 0$ and $[b', d'] = [b', d] - [b', s] = -1$, so $-d' = s - d$ is a special contraction for the b' complex.

Similarly for $-\lambda^{-1}s + d$

Recall that $d = s - \delta + \lambda^{-1}s = \sum_{i=0}^{n+1} \lambda^i s \lambda^{-i}$ on $A^{\otimes n+1}$, so that $-\lambda^{-1}s + d = \sum_{i=0}^{n} \lambda^i s \lambda^{-i}$.

Suppose we take $h = s - d$ or $-\lambda^{-1}s + d$, and put $B_0 = (1-\lambda)h$. One gets

$$B_0 = (1-\lambda)s - \underbrace{(1-\lambda)d}_{d'(1-\lambda)} \implies B_0 N_\lambda = (1-\lambda)s N_\lambda$$

or

$$B_0 = (1-\lambda^{-1})s + (1-\lambda)d \implies B_0 N_\lambda = (1-\lambda^{-1})s N_\lambda$$

so we end up with the Connes B-operators.

We might also try to use these contractions to make a Karoubi operator, but

$$[b, -d'] = [b, s] - [b, d] = 1 - K - (1 - K) = 0$$

$$\begin{aligned}[b, -\lambda^{-1}s + d] &= [b', -\lambda^{-1}s] + [c, -\lambda^{-1}s] + 1 - K \\ &= 1 - 1 - \lambda^{-1}sc + 1 - K \\ &= 1 - K - \lambda^{-1}sc\end{aligned}$$

($\lambda^{-1}sc$ is a projection on $A^{\otimes n} \otimes 1$)

$$\begin{aligned}[b, -\delta] &= [b, -s + d - \lambda^{-1}s] = [b, -\lambda^{-1}s] \\ &= 1 + [c, -\lambda^{-1}s] = 1 - 1 - \lambda^{-1}sc = -\lambda^{-1}sc\end{aligned}$$

None of this seems useful.

Note added March 8, 1994.

Observation: The condition

$$\iota(C') \cap B\bar{\Omega}\tilde{A} = 0 \qquad \iota = \begin{pmatrix} -B_0 \\ 1 \end{pmatrix}$$

is equivalent to $\operatorname{Ker} B = (1-\lambda)C$. Indeed

$$\iota(C') \cap B\bar{\Omega}\tilde{A} = \left\{ \begin{pmatrix} 0 \\ N_\lambda y \end{pmatrix} \middle| \overbrace{B_0 N_\lambda}^{B} y = 0 \right\}$$

is equal to zero $\Longleftrightarrow$ $(By = 0 \Longrightarrow N_\lambda y = 0)$

$\Longleftrightarrow$ $\operatorname{Ker} B \subset \operatorname{Ker} N_\lambda = (1-\lambda)C$ $\Longleftrightarrow$ $\operatorname{Ker} B = (1-\lambda)C$.

Thus if we take ~~[illegible]~~ the special contraction $-d' = s - d$ (resp. $d'' = d - \lambda^{-1}s$) instead of s (resp. $-\lambda^{-1}s$) we also get $\iota(C') \cap B\bar{\Omega}\tilde{A} = 0$. Why: because B is the same for $-d'$ and s (resp. d'' and $-\lambda^{-1}s$).

Alternatively, Connes property for $(C(A), b, B)$ depends on $B =$ [illegible] $(1-\lambda)hN_\lambda$ and not on h.

The only useful thing here might be the fact that we can assume h is special.

February 26, 1994

I want now to try to use the special contractions found yesterday to lift $A * \mathbb{C}[\varepsilon]$ into $A * \mathbb{C}[h]$.

Recall $d = s - \delta + \lambda^{-1}s$, where d, δ are the degree +1 derivations of $A * \mathbb{C}[h]$ [redacted] given by $d(a) = [h,a]$, $d(h) = h^2$, $\delta(a) = 0$, $\delta(h) = h^2$ and $s\alpha = h\alpha$, $(-\lambda^{-1}s)\alpha = (-1)^{|\alpha|}\alpha h$. From this we see that

$$[\delta, s]\alpha = \delta(h\alpha) + h\delta\alpha = h^2\alpha = s^2\alpha$$

[redacted] $$\implies (s-\delta)^2 = s^2 - [\delta, s] = 0$$

$$[d, s]\alpha = d(h\alpha) + h(d\alpha) = h^2\alpha = s^2\alpha$$

$$\implies (\underbrace{s-d}_{\delta-\lambda^{-1}s})^2 = s^2 - [d,s] = 0$$

[redacted] also $s - d = -d'$.

Thus $\begin{cases} s-\delta = h-\delta \\ s-d = \delta - \lambda^{-1}s \end{cases}$ are *special contractions* for b'.

Let's now use $h - \delta$ instead of the special contraction $hb'h = h - h^2b'$ used previously (see p.92, 281, 302) to define [redacted] a left action of $A * \mathbb{C}[\varepsilon]$ on $A * \mathbb{C}[h]$ extending the obvious lift A-module structure by putting

$$\varepsilon\alpha = h\alpha - \delta(\alpha).$$

Then $$[\varepsilon, a] = [h-\delta, a] = [h,a],$$

in more detail: $$\varepsilon(a\alpha) - a\varepsilon\alpha = \begin{matrix} ha\alpha - ah\alpha \\ +\delta(a\alpha) - a\delta(\alpha) \end{matrix} = [h,a]\alpha$$

so

$$(a_0 \varepsilon a_1 \cdots \varepsilon a_n)\alpha = a_0[\varepsilon, a_1]\cdots[\varepsilon, a_{n-1}]\varepsilon a_n \alpha$$

$$= a_0[h,a_1]\cdots[h,a_{n-1}](ha_n\alpha - a_n\delta(\alpha))$$

Note that this left action of $A * \mathbb{C}[\varepsilon]$ on $A*\mathbb{C}[h]$ is compatible with the diffl b' because $[b', h-\delta] = 1 = b'(\varepsilon)$.

Notice that for the old action $\varepsilon * \alpha = h\alpha - h^2 b'(\alpha)$ one has

$$\varepsilon * (a\alpha) = ha\alpha - h^2 b'(a\alpha)$$
$$a(\varepsilon * \alpha) = a(h\alpha - h^2 b'(\alpha))$$

so $\quad [\varepsilon, a] * \alpha = [h,a]\alpha - [h^2, a]\, b'(\alpha)$

in contrast to $[\varepsilon, a]\alpha = [h,a]\alpha$ above.

Next consider the right action of $A * \mathbb{C}[\varepsilon]$ on $A * \mathbb{C}[h]$ extending the obvious right A-mod. structure such that $\quad \alpha\varepsilon = \alpha h + (-1)^{|\alpha|}\delta(\alpha)$

Note that $\quad (-1)^{|\alpha|}\alpha\varepsilon = \delta\alpha + (-1)^{|\alpha|}\alpha h$

$$= (\delta - \lambda^{-1}s)\alpha = \underbrace{(s-d)}_{-d'}\alpha$$

Then $\quad (\alpha\varepsilon)a - (\alpha a)\varepsilon =$

$$(\alpha h + (-1)^{|\alpha|}\cancel{\delta(\alpha)})a - (\alpha a h + (-1)^{|\alpha|}\overbrace{\cancel{\delta(\alpha a)}}^{(\delta\alpha)a})$$

$$= \alpha[h,a]. \qquad \therefore \ \alpha[\varepsilon, a] = \alpha[h,a]$$

so $\quad \alpha(a_0[\varepsilon,a_1]\cdots[\varepsilon,a_{n-1}]\varepsilon a_n) =$

$$\alpha a_0[h,a_1]\cdots[h,a_{n-1}]ha_n + (-1)^{|\alpha|+n-1}\delta(\alpha a_0[h,a_1]\cdots[h,a_{n-1}])a_n$$

Beware that $\delta[h,a] = [h^2, a]$ although $b'[h,a] = 0$
$d[h,a] = 0.$

Instead proceed as follows

$$\alpha\left(a_0\varepsilon a_1\cdots\varepsilon a_n\right) = \alpha\left(a_0\varepsilon[a_1,\varepsilon]\cdots[a_{n-1},\varepsilon]a_n\right)$$

$$= \left(\alpha a_0 h + (-1)^{|\alpha|}\delta(\alpha a_0)\right)[a_1,h]\cdots[a_{n-1},h]a_n$$

$$= \left(\alpha a_0 h + (-1)^{|\alpha|}\delta(\alpha)a_0\right)[a_1,h]\cdots[a_{n-1},h]a_n$$

So now take $\alpha = 1$ and we get the liftings

$$a_0\varepsilon a_1\cdots\varepsilon a_n \longmapsto a_0[h,a_1]\cdots[h,a_{n-1}]ha_n$$

$$\longmapsto a_0h[a_1,h]\cdots[a_{n-1},h]a_n$$

These are the same ones encountered before, except that the former right action $\alpha*\varepsilon = \alpha h + (-1)^{|\alpha|}b'(\alpha)h^2$ yields the same lifting as the present left action $\varepsilon\alpha = (h-\delta)\alpha$, and similarly for left + right interchanged!

February 27, 1994

Define two sets of left and right $A * \mathbb{C}[\varepsilon]$ module structures on $A * \mathbb{C}[h]$ by

$$\varepsilon \circ \alpha = h\alpha - h^2 b'(\alpha)$$
$$\alpha \circ \varepsilon = \alpha h + (-1)^{|\alpha|} b'(\alpha) h^2$$
$$\varepsilon * \alpha = h\alpha - \delta(\alpha)$$
$$\alpha * \varepsilon = \alpha h + (-1)^{|\alpha|} \delta(\alpha)$$

Then

$$(\varepsilon \circ \alpha) * \varepsilon = (h\alpha - h^2 b'(\alpha))h + (-1)^{|\alpha|+1} \delta(h\alpha - h^2 b'(\alpha))$$
$$= \overset{\textcircled{1}}{h}\alpha h - h^2 \overset{\textcircled{2}}{b'}(\alpha) h - (-1)^{|\alpha|} \overset{\textcircled{3}}{h^2}\alpha + (-1)^{|\alpha|} \overset{\textcircled{4}}{h}\delta(\alpha) + (-1)^{|\alpha|} h^2 \delta \overset{\textcircled{5}}{b'}(\alpha)$$

$$\varepsilon \circ (\alpha * \varepsilon) = h(\alpha h + (-1)^{|\alpha|}\delta(\alpha)) - h^2 b'(\alpha h + (-1)^{|\alpha|}\delta(\alpha))$$
$$= \overset{\textcircled{1}}{h}\alpha h + (-1)^{|\alpha|} \overset{\textcircled{4}}{h}\delta(\alpha) - h^2 \overset{\textcircled{2}}{b'}(\alpha) h - (-1)^{|\alpha|} \overset{\textcircled{3}}{h^2}\alpha - (-1)^{|\alpha|} h^2 b' \overset{\textcircled{5}}{\delta}(\alpha)$$

where we have used $[b', \delta] = 0$. Thus $\boxed{(\varepsilon \circ \alpha) * \varepsilon = \varepsilon \circ (\alpha * \varepsilon)}$

Also

$$\varepsilon * (\alpha \circ \varepsilon) = h(\alpha h + (-1)^{|\alpha|} b'(\alpha) h^2) - \delta(\alpha h + (-1)^{|\alpha|} b'(\alpha) h^2)$$
$$= \overset{\textcircled{1}}{h}\alpha h + (-1)^{|\alpha|} h \overset{\textcircled{2}}{b'}(\alpha) h^2 - \delta(\overset{\textcircled{3}}{\alpha}) h - (-1)^{|\alpha|} \overset{\textcircled{4}}{\alpha} h^2 - (-1)^{|\alpha|} \delta b'(\overset{\textcircled{5}}{\alpha}) h^2$$

$$(\varepsilon * \alpha) \circ \varepsilon = (h\alpha - \delta(\alpha)) h + (-1)^{|\alpha|+1} b'(h\alpha - \delta(\alpha)) h^2$$
$$= \overset{\textcircled{1}}{h}\alpha h - \delta(\overset{\textcircled{3}}{\alpha}) h - (-1)^{|\alpha|} \overset{\textcircled{4}}{\alpha} h^2 + (-1)^{|\alpha|} h \overset{\textcircled{2}}{b'}(\alpha) h^2 + (-1)^{|\alpha|} b'\delta(\overset{\textcircled{5}}{\alpha}) h^2$$

Thus $\boxed{\varepsilon * (\alpha \circ \varepsilon) = (\varepsilon * \alpha) \circ \varepsilon}$.

Note
$$\boxed{\begin{aligned} \varepsilon \circ (\alpha a) &= (\varepsilon \circ \alpha) a \\ \varepsilon * (\alpha a) &= (\varepsilon * \alpha) a \\ (a\alpha) \circ \varepsilon &= a(\alpha \circ \varepsilon) \\ (a\alpha) * \varepsilon &= a(\alpha * \varepsilon) \end{aligned}}$$

This means that [redacted] we have two $A * \mathbb{C}[\varepsilon]$ - bimodule structures on $A * \mathbb{C}[h]$. Since 1 is central for either [redacted] bimodule structure: $\varepsilon \circ 1 = h = 1 * \varepsilon$, $\varepsilon * 1 = h = 1 \circ \varepsilon$, this explains why acting on 1 yields only two liftings of $A * \mathbb{C}[\varepsilon]$ into $A * \mathbb{C}[h]$.

March 2, 1994

On the standard bimodule resolution $A * \mathbb{C}[h]$ we have the differentials b', d, δ which are derivations relative to the product structure. On the tensor coalgebra $T(A)$ the differentials b' and d are coderivations. The point to investigate concerns the mixed complexes arising from the fact that $[b', d] = 0$ and $[b', \delta] = 0$.

Let's focus on the coalgebra $T(A)$. We have the bicomplexes

$$\begin{tikzcd}
 & & \arrow[d, "b"] & \arrow[d, "-b'"] & \\
0 & A_\lambda^{\otimes 2} \arrow[l] \arrow[d, "\lambda"'] & A^{\otimes 2} \arrow[l] \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "1-\lambda"'] \arrow[d, "-b'"] & \arrow[l, "N_\lambda"'] \\
0 & A \arrow[l] & A \arrow[l] & A \arrow[l, "1-\lambda"'] & \arrow[l, "N_\lambda"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & \arrow[from=d] & \arrow[from=d, "-d"] & \\
A_\lambda^{\otimes 2} \arrow[d] & A^{\otimes 2} \arrow[l] & A^{\otimes 2} \arrow[l, "1-\lambda"'] & \arrow[l, "N\lambda"'] \\
A & A \arrow[l] \arrow[u, "d'"'] & A \arrow[l, "1-\lambda"'] \arrow[u, "-d"'] & \arrow[l, "N\lambda"']
\end{tikzcd}$$

and we know that $[b', d] = 0$ and $[b, d'] = 0$. The latter follows from the former because $(C(A), b, d')$ can be interpreted as $\Omega_1^{\mathrm{coalg}} T(A) = A \otimes T(A)$ with b, d' induced by the coderivations b', d on $T(A)$.

So the conclusion is that we have an exact sequence of mixed complexes

$$0 \leftarrow \underset{(b,d)}{C_\lambda(A)} \leftarrow \underset{(b,d')}{C(A)} \xleftarrow{1-\lambda} \underset{(b',d)}{C(A)} \xleftarrow{\bar{N}_\lambda} \underset{(b,d)}{C_\lambda(A)} \leftarrow 0$$

It's natural to find the cyclic homology of these mixed complexes.

Now $(C(A), b', d)$ is quis to 0 and we have a quis $(C(A), b, d') \xrightarrow{p} (\Omega A, b, 0)$.

Note that $d' = d - s$ has image $\subset$ the degenerate subcomplex for the standard semi-simplicial structure on $(C(A), b)$.

For a mixed complex M with $B=0$ the cyclic homology is

$$HC_n((M, b, 0)) = H_n\left(\begin{tikzcd} {} \arrow[d] & {} \arrow[d] & \\ M_1 \arrow[d] & {} \arrow[l, "0"'] \arrow[d] & {} \arrow[l, "0"'] \\ M_0 \arrow[d] & M_{-1} \arrow[l, "0"'] & \\ {} & & \end{tikzcd} \right)$$

$$= \bigoplus_{p \geq 0} HH_{n-2p}(M) = HH(M)[u]_n$$

Thus we have

$$HC((C(A), b', d)) = 0$$

$$HC((C(A), b, d')) = HH(A)[u]$$

and we would like to find $HC((C_\lambda(A), b, d))$.

Let us use

$$\begin{tikzcd} 0 \arrow[r] & \underset{b=d=0}{C_\lambda(\mathbb{C})} \arrow[r] & \underset{(b,d)}{C_\lambda(A)} \arrow[r] & C_\lambda(A)/C_\lambda(\mathbb{C})_{(b,d)} \arrow[r] \arrow[d, "\text{quis wrt } b"] & 0 \\ & & & \bar{C}_\lambda(A)_{(b,0)} & \end{tikzcd}$$

This gives a long exact sequence

$$\begin{tikzcd} {} \arrow[r] & HC(C_\lambda(\mathbb{C})) \arrow[r] \arrow[d, equal] & HC(C_\lambda(A)) \arrow[r] & HC(\bar{C}_\lambda(A)) \arrow[r] \arrow[d, equal] & {} \\ & C_\lambda(\mathbb{C})[u] & & \overline{HC}(A)[u] & \end{tikzcd}$$

The obvious guess at this point is

$$HC((C_\lambda(A), b, d)) = HC(A)[u]$$

and moreover that the above long exact sequence is the standard one

$$\cdots\cdots \longrightarrow HC(\mathbb{C}) \longrightarrow HC(A) \longrightarrow \overline{HC}(A) \longrightarrow \cdots\cdots$$

tensored with $\mathbb{C}[u]$. In other words the conjecture is that $(C_\lambda(A), b, d)$ is like mixed complexes arising for group algebras and the identity conjugacy class, where $B = 0$ and so S is surjective.

There's an S operator of degree $+2$ on $C_\lambda(A)$ given by diagram chasing in the bicomplex

$$\begin{tikzcd}
 & & A^{\otimes n+2} \arrow[r, leftrightarrow, "\bar{N}_\lambda"] & A_\lambda^{\otimes n+2} \\
\arrow[u, "d"] & A^{\otimes n+1} & A^{\otimes n+1} \arrow[l, "1-\lambda"'] \arrow[u, "-d"] & \arrow[l] \\
A_\lambda^{\otimes n} \arrow[u, "d"] & A^{\otimes n} \arrow[l, twoheadrightarrow, "\pi"'] \arrow[u, "d'"] & A^{\otimes n} \arrow[l, "1-\lambda"'] \arrow[u, "-d"] & \arrow[l] \\
 & \arrow[u] & \arrow[u] &
\end{tikzcd}$$

Thus

$$S = \bar{N}_\lambda^{-1}(-dG_\lambda + G_\lambda d')(d'\bar{P}_\lambda - \bar{P}_\lambda d)$$

$$= \bar{N}_\lambda^{-1} P_\lambda (-d) G_\lambda d' \bar{P}_\lambda$$

$$= \bar{N}_\lambda^{-1} P_\lambda d G_\lambda (-d') \bar{P}_\lambda \qquad \text{(better signs)}$$

$$= \bar{N}_\lambda^{-1} P_\lambda s G_\lambda s \bar{P}_\lambda = \bar{N}_\lambda^{-1} P_\lambda s G_\lambda s P_\lambda \bar{P}_\lambda$$

because $P_\lambda d' G_\lambda \sim N_\lambda d' G_\lambda = dN_\lambda G_\lambda = 0$, similarly $G_\lambda d P_\lambda = 0$ and $d = s + d'$.

Now

$$P_\lambda s G_\lambda s P_\lambda = P_\lambda s \frac{1}{n+1} \sum_{k=0}^{n} \left(\frac{n}{2}-k\right) \lambda^k s P_\lambda \qquad \text{on } A^{\otimes n}$$

$$= \frac{1}{n+1} \sum_{k=0}^{n} (-k) P_\lambda s \lambda^k s P_\lambda$$

since $P_\lambda s N_\lambda s P_\lambda = 0$. Same argument: $N_\lambda d' = N_\lambda s N_\lambda$ so $N_\lambda s N_\lambda s N_\lambda = N_\lambda s N_\lambda d' = N_\lambda d' d' = 0$. Finally if we define

$$d'^{[2]} = \sum_{n+1 \geq i > j \geq 1} \lambda^i s \lambda^{-i} \lambda^j s \lambda^{-j} \qquad \text{on } A^{\otimes n}$$

Then

$$P_\lambda d'^{[2]} P_\lambda = \sum_{n+1 \geq i > j \geq 1} P_\lambda s \lambda^{-(i-j)} s P_\lambda$$

$$= \sum_{k=1}^{n} (n+1-k) P_\lambda s \underset{\overset{\shortparallel}{\lambda^{n+1-k}}}{\boxed{\lambda^{-k}}} s P_\lambda \qquad \cdot \quad l = n+1-k$$

$$= \sum_{0 \leq l \leq n} l \, P_\lambda s \lambda^l s P_\lambda$$

Thus

$$S = -\frac{1}{(n+2)(n+1)} \pi \, d'^{[2]} \overline{P_\lambda} \qquad \text{on } A_\lambda^{\otimes n}$$

March 4, 1994

Here's another angle. Consider the tensor algebra $T(A)$ with A in degree 1. Let ξ be the identity element of A. Then we have degree 1 derivations on $T(A)$ defined by

$$d_\ell(a) = \xi a \qquad d_r a = -a\xi$$

Since $d_\ell^2(a) = d_\ell(d_\ell a) = d_\ell(\xi a) = d_\ell(\xi)a - \xi\, d_\ell(a)$ $= \xi^2 a - \xi\xi a = 0$, $d_r^2(a) = d_r(-a\xi) = -(-a\xi)\xi + a(-\xi^2) = 0$ these are differentials: $\boxed{d_\ell^2 = d_r^2 = 0}$. Also $(d_\ell - d_r)(a) = \xi a + a\xi = ad(\xi)a$ so that

$$\boxed{\begin{array}{c} d_\ell - d_r = ad(\xi) \\ \hline -[d_\ell, d_r] = ad(\xi^2 \end{array}}$$

Concretely

$$d_\ell(a_1 \cdots a_n) = \sum_{i=1}^{n} (-1)^{i-1} a_1 \cdots a_{i-1}\xi a_i \cdots a_n$$

$$d_r(a_1 \cdots a_n) = \sum_{i=1}^{n} (-1)^{i} a_1 \cdots a_i \xi a_{i+1} \cdots a_n$$

Thus $\boxed{d_r = d'}$ (i.e. $d - s = -\delta + \lambda^{-1}s$)

and $\boxed{d_\ell = d - \lambda^{-1}s = s - \delta}$ (i.e. d'')

Next consider the bicomplex

$$0 \leftarrow T(A)_\natural \longleftarrow T(A) \xleftarrow{\beta} \Omega^1 T(A)_\natural \xleftarrow{\partial} \cdots$$

For d_r we have

$$d_r \natural(a_1 \cdots a_{n-1}\partial a_n) = \natural\left\{ d_r(a_1 \cdots a_{n-1})\partial a_n +_n \overset{(-1)^n}{a_1} \cdots a_{n-1}\left(\partial a_n \xi + a_n \partial\xi\right)\right\}$$

$$=_\natural \xi a_1 \cdots a_{n-1} \partial a_n + \sum_{i=1}^{n-1} (-1)^i a_1 \cdots a_i \xi a_{i+1} \cdots a_{n-1} \partial a_n$$

$$+ (-1)^n a_1 \cdots a_n \partial \xi$$

Thus under the identification

$$\Omega^1 T(A)_\natural = T(A) \otimes A$$

$$\natural(a_1 \cdots a_{n-1} \partial a) \longleftrightarrow a_1 \cdots a_{n-1} \otimes a_n$$

we see d_r on $\Omega^1 T(A)_\natural$ is d on chains, yielding

$$\begin{tikzcd}
T(A)_\natural & T(A) \arrow[l] & T(A) \arrow[l, "1-\lambda"'] & \arrow[l, "N_\lambda"'] \\
 & T(A) \arrow[u, "d'"] & T(A) \arrow[l, "1-\lambda"'] \arrow[u, "d"] & \arrow[l, "N_\lambda"']
\end{tikzcd}$$

If we use d_ℓ and the identification $\Omega^1 T(A)_\natural = A \otimes T(A)$
$a_1 \otimes a_2 \otimes \cdots \otimes a_n \longrightarrow -\natural(\partial a_1\, a_2 \cdots a_n)$, then we get instead

$$\begin{tikzcd}
T(A) & A \otimes T(A) \arrow[l, "1-\lambda^{-1}"'] & \arrow[l] \\
T(A) \arrow[u, "d''"] & A \otimes T(A) \arrow[l, "1-\lambda^{-1}"'] \arrow[u, "d"] & \arrow[l]
\end{tikzcd}$$

March 5, 1994

Review: Consider the tensor algebra $T(A)$ with A in degree one, where A is a vector space equipped with a distinguished elt. $\xi \neq 0$. Derivations of degree -1 correspond to linear functionals on A: If $f \in A^*$, the corresponding derivation is

$$\iota_f(a_1 \cdots a_n) = \sum_{i=1}^{n} (-1)^{i-1} a_1 \cdots a_{i-1} f(a_i) a_{i+1} \cdots a_n$$

Derivations of degree $+1$ correspond to linear maps $A \longrightarrow A^{\otimes 2}$. The only obvious ones in the present situation are linear combinations of $a \mapsto \xi a, a\xi$.

Let d_ℓ, d_r be the derivations

$$d_\ell(a) = \xi a \quad , \quad d_r(a) = -a\xi$$

Then $\quad d_\ell - d_r = \mathrm{ad}\,\xi \quad$ and

$$-[d_\ell, d_r] = (d_\ell - d_r)^2 = \mathrm{ad}(\xi)^2 = \mathrm{ad}(\xi^2).$$

Calculate

$$[\iota_f, d_\ell](a) = \iota_f(\xi a) = f(\xi)a - \xi f(a)$$

$$[\iota_f, d_r](a) = \iota_f(-a\xi) = -f(a)\xi + a f(\xi)$$

which checks since $\quad [\iota_f, d_\ell - d_r] = [\iota_f, \mathrm{ad}(\xi)] = \mathrm{ad}(f(\xi)) = 0$

Suppose we choose $\rho \in A^*$ such that $\rho(\xi) = 1$, equivalently a complement: $A = \mathbb{C}\xi \oplus V$. Then

$$[\iota_\rho, d_\ell](a) = [\iota_\rho, d_r](a) = a - \rho(a)\xi$$

$$= \begin{cases} 0 & a \in \mathbb{C}\xi \\ a & a \in V \end{cases}$$

Thus $[\iota_\rho, d_r]$ is the derivation of degree zero which gives the degree in V relative to $T(A) = \mathbb{C}[\xi] * T(V)$.

Since derivations induce operators on the commutator quotient space, it follows that the homology of $T(A)_\natural$ with respect to either ι_ρ or d_r (or d_ℓ) is supported in V-degree 0. Thus

$$H(T(A)_\natural, d) = H(\mathbb{C}[\xi]_\natural, d) = \mathbb{C}[\xi]_\natural$$

where d on $T(A)_\natural$ means the differential induced by either d_ℓ or d_r on $T(A)$. Since $d_r = \sum_1^n \lambda^i s \lambda^{-i}$ on $A^{\otimes n}$, this means $\pi d_r = \pi s \sum_1^n \lambda^{-i} = \pi s N_\lambda$.

This calculation is related to triviality of reduced cyclic homology for free algebras. The reason is that $T(A) = \mathbb{C}[\xi] * T(V)$ with differential $\iota_\rho(\xi) = 1$ is the standard resolution over $T(V)$, except for the fact that V is here located in degree 1.

Consider however $(T(V) * \mathbb{C}[h], b')$ with V in degree zero, where as usual $b'(T(V)) = 0$ and $b'(h) = 1$. Define d_ℓ on $T(V) * \mathbb{C}[h]$ by $d_\ell(v) = \xi v$, $d_\ell(\xi) = \xi^2$. Then $[b', d_\ell](v) = b'(\xi v) = v$ and $[b', d_\ell](\xi) = b'(\xi^2) + d_\ell(1) = 0$, so that $[b', d_\ell]$ is the V degree operator. This shows that $(T(V) * \mathbb{C}[h])_\natural$ and $\mathbb{C}[h]_\natural$ have the same homology wrt ~~the~~ b' differential.

But $(A * \mathbb{C}[h])_\natural = (A \oplus AhA \oplus \cdots)_\natural$ with differential induced by b' is

$$A_\natural \xleftarrow{\pi} A \xleftarrow{b} A_\lambda^{\otimes 2} \xleftarrow{b} A_\lambda^{\otimes 3} \longleftarrow \cdots$$

so it follows that for $A = T(V)$ one has

$$HC_n(A) = \begin{cases} A_\natural & n=0 \\ HC_n(\mathbb{C}) & n \geq 1 \end{cases}$$

What this proof amounts to is using the formula

$$T(V) * \mathbb{C}[h] = \mathbb{C}[h] \oplus \mathbb{C}[h] \otimes V \otimes \mathbb{C}[h] \oplus \cdots$$

$$(T(V) * \mathbb{C}[h])_\natural = \mathbb{C}[h]_\natural \oplus (\mathbb{C}[h] \otimes V) \oplus (\mathbb{C}[h] \otimes V)_\lambda^{\otimes 2} \oplus \cdots$$

and the triviality of the [redacted] homology of $\mathbb{C}[h]$ wrt b'.

(Alternative formulas: Define d_r on $T(V) * \mathbb{C}[h]$ by $d_r(v) = v\xi$ (recall $|v| = 0$), $d_r(\xi) = -\xi^2$.
Then $d_r^2(v) = d_r(v\xi) = v\xi^2 + v(-\xi^2) = 0$, $d_r(\xi^2) = 0$.
Also $[b', d_r](v) = b'(v\xi) + \cancel{d_r(b'(v))} = v\,b'(\xi) = v$
and $[b', d_r](\xi) = b'(-\xi^2) + d_r(1) = 0$.)

[redacted]

March 8, 1994

Summarize some points. Consider the tensor algebra $T(A) = T(\mathbb{C}\xi \oplus V)$ where A is in degree 1. Then

$$T(A) = \mathbb{C}[\xi] * T(V)$$
$$= \mathbb{C}[\xi] \oplus \mathbb{C}[\xi] \otimes V \otimes \mathbb{C}[\xi] \oplus \cdots$$

$$T(A)_\xi = \mathbb{C}[\xi]_\xi \oplus \left(\mathbb{C}[\xi]\otimes V\right) \oplus \left(\mathbb{C}[\xi]\otimes V\right)_\lambda^{\otimes 2} \oplus \cdots$$

On $T(A)$ we have several (graded) derivations:

$$d_\ell(v) = \xi v \qquad d_\ell(\xi) = \xi^2$$
$$d_r(v) = -v\xi \qquad d_r(\xi) = \xi^2$$
$$d(v) = [\xi, v] \qquad d(\xi) = \xi^2$$
$$\delta(v) = 0 \qquad \delta(\xi) = \xi^2$$
$$b'(v) = 0 \qquad b'(\xi) = 1$$

These are differentials. ~~[illegible]~~ d_ℓ and d_r depend only on A and ξ but not the complement V. We have

$$d_\ell - d_r = d + \delta = \mathrm{ad}(\xi)$$
$$-[d_\ell, d_r] = [d, \delta] = \mathrm{ad}(\xi^2)$$

$$[b', d_\ell] = [b', d_r] = \text{the } V \text{ degree operator}$$
$$[b', d] = [b', \delta] = 0.$$

From $d_\ell - d_r = d + \delta = \mathrm{ad}(\xi)$ we have that on $T(A)_\xi$ the induced differentials satisfy $(d_\ell)_* = (d_r)_*$ and $d_* = -\delta_*$.

On $\mathbb{C}[\xi]\otimes V$ we have

$$d_\ell v = \xi v \;, \quad d_\ell(\xi v) = 0$$

$$d_\ell(\xi^2 v) = \xi^3 v \;, \quad d_\ell(\xi^3 v) = 0, \quad \text{etc}$$

so the homology is trivial hence

$$\boxed{H_\cdot\left(T(A)_\natural, \begin{pmatrix} d_\ell \\ d_r \end{pmatrix}\right) = \mathbb{C}[\xi]_\natural}$$

Also on $\mathbb{C}[\xi]\otimes V$ we have

$$\delta(v) = 0 \qquad \delta(\xi v) = \xi^2 v$$

$$\delta(\xi^2 v) = 0 \qquad \delta(\xi^3 v) = \xi^4 v \quad \text{etc.}$$

so the homology is V and we have

$$\boxed{H_\cdot\left(T(\mathbb{C}\oplus V), \begin{pmatrix} d \\ \delta \end{pmatrix}\right) = \mathbb{C}[\xi]_\natural \oplus C_\lambda(V)}$$

In the former situation b' furnishes a [redacted] contraction for the homology of positive V degrees. However, it's the latter situation I would really [redacted] like a good way to handle [redacted] the evident contraction: $\xi^{2n}v \longmapsto \xi^{2n-1}v$, $n>0$.

This is a coderivation on $\mathbb{C}[\xi]$ for the coalgebra structure.

This suggests looking at [redacted] Laplacians, where the homotopy is not a derivation, i.e. like the adjoint of ~~the~~ de Rham differential.

List ideas from scratch work:

Classify contraction on $(\mathbb{C}[h], \partial)$:

$$\begin{tikzcd} \mathbb{C} \arrow[r, dashed, bend left, "1"] & \mathbb{C} \arrow[l, "1"] \arrow[r, dashed, bend left, "\text{arb}"] & \mathbb{C} \arrow[l, "0"] \arrow[r, dashed, bend left, "1"] & \mathbb{C} \arrow[l, "1"] \arrow[r, dashed, bend left, "\text{arb.}"] & {} \arrow[l, "\partial"] \end{tikzcd}$$

Recall that if E is an A-bimodule with A-coalgebra structure $E \to A$, $E \longrightarrow E \otimes_A E$ then

$$T_A(E): \quad A \leftarrow E \leftleftarrows\!\!\rightarrow E \otimes_A E \overset{\rightrightarrows}{\leftleftarrows\!\!\leftarrow} E \otimes_A E \otimes_A E \cdots$$

has a simplicial structure. It seems there is a differential δ which is a derivation in this context, anti-commuting with $\partial = b'$.

I have a conjecture that $(C_\lambda(A), b, d)$ has cyclic homology $= HC(A)[u]$, i.e. as if $d = 0$. Methods that might work:

Burghelea group algebra situation, the identity conjugacy class contributes $H(BG)[u]$ to the cyclic cohomology. I've looked at this from the viewpoint of G torsors over S^1, but haven't got around the difference between the "cyclic groupoid" of G-torsor and the cyclic set given by the cyclic bar construction.

Direct algebraic method (exercise in [L] - corrected): Given (M, b, B) such that $\exists\ \alpha$ of degree 2 on M such that $[b, \alpha] = B$ and $[B, \alpha] = 0$. Then

$$[b, e^{S\alpha}] = \int_0^1 e^{(1-t)S\alpha} \underbrace{[b, S\alpha]}_{SB} e^{tS\alpha}\, dt \quad \text{which commutes with } S\alpha$$

$$= e^{S\alpha} SB \qquad \text{so} \qquad \boxed{b e^{S\alpha} = e^{S\alpha}(b + SB)}$$

which means that the mixed complexes (M, b, B) and $(M, b, 0)$ become isomorphic *after* applying B.

Example. $W_\natural$ where $W = \mathbb{C}\langle A, F\rangle$ and $dA = F - A^2$, $dF = -[A, F]$; dual bar construction for $\mathbb{C}[\varepsilon]$, $d(\varepsilon) = 1$.

W is filtered by the (F)-adic filtration. Problem: to find a minimal model for the filtered complex $W_\natural$. This example is similar to $C_\lambda(A)^* = T(A^*)_\natural = {}^\natural(T(\bar{A}^*) * \mathbb{C}[\rho])_\natural$.

For this problem it seems one needs a vertical SDR i.e. for $\mathrm{gr}\, W_\natural$ which is $W_\natural$ with the vertical differential $\delta A + A^2 = 0$, $\delta F + [A, F] = 0$.

The constructeon of Chern-Simons forms uses a 'horizontal' contraction. Specifically let h be the derivation of degree -1 on $W = \mathbb{C}\langle A, F\rangle = \mathbb{C}\langle A, dA\rangle$ given by $h(A) = 0$, $h(dA) = A$. Then $[d, h]$ gives the A-degree: $[d, h] = 1$ on A and dA.

Note that $h(F) = h(dA + A^2) = A$.

Let $D = [d, h]$ be the A-degree operator. Then $t^D A = tA$ is the Chern-Simons deformation.

We have a contraction $\frac{1}{D} h$ for $W_\natural / \mathbb{C}$ and applying this to $\mathrm{tr}\left(\frac{F^n}{n!}\right)$ yields the corresponding Chern-Simons forms.

March 17, 1994

I want to organize matters concerning the construction used by J. Brodzki which he learned from Cuntz. Joachim pointed out in his [redacted] visit last week that the deformation of $(\bar{Q}\tilde{A})^\wedge$ to $\hat{Q}A$ commutes with γ, hence it restricts to a deformation of $(\bar{R}\tilde{A})^\wedge$ [redacted] to $\hat{R}A$. This means, in particular that there is a lifting for [redacted] $TA = R\tilde{A} \longrightarrow \mathbb{C} \times RA$ after completing suitably which doesn't depend on a choice of $\rho : A \to \mathbb{C}$, so it seems.

First concentrate on the case $A = \tilde{\mathbb{C}} = \mathbb{C}[F]$ where $F^2 = 1$.

Let $A = \mathbb{C}[F]$, $F^2 = 1$.

A is separable + commutative so there is a unique $Y \in \Omega^1 A$ such that $da = [a, Y]$, $\forall a \in A$, i.e. $dF = [F, Y]$, namely $\boxed{Y = \frac{1}{2} F dF}$, [redacted]

Let $\boxed{\phi(F) = -\frac{1}{2} F dF^2}$. Then ϕ satisfies $\delta(\phi) = -d \cup d$, i.e. $\phi(a_1 a_2) = \phi(a_1) a_2 + a_1 \phi(a_2) + da_1 da_2$, which amounts to $0 = \phi(1) = \phi(F)F + F\phi(F) + dF^2$.

We know from CQ1 that there is a derivation D of QA commuting with γ, which is defined by

$$D(F + dF) = (F + dF) \circ Y - Y \circ (F + dF)$$

$$= \left(\begin{array}{l} FY - YF \\ + dFY - YdF \\ - dFdY \underset{\uparrow}{-} dYdF \end{array} \right) = \left(\begin{array}{l} dF \\ -FdF^2 \\ -dF^3 \end{array} \right)$$

note sign since Y is odd

Thus we have

$$* \begin{cases} D(F) = -FdF^2 & (= 2\phi(F)) \\ D(dF) = dF - dF^3 & (= dF + 2d\phi(F)) \end{cases}$$

So far we have used Fedosov product notation, but suppose we shift to "ρ-notation" and describe QA via the generators $p(a) = \rho(a)$ and $g(a)$ subject to the relations

$$\rho(a_1a_2) = \rho(a_1)\rho(a_2) + g(a_1)g(a_2)$$
$$g(a_1a_2) = \rho(a_1)g(a_2) + g(a_1)\rho(a_2).$$

In the present case: $A = \mathbb{C}[F]$ we have the generators $p = p(F)$, $g = g(F)$ subject to the relations

$$p^2 + g^2 = 1$$
$$pg + gp = 0.$$

The derivation D is then given by

$$\begin{cases} Dp = -pg^2 = p(p^2-1) \\ Dg = g - g^3 = g(1-g^2). \end{cases}$$

We now would like to interpret this derivation in Cayley transform terms. (The idea comes from June 27, 1991 - p 422.)

Consider a $\mathbb{Z}/2$-graded vector space $V = V^+ \oplus V^-$ and let $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. A representation $QA \longrightarrow \mathrm{End}(V)$ compatible with the superalg structure is the same thing as an ordinary homomorphism $A \longrightarrow \mathrm{End}(V)$, i.e. an involution F on V. I suppose inner products on V such that γ, F unitary. Such representations correspond to points in the Grassmannian $Gr(V)$.

Suppose that F, γ are close in the sense that the spectrum of $F\gamma$ is in the open right unit semi-circle. Then there is a unique skew-adjoint operator X on V such that

$$F\gamma = \frac{1+X}{\sqrt{1-X^2}} \qquad \gamma X = -X\gamma$$

Then $p\gamma = \frac{F+\gamma F\gamma}{2}\gamma = \frac{F\gamma+\gamma F}{2} = \frac{1}{\sqrt{1-X^2}}$

and $q\gamma = \frac{F\gamma - \gamma F}{2} = \frac{X}{\sqrt{1-X^2}}$. Now let us consider the flow on the spase of F close to γ which replases X by tX, and let us calculate the [redacted] derivative $\dot{F}$. We have

$$\dot{p}\gamma = \frac{d}{dt}\Big|_{t=1} (1-t^2X^2)^{-1/2}$$

$$= \left(-\tfrac{1}{2}\right)(1-X^2)^{-3/2}(-2X^2) = (1-X^2)^{-1/2}\frac{X^2}{1-X^2}$$

$$= p\gamma(q\gamma)^2 = -pq^2\gamma$$

$$\dot{q}\gamma = X(1-X^2)^{-1/2} + X\left(-\tfrac{1}{2}\right)(1-x^2)^{-3/2}(-2X^2)$$

$$= \frac{X}{\sqrt{1-X^2}}\left(1+\frac{X^2}{1-X^2}\right) = q\gamma(1+(q\gamma)^2)$$

$$= q\gamma(1-q^2) = q(1-q^2)\gamma.$$

Thus the derivation D corresponds to the Cayley transform flow.

March 18, 1994

Consider a hermitian vector space V and the space of pairs of [redacted] unitary involutions (F_1, F_2) on it. We will construct a vector field on this space.

Consider first pairs such that $F_1 F_2$ are close in the sense that the unitary $g = F_1 F_2$ does not have the eigenvalue -1. Then

$$g = \frac{1+X}{1-X} \qquad \text{where} \qquad X = \frac{g-1}{g+1} \quad \text{is}$$

a skew hermitian operator anticommuting with F_1 and F_2:

$$F_1 X F_1^{-1} = \frac{F_1 g F_1^{-1} - 1}{F_1 g F_1^{-1} + 1} = \frac{g^{-1}-1}{g^{-1}+1} = \frac{1-g}{1+g} = -X$$

Set $\quad h = \frac{1+X}{\sqrt{1-X^2}} \quad$ so that $\quad h^2 = g,$ [redacted] and h is inverted by both F_1 and F_2. Put

$$\gamma = F_1 h = h F_2$$

Then γ is the involution which is the midpoint of the geodesic $F_1 g^t$, $0 \leq t \leq 1$, joining F_1 to F_2. We [redacted] define the vector field on the open set of pairs (F_1, F_2) which are close by

$$\dot{\gamma} = 0 \qquad \dot{X} = X$$

We now calculate $(\dot{F}_1, \dot{F}_2)$ in terms of (F_1, F_2).

$$\dot{h} = \left((1+X)(1-X^2)^{-1/2}\right)^{\bullet}$$

$$= X(1-X^2)^{-1/2} + (1+X)(-1/2)(1-X^2)^{-3/2}(\cancel{-}2XX)$$

$$= X(1-X^2)^{-3/2}\left\{(1-\cancel{X}^2) + (1+\cancel{X})X\right\} = \frac{X}{\sqrt{1-X^2}} \; \frac{1+X}{1-X^2}$$

where we have used that $\mathring{X} = X$ commutes with X. Thus

$$h^{-1}\mathring{h} = \frac{\cancel{\sqrt{1-X^2}}}{\cancel{1+X}} \cdot \frac{X}{\cancel{\sqrt{1-X^2}}} \; \frac{\cancel{1+X}}{1-X^2} = \frac{X}{1-X^2}$$

$$= \frac{1}{2}\left(\frac{1}{1-X} - \frac{1}{1+X}\right)$$

But $g = \frac{1+X}{1-X} = -1 + \frac{2}{1-X} \implies \frac{g+1}{2} = \frac{1}{1-X}$

and $\frac{g^{-1}+1}{2} = \frac{1}{1+X}$, so

$$h^{-1}\mathring{h} = \frac{1}{2}\left(\frac{g+1}{2} - \frac{g^{-1}+1}{2}\right) = \frac{1}{4}(g-g^{-1}) = \frac{1}{4}(F_1F_2 - F_2F_1)$$

This anti-commutes with F_1 and F_2.

Let $p = \frac{F_1+F_2}{2}$, $q = \frac{F_1-F_2}{2}$. Then

$$qp = \left(\frac{F_1-F_2}{2}\right)\left(\frac{F_1+F_2}{2}\right) = \frac{1}{4}(\cancel{1} + F_1F_2 - F_2F_1 \cancel{-1}) = h^{-1}\mathring{h}$$

Now $F_1 = \gamma h^{-1} = h\gamma$ so $\mathring{F}_1 = \mathring{h}\gamma = \mathring{h}h^{-1}F_1$

and $F_2 = h^{-1}\gamma = \gamma h$ so $\mathring{F}_2 = \gamma\mathring{h} = F_2 h^{-1}\mathring{h} = -\mathring{h}h^{-1}F_2$

Thus we have

$$\mathring{F}_1 = qpF_1 \qquad \mathring{F}_2 = -qpF_2$$

so

$$\mathring{p} = qpq = -pq^2 = -p(1-p^2) = p(p^2-1)$$

$$\mathring{q} = -qp^2 = -q(q^2-1)$$

This shows the flow is globally defined and that it corresponds to the derivation D on $Q(\mathbb{C}[F])$.

Here's a similar calculation (compare June 1991 p. 422). It concerns $R(\mathbb{C}[F]) = \mathbb{C}[x]$ where $= \rho(F)$. In the (f.d.) Hilbert space context x will be a self-adjoint contraction, which if it doesn't have the the eigenvalues ±1 has the form $p = \frac{\alpha}{\sqrt{1+\alpha^2}}$. Consider then the flow $\mathring{\alpha} = \alpha$. One has

$$\mathring{p} = \alpha(1+\alpha^2)^{-1/2} + \alpha(-\tfrac{1}{2})(1+\alpha^2)^{-3/2}\, 2\alpha^2$$

$$= \alpha(1+\alpha^2)^{-3/2}\left((1+\alpha^2) - \alpha^2\right)$$

$$= \frac{\alpha}{\sqrt{1+\alpha^2}}\left(\frac{1}{1+\alpha^2}\right) = \frac{\alpha}{\sqrt{1-\alpha^2}}\left(1 - \frac{\alpha^2}{1+\alpha^2}\right)$$

$$= p(1-p^2)$$

This corresponds to the negative of

$$D(p) = p(p^2-1).$$

The sign comes from the fact that D moves the eigenvalues away from ±1 to 0, while $\alpha \mapsto t\alpha$ with $t \longrightarrow +\infty$ moves them in the opposite direction.

March 20, 1994

Consider $A = A_+ \oplus A_-$ a $\mathbb{Z}/2$ graded vector space equipped with a distinguished element $1 = e_+ + e_-$ where $e_+ \in A_+$, $e_- \in A_-$ are both nonzero. Let K be the ideal in RA generated by the elements

$$\rho(e_-)\rho(a_+) = \rho(a_+) - \rho(e_+)\rho(a_+) \qquad \overset{\rho(e_+)\rho(a_-)}{\|}\ \ \rho(a_-) - \rho(e_-)\rho(a_-)$$

$$\rho(a_+)\rho(e_-) = \rho(a_+) - \rho(a_+)\rho(e_+) \qquad \rho(a_-) - \rho(a_-)\rho(e_-) \overset{\rho(a_-)\rho(e_+)}{\|}$$

where RA is generated by the linear map $\rho : A \to RA$ with relation $\rho(1) = 1$, i.e. $\rho(e_+) + \rho(e_-) = 1$.

This construction is functorial, so to understand it let's restrict to the case where $A_- = \mathbb{C}e_-$ and $A_+ = \mathbb{C}e_+ + \mathbb{C}a$. (We want to allow a to belong to $\mathbb{C}e_+$ as a degenerate case.)

Then $R = RA$ has generators $x_\pm = \rho(e_\pm)$ and $y = \rho(a)$ subject to the relation $x_+ + x_- = 1$. The ideal K has the generators

$$x_- y \quad , \qquad y x_- \ , \qquad x_+ x_- = x_- x_+$$

We consider the derivation D on R defined by

$$D(x) = x(x^2 - 1) \qquad \text{where} \quad x = x_+ - x_-$$

$$D(y) = \frac{x^2 - x}{2}\, y + y\, \frac{x^2 - x}{2}$$

Let's work this out using x_+, x_- instead of x.

$$\begin{aligned} 1 &= x_+ + x_- \\ x &= x_+ - x_- \end{aligned} \implies \begin{aligned} x_+ &= \frac{1+x}{2} \\ x_- &= \frac{1-x}{2} \end{aligned} \qquad x_+ x_- = \frac{1 - x^2}{4}$$

$$D(x_-) = D\left(\frac{1-x}{2}\right) = {}_{\wedge}^{(-\frac{1}{2})}(x_+ - x_-)(-4x_+x_-)$$

$$= 2(x_+ - x_-)x_+x_-$$

$$\boxed{D(x_-) = 2x_+^2x_- - 2x_+x_-^2}$$

$$D(y) = -x x_- y - y x_- x$$

$$\boxed{D(y) = x_-^2 y - x_+x_-y + yx_-^2 - yx_-x_+}$$

Observe that $D(R) \subset K$ so that $D = 0$ on R/K. We want to compute D on K/K^2 which is a R/K bimodule. We have

$$\begin{array}{ccc} R/K & \xrightarrow{\sim} & \cancel{\mathbb{C}[x]/(x^2)}\,\mathbb{C} \times \mathbb{C}[y] \\ x_- & \longmapsto & (1, 0) \\ y & \longmapsto & (0, y) \end{array}$$

Put another way $R/K = \mathbb{C}x_- \oplus \mathbb{C}[y]x_+$ where $[y, x_+] = 0$. Thus K/K^2 decomposes into four pieces $x_\pm(K/K^2)x_\pm$, which are stable under D.

Consider the generator x_-y for K. Its four components in K/K^2 are $x_+(x_-y)x_+$, $x_-(x_-y)x_+$, $x_+(x_-y)x_- = (x_+x_-)(yx_-) \in K^2 = 0$ and $x_-(x_-y)x_- = x_-^2(yx_-) = x_-yx_-$, because $x_-^2 = x_-$ in R/K. Now

$$D(x_+(x_-y)x_+) = x_+D(x_-)yx_+ \qquad \left(\begin{array}{l}\text{since } x_+x_- \in K \\ \text{and } x_-y \in K\end{array}\right)$$

$$= x_+2(x_+^2x_- - \cancel{x_+x_-^2})yx_+ = 2x_+^3x_-yx_+ = 2x_+(x_-y)x_+$$

$$D(x_- y x_-) = x_- D y\, x_-$$
$$= x_-(x_-^2 y - x_+ \cancel{x_-} y + y x_-^2 - y \cancel{x_-} x_+) x_-$$
$$= x_-^3 (y x_-) + (x_- y) x_-^3 = 2 x_- y x_-$$

$$D(x_-(x_- y) x_+) = x_- D(x_-) y x_+ + x_-^2 D(y) x_+$$
$$= x_- 2(x_+^2 \cancel{x_-} - x_+ \cancel{x_-^2}) y x_+ + x_-^2 (x_-^2 y - x_+ \cancel{x_-} y + \cancel{y x_-^2} - \cancel{y x_-} x_+) x_+$$
$$= x_-^3 (x_- y) x_+ = x_-(x_- y) x_+ .$$

Next take the relation $y x_-$

$$D(x_+ (y x_-) x_+) = x_+ y D(x_-) x_+$$
$$= x_+ y\, 2(x_+^2 x_- - \cancel{x_+ x_-^2}) x_+$$
$$= 2 x_+ (y x_-) x_+^3 = 2 x_+ (y x_-) x_+$$

$D(x_- y x_-) = 2 x_- y x_-$ as above; $x_-(y x_-) x_+ = 0.$

$$D(x_+ (y x_-) x_-) = x_+ D(y) x_-^2 + x_+ y D(x_-) x_-$$
$$= x_+ (\cancel{x_-^2} y - x_+ \cancel{x_-} y + y x_-^2 - y \cancel{x_-} x_+) x_-^2 + x_+ y\, 2(\cancel{x_+^2} x_- - x_+ \cancel{x_-^2}) x_-$$
$$= x_+ y x_-^2 = x_+ (y x_-) x_-$$

Finally

$$D(x_+ x_-) = \overset{-D(x_-)}{D(x_+)} x_- + x_+ D(x_-)$$
$$= -2(\cancel{x_+^2} x_- - x_+ x_-^2) x_- + x_+\, 2(x_+^2 x_- - \cancel{x_+ x_-^2})$$
$$= 2 x_+ x_- (x_-^2 + x_+^2) = 2 x_+ x\, (x_- + x_+) = 2 x_+ x_-$$

simpler $D(x^2 - 1) = D x x + x D x = x(x^2-1) x + x^2 (x^2 - 1)$
$$= 2 x^2 (x^2 - 1) = 2(x^2 - 1) + \cancel{(x^2-1)^2}$$

So the conclusion is that D on K/K^2 has eigenspace decomposition with the eigenvalues $1, 2$. The space where $D=2$ is $x_+(K/K^2)x_+ + x_-(K/K^2)x_-$ and where $D=1$ is $x_-(K/K^2)x_+ + x_+(K/K^2)x_-$.

See $\overbrace{\text{June 1991}}^{p396-438}$ for improvements on the above calculation. I recall that D above arises from $D(a) = 2\phi(a) = [da, Y]$ (ordinary bracket). However a more sophisticated choice is

$$D(a) = [da, Y] - \tfrac{1}{2}[[a,Y],Y]$$

Basically you understood things better three years ago concerning $RA \longrightarrow R_5A$. Some new ideas are: Functoriality which reduces calculations to the case $A_- = \mathbb{C}x_-$, $A_+ = \mathbb{C}x_+ + \mathbb{C}y$. Also the Cayley interpretation of the flow D which is slightly better than before.