June 10, 1992

I need to work on some sort of example. Suppose we consider the case of $\mathbb{C}[G]$, G discrete group. In this case there is a cyclic ~~space~~ set situation lying behind the cyclic theory of $\mathbb{C}[G]$, such that the latter is obtained by applying the functor $\mathbb{C}[?]$. Furthermore the cyclic set situation can be analyzed in terms of torsors over the circle.

Instead of bimodule complexes we use simplicial (objects in the category of) G-bisets, i.e. sets with a left and a right action of G which commute. Then our projective bimodule resolution of A corresponds to a free simp. G-biset ($= G\times G$ set) resolution of G. To obtain such a thing start with a free G-biset mapping onto E, say E, and form

1) $$E \leftleftarrows E\times^G E \Lleftarrow E\times^G E\times^G E \cdots$$

We know this is isomorphic to

2) $$E \leftleftarrows E\times_G E \Lleftarrow E\times_G E\times_G E \cdots$$

where $\times_G$ means fibre product over G. In 1) G acts on the left and on the right; in 2) G diagonally on the left and right.

The simplest choice for E is $G\times G$, whence 1) is

3) $$G\times G \leftleftarrows G\times G\times G \Lleftarrow G\times G^2\times G \cdots$$

the nerve of the groupoid given by G acting internally on $G\times G$. Let 3) be denoted P.

Then we can form the bisimplicial set

4) $$P \leftleftarrows P \times^G P \; \overset{\leftarrow}{\leftleftarrows} \; \cdots$$

and the cyclic object in the category of simplicial sets

5) $$P \times^G \leftleftarrows P \times^G P \times^G \; \overset{\leftarrow}{\leftleftarrows} \; \cdots\cdots$$

The main problem is to link 5) to the study of G torsors on the circle.

Remark: Because 1) and 2) are isomorphic canonically, ~~[illegible]~~ there is a unique way to put degeneracies into the above pre-simplicial objects.

~~Suppose I want to link P with torsors on $I = [0,1]$. The idea is that given a G-torsor P over $X \times [0,1]$ we trivialize it over the open covering of the form $\mathcal{U} \times \{[0,1), (0,1]\}$, where $\mathcal{U}$ is an open covering of X. Then for each $U \in \mathcal{U}$ we have two maps from U to G~~

The problem is to link the cyclic simplicial set 5) to G torsors on S^1. Now what one does (Burghelea) is to link $P \times^G$ to torsors on S^1. $P \times^G$ is the cyclic bar construction of G, and it is a cyclic set, so it's geometric realization has an S^1-action. One obtains an S^1-equivariant equivalence

$$|P \times^G| \sim BG^{S^1}.$$

(Observe $|P\times^G|$ is an S^1-space equipped with a non equivariant map $|P \times^G| \longrightarrow BG = |\text{nerve of } G|$, whence we get a canonical equivariant map

$$|P\times^G| \longrightarrow BG^{S^1}$$

which turns out to be a homotopy equivalence. Furthermore, as shown by Burghelea, one gets a h eq:

$$PS^1 \times^{S^1} BG^{S^1} \sim \underset{[n]\in\Lambda}{\xrightarrow{\text{holim}}} G^{n+1}$$

Remark that P is the nerve of the groupoid with objects $G\times G$ and with morphisms determined by the internal action: $g(g_0,g_1) = (g_0g^{-1}, gg_1)$. Similarly $\overbrace{P\times^G\cdots\times^G P}^{n+1}$, where the product is calculated in the category of simplicial sets:

$$(P\times^G\cdots\times^G P)_q = P_q\times^G P_q\times^G\cdots\times^G P_q$$

$$= (G\times G^q\times G)\times^G\cdots\times^G(G\times G^q\times G)$$

$$= G^{n+2}\times (G^{n+1})^q$$

should be the nerve of the groupoid with objects $\underbrace{G\times\cdots\times G}_{n+2}$ and G^{n+1} acting internally at the different $\times$ positions. Put another way P^{n+1} is the nerve of the $(n+1)$ fold product of the groupoid $(G\times G, G_{int})$ which should be

$$\Big(\underbrace{(G\times G)\times\cdots\times(G\times G)}_{n+1},\ G^{n+1}\Big)$$

Then taking the quotient by the internal G^n action at the $\times$ positions between the $G\times G$ factors gives the groupoid $(\underbrace{G\times\cdots\times G}_{n+2},\ G^{n+1})$.

We conclude therefore that

$$\underbrace{P \times^G \cdots \times^G P \times^G}_{n+1}$$

is the nerve of the groupoid (G^{n+1}, G^{n+1}) where G^{n+1} acts by conjugation twisted by the cyclic shift. Thus the cyclic object in simplicial sets

$$[n] \longmapsto [P \times^G]^{(n+1)}$$

is obtained by applying the nerve functor to the cyclic object in groupoids

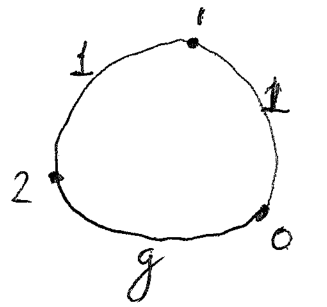
$$[n] \longmapsto (G^{n+1}, G^{n+1}_{\text{twisted conj.}})$$

Now the idea is to consider the scinded fibred category over Λ associated to this functor, and then replace the groupoids by equivalent smaller groupoids. For example $(G^{n+1}, G^{n+1}_{\text{twisted conjugation}})$ is equivalent to (G, G_{conj}). Specifically take $n=1$ and consider

$$(G, G_{conj}) \longrightarrow (G \times G, G_{int} \times G_{conj})$$
$$g \longmapsto (g, 1)$$

Note that a map $(g, 1) \longrightarrow (g', 1)$ in the latter is a pair (g_1, g_2) such that $(g_2 g g_1^{-1}, g_1 g_2^{-1}) = (g', 1)$, i.e. it is a pair (g_1, g_1) such that $g_1 g g_1^{-1} = g'$.

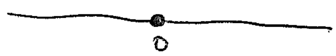
In terms of G-torsors over cyclic graphs we describe a torsor by 1-cocycles acted on by 0 cochains. A 1-cocycle assigns a group element to each edge of the graph (note 1-cocycle = 1 cochain), and a 0-cochain gives a group elts for each vertex. This gives the groupoid with objects G^{edges} acted on by G^{vertices}. In order to get the equivalent groupoid (G, G_{conj}) restrict the 1-cocycles to be the identity except for one edge:



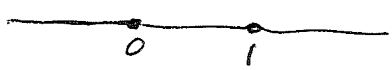
Then one must restrict the 0-chains to be equal at the vertices. Observe that the objects sets are preserved under the faces except for d_0.

Consider $P \leftleftarrows P\times^G P \;\overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}}\; \cdots$

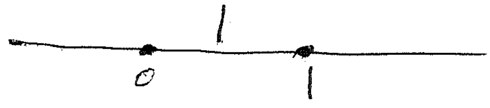
Here P is the groupoid $(G\times G, G_{\text{int}})$ which we can think of as 1-cochains ~~with~~ the action of 0-cochains on



(think of this as a poset maybe.) Similarly $P\times^G P$ is the groupoid $(G\times G\times G, G\times G)$ corresponding to cochains on



We can reduce this to the groupoid $(G\times G, G_{\text{int}})$ by restricting the 1 cochains to be the identity on the middle edge



Thus we use $G\times G \longrightarrow G\times G\times G$, $(g_0, g_1) \mapsto (g_0, 1, g_1)$

which is the degeneracy s_0.

June 11, 1992

Review: G group, let E be the simplicial set

$$G\times G \underset{1\times m}{\overset{m\times 1}{\leftleftarrows}} G\times G\times G \;\overset{\leftarrow}{\leftleftarrows}\; \cdots$$

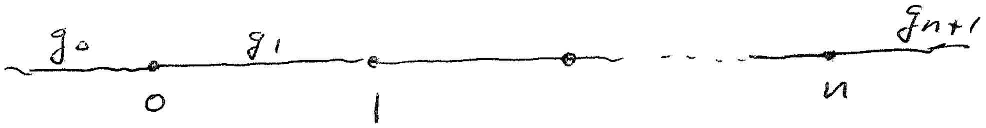
which is the nerve of the groupoid $(G\times G, G_{int})$. E is a free G biset resolution of G. Consider the bisimplicial set

$$E \leftleftarrows E\times^G E \;\overset{\leftarrow}{\leftleftarrows}\; E\times^G E\times^G E$$

It is obtained by applying the nerve functor to the simplicial groupoid

$$[n] \longmapsto (G^{n+1}, (G^n)_{int})$$

This groupoid of degree n describes torsors on the graph with open ends



i.e. it is the group of 1-chains with values in G acted on by 0-cochains.

$$C^1 = G^{n+2} = \{(g_0, \ldots, g_{n+1})\}$$

$$C^0 = G^{n+1} = \{(h_0, \ldots, h_n)\}$$

Action: $\quad h\cdot g = (g_0 h_0^{-1}, h_0 g_1 h_1^{-1}, \ldots, h_n g_{n+1})$

Face maps: $\quad d_i(g_0, \ldots, g_{n+1}) = (\ldots, g_i g_{i+1}, \ldots)$

$\qquad\qquad\quad d_i(h_0, \ldots, h_n) = (\ldots, \hat{h}_i, \ldots)$

$G\times G$ acts on the outside on these groupoids compatible with the faces.

We also have degeneracies

$$s_i(g_0, \dots, g_{n+1}) = (g_0, \dots, g_i, 1, g_{i+1}, \dots, g_n)$$
$$s_i(h_0, \dots, h_n) = (h_0, \dots, h_i, h_i, h_{i+1}, \dots, h_n)$$

Faces correspond to deleting a vertex in the graph whereas degeneracies correspond to collapsing an edge.

Consider this [redacted] simplicial groupoid as [redacted] a fibred category over Δ' (or Δ) and replace the fibres by equivalent groupoids. In the present case we can use the degeneracy operators

$$(s_0)^n : (G\times G, G_{int}) \longrightarrow (G^{n+1}, G^n_{int})$$
$$(g, g') \longmapsto (g, 1, \dots, 1, g')$$
$$h \longmapsto (h, h, \dots, h)$$

which ~~commutes with~~ are compatible with the faces:

$$d_i s_0^n = s_0 d_{i-1} s_0^{n-1} = \dots\dots = s_0^{i-1} d_1 s_0^{n-i+1}$$
$$= s_0^{i-1} s_0^{n-i} = s_0^{n-1} \qquad \text{if } 0 \le i \le n$$

Notice that this is an equivalence of groupoids, but there is not a canonical inverse. In general if $\mathcal{G}$ is a connected groupoid, [redacted] and X is an object of $\mathcal{G}$, [redacted] $G = \mathrm{Hom}_{\mathcal{G}}(X, X)$, then we have an equivalence of categories

$$(pt, G) \longrightarrow \mathcal{G}$$

[redacted] (i.e. fully faithful essentially surjective functor). However there is no canonical inverse functor. One must choose for each [redacted] object Y of $\mathcal{G}$ an isomorphism $Y \simeq X$. This is like choosing a maximal tree in a graph.

In our [redacted] situation

$$(G\times G,\ G_{int}) = \mathcal{G}_0 \longrightarrow (G\times G\times G,\ G_{int}^2) = \mathcal{G}_1$$
$$(g, g') \longmapsto (g, 1, g')$$

we have functors the other way given by d_0 and d_1.

$$\begin{tikzcd}
(g_0, g_1, g_2) \arrow[r, mapsto, "d_0"] \arrow[dr, mapsto, "d_1"'] & (g_0 g_1, g_2) \\
 & (g_0, g_1 g_2)
\end{tikzcd}$$

I also want a contraction to go with each of these. Thus in the case of d_0 we have

$$\mathcal{G}_1 \xrightarrow{d_0} \mathcal{G}_0 \xrightarrow{s_0} \mathcal{G}_1$$

$$\begin{tikzcd}
(g_0, g_1, g_2) \arrow[r, mapsto] \arrow[rr, mapsto, bend right, "id"'] & (g_0 g_1, g_2) \arrow[r, mapsto] & (g_0 g_1, 1, g_2) \arrow[d, "{[redacted]\ (g_1, 1)}"] \\
 & & (g_0, g_1, g_2)
\end{tikzcd}$$

Actually the groupoids being considered are "discrete", i.e. there is at most map between objects. Thus there is no choice in the isomorphism [redacted] $s_0 d_0 \simeq id$.

So at the moment we have a simplicial groupoid $[n] \longrightarrow \mathcal{G}_n$ and inclusions $s_0^n : \mathcal{G}_0 \hookrightarrow \mathcal{G}_n$ which are compatible with faces and degeneracies, and also are equivalences. There is however no natural retraction $\mathcal{G}_n \longrightarrow \mathcal{G}_0$ but rather $n+1$ of them corresponding to the vertices of $[n]$. For example if I use [redacted] the vertex $0 \in [n]$ I have the retraction

$$(g_0, \cdots, g_{n+1}) \longmapsto (g_0, g_1 \cdots g_{n+1})$$

and this is compatible with all the faces but d_0.

June 12, 1992

Review. If P is a projective bimodule resolution of A, then we have a pre-cyclic object of complexes

$$[n] \longmapsto [P\otimes_A]^{(n+1)}$$

such that face maps are quis. The problem is to reduce the hyperhomology

$$\mathbb{L}\varinjlim_{\Lambda'} \left\{ [n] \longmapsto [P\otimes_A]^{(n+1)} \right\}$$

to something involving the complex $P\otimes_A$. Specifically I would like to end up with a perturbed differential on $[P\otimes_A]\otimes\mathbb{C}[u]$ of the form

$$b + SB_1 + S^2B_2 + \cdots$$

Here $S(u^n) = u^{n-1}$.

For insight we consider the case $A = \mathbb{C}[G]$, and we work simplicially with $P = \mathbb{C}[?]$ applied to the nerve of the groupoid $(G\times G, G_{int})$. This means that in $[P\otimes_A]^{(n+1)}$ we use the tensor product of simplicial modules. Then we find

$$[P\otimes_A]^{(n+1)} = \mathbb{C}[?] \text{ applied to the nerve of } (G^{n+1}, G^{n+1}_{\text{twisted conjugation}})$$

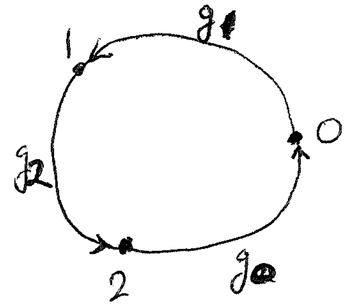
So we end up with a contravariant functor from Λ' to groupoids. The n-th groupoid describes G-torsors over the cyclic graph of size $[n]$.

The idea we wish to exploit is that we should be able to replace this contravariant functor $\Lambda' \longrightarrow$ Groupoids by the corresponding fibred category in groupoids over Λ', and then replace the n-th groupoid by the equivalent 0-th groupoid (G, G_{conj}). What really matters is the fibred category over Λ' of G-torsors over cyclic graphs.

We have over Λ' a specific fibred category with all fibres the groupoid (G, G_{conj}). Recall the groupoid

$$\mathcal{G}_n = \left(G^{n+1}, G^n_{int} \times G_{conj}\right)$$

can be identified with 1-cochains on the cyclic graph $[n]$ ~~equipped~~ with the action of 0-cochains



$$g = (g_0, g_1, g_2) \in C^1$$
$$h = (h_0, h_1, h_2) \in C^2$$
$$h \cdot g = \left(h_2 g_0 h_0^{-1}, h_0 g_1 h_1^{-1}, h_1 g_2 h_2^{-1}\right)$$

We use $s_0^n : \mathcal{G}_0 \hookrightarrow \mathcal{G}_n$ given by

$$g \longmapsto (g, 1, \dots, 1)$$
$$h \longmapsto (h, h, \dots, h)$$

Since s_0^n is an equivalence, we get an equivalence between the fibred sub category with fibres $s_0^n \mathcal{G}_0$ and the fibred category with fibres $\mathcal{G}_n$.

Observe that this gives us an explicit fibred category over Λ' (even Λ). Now our problem is to find a suitable way to compute its homology. We want the B operator to occur naturally.

Observe that $s_0^n \mathcal{G}_0 \subset \mathcal{G}_n$ is the subgroupoid resulting by restricting our 1-cochains to be identity except on the edge ending with the 0-th vertex. Observe also that we have $d_i s_0^n = s_0^{n-1}$.

Thus restricting to Δ' (or Δ) we get the product fibred category

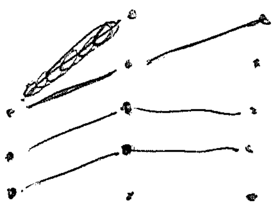
$$\Delta' \times \mathcal{G}_0$$

June 13, 1992

Recall the category Δ' has objects the linearly ordered sets $[n] = \{0, 1, \dots, n\}$, $n \geq 0$ with injective monotone maps. Presimplicial objects, i.e. contravariant functors on Δ', are sequences F_n, $n \geq 0$ with face maps $d_i : F_n \longrightarrow F_{n-1}$, $0 \leq i \leq n$ satisfying $\boxed{d_i d_j = d_j d_{i+1} \quad \text{if} \quad i \geq j}$. Canonical form for a simplicial operation is

$$d_{i_1} d_{i_2} \cdots d_{i_k} \qquad i_1 < i_2 < \dots < i_k$$

where $i_1, \dots, i_k$ are the vertices omitted:



$\partial_0 \quad \partial_3$

Λ' has objects the cyclically ordered sets $[n] = \{0, 1, \dots, n\}$, $n \geq 0$ with injective maps compatible with the orderings. Any map $[p] \hookrightarrow [q]$ in Λ' factors uniquely

$$[p] \xrightarrow{\sim} [p] \hookrightarrow [q]$$

where the first is an automorphism in the cyclic group and the second is a morphism in Δ'. Let $\tau : [p] \xrightarrow{\sim} [p]$ be the backward shift and t the transpose operator on contravariant functors (precyclic objects). Then a precyclic operator has the canonical form

$$t^j d_{i_1} \cdots d_{i_p}$$

with $i_1 < \cdots < i_p$ and $j <$ the order of the cyclic permutation in the appropriate degree. Let's work out $d_i t$:

$$\begin{aligned}(d_i t)(x_0, \dots, x_n) &= d_i(x_n, x_0, \dots, x_{n-1}) \\ &= (x_n, x_0, \dots, \hat{x}_{i-1}, \dots) \\ &= t(x_0, \dots, \hat{x}_{i-1}, \dots, x_n) \\ &= t\, d_{i-1}(x_0, \dots, x_n)\end{aligned}$$

for $1 \le i \le n$. If $i = 0$ we get $d_0 t = d_n$.
Thus the relations are

$$\boxed{\begin{aligned} d_i t &= t d_{i-1} & 1 \le i \le n \qquad & \text{in degree } n \\ &= d_n & i = 0 \qquad & \end{aligned}}$$

as well as $\boxed{t^{n+1} = 1}$

Check the standard complex

$$\begin{tikzcd} M_n \arrow[d, "b"] & M_n \arrow[l, "1-\lambda"'] \arrow[d, "b'"] \\ M_{n-1} & M_{n-1} \arrow[l, "1-\lambda"'] \end{tikzcd}$$

where $\lambda = (-1)^n t$ on M_n.

d_n for $i=0$
$t d_{i-1}$ for $i \ge 1$

$$\Big(\sum_{i=0}^{n} (-1)^i d_i\Big)\big(1 - (-1)^n t\big) = \sum_{i=0}^{n} (-1)^n d_i - \sum_{i=0}^{n} (-1)^{n+i} d_i t$$

$$= \sum_{i=0}^{n-1} (-1)^n d_i - \sum_{\cancel{i=1}\; i=0}^{n-1} (-1)^{n+i-1} t\, d_{i\cancel{-1}} + \cancel{(-1)^n d_n} - \cancel{(-1)^n d_n}$$

$$= (1-\lambda) \sum_{i=0}^{n-1} (-1)^i d_i$$

Consider the fibred category over Δ' defined by the functor which associates to $[n]$ considered as a cyclic graph the category of edges and vertices in the graph, where the ordering is the opposite of inclusion, so that from an edge there are two maps, one to the vertex at the beginning and the other the vertex at the end of the edge. The total category is equivalent to the categories of cyclic graphs equipped with either edge or vertex. The objects can be taken to be

for each $n \geq 0$ where the x marks the distinguished edge and vertex. Objects of the first form form a full subcategory isomorphic to Δ'. Objects of the second kind form a full subcategory isomorphic to Δ' with an initial object adjoined. More precisely if the distinguished vertex is removed we get the totally ordered sets $\{0, 1, \ldots, n-1\}$ for $n \geq 0$. There are morphisms where the edge is allowed to specialize to a vertex.

There are two maps from the n-th graph with edge to the n-th graph with vertex. One is the identity on $[n]$ the other is the backward shift. It should be possible to describe contravariant functors on the category of "pointed cyclic graphs" (i.e. cyclic graphs equipped with edge or vertex) as a collection of objects F_n^e, F_n^v $n \geq 0$ with face maps

$$d_i : F_n^e \longrightarrow F_{n-1}^e \qquad i = 0, \dots, n .$$

$$d_i : F_n^v \longrightarrow F_{n-1}^v \qquad i = 0, \dots, n-1$$

for $n \geq 1$ satisfying the relations

$$d_i d_j = d_j d_{i+1} \quad \text{for } i \geq j$$

and also maps

$$\iota, \tau : F_n^v \longrightarrow F_n^e \qquad n \geq 0$$

satisfying

$$\begin{cases} d_j \tau = \tau d_{j-1} & j = 1, \dots, n \\ & d_0 \tau = d_n \iota \\ d_j \iota = \iota d_j & j = 0, \dots, n-1 \end{cases}$$

on F_n^v.

Review: Let E be the simplicial set which is the nerve of $(G \times G, G_{int})$. It's a free G-biset resolution of G. Form precyclic object

$$[n] \longmapsto [E \times^G]^{(n+1)}$$

of simplicial sets, and apply $\mathbb{C}[?]$ to get a precyclic object of complexes. The homology (really hyperhomology) of this complex of Λ' modules should give the cyclic homology of $\mathbb{C}[G]$.

Next $[E \times^G]^{(n+1)} =$ nerve of $(G^{n+1}, G^{n+1}_{\text{twisted conj}}) = \mathcal{G}_n$, and $\mathcal{G}_n$ describes G-torsors over $[n]$ viewed as cyclic graph. By G-torsor we mean a covering graph and such a thing can be described by giving the fibres over the vertices and the monodromy all the edges. If we pick points in the torsor over the vertices, i.e. we identify the fibres over vertices with G, then

The monodromy for each edge is left mult by a group element, so this gives an object of G^E, E = edges. Then morphisms between torsors are given by action of G^V, V = vertices.

So $[n] \longmapsto \mathcal{G}_n$ is a precyclic groupoid (even cyclic). Now I am interested in applying $\mathbb{C}[\ ]\circ$ nerve to get a precyclic complex and taking $L \varinjlim_{\Lambda}$. The result should be the homology of the fibred category over Λ defined by this functor. This homology is unchanged if we replace this fibred category, denote it $(\Lambda', \mathcal{G})$ by an equivalent category.

I want to understand the homology of the fibred category of G-torsors over cyclic graphs. One can reduce to the case of G torsors with monodromy z, where z is a central element of G. Let's consider the case where $G = \mathbb{Z}$ with $z = 1$. This is in some sense the critical example.

A $\mathbb{Z}$-torsor with monodromy 1 over the cyclic graph $[n]$ can be identified with an infinite cyclic graph with translation $n+1$.

June 14, 1992

Notation. $\mathcal{C}$ for the category of cyclic graphs, $\mathcal{C}G$ for the category of cyclic graphs equipped with G-torsor.

$\mathcal{C}_\infty$ for the category of infinite cyclic graphs equipped with translation.

$\mathcal{C}(G,z)$ category of cyclic graphs equipped with G torsor having monodromy the central element z.

Note that $\mathcal{C}_\infty = \mathcal{C}(\mathbb{Z},1)$ as fibred categories over $\mathcal{C}$.

$\mathcal{C}_\infty \times_{\mathcal{C}} \mathcal{C}G$ describes infinite cyclic graph equipped with translation and a G-torsor over the graph with a compatible translation. There is a functor of fibred categories over $\mathcal{C}_\infty$

$$\mathcal{C}_\infty \times_{\mathcal{C}} \mathcal{C}G \longrightarrow \mathcal{C}_\infty \times \begin{Bmatrix} G \text{ torsors} \\ \text{with auto.} \end{Bmatrix}$$

assigning to a G torsor its set of sections over the infinite cyclic graph. This is an equivalence of fibred categories.

In the inverse direction given a G-torsor E with automorphism (i.e. left action of $\mathbb{Z}$) and an infinite cyclic graph Γ with translation, then we can form $\Gamma \times^{\mathbb{Z}} E$ which is a G-torsor over the cyclic graph $\Gamma/\mathbb{Z}$. In this way we obtain a map

$$\begin{Bmatrix} G \text{ torsors} \\ \text{with auto} \end{Bmatrix} \longrightarrow \mathrm{Homcart}_{/\mathcal{C}}(\mathcal{C}_\infty, \mathcal{C}G)$$

This general nonsense is fine but how does it help us deal with the original problem?

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Let us return to the model Λ' for $\mathcal{C}$ consisting of cyclic graphs C_n associated to $[n] = \{0, \dots, n\}$ with the standard cyclic ordering. Similarly use the model $\tilde{\Lambda}'$ for C_∞ consisting of $\tilde{C}_n =$ graph associated to $\mathbb{Z}$ equipped with the translation by $n+1$. We have a functor $C_n \longmapsto \mathcal{G}_n = (G^{n+1}, G^n \times G_{\mathrm{int}})$ from Λ' to groupoids; $\mathcal{G}_n$ can be identified with the full subcategory of G-torsors over C_n which are trivialized over the vertices of C_n.

June 16, 1982

Review equivalence of categories.
First recall adjoint functors

$$\mathcal{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{Y}$$

Adjunction maps $\alpha : FG \longrightarrow id_{\mathcal{X}}$, $\beta : id_{\mathcal{Y}} \longrightarrow GF$ must satisfy condition that

$$FX \xrightarrow{F(\beta_X)} FGF(X) \xrightarrow{\alpha_{F(X)}} F(X)$$

$$G(Y) \xrightarrow{\beta_{G(Y)}} GFG(Y) \xrightarrow{G(\alpha_Y)} G(Y)$$

are the identity. Then α, β define inverse bijections

$*$
$$\operatorname{Hom}_{\mathcal{Y}}(FX, Y) \underset{\beta^*}{\overset{\alpha_*}{\rightleftarrows}} \operatorname{Hom}_{\mathcal{X}}(X, GY)$$
$$u \in \operatorname{Hom}_{\mathcal{Y}}(FX, Y), \qquad v \in \operatorname{Hom}_{\mathcal{X}}(X, GY)$$

$$\alpha_*(v) : \quad FX \xrightarrow{F(v)} FGY \xrightarrow{\alpha_Y} Y$$

$$\beta^*(u) : \quad X \xrightarrow{\beta} GFX \xrightarrow{G(u)} GY$$

To check $\beta^* \alpha_* = id$, it suffices to consider when $X = GY$ and $v = id_{GY} : GY \longrightarrow GY$. Then $\alpha_*(v)$ is

$$FGY \xrightarrow{id} FGY \xrightarrow{\alpha_Y} Y$$

$\therefore$ $\alpha_*(v) = \alpha_Y$, and $\beta^* \alpha_Y$ is

$$GY \xrightarrow{\beta_{G(Y)}} GFGY \xrightarrow{G(\alpha_Y)} GY$$

which by assumption is id_{GY}

Now an equivalence of categories is $\mathcal{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{Y}$ together with

$$FG \xrightarrow[\sim]{\alpha} id \qquad id \xrightarrow[\sim]{\beta} GF$$

such that

$$F \xrightarrow{F(\beta)} FGF \xrightarrow{\alpha_F} F$$
$$G \xrightarrow{\beta_G} GFG \xrightarrow{G(\alpha)} G$$

are the identity.

Consider the corresponding game for complexes

$$X \underset{i}{\overset{p}{\rightleftarrows}} Y \qquad pi \xrightarrow{h_X} 1 \qquad [d,h_X] = 1 - pi$$
$$1 \xrightarrow{-h_Y} ip \qquad [d,h_Y] = 1 - ip$$

Then we want

$$i \xrightarrow{-h_Y i} ipi \xrightarrow{ih_X} i$$

$$p \xrightarrow{h_X p} pip \xrightarrow{-ph_Y} p$$

to be zero, which amounts to the conditions

$$[i,h] = [p,h] = 0$$

we have already encountered.

Here's a picture of an equivalence of groupoids. Let $\mathcal{G}$ be a connected groupoid, let x be an object and $G = Hom_{\mathcal{G}}(x,x)$. Then we have an inclusion of groupoids

$$(pt, G) \subset \mathcal{G}$$

which is an equivalence. To obtain the inverse functor we choose for each object y an arrow $\varphi_y^x : x \to y$. For $y = x$ we take the identity.

Then we can extend the collection of φ^x_y, $y \in Ob\,\mathcal{G}$, to a collection of arrows $\varphi^y_z : y \to z$ for $y, z \in Ob\,\mathcal{G}$ by setting $\varphi^y_z = \varphi^x_z \cdot (\varphi^x_y)^{-1}$. Let $S = Ob\,\mathcal{G}$ and let $\tilde{S}$ be the category with object set S and a unique arrow from x to y for every $(x,y) \in S \times S$. Then we have an isomorphism of categories

$$\mathcal{G} \simeq (pt, G) \times \tilde{S}$$

The nerve of $\mathcal{G}$ is thus the nerve of (pt, G) times the contractible simplicial set

$$S \leftleftarrows S \times S \; \overset{\textstyle\leftarrow}{\leftleftarrows} \; S \times S \times S \; \leftleftarrows\!\!\leftleftarrows \cdots$$

Observe there is a close link between equivalence of categories and Morita equivalence. If two categories $\mathcal{C}, \mathcal{C}'$ are equivalent then the categories $\underline{Hom}(\mathcal{C}, \mathcal{D})$, $\underline{Hom}(\mathcal{C}', \mathcal{D})$ are equivalent. When $Ob\,\mathcal{C}$ is finite and $\mathcal{D}$ = Vector spaces, then $\underline{Hom}(\mathcal{C}, \mathcal{D})$ is the category of modules over the incidence (arrow) algebra. So when both $Ob\,\mathcal{C}$ and $Ob\,\mathcal{C}'$ are finite, $\mathcal{C}$ equiv to $\mathcal{C}'$ implies the incidence algebras are Morita equivalent.

Ex. $\mathcal{C}' = \tilde{S}$ as above, $\mathcal{C} = pt$. A functor $\tilde{S} \longrightarrow$ vector spaces is a family of vector spaces V_x, $x \in S$ together with compatible isomorphisms $V_x \xrightarrow{\sim} V_y$, $\forall x, y \in S$. This functor category is equivalent to vector spaces.

June 17, 1992

Consider the cyclic groupoid with $\mathcal{G}_n = (G^{n+1}, G^n_{int} \times G_c)$ describing G-torsors over the cyclic graph C_n. We have then a cyclic object of complexes $\mathbb{C}[\mathrm{Nerv}\, \mathcal{G}_n]$, whence a mixed complex (M, b, B), where M is the total complex of the first two columns in the bicomp of complexes

$$\begin{array}{ccccccc} \downarrow & & \downarrow & & & & \\ K_1 & \xleftarrow{1-\lambda} & K_1 & \xleftarrow{N_\lambda} & & \leftarrow & \\ \downarrow b & & \downarrow -b' & & & & \\ K_0 & \xleftarrow{1-\lambda} & K_0 & \xleftarrow{N_\lambda} & & \leftarrow & \end{array} \qquad K_n = \mathbb{C}[\mathrm{Nerv}\, \mathcal{G}_n]$$

In this example we have degeneracies, so the b' column should have a canonical contraction $s = s_0$. This means we can take M = total complex associated to the simplicial complex $\mathbb{C}[\mathrm{Nerv}\, \mathcal{G}_n]$.

Now we have $K_0 \hookrightarrow M$ which is a quis, and we have a B operator on K_0. The problem is to find what we need in order to show that K_0 and M as mixed complexes give the same cyclic theory. Here's a possible argument.

Consider double complexes X, Y

$$\begin{array}{ccccccccc} \xleftarrow{\theta} & X_0 & \xleftarrow{\theta} & X_1 & \xleftarrow{\theta} & X_2 & \xleftarrow{\theta} \\ & \uparrow i & \nearrow k & \uparrow i & \nearrow k & \uparrow i & \nearrow k \\ \xleftarrow{\theta} & Y_0 & \xleftarrow{\theta} & Y_1 & \xleftarrow{\theta} & Y_2 & \xleftarrow{\theta} \end{array}$$

Here we view a double complex as a complex of complexes, so the X_n, Y_n are complexes in the vertical direction, and θ (maybe $-\theta$) is the differential in the horizontal direction. i is a map compatible with the vertical differentials, and $k: X_n \to X_{n+1}$ is a homotopy from $i\theta$ to θi

$$[d,k] = -i\theta + \theta i = [\theta, i]$$

When is $i+k$ a map of double complexes?

$$[d+\theta,\ i+k] = \overset{0}{\cancel{[d,i]}} - \overbrace{[\theta,i]}^{+[d,k]} + [d,k] - [\theta,k]$$
$$= -[\theta,k]$$

In the HPT situation

$$\begin{array}{c} X \\ p\downarrow\uparrow i \\ Y \end{array} \qquad \begin{array}{l} [d,h] = 1 - ip \\ pi = 1. \end{array}$$

we have $\displaystyle i' = \frac{1}{1-h\theta}\, i = i + h\theta i + h\theta h\theta i + \cdots$

Thus if $i' = i + k$ where $k: Y_n \longrightarrow X_{n-1}$, it seems we want to have $h\theta h\theta i = 0$.

A fibred category whose base change functors are equivalences and whose fibres are connected groupoids is, from the homotopy viewpoint, a fibration with fibre $K(G,1) = \mathrm{Nerv}(pt, G)$. This is a minimal simplicial complex so such fibrations are classified by bundles for the simplicial group $\mathrm{Aut}(K(G,1))$. This should be essentially the same as the category $\underline{\mathrm{Aut}}_{Cat}(pt, G)$. In general $\underline{\mathrm{Hom}}_{Cat}((pt,G),(pt,G'))$ has objects $\mathrm{Hom}_{gps}(G,G')$ and the morphisms are given by the conjugation action of G'. Thus the simplicial group $\mathrm{Aut}(K(G,1))$ should correspond to the complex

$$G \longrightarrow \mathrm{Aut}(G)$$

having $\pi_0 = \mathrm{Out}(G)$, $\pi_1 = Z_G$ (center). Apparently this complex is what one calls a crossed module.

June 18, 1992

Consider the following set-up.

1) A base category $\mathcal{Y}$

2) a contravariant functor $\mathcal{Y} \to Cat$, $y \mapsto \mathcal{X}_y$

3) a family of equivalences $\mathcal{X}'_y \simeq \mathcal{X}_y$, say that $i : \mathcal{X}'_y \hookrightarrow \mathcal{X}_y$ is a full embedding, and that one is given a retraction $p : \mathcal{X}_y \to \mathcal{X}'_y$, $pi = id_{\mathcal{X}'_y}$ such that ip is isomorphic to the identity on $\mathcal{X}_y$. (In this case there should be a unique isomorphism $id_{\mathcal{X}_y} \xrightarrow[\sim]{\alpha} ip$ such that $p(\alpha) : p \to pip = p$ and $\alpha_i : i \to ipi = i$ are the identity.)

We consider the contravariant functor $y \mapsto C(\mathcal{X}_y)$ $\mathbb{Z}[\mathrm{Nerv}\,\mathcal{X}_y]$ from $\mathcal{Y}$ to complexes and form the double complex

$$C_p(\mathcal{Y}, y \mapsto C_q(\mathcal{X}_y)) = \bigoplus_{y_0 \to \cdots \to y_p} C_q(\mathcal{X}_{y_p})$$

Since we are given a SDR $C(\mathcal{X}'_y) \underset{i}{\overset{p}{\rightleftarrows}} C(\mathcal{X}_y)$ we can apply HPT to the total complex of this double complex and obtain a twisted diffl on

$$\bigoplus_{p,q} \bigoplus_{y_0 \to \cdots \to y_p} C_q(\mathcal{X}'_{y_p})$$

Let us look at this over a 1-simplex $y_0 \to y_1$. Recall the HPT formula for the twisted (or perturbed) differential $d - \theta'$ where $\theta' = p\theta\frac{1}{1-h\theta}i$, $d-\theta$ being the original differential in the double complex. Associated to this arrow $y_0 \xrightarrow{u} y_1$ is the map

$$C(\mathcal{X}'_{y_1}) \xrightarrow{i} C(\mathcal{X}_{y_1}) \xrightarrow{u^*} C(\mathcal{X}_{y_0}) \xrightarrow{p} C(\mathcal{X}'_{y_0})$$

At this point it is clear that we are treating the retraction equivalence (α, β) as ~~giving~~ a connection. We use this to obtain a cleavage in the fibred category over Y with fibres $\mathcal{X}'_y$. In fact perhaps we should regard a cleavage in a fibred category as the analogue of a connection. Then the cocycle $c_{fg} : g^* f^* \xrightarrow{\sim} (fg)^*$ is the curvature.

June 19, 1992

Suppose we have a base category $\mathcal{Y}$, a functor $y \longmapsto \mathcal{X}_y$ from $\mathcal{Y}$ to Cat an equivalence of categories

$$1) \qquad \mathcal{X}'_y \underset{i}{\overset{p}{\rightleftarrows}} \mathcal{X}_y \qquad \begin{array}{l} ip \simeq id \\ pi \simeq id \end{array}$$

for each object y. We have a scinded cofibred category $\mathcal{X}$ over $\mathcal{Y}$ with fibres $\mathcal{X}_y$, and a cofibred category $\mathcal{X}'$ over $\mathcal{Y}$ with fibres $\mathcal{X}'_y$, and an equivalence of cofibred categories. To construct $\mathcal{X}'$ define for each arrow f in $\mathcal{Y}$ the cobase change functor $\mathcal{X}'_{s(f)} \longrightarrow \mathcal{X}'_{t(f)}$ to be pf_*i. Given composable arrows $\xrightarrow{f}\xrightarrow{g}$ we have

$$2) \qquad (pf_*i)(pg_*i) \xrightarrow{\sim} p(fg)_*i$$

obtained from the isomorphism $ip \xrightarrow{\alpha} id$. Let's check this satisfies the cocycle condition. The square

$$\begin{array}{ccc} pf_*i\,pg_*i\,ph_*i & \longrightarrow & pf_*i\,pg_*h_*i \\ \downarrow & & \downarrow \\ pf_*g_*i\,ph_*i & \longrightarrow & pf_*g_*h_*i \end{array}$$

commutes because

$$\begin{array}{ccc} ipg_*ip & \xrightarrow{ipg_*\alpha} & ipg_* \\ {\scriptstyle \alpha g_* ip}\downarrow & & \downarrow{\scriptstyle \alpha g_*} \\ g_*ip & \xrightarrow[g_*\alpha]{} & g_* \end{array}$$

commutes; this is the fact that composition of functors is compatible with natural transformations.

Next we have $p\,id_*i = pi = id$ provided we assume $pi = id$ i.e. p is a retraction with respect to the embedding i. Also when $f = id$, the isomorphism 2)

$$pg_* i = (p\, id_* i)(p g_* i) \xrightarrow{\alpha} p\, id_* g_* i = pg_* i$$

will be the identity when $\alpha : ip \simeq id$ satisfies the condition that $p = pip \xrightarrow{p(\alpha)} p$ is the identity. Similarly 2) when $g = id$ will be the identity when $i = ipi \xrightarrow{\alpha \cdot i} i$ is the identity. (These conditions on α mean that α is the isom. compatible with the identity isom of pi and 1, and they correspond to the SDR conditions $ph = hi = 0$.)

Suppose we replace the categories X_y, X'_y with complexes. In fact to simplify, let us suppose that Y has a single object, say Y is (pt, G). Thus we suppose given a complex K on which G acts, and an SDR

$$K \underset{i}{\overset{p}{\rightleftarrows}} L \qquad \begin{matrix} [d,h] = 1 - ip \\ pi = 1 \end{matrix} \qquad ph = hi = h^2 = 0$$

To $g \in G$ we associate pgi on L, and to a pair $g_1, g_2 \in G$ we associate the homotopy

$$pg_1 i \; pg_2 i \xrightarrow{pg_1 h g_2 i} pg_1 g_2 i$$

Do we have the cocycle condition, i.e. does

$$\begin{tikzcd}
(pg_1 i)(pg_2 i)(pg_3 i) \arrow[r] \arrow[d] & (pg_1 i)(pg_2 g_3 i) \arrow[d] \\
(pg_1 g_2 i)(pg_3 i) \arrow[r] & pg_1 g_2 g_3 i
\end{tikzcd}$$

commute. Thus look at

$$\begin{tikzcd} \iota p g \iota p \arrow[r, "\iota p g h"] \arrow[d, "h g \iota p"'] & \iota p g \arrow[d, "h g"] \\ g \iota p \arrow[r, "g h"'] & g \end{tikzcd}$$

$$[d, hg\iota p] = [d,h]g\iota p = g\iota p - \iota p g \iota p$$

$$[d, gh] = g - g\iota p$$

$$[d, \iota p g h] = \iota p g - \iota p g \iota p$$

$$[d, hg] = g - \iota p g$$

Is $\quad hg + \iota p g h \stackrel{?}{=} gh + hg\iota p$

i.e $\quad hg[d,h] \stackrel{?}{=} [d,h]gh.$ But

$$[d, hgh] = [d,h]gh - hg[d,h]$$

[redacted] Thus we see that the cocycle condition is probably not satisfied in general. One has to strengthen the condition $h^2 = 0$ to $hgh = 0$, at least to hgh being closed. What this probably means is that we have higher order homotopies to consider. If we [redacted] work with chain complexes of length 1 (these correspond to groupoids), then $hgh = 0$ is automatic for degree reasons.

Observe that in the setup just discussed we have G acting on the complex K and L a homotopy equivalent complex, thus L has an action up to higher homotopy of G. Another way to say this is that we have a homomorphism up to higher homotopy from $A = \mathbb{Z}[G]$ to [redacted].

End(L). One already has a notion of what this should mean, namely, a twisting cochain

$$\tau : \mathcal{B}(A) \longrightarrow \mathrm{End}(L)$$

where $\mathcal{B}$ is the bar construction. Twisting cochain means that in the DG algebra <u>Hom</u> $(\mathcal{B}(A), \mathrm{End}(L))$ we have

$$d_{tot}\tau + \tau^2 = 0.$$

So the natural question is whether HPT is compatible with the twisting cochain idea.

Let $\theta : \mathcal{B}(A) \longrightarrow A$ be the canonical twisting cochain, so that $\delta\theta + \theta^2 = 0$. We can consider θ as an element of degree $(1,0)$ in $\mathrm{Hom}(\mathcal{B}(A), \mathrm{End}(K))$ such that $\delta\theta + \theta^2 = 0$ and $d\theta = 0$, d being the differential of $\mathrm{End}(K)$ essentially. We have $e = \iota p \in \mathrm{End}(K)$ is an idempotent such that

$$\circledast \qquad e\,\mathrm{End}(K)\,e \simeq \mathrm{End}(L).$$

Finally the homotopy h is an element of degree $(0,1)$ such that $dh = 1 - e$ and $\delta h = 0$.

We look for a twisting cochain τ in

$$\mathrm{Hom}(\mathcal{B}(A), \mathrm{End}(L)) = e\,\mathrm{Hom}(\mathcal{B}(A), \mathrm{End}(K))\,e$$

of the form $\tau = e\theta e + \text{higher}$. Note that $\alpha \longmapsto p\alpha\iota$ is just $\alpha \longmapsto e\alpha e$ relative to the identification $\circledast$. Thus HPT tells us that

$$\boxed{\tau = e\theta e - e\theta h\theta e + \cdots = e\theta\frac{1}{1+h\theta}e}$$

should be our twisting cochain.
Check this using

$$d_{tot}\theta + \theta^2 = 0 \quad , \quad d_{tot}h = 1-e$$

We have

$$d_{tot}\,\tau = e\left\{(-\theta)^2\frac{1}{1+h\theta} + \theta\frac{1}{1+h\theta}\left((1-e)\theta + h\theta^2\right)\frac{1}{1+h\theta}\right\}e$$

$$\tau^2 = e\left\{\theta\frac{1}{1+h\theta}\,e\theta\frac{1}{1+h\theta}\right\}e$$

$$\therefore\ d_{tot}\tau + \tau^2 = e\left\{-\theta^2\frac{1}{1+h\theta} + \theta\frac{1}{1+h\theta}(\theta + h\theta^2)\frac{1}{1+h\theta}\right\} = 0$$

<u>June 21, 1992</u> (in Strasbourg)

Problem: Given an SDR $X \underset{i}{\overset{p}{\rightleftarrows}} Y$, $[d,h]=1-ip$ and an A-module structure on X, we have [redacted] seen there are two examples of HPT.

1) Operators on $C\otimes X$, where C is the bar construction of A

2) Elements of [redacted] the DG alg $\mathrm{Hom}(C, \mathrm{End}_C X)$.

The problem is to relate these two instances of HPT.

Ideas: 1) Instead of $C\otimes X$, which is linked to homology, consider $\mathrm{Hom}(C,X)$, which is linked to cohomology. $\mathrm{Hom}(C,X)$ is the complex of cochains with values in X; this is like $\Omega(M,E)$. It is naturally a module over $\mathrm{Hom}(C,\mathrm{End}\,X)$ which is like $\Omega(M,\mathrm{End}\,E)$.

2) $\mathrm{Hom}(C,\mathrm{End}\,X)$ acts on $C\otimes X$. In general if C is a coalgebra, [redacted] N is a right C-comodule, R is [redacted] an algebra and X is a left R-module, then given $u\in \mathrm{Hom}(C,R)$ we have

$$N\otimes X \xrightarrow{\Delta_N\otimes 1} N\otimes C\otimes X \xrightarrow{1\otimes u\otimes 1} N\otimes R\otimes X \xrightarrow{1\otimes m_\ell} N\otimes X$$

making $C\otimes X$ a module over $\mathrm{Hom}(C,R$

3) GNS [redacted]. Observe that when you have an SDR $X \underset{i}{\overset{p}{\rightleftarrows}} Y$ with A-acting on X, then the GNS algebra $\Gamma(A\to RA)$ acts.

(In Heidelberg).

<u>Observation</u>: Recall that I would like to be able to treat covariant (contravariant) functors from a category $\mathcal{C}$ to vector spaces as left (right) modules over an algebra. This doesn't work when there are infinitely many objects. For example let us consider the category with object set S and where the only maps are identity maps. A functor $S \to$ Vector spaces is just a family of vector spaces $(V_s, s \in S)$. Equivalently we have a vector space V equipped with a grading $V = \bigoplus_{s \in S} V_s$. When S is finite, such a vector space with grading is the same this as a module over $A = \mathbb{C}^S$. It's probably true that in general A is the endomorphism ring of the functor

$$(V_s, s\in S) \longmapsto \oplus V_s$$

but when S is infinite not every $\mathbb{C}^S$ module M is of this form. One has projectors $e_s \in \mathbb{C}^S$ which are orthogonal, and one has an injection

$$\bigoplus_{s \in S} e_s M \hookrightarrow M$$

which is not surjective in general, e.g. $M = \prod_{s\in S} \mathbb{C} = \mathbb{C}^S$.

The observation is that instead of the algebra $\mathbb{C}^S$ we should use the coalgebra $\mathbb{C}[S]$ where $\Delta s = s \otimes s$. Then the category of comodules for $\mathbb{C}[S]$ is exactly the category of graded vector spaces: $V = \bigoplus_{s\in S} V_s$. In effect the dual of $\mathbb{C}[S]$ is $\mathbb{C}^S$, but this times one has $M = \bigoplus_{s \in S} e_s M$ because the counit condition on

$$M \xrightarrow{\Delta} \mathbb{C}[S] \otimes M$$

says that composing with $\mathbb{C}[S] \otimes M \longrightarrow M$, where $s \otimes m \longmapsto m$, gives the identity.

June 22, 1992 (52 years old)

Notation: In general let $\mathcal{B}$ be the standard normalized resolution

$$A\otimes A \xleftarrow{b'} A\otimes \bar{A}\otimes A \xleftarrow{b'} A\otimes \bar{A}^{\otimes 2}\otimes A \longleftarrow$$

When A is augmented let C be its bar construction

$$\mathbb{C} \xleftarrow{0} \bar{A} \xleftarrow{m} \bar{A}^{\otimes 2} \longleftarrow \cdots$$

Then $\mathcal{B} = A\otimes C\otimes A$ where the differential uses the twisting cochain $\tau : C \to A$ at the beginning and end.

Let X be a complex of (left) A-modules. Consider $\mathcal{B}\otimes_A X$. This is a standard resolution of X which is a complex of free modules. Suppose we have a SDR situation $X \underset{i}{\overset{p}{\rightleftarrows}} Y$, $[d,h] = 1-ip$, which is not compatible with the A-module structure on X. We want to use HPT to construct a corresponding SDR of $\mathcal{B}\otimes_A X$.

We have $\mathcal{B}\otimes_A X = A\otimes C\otimes X$ with $C = T(\Sigma\bar{A})$.
We also have the DG algebra

$$\mathrm{Hom}_{A^e}(\mathcal{B}, \mathrm{End}(X)) = \mathrm{Hom}(C, \mathrm{End}(X)) = C_N^{\cdot}(A, \mathrm{End}(X))$$

operating on $\mathcal{B}\otimes_A X$. This is a bigraded DG alg. in general with Hochschild diffl on cochains and the diffl d on $\mathrm{End}(X)$.

Let's consider the augmented case and write δ for the differential on $A\otimes C$, d for the differential in X and extend these to $A\otimes C\otimes X$ so they anti-commute. On the double complex $\mathcal{B}\otimes_A X = A\otimes C\otimes X$ the horizontal diffl is $\delta+\tau$, the vertical diffl is d.

We can ~~treate~~ treat the total differential $\delta+\tau+d$ as a perturbation of $\delta+d$ or as a perturbation of d. It would be nice to know they lead to the same answer.

HPT in the case of $\delta + d$ and the perturbation τ leads to the differential $\delta + d + \tau'$ on $A \otimes C \otimes Y$, where

$$\tau' = p\tau \frac{1}{1+h\tau} i$$

HPT in the case of d and the perturbation $\delta + \tau$ leads to the differential

$$d + p(\delta+\tau)\frac{1}{1+h(\delta+\tau)} i$$

Now $(\delta+\tau)^2 = 0$ implies

$$(\delta+\tau)\,(h(\delta+\tau))^n = (\delta+\tau)[h,\delta+\tau]^n = (\delta+\tau)[h,\tau]^n$$

Thus

$$d + p(\delta+\tau)\frac{1}{1+h(\delta+\tau)} i = d + p(\delta+\tau)\frac{1}{1+[h,\tau]} i$$

$$= d + \delta p \frac{1}{1+[h,\tau]} i + p\tau\frac{1}{1+[h,\tau]} i$$

Assume $h^2 = ph = hi = 0$. Then

$$\begin{aligned} p[h,\tau]^n i &= p\tau h[h,\tau]^{n-1} i \\ &= p\tau h\tau h[h,\tau]^{n-2} i \\ &= p\underbrace{\tau h \ldots \tau h}_{n} i = 0 \end{aligned}$$

similarly

$$\begin{aligned} \tau(h\tau)^n i &= \tau h\tau h\tau \ldots h\tau i \\ &= \tau[h,\tau]^{n-1}\underbrace{h\tau i}_{[h,\tau]i} = \tau[h,\tau]^n i \end{aligned}$$

as $hi = 0$.

Conclude

$$\delta p \frac{1}{1+[h,\tau]} i = \delta p i = \delta$$

$$p\tau \frac{1}{1+[h,\tau]} i = p\tau\frac{1}{1+h\tau} i$$

so the results are the same.

Here's a better calculation

First we have for the perturbation τ of $\delta + d$ that

$$p' = p\frac{1}{1+\tau h} = p\frac{1}{1+h\tau}\frac{1}{1+\tau h} \qquad \text{as } ph=0$$

$$= p\frac{1}{(1+\tau h)(1+h\tau)} \qquad \text{as } x^{-1}y^{-1} = (yx)^{-1}$$

$$= p\frac{1}{1+[\tau,h]} \qquad \text{as } h^2 = 0.$$

Thus $p(\tau h)^n = p[\tau,h]^n$ for all n. Similarly

$$i' = \frac{1}{1+h\tau}\,i = \frac{1}{1+h\tau}\frac{1}{1+\tau h}\,i \qquad \text{as } hi=0$$

$$= \frac{1}{(1+\tau h)(1+h\tau)}\,i = \frac{1}{1+[\tau,h]}\,i \qquad \text{as } h^2=0.$$

and

$$h' = h\frac{1}{1+\tau h} = h\frac{1}{1+h\tau}\frac{1}{1+\tau h}$$

$$= h\frac{1}{(1+\tau h)(1+h\tau)} = h\frac{1}{1+[h,\tau]}$$

and

$$\tau' = p\tau\underbrace{\frac{1}{1+h\tau}\,i}_{i'} = p\tau\frac{1}{1+[\tau,h]}\,i$$

Second for the perturbation $\delta+\tau$ of d we have

$$p' = p\frac{1}{1+(\delta+\tau)h} = p\frac{1}{1+h(\delta+\tau)}\frac{1}{1+(\delta+\tau)h} \qquad \text{as } ph=0$$

$$= p\frac{1}{1+[h,\delta+\tau]} \qquad \text{as } h^2=0$$

$$= p\frac{1}{1+[h,\tau]} \qquad \text{as } [h,\delta]=0$$

Similarly

$$i' = \frac{1}{1+h(\delta+\tau)}\,i = \frac{1}{1+[\delta+\tau,h]}\,i \qquad \text{as } hi=0 \text{ and } h^2=0$$

$$= \frac{1}{1+[\tau,h]}\,i$$

$$h' = h\frac{1}{1+(\delta+\tau)h} = h\frac{1}{1+[\delta+\tau,h]} \qquad \text{as } h^2=0$$

$$= h\frac{1}{1+[\tau,h]}$$

and

$$\delta+\tau' = p(\delta+\tau)\,\iota'$$

$$= p(\delta+\tau)\frac{1}{1+[h,\tau]}\,i$$

$$= \delta\,\underbrace{p\frac{1}{1+[h,\tau]}\,i}_{p\frac{1}{(1+\tau h)(1+h\tau)}i \;=\; p\frac{1}{1+h\tau}\frac{1}{1+\tau h}i \;=\; pi = 1} + \; p\tau\frac{1}{1+[h,\tau]}\,i$$

$$= \delta + p\tau\frac{1}{1+[h,\tau]}\,i$$

<u>Conclusion:</u> $B\otimes_A X = A\otimes C\otimes X$ is a double complex in general whose horizontal differential $d_B\hat{\otimes}_A 1$ can be written $\delta+\tau$ in the augmented case. We get the same result from HPT if we regard $\delta+\tau+d$ as

- $\delta+d$ with perturbation τ or
- d with perturbation $\delta+\tau$

provided we assume that $ph=h^2=hi=0$.

June 23, 1992

Consider a chain complex (X, d) of A-modules equipped with a contraction $[d, h] = 1$, where h is not necessarily compatible with [redacted] the A-module structure. Look at what we have in $\mathrm{End}(X)$. We have a homomorphism $A \longrightarrow \mathrm{End}^0(X)$ and an element $h \in \mathrm{End}^{-1}(X)$ such that [redacted] $d(h) = 1$. We thus get a map of DG algebras

$$T_A(A \otimes A) \longrightarrow \mathrm{End}(X)$$

where the image of $1 \otimes 1$ is h. Here $T_A(A \otimes A)$ is the DG algebra

$$\xrightarrow{b'} A \otimes A \otimes A \xrightarrow{b'} A \otimes A \xrightarrow{b'} \overset{0}{A} \longrightarrow 0$$

If we further require $h^2 = 0$, then we get the quotient algebra

$$\xrightarrow{b'} A \otimes \bar{A} \otimes A \xrightarrow{b'} A \otimes A \xrightarrow{b'} \overset{0}{A} \longrightarrow 0$$

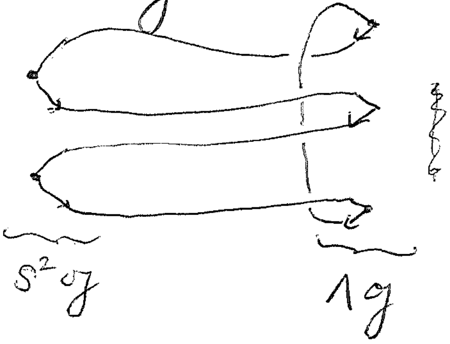
Thus we find that the standard annormalized and [redacted] normalized resolutions occur as universal gadgets for pairs consisting of a A-module chain complex equipped with not necessarily linear contraction.

One can pose a similar problem where one has an SDR instead of contraction. Lets defer this problem on the principal that the coalgebra structure on the standard norm. [redacted] resolution $\mathcal{B}$ is more important [redacted] than the DG algebra structure on $\mathrm{Cone}(\mathcal{B} \to A)$.

June 27, 1992

Ideas: Recall the following way to look at the character map from algebraic K-theory to negative cyclic theory. K-theory of A is built out of representations of groups over A, a representation $G \longrightarrow \mathrm{Aut}(P)$, P finitely gen. proj. A-module is equivalent to a "Morita map" $\mathbb{C}[G] \longrightarrow A$. Such a map maps the cyclic theory of $\mathbb{C}[G]$ to that of A. But for $\mathbb{C}[G]$, the identity conjugacy class gives a lift of group homology $H_\bullet(G)$ into the negative cyclic homology. This gives a canonical map $H_\bullet(G) \longrightarrow HC^-_\bullet(A)$ associated to any representation of G over A. <u>Problem</u>: How to get control of this, say in the smooth commutative case?

Instead of groups consider Lie algebras $\mathfrak{g}$. A representation $U(\mathfrak{g}) \xrightarrow{\text{Morita}} A$ maps the cyclic theory of $U(\mathfrak{g})$ to that of A. But the cyclic theory of $U(\mathfrak{g})$ is described by a mixed complex $S(\mathfrak{g}) \otimes \Lambda(\mathfrak{g})$ with b map given by adjoint action of $\mathfrak{g}$ on $S(\mathfrak{g})$, and B map the de Rham d associated to $S(\mathfrak{g})$. The universal case should be $\mathfrak{g} = \mathfrak{gl}(A)$, and we should be able to apply invariant theory. We get from $\Lambda\mathfrak{g}$ the cyclic complex, from $\mathfrak{g} \otimes \Lambda\mathfrak{g}$ the Hochschild complex, and perhaps from $S^2(\mathfrak{g}) \otimes \Lambda\mathfrak{g}$ the complex $[A\overset{!}{\otimes}]^{(2)}_\sigma$, etc. This is clear more or less from the graphs resulting from invariant theory:



Now we have encountered $S(\mathfrak{g}) \otimes \Lambda\mathfrak{g}$ in a different way as the chains on the DG

Lie algebra $\mathfrak{g} \oplus \mathfrak{g}[1]$ which occurs when we look at equivariant forms. This is the DG Lie algebra of operators L_X, ι_X for $X \in \mathfrak{g}$. Then the chains are $S(\mathfrak{g}) \otimes \Lambda \mathfrak{g}$, where the $\mathfrak{g}$ in $S(\mathfrak{g})$ has degree 2.

In general for a Lie algebra we have $S(\mathfrak{g}) \otimes \Lambda \mathfrak{g}$ occurring in two ways.

1) As the mixed complex describing the cyclic theory of $U(\mathfrak{g})$.

2) As chains on the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}[1]$ occurring in the study of equivariant differential forms.

These differ in two respects: First the grading, and secondly there are twisted versions of enveloping algebras associated to a central extension $0 \to \mathbb{C} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$ for which in 1) the $S(\mathfrak{g})$ in $S(\mathfrak{g}) \otimes \Lambda \mathfrak{g}$ is really the polynomial functions on an affine space. This means one can't ~~[illegible]~~ identify 1) and 2) canonically

July 1, 1992

It might be possible to view a projective bimodule resolution P of A as analogous to a fibre functor. [redacted] So we should look at [redacted] the DG alg $\mathrm{Hom}_{A^e}(P,P)$ of its endomorphisms. Also we have seen P itself is a sort of coalgebra up to homotopy in the category of bimodules, so P might correspond to a sort of DG algebra by a version of the cobar construction.

One has

$$H_n\left(\mathrm{Hom}_{A^e}(P,P)\right) \xrightarrow{\sim} H_n\left(\mathrm{Hom}_{A^e}(P,A)\right) = H^{-n}(A,A)$$

and in general if Q is another proj. bimod. resolution

$$H_n\left(\mathrm{Hom}_{A^e}(P,Q)\right) = \begin{cases} 0 & n>0 \\ H^{-n}(A,A) & n\leq 0 \\ A^{?} & n=0 \end{cases}$$

Notice that in the case of the standard resolution $P = B$, $\mathrm{Hom}_{A^e}(B,A) = C^{\cdot}(A,A)$ is the DG algebra of cochains with values in A. This contains the Lie algebra $\mathrm{Der}(A)$ as the subspace of 1-cocycles, which we want to act with suitable homotopies on the (b,B) complex.

[redacted]

When a Lie algebra $\mathfrak{g}$ acts on a manifold M, one has an action of the DG Lie algebra $\mathfrak{g}\oplus\mathfrak{g}[+1]$ (with diff given by the identity map) on $\Omega(M)$ given by the operator L_X, ι_X for $X\in\mathfrak{g}$. This allows us to construct a BRS algebra

$$\Omega(M) \otimes \Lambda \mathfrak{g}^* \otimes S\mathfrak{g}^*$$

which is the DG algebra of cochains on $\mathfrak{g} \oplus \mathfrak{g}[1]$ with values in $\Omega(M)$.

When it comes to the action of $\mathfrak{g} = \mathrm{Der}(A)$ on the b, B complex, it seems one does not get an action of $\mathfrak{g} \oplus \mathfrak{g}[1]$, but rather an action up to higher homotopy. This should mean that we should have a complex of chains on $\mathfrak{g} \otimes \mathfrak{g}[1]$ with values in this b, B complex:

$$\{(b,B)\text{-cx}\} \otimes \underline{S\{\overbrace{\mathfrak{g}[1] \oplus \mathfrak{g}[2]}^{(\mathfrak{g}+\mathfrak{g}[1])[1]}\}}$$

which is a comodule over the DG algebra of chains. This is apparently what Ness-Tsygan do.

The point is that when one has a twisting cochain $\tau : C \longrightarrow L$ from a comm. DG coalg to a DG Lie algebra such that $C \otimes_\tau U(L)$ is acyclic, then one has an equivalence of the homotopy categories of DG comodules over C and DG modules over L.

Puzzle: The Lie algebra structure on $\mathrm{Der}(A) = Z^1(A, A)$ does not come from the DG algebra structure of $C^\cdot(A, A)$. Apparently there is a Gerstenhaber bracket operation on $C^\cdot(A, A)$ such that $[f, g]$ has degree $|f| + |g| - 1$. This arises naturally from deformation theory.

The puzzle is how to how to understand this in terms of bimodule resolutions.

July 3, 1992

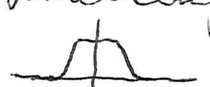
Teleman claims to have a way to deform the Hochschild complex (unnormalized) to something like infinite jets along the diagonal. Consider a smooth manifold M let $A = C^\infty(M)$ and consider the Hochschild complex constructed with the topological tensor products $A \hat{\otimes} \cdots \hat{\otimes} A = C^\infty(M^{n+1})$. Observe that multiplication $A \hat{\otimes} A \longrightarrow A$ is $f(x_0,x_1) \longmapsto f(x_0,x_0)$, restriction to the diagonal. Thus the face operators are

$$(d_j f_n)(x_0,\dots,x_{n-1}) = \begin{cases} f_n(x_0,\dots,x_j,x_j,\dots,x_{n-1}) & 0 \le j < n \\ f_n(x_0,\dots,x_{n-1},x_0) & j = n \end{cases}$$

and degeneracies are

$$(s_j f_n)(x_0,\dots,x_{n+1}) = f_n(x_0,\dots,\hat{x}_{j+1},\dots,x_{n+1}) \qquad 0 \le j \le n$$

Thus the subspace of $C^\infty(M^{n+1})$ of degenerate chains is spanned by the functions $f_n(x_0,x_1,\dots,x_n)$ which are independent of x_j for some $j \ge 1$.

Teleman chooses a smooth function $\lambda(x_0,x_1)$ supported in a small neighborhood of the diagonal and identically one near the diagonal, something like $\lambda(x_0,x_1) = \varphi(|x_0 - x_1|)$ where φ: . He considers the homotopy operator

$$(hf_n)(x_0,\dots,x_{n+1}) = \lambda(x_0,x_1)\, f_n(x_1,\dots,x_{n+1})$$

I have computed that

$$\boxed{\begin{aligned}\big((bh+hb)f_n\big)(x_0,\dots,x_n) &= f_n(x_0,\dots,x_n)\\ &+ (-1)^{n+1}\lambda(x_0,x_1)\,f_n(x_1,\dots,x_n,x_0)\\ &+ (-1)^n\lambda(x_0,x_1)\,\underbrace{\cancel{f_n(x_n,x_1,\dots,x_n)}}_{f_n(x_1,\dots,x_n,x_1)}\end{aligned}}$$

For example $n=1$.

$$\begin{aligned}\big((bh+hb)f_1\big)(x_0,x_1) &= f(x_0,x_1)\\ &+\lambda(x_0,x_1)f(x_1,x_0)\\ &-\lambda(x_0,x_1)f(x_1,x_1)\end{aligned}$$

and $n=2$.

$$\begin{aligned}\big((bh+hb)f_2\big)(x_0,x_1,x_2) &= f(x_0,x_1,x_2)\\ &-\lambda(x_0,x_1)f(x_1,x_2,x_0)\\ &+\lambda(x_0,x_1)f(x_1,x_2,x_1)\end{aligned}$$

Here's how to understand his construction. [redacted] Let us replace $C^\infty(M^{n+1})$ by $A^{\otimes n+1}$, A any algebra. [redacted] and let $\lambda \in A\otimes A$ be such that $m(\lambda)=1$. Then we can consider the homotopy operator h:

$$A^{\otimes n+1} \longrightarrow (A\otimes A)\otimes_A A^{\otimes n+1} = A^{\otimes n+2}$$

$$a_0\otimes\cdots\otimes a_{n+1} \longmapsto \lambda\otimes_A(a_0\otimes\cdots\otimes a_{n+1}) = \underbrace{\lambda a_0\otimes a_1\otimes\cdots\otimes a_{n+1}}$$

Notice that if $a_i = 1$ for some $i\geq 1$, then lies in the degenerate subcomplex. Hence we have an induced operator on ΩA. To compute it we [redacted] look at the case $n=0$. [redacted] Let $\lambda = \sum x_i\otimes y_i \in A\otimes A$ so that $\sum x_i y_i = 1$. Then

$$a_0 \xmapsto{h} \Big(\sum x_i\otimes y_i\Big)\otimes_A a_0 = \sum x_i\otimes y_i a_0 \in A\otimes A$$

$$\xmapsto{1\otimes d} \sum x_i\, d(y_i a_0) \in \Omega^1 A.$$

But

$$\sum x_i\, d(y_i a_0) = \sum x_i y_i\, da_0 + \Big(\sum x_i dy_i\Big) a_0$$
$$= (d+Y)a_0, \qquad Y = \sum_i x_i dy_i$$

The conclusion is that we obtain [redacted]

$$h(a_0 da_1 \cdots da_n) = \big((d+Y)a_0\big) da_1 \cdots da_n$$

i.e. $\quad h = d + Y \quad$ on ΩA. Compute

$$(bh+hb)(\omega da) = b(d\omega da + Y\omega a) + (-1)^{|\omega|} h([\omega, a])$$
$$= (-1)^{|\omega|+1}\big(\cancel{[d\omega, a]} + [Y\omega, a]\big) + (-1)^{|\omega|}\big(\underbrace{d[\omega, a]}_{[\cancel{d\omega}, a] + (-1)^{|\omega|}[\omega, da]} + Y[\omega, a]\big)$$
$$= \underbrace{\omega da - (-1)^{|\omega|} da\,\omega}_{\kappa(\omega da)} + (-1)^{|\omega|+1}\Big\{\underbrace{[Y\omega, a]}_{\cancel{Y[\omega, a]} + [Y, a]\omega} - \cancel{Y[\omega, a]}\Big\}$$

Thus
$$\boxed{(bh+hb)(\omega da) = \omega da + (-1)^{|\omega|+1}\big((da + [Y,a])\omega\big)}$$

What we have is a kind of modified Karoubi operator: $\quad bh + hb = 1 - \tilde{\kappa} \quad$ where
[redacted]

$$\tilde{\kappa}(\omega da) = (-1)^{|\omega|}(da + [Y,a])\,\omega$$

Notice that in the case of a separability element i.e. $\lambda \in (A \otimes A)^{\natural}$, $m(\lambda) = 1$ (in which case we formerly wrote Z for λ) we have

$$da + [Y, a] = 0,$$

and so $\quad bh + hb = 1 \quad$ in this case.

[redacted] Recall that if $\lambda = Z = \sum x_i y_i$ and $Y = \sum_i x_i dy_i$, then

$$j(Y) = \sum_i x_i(y_i \otimes 1 - 1 \otimes y_i) = 1 \otimes 1 - Z$$

so

$$j(da + [Y, a]) = a\otimes 1 - 1\otimes a + [(1\otimes 1 - Z), a]$$
$$= [a, Z].$$

July 4, 1992

Consider the standard $\underbrace{\text{resolution}}^{\text{unnormalized}}$ of the DG algebra $T_A(A\otimes A)$ with $A\otimes A$ in degree 1 and $d : A\otimes A \to A$ the multiplication map m. Given $\lambda \in A\otimes A$ such that $m(\lambda) = 1$ we get a bimodule map

$$\begin{tikzcd} A\otimes A \arrow[rr, "a_1\otimes a_2 \mapsto a_1\lambda a_2"] \arrow[dr, "m"'] & & A\otimes A \arrow[dl, "m"] \\ & A & \end{tikzcd}$$

and hence a map ~~of~~ DG algebras $\tilde{\lambda} : T_A(A\otimes A) \to T_A(A\otimes A)$, such that $1\otimes 1 \longmapsto \lambda$. We want to construct a homotopy between $\tilde{\lambda}$ and $\tilde{\mu}$, where $\mu \in A\otimes A$ is also such that $m(\mu) = 1$.

Actually we might as well generalize to the case of $T_A(E)$, or more generally when the target is a DG chain algebra R with $R_0 = A$. We have

$$\tilde{\lambda}(a_0\otimes \cdots \otimes a_n) = a_0\lambda a_1 \lambda \cdots a_{n-1}\lambda a_n$$

Note that

$$\Omega^1_A(T_A(E)) = T_A(E) \otimes_A E \otimes_A T_A(E)$$
$$\Omega^1_A(T_A(A\otimes A)) = T_A(A\otimes A) \otimes T_A(A\otimes A)$$

hence we get a derivation $T_A(A\otimes A) \to R$ relative to the bimodule structure on R over $T_A(A\otimes A)$ given by ~~two~~ homomorphisms $T_A(A\otimes A) \rightrightarrows R$, say $\tilde{\lambda}$ and $\tilde{\mu}$, by giving the derivation on $1\otimes 1$.

Let's choose $\eta \in R_2$ such that $d(\eta) = \lambda - \mu \in R_1$, and consider the derivation h relative to $\tilde{\lambda}$ for the left and $\tilde{\mu}$ for the right $T_A(A\otimes A)$ module structure on R such that $1\otimes 1 \longmapsto \eta$. Thus

$$
\begin{aligned}
h(a_0\otimes\cdots\otimes a_n) = {} & a_0\eta a_1\mu a_2\cdots\mu a_n \\
& - a_0\lambda a_1\eta a_2\mu\cdots\mu a_n \\
& + a_0\lambda a_1\lambda a_2\eta a_3\mu\cdots \\
& \cdots\cdots \\
& + (-1)^{n-1} a_0\lambda a_1\cdots\lambda a_{n-1}\eta a_n
\end{aligned}
$$

where the signs are because λ is of degree 1. Then $[d,h] : T_A(A\otimes A) \longrightarrow R$ is a derivation relative to $\tilde{\lambda}$ on the left and $\tilde{\eta}$ on the right such that

$$
\begin{aligned}
[d,h](1\otimes 1) &= d\eta + h\underbrace{d(1\otimes 1)}_{=1\in A} = d\eta = \lambda-\mu \\
&= (\tilde{\lambda}-\tilde{\mu})(1\otimes 1).
\end{aligned}
$$

But $\tilde{\lambda}-\tilde{\mu} : T_A(A\otimes A) \longrightarrow R$ is a derivation with the same properties, whence we have

$$[d,h] = \tilde{\lambda}-\tilde{\mu}$$

Now apply this in the case $A = C^\infty(M)$, where $A^{\otimes n}$ is the topological tensor product: $A^{\otimes n} = C^\infty(M^n)$

July 9, 1992

Consider the problem of linking the tower $\mathcal{X}^{\delta}(R,I)$, where $A=R/I$ with R quasi-free, to Jones-Kassel bivariant cyclic cohomology. In Jones-Kassel one uses the $\mathbb{C}[u]$-comodule complex associated to the mixed complex $(\Omega A, b, B)$. So the essential point is to connect up the tower of $\mathbb{Z}/2$ graded complex picture with the $\mathbb{C}[u]$-comodule picture of the cyclic theory of A.

As in previous work let's consider the dual picture, where we deal with cochain complexes $X^\bullet$ equipped with an injective endomorphism of degree 2:

$$\begin{tikzcd} X^0 \arrow[r, hook] \arrow[dr] & X^2 \arrow[r, hook] \arrow[dr] & X^4 \arrow[r, hook] \arrow[dr] & {} \\ & X^1 \arrow[r, hook] \arrow[u] & X^3 \arrow[r, hook] \arrow[u] & X^5 \arrow[r, hook] \arrow[u] & {} \end{tikzcd}$$

To such a thing we can associate an injective inductive system of $\mathbb{Z}/2$ graded complexes:

$$\begin{array}{lcc}
F_{-1}: & 0 & 0 \\
 & \cap & \| \\
F'_0: & {}_dX^0 & 0 \\
F_0: & X^0 & dX^0 \\
 & \| & \cap \\
F'_1: & X^0 & d^{-1}(SX^0) \\
F_1: & X^0+dX^1 & X^1 \\
 & \cap & \| \\
F'_2: & X^0+d^{-1}(SX^1) & X^1 \\
F_2: & X^2 & X^1+dX^2
\end{array}$$

Here F_n/F'_n has diff'l from parity n to parity $n+1$ an isomorphism, and F'_n/F_{n-1} is supported in parity n.

Next suppose we have an injective system of $\mathbb{Z}/2$ graded complexes

$$0 = M_{-1} \subset M_0 \subset M_1 \subset \cdots$$

such that M_n/M_{n-1} has homology supported in parity n. We would like to associate to this an $(X^\cdot; S)$ as above. We first have to understand ~~the~~ case where $M_0 = M_1 = \cdots = M$.

For insight consider the family $\cdots F_n' \subset F_n \subset F_{n+1}' \cdots$ before and look at what happens around a place where homology occurs. Thus look at the interval

$$F_{n-1}' \subset F_{n-1} \subset F_n' \subset F_n.$$

We have

$$\begin{array}{lll} F_{n-1}/F_{n-1}' & \text{has} & (n+2\mathbb{Z}) \xleftarrow{\sim} (n-1+2\mathbb{Z}) \\ F_n'/F_{n-1} & \text{has} & (n+2\mathbb{Z}) \qquad 0 \\ F_n/F_n' & \text{has} & (n+2\mathbb{Z}) \xrightarrow{\sim} (n-1+2\mathbb{Z}) \end{array}$$

This suggests taking M with $H_{1+2\mathbb{Z}}(M) = 0$ and finding a 3 stage filtration of this ~~type~~.

$$\begin{array}{ccc} M^{ev} & & M^{od} \\ \cup & & \cup \\ Z^{ev} & \xrightarrow{\;\simeq\;} & Z^{od} = B^{od} \\ \cup & & \\ B^{ev} & \xleftarrow{\;\simeq\;} & \end{array}$$

All we have to do is choose a complement:

$$M^{od} = Z^{od} \oplus K$$

and then we have the three stage filtration

$$\begin{array}{ccc} B^{ev} & \xleftarrow[\circ]{\sim} & K \\ \cap & & \| \\ Z^{ev} & \underset{\circ}{\leftrightarrows} & K \\ \cap & & \cap \\ M^{ev} & & M^{od} \end{array} \quad \begin{array}{l} Z^{ev}/B^{ev} \leftrightarrows 0 \\ \\ M^{ev}/Z^{ev} \overset{\circ}{\underset{\sim}{\rightleftarrows}} M^{od}/K = Z^{od} \end{array}$$

In general given

$$0 = M_{-1} \subset M_0 \subset M_1 \subset \cdots$$

we apply this construction to each quotient M_n/M_{n-1}. This gives a refined injective system

$$\cdots \subset M_{n-1} \subset \underset{F_{n-1}}{M'_{n-1}} \subset \underset{F'_n}{M''_n} \subset M_n \subset M'_n \subset M''_{n+1} \subset M_{n+1} \subset \cdots$$

such that M''_n/M'_{n-1} is 0 in parity $n+1$

M'_{n}/M_{n-1} has form $(n+2\mathbb{Z}) \xleftarrow{\sim} (n-1+2\mathbb{Z})$

M_n/M''_n ——— $(n+2\mathbb{Z}) \xrightarrow{\sim} (n+1+2\mathbb{Z})$

Thus M'_n/M_n ——— $(n+2\mathbb{Z}) \xrightarrow{\sim} (n+1+2\mathbb{Z})$

so if we drop M_n then we obtain a system F'_n, F_n with $F_{n-1} = M'_{n-1}$, $M''_n = F'_n$. I think it then follows from previous work that the F'_n, F_n system is equivalent to an $(X^{\cdot}, S)$ with S injective.

Next consider the tower associated to (R,I) with R quasi-free.

$$\begin{array}{llcl}
X^0 : & R/I+[R,R] & \rightleftarrows & 0 \\
 & \uparrow & & \uparrow \\
X^1 : & R/I & \rightleftarrows & \Omega^1R/[\,,\,]+IdR+RdI \\
 & \uparrow & & \uparrow \\
X^2 : & R/I^2+[I,R] & \rightleftarrows & \Omega^1R/[\,,\,]+IdR \\
 & \uparrow & & \uparrow \\
X^3 : & R/I^2 & \rightleftarrows & \Omega^1R/[\,,\,]+I^2dR+IdI \\
 & \uparrow & & \uparrow \\
X^4 : & R/I^3+[I^2,R] & \rightleftarrows & \Omega^1R/[\,,\,]+I^3dR+I^2dI
\end{array}$$

Write $X^q = X(R)/F_I^q X(R)$. The ~~quotients~~ are

$$F_I^{2n}/F_I^{2n+1} : \quad \frac{I^{n+1}+[I^n,R]}{I^{n+1}} \underset{0}{\rightleftarrows} \frac{\natural(I^n dR)}{\natural(I^{n+1}dR+I^n dI)}$$

$$F_I^{2n-1}/F_I^{2n} : \quad \frac{I^n}{I^{n+1}+[I^n,R]} \overset{b}{\underset{d}{\rightleftarrows}} \frac{\natural(I^n dR+I^{n-1}dI)}{\natural(I^n dR)}$$

homology concentrated here

Observe $\dfrac{I^n}{I^{n+1}+[I^n,R]} = [I/I^2\otimes_R]^{(n)}$, that we have a surjection

$$[I/I^2\otimes_R]^{(n)} \xrightarrow{\ \alpha\ } \frac{\natural(I^{n}dR+I^{n-1}dI)}{\natural(I^n dR)}$$

$$x_1\otimes\cdots\otimes x_n \longmapsto x_1\cdots x_{n-1}dx_n$$

Such that $b\alpha = 1-\sigma$ on $[I/I^2\otimes_R]^{(n)}$.

It follows that b

$$\alpha\left(P_\sigma^\perp[I/I^2\otimes_R]^{(n)}\right) \xrightarrow{\ \sim\ } \frac{I^{n+1}+[I^{n-1},I]}{I^{n+1}+[I^n,R]} = P_\sigma^\perp[I/I^2\otimes_R]^{(n)}$$

Thus by exactness in degree odd for F_I^{2n-1}/F_I^{2n}, we see that $\alpha\left(P_\sigma^\perp[I/I^2\otimes_R]^{(n)}\right)$ is a complement to $\operatorname{Im} d = \operatorname{Ker} b$. This means the layer F_I^{2n-1}/F_I^{2n} ~~has~~ be a three stage filtration

$$
\begin{tikzcd}
I^n \arrow[d, dash] \arrow[dr, "\simeq"] & \natural(I^n dR + I^{n-1} dI) \arrow[d, dash] \\
Z \arrow[d, dash] \arrow[r, "0"] & {} \arrow[d, dash] \\
I^{n+1} + [I^{n-1}, I] \arrow[d, phantom, "\cup"] & \natural(I^n dR + \alpha P_\sigma^\perp[\ ]^{(n)}) \arrow[d, phantom, "\cup"] \\
{} & {} \arrow[l, "\sim"'] \\
I^{n+1} + [I^n, R] \arrow[u, phantom, "\cup"] & \natural(I^n dR) \arrow[u, phantom, "\cup"]
\end{tikzcd}
$$

From this we get the following guess for the $\mathbb{C}[u]$-comodule:

$$X_{2n} = R/I^{n+1} \qquad X_{2n-1} = \Omega^1 R_\natural / \natural(I^n dR + \alpha P_\sigma^\perp \cdots)$$

One might also write $X_{2n-1} = \Omega^1 R_\natural / \natural(I^n dR + (1-\sigma) I^{n-1} dI)$

For earlier work see March 21, 1989, p. 221 for

$$\longrightarrow \Omega^1 R_\natural / \natural(I^{n+1} dR) \longrightarrow R/I^{n+1,\sigma} \longrightarrow \Omega^1 R_\natural / \natural(I^n dR) \longrightarrow R/I^{n,\sigma} \longrightarrow$$

and July 16, 1990, p. 506 for

$$\longrightarrow \left(\Omega^1 R / I^{n+1}\Omega^1 R\right)_{\natural,\sigma} \longrightarrow R/I^{n+1} \longrightarrow \left(\Omega^1 R / I^n \Omega^1 R\right)_{\natural,\sigma} \longrightarrow R/I^n \longrightarrow$$

which should be the same as above.

These can be compared as follows:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & (1-\sigma)\left[I/I^2 \otimes R\right]^{(n)} \arrow[r, "\sim"] \arrow[d] & (1-\sigma)\left[I/I^2 \otimes R\right]^{(n)} \arrow[d] & \\
 \arrow[r] & R/I^{n+1,\sigma} \arrow[r] \arrow[d] & \Omega^1_{R/k}/(I^n dR) \arrow[r] \arrow[d] & R/I^{n,\sigma} \arrow[r] \arrow[d] & {} \\
 \arrow[r] & R/I^{n+1} \arrow[r] \arrow[d] & \Omega^1_{R/k}/(I^n dR + (1-\sigma)I^{n-1}dI) \arrow[r] \arrow[d] & R/I^n \arrow[r] \arrow[d] & {} \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

July 11, 1992

Consider $\mathbb{Z}/2$ graded complexes M equipped with filtration: $0 = F_{-1}M \subset F_0 M \subset \cdots$, $M = \bigcup F_n M$ such that $F_nM/F_{n-1}M$ has zero homology in parity $n-1$. Choose splittings $F_n M = F_{n-1}M \oplus M_n$, whence $F_nM = \bigoplus_{k=0}^{n} M_k$. The differential of M has the form $d - \theta$ where $d(M_n) \subset M_n$ for all n and $\theta(M_n) \subset F_{n-1}M$. Choose SDRs: $M_n \underset{i}{\overset{p}{\rightleftarrows}} \bar{M}_n$, $pi = 1$, $[d,h] = 1 - ip$, $ph = h^2 = hi = 0$ with $d = 0$ on $\bar{M}_n$. Apply HPT to get an SDR

$$M \underset{i'}{\overset{p'}{\rightleftarrows}} \bar{M}, \quad h'$$

compatible with the filtrations, where $\bar{M}$ is minimal: diff $= 0$ on $F_n\bar{M}/F_{n-1}\bar{M}$ for all n.

Notice that we haven't used the condition that the homology of $F_nM/F_{n-1}M$ be supported in parity n. We can make a category out of filtered $\mathbb{Z}/2$ graded complexes using maps $f : M \to M'$ of complexes modulo homotopies $h : M \xrightarrow{\text{odd}} M'$ which are compatible with the filtration: $f(F_nM) \subset F_nM'$, similarly for h. In this category M is isomorphic to the minimal complex $\bar{M}$. I think this homotopy category is triangulated, with triangles arising from short exact sequences of filtered complexes. However, it is not clear that the subcategory of M satisfying the condition that $H(F_nM/F_{n-1}M)$ be supported in parity n for all n is a triangulated category.

Next consider cochain complexes $X^{\cdot}$ equipped with an endomorphism S of degree $+2$. Up to quis one ought to be able to assume S injective. Choose splittings $X^n = SX^{n-2} \oplus Y^n$, whence $X_n^n = \bigoplus_{k\geq 0} S^k Y^{n-2k}$, and the differential has the form $d-\theta$, where $d(Y^n) \subset Y^n$ and $\theta(Y^n) \subset \bigoplus_{k\geq 0} S^k Y^{n-1-2k}$:

$$\begin{tikzcd}
Y^3 \arrow[r, dashed] \arrow[rrd, dashed, bend left] & Y^2 & \\
Y^2 \arrow[u] \arrow[r, dashed] & Y^1 \arrow[u] \arrow[r, dashed] & Y^0 \\
Y^1 \arrow[u] \arrow[r, dashed] & Y^0 \arrow[u] & \\
Y^0 \arrow[u] & &
\end{tikzcd}$$

Again if we choose an SDR: $\circlearrowleft h \; Y \underset{\iota}{\overset{p}{\rightleftarrows}} \bar{Y}$ with $d = 0$ on $\bar{Y}$ we ought to obtain from HPT a SDR: $X \underset{\iota'}{\overset{p'}{\rightleftarrows}} \bar{X}$, h' with $\bar{X}$ minimal in the sense that $\bar{X}/S\bar{X}$ has zero differential.

The advantage of the DG-S-module situation is that it gives rise to a triangulated category, where the triangles result from exact sequenses of DG-S-modules. The conclusion I draw is that probably the filtered $\mathbb{Z}/2$ graded complex situation is somewhat artificial caused by the urge to make S into the identity.

<u>July 13, 1992</u>

It appears that there are interesting things to be learned from studying bivariant cyclic cohomology. For example, Goodwillie's theorem says that if $A = R/I$ is a nilpotent extension, then $HP(A) \xleftarrow{\sim} HP(R)$. We can ask whether the inverse isomorphism is given by an element of $HC^{2n}(A,R)$ for some n. The critical case is where $I^2 = 0$, and here one might hope for an element in $HC^2(A,R)$. But [redacted] this might not be true. In effect, if a class c [redacted] $\in HC^{2k}(R)$ is represented by $\overbrace{\tau}^{\text{a trace}}: S/J^{k+1} \longrightarrow \mathbb{C}$, $R = S/J$, then as S/J^{k+1} is a nilpotent extension of A or order [redacted] $2k+1$, all we know is that c comes from [redacted] a class in $HC^{2(2k+1)}(A)$. (Check: Let $I' = \mathrm{Ker}(S \to R \to A)$, so that $I = I'/J$. Then $I^2 = 0 \implies I'^2 \subset J$ $\implies S/I'^{2(k+1)} \twoheadrightarrow S/J^{k+1}$, so S/J^{2k+1} is a nilpotent extension of order $\leq 2k+1$ of A.)

My proof of Goodwillie's theorem does not yield a bivariant cyclic class, [redacted] but perhaps Goodwillie's proof can be modified to do this.

Let's now return to [redacted] the task of relating my filtered $\mathbb{Z}/2$ graded complexes to DG-S-modules, but this time let's adopt the bivariant viewpoint. I will work dually with $(X^\bullet; S)$ where S is injective.

Consider first $\mathbb{Z}$-graded $\mathbb{C}[S]$-modules where degree $S=1$:

$$\xrightarrow{S} X^0 \xrightarrow{S} X^1 \xrightarrow{S} X^2 \longrightarrow \cdots$$

To such an $X = \oplus X^n$ we can associate a vector space

$$\vec{X} = \varinjlim X^n$$

equipped with an increasing exhaustive filtration $F_n \vec{X} = \mathrm{Im}(X^n \longrightarrow \vec{X})$. We can identify X which are torsion-free (i.e. S injective) with such filtered vector spaces.

Given graded $\mathbb{C}[S]$-modules $X = \oplus X^n$, $Y = \oplus Y^n$ we have another $\mathrm{Hom}^n_S(X,Y) = \left\{\begin{array}{c}\text{degree } n \text{ maps } X \to Y \\ \text{commuting with } S\end{array}\right\}$.
If S is injective on Y, the same holds for $\mathrm{Hom}_S(X,Y)$, & the corresponding filtered vector space consists of linear maps $f : \vec{X} \longrightarrow \vec{Y}$ such that $\exists\, n$ with $f(X_k) \subset Y_{k+n}$, $\forall k$.

Now we want to consider two types of complexes. First ignore differentials. The first type is $\mathbb{Z}$-graded $\mathbb{C}[S]$-modules where now degree $S = 2$:

$$\longrightarrow X^0 \xrightarrow{S} X^2 \xrightarrow{S} X^4 \longrightarrow$$

$$\longrightarrow X^1 \xrightarrow{S} X^3 \longrightarrow X^5$$

The second type is $\mathbb{Z} \times (\mathbb{Z}/2)$ graded $\mathbb{C}[S]$-modules where degree $S = (1,0)$.

$$\longrightarrow M^0_+ \xrightarrow{S} M^1_+ \longrightarrow$$

$$\longrightarrow M^0_- \xrightarrow{S} M^1_- \longrightarrow$$

These two categories are the same, only the indexing changes.

Observe that the splitting of X and Y into even and odd pieces leads to a splitting of $\mathrm{Hom}_S(X,Y)$ into four pieces.

Next we add differentials. For the first type we require $d(X^n) \subset X^{n+1}$ for all n and for the second we require $d(M^n_\pm) \subset M^n_\mp$ for all n.

The preceding is a bit misguided for, although we can identify $\mathbb{Z}$-graded S modules with $\deg(S) = 2$ and S injective with $\mathbb{Z}/2$ graded vector spaces equipped with exhaustive increasing filtration, this identification doesn't seem to work when we introduce differentials.

Recall that if we have a complex $(X; d)$ with injective S we get a canonical filtration of the $\mathbb{Z}/2$ graded complex $(\bar{X}, d)$:

$$
\begin{array}{lcc}
 & 0 & 0 \\
 & \cap & \parallel \\
F_0' & dX^0 & 0 \\
 & & \\
F_0 & X^0 & dX^0 \\
 & \parallel & \cap \\
F_1' & X^0 & d^{-1}(SX^0) \\
 & & \\
F_1 & X^0 + dX^1 & X^1 \\
 & \cap & \parallel \\
F_2' & X^0 + d^{-1}(SX^1) & X^1
\end{array}
$$

F_n / F_n' acyclic

F_n' / F_{n-1} concentrated in parity n

We have two functors from DG S-modules (X, d, S) with injective S to filtered $\mathbb{Z}/2$ graded complexes, and a natural transformation which is injective

$$X \mapsto \begin{pmatrix} dX^0 & 0 \\ X^0 & d^{-1}(SX^0) \\ X^0 + d^{-1}(SX^1) & X^1 \\ \vdots & \end{pmatrix} \overset{\alpha}{\hookrightarrow} \begin{pmatrix} X^0 & dX^0 \\ X^0 + dX^1 & X^1 \\ X^2 & X^1 + dX^2 \\ \vdots & \end{pmatrix}$$

Note that the cokernel of α is acyclic, and that α is an isomorphism $\Longleftrightarrow$ X is minimal (i.e. $dX^n \subset SX^{n-1}$ for all n).

We have seen that ~~[illegible]~~ the second functor gives an equivalence between the cat of (X, d, S), S injective and the subcategory of $\mathbb{Z}/2$ graded complexes M equipped with exhaustive filtration $F_n M$ such that the diff'l in $F_n M / F_{n-1} M$ is surjective from parity n to parity $n+1$, for all n. Similarly it should be true that the first functor is an equivalence with those $(M, F_n M)$ such that the differential on $F_n M / F_{n-1} M$ from parity $n-1$ to parity n is injective.

I guess the problem to handle is the behavior of the bivariant groups. Thus on the DG S-module side we have a DG S-module $\mathrm{Hom}_S^{\cdot}(X, Y)$ consisting in degree n of maps $f : \vec{X} \longrightarrow \vec{Y}$ such that $f(X_k) \subset Y_{k+n}$ for all k. What happens on the filtered $\mathbb{Z}/2$ graded complex side? In general given $(M, F_n M)$ $(N, F_n N)$

we can form the $\mathbb{Z}/2$ graded complex $\mathrm{Hom}(M,N)$ and filter it by $F_n \mathrm{Hom}(M,N) = \{ f : M \to N \mid f(F_k M) \subset F_{k+n} N \text{ for all } k \}$.

Serre fibring

$$\begin{tikzcd} & Z \arrow[dl, shift left, "p"] \arrow[dr, shift left, "q"] & \\ X \arrow[ur, shift left, "i"] \arrow[rr, "f"] & & Y \arrow[ul, shift left, "j"] \end{tikzcd}$$

$Z \text{ "=" } X \times_Y Y^I$

$$Z_n = X_n \times Y_{n+1} \times Y_n \qquad d_Z = \begin{pmatrix} d_X & & \\ f & -d_Y & -1 \\ & & d_Y \end{pmatrix}$$

$$i = \begin{pmatrix} 1 \\ 0 \\ f \end{pmatrix} \qquad p = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

$$j = \begin{pmatrix} g \\ -h_Y \\ 1 \end{pmatrix} \qquad q = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

$$k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \qquad [d_Z, k] = 1 - ip$$

also $k^2 = ki = pk = 0$

Here $1 - fg = [d_Y, h_Y]$, $1 - gf = [d_X, h_X]$.

To get a homotopy $[d_Z, h] = 1 - jq$, we have to make an assumption like $f h_X = h_Y f$, since

$$\left[\begin{pmatrix} d_X & \\ f & -d_Y \end{pmatrix}, \begin{pmatrix} h_X & g \\ & -h_Y \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 \\ [f,h] & 1 \end{pmatrix}$$

July 25, 1992

What is a homotopy equivalence of complexes?

It is a map of complexes $f: X \longrightarrow Y$ such that there exists a homotopy inverse for f.

What is a homotopy inverse for f?

It is a map of complexes $g: Y \longrightarrow X$ such that gf and fg are homotopic to the identity.

Suppose we want to make this more precise. We give homotopies h_X, h_Y such that

$$1 - gf = [d_X, h_X] \qquad 1 - fg = [d_Y, h_Y].$$

Notice that this isn't completely satisfactory, because as in the case of an equivalence of categories (or adjoint functors) these homotopies should be compatible when compared using f + g. Thus

$$d_Y(fh_X - h_Y f) - (fh_X - h_Y f)d_X$$

$$= f[d_X, h_X] - [d_Y, h_Y]f$$

$$= f(1-gf) - (1-fg)f = 0,$$

so we want $fh_X - h_Y f$ and similarly $gh_Y - h_X g$ to be null-homotopic. The choice of such homotopies should lead to higher compatibility questions. The problem is to find the way to organize this efficiently.

Adopt a symmetrical viewpoint

$$d = \begin{pmatrix} d_X & \\ & d_Y \end{pmatrix}, \quad \alpha_0 = \begin{pmatrix} & g \\ f & \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} h_X & \\ & h_Y \end{pmatrix}$$

Then we have

$$[d, \alpha_0] = 0$$

$$[d, \alpha_1] = 1 - \alpha_0^2$$

and $[d, [\alpha_0, \alpha_1]] = [\alpha_0, 1 - \alpha_0^2] = 0.$

Our first compatibility condition is that there should exist α_2 such that $[d, \alpha_2] = [\alpha_0, \alpha_1]$. Let's write $d(\phi)$ for $[d, \phi]$; (thus write as if we are in a DG algebra). Then we calculate

$$d(\alpha_0\alpha_2 + \alpha_1^2 + \alpha_2\alpha_0) = 0$$

and we obtain the next compatibility condition, which says $\alpha_0\alpha_2 + \alpha_1^2 + \alpha_2\alpha_0$ is d of something. The list of equations is the following

$$d(\alpha_1) = 1 - \alpha_0^2$$

$$d(\alpha_2) = \alpha_0\alpha_1 - \alpha_1\alpha_0$$

$$(*) \qquad d(\alpha_3) = -(\alpha_0\alpha_2 + \alpha_1^2 + \alpha_2\alpha_0)$$

$$d(\alpha_4) = \alpha_0\alpha_3 - \alpha_1\alpha_2 + \alpha_2\alpha_1 - \alpha_3\alpha_0$$

$$d(\alpha_5) = -(\alpha_0\alpha_4 + \alpha_1\alpha_3 + \alpha_2^2 + \alpha_3\alpha_1 + \alpha_4\alpha_0)$$

The way I found to organize these equations is as follows. The idea is that we are working in a <u>free DG algebra resolution of $\mathbb{C}[F]/(F^2-1)$</u>.

Now this suggests looking at the bar construction of the algebra. The cobar construction of the bar construction might be a free resolution of the algebra (this is at the edge of the convergence problem - recall that F+T in the Manin seminar assume it's true for $U(\mathfrak{g})$).

The upshot is the elements α_n, $n \geq 0$ might be the same as a twisting cochain from the bar construction to our graded algebra.

A problem with this is that $\mathbb{C}[F]/(F^2-1) = \mathbb{C} + \mathbb{C}F$ is not an augmented algebra in an obvious way. There are two possible augmentations. Picking one amounts to working with $\mathbb{C} + \mathbb{C}e$, $e^2 = e$ with the augmentation $e \mapsto 0$. I found

$$C^{\cdot}(\mathbb{C}e, R) = R \otimes \mathbb{C}[x]$$

where $|x| = 1$ and $\delta x = -x^2$. Here $x : \mathbb{C}e \longrightarrow \mathbb{C}$ is $x(e) = 1$ and

$$(\delta x)(e, e) = x(e^2) = 1$$

$$x^2(e, e) = -x(e)\,x(e) = -1$$

Let $\tau \in C^1(\mathbb{C}e, R)$ be a twisting cochain.

$$\tau = \sum_{n \geq 0} \alpha_n x^{n+1} \quad \text{with} \quad \alpha_n \in R^n$$

$$d_{tot}\tau + \tau^2 = 0$$

Now
$$\tau^2 = \sum_{n \geq 1} \Big(\sum_{p+q=n-1} (-1)^{(p+1)q} \alpha_p \alpha_q \Big) x^{n+1}$$

$$(-1)^{(p+1)q} = (-1)^{(n-q)q} = (-1)^{(n+1)q}$$

The rest of the calculations do not work. We get

$$d(\alpha_n) = \underbrace{\begin{cases} 0 & n \text{ even} \\ \alpha_{n-1} & n \text{ odd} \end{cases}}_{\text{arises from } \delta x = -x^2} - \sum_{p+q=n-1} (-1)^{(n+1)q} \alpha_p \alpha_q$$

i.e.

$$d\alpha_0 = 0$$
$$d\alpha_1 = \alpha_0 - \alpha_0^2$$
$$d\alpha_2 = \alpha_0\alpha_1 - \alpha_1\alpha_0$$
$$d\alpha_3 = \alpha_2 - (\alpha_0\alpha_2 + \alpha_1^2 + \alpha_2\alpha_0)$$
$$\cdots\cdots\cdots$$

Things are better if we use instead $\delta x = 0$. In this case (1) $\boxed{d\tau + \tau^2 = 0}$ amounts to the equations

Note $d_{tot} = d$

(1.)′
$$d\alpha_0 = 0$$
$$d\alpha_1 = -\alpha_0^2$$
$$d\alpha_2 = \alpha_0\alpha_1 - \alpha_1\alpha_0$$
$$d\alpha_3 = -(\alpha_0\alpha_2 + \alpha_1^2 + \alpha_2\alpha_0)$$

So the equation (2) $\boxed{d\tau + \tau^2 = x^2}$ is equivalent to the system

(2)′
$$d\alpha_0 = 0$$
$$d\alpha_1 = 1 - \alpha_0^2$$
$$d\alpha_2 = \alpha_0\alpha_1 - \alpha_1\alpha_0$$
$$d\alpha_3 = -(\alpha_0\alpha_2 + \alpha_1^2 + \alpha_2\alpha_0)$$
$$\cdots\cdots$$

we have found.

However we can write (2) in the form (1) as follows. Notice that in the original situation $\alpha_0 = \begin{pmatrix} & g \\ f & \end{pmatrix}$, $\alpha_1 = \begin{pmatrix} h_x & \\ & h_y \end{pmatrix}$, etc. the α_{2n} anti-commute (resp. the α_{2n+1} commute) with $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Thus

$$\varepsilon x \tau = \sum_{n \geq 0} \varepsilon x \, \alpha_n x^{n+1}$$

$$= \sum_{n\geq 0} \varepsilon \alpha_n (-1)^n x^{n+2} = -\sum_{n \geq 0} \alpha_n \varepsilon x^{n+2}$$

$$= -\tau(\varepsilon x)$$

so we have

$$(3) \qquad \boxed{d(\tau + i\varepsilon x) + (\tau + i\varepsilon x)^2 = 0}$$

We observe that $\tau + i\varepsilon x = (\alpha_0 + i\varepsilon)x + \alpha_1 x^2 + \cdots$, so (3) is equivalent to

$$d\alpha_0 = 0$$

$$d\alpha_1 = -(\alpha_0 + i\varepsilon)^2 = 1 - \alpha_0^2$$

$$d\alpha_2 = (\alpha_0 + i\varepsilon)\alpha_1 - \alpha_1(\alpha_0 + i\varepsilon) = \alpha_0\alpha_1 - \alpha_1\alpha_0$$

$$d\alpha_3 = -\big((\alpha_0 + i\varepsilon)\alpha_2 + \alpha_1^2 + \alpha_2(\alpha_0 + i\varepsilon)\big)$$

$$= -(\alpha_0\alpha_2 + \alpha_1^2 + \alpha_2\alpha_0)$$

which is the system (2)' above.

Ideas occurring during the above discussion

1) Recall we tried once to understand what it means for $f: X \to Y$ to be a heq by using the mapping fibre $X_n \oplus Y_{n+1}$, $d = \begin{pmatrix} d_X & \\ f & -d_Y \end{pmatrix}$. $\exists$ analogy with parametrices for ψDO's, rather operators invertible modulo an ideal. A similar asymmetry seems to be present.

2) What should one mean by an idempotent up to higher homotopy? Perhaps this is the same as a twisting cochain on the non-unital algebra $\mathbb{C}e$, $e^2 = e$. Can this be related to the homotopy inductive limit of $X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \cdots$, i.e. the telescope?

3) The bar construction seems to handle nonunital algebras up to higher homotopy (A_∞ algebras). Is there something different for unital algebras up to higher homotopy?

4) Once one understands homotopy equivalence up to higher homotopy, there arises the question of composition. Is any heq up to higher homotopy replaceable by SDR's:

$$\begin{tikzcd} & Z \arrow[dl, shift right] \arrow[dr, Rightarrow] & \\ X \arrow[ur, shift right] \arrow[rr, dashed, leftrightarrow] & & Y \end{tikzcd}$$

<u>July 27, 1992</u>

Technical addition to HPT

Recall that if we have a perturbed diff'l $d-\theta$, hence $d\theta=\theta^2$ and a contraction h: $[d,h]=1$, then

$$\left[d-\theta,\ h\frac{1}{1-\theta h}\right] = (1-\theta h-h\theta)\ \text{[redacted]}\ \frac{1}{1-\theta h}$$

$$- h(-1)\frac{1}{1-\theta h}\underbrace{[d-\theta, 1-\theta h]}_{\substack{-\cancel{[d,\theta]}h+\theta[d,h] \\ +\cancel{\theta^2 h}-\theta h\theta}}\frac{1}{1-\theta h}$$

$$= (1-\theta h-h\theta)\frac{1}{1-\theta h} + h\frac{1}{\cancel{1-\theta h}}(\cancel{1-\theta h})\theta\frac{1}{1-\theta h}$$

$$= 1.$$

However if we assume h special: $h^2=0$, then

$$h\frac{1}{1-\theta h} = h\frac{1}{1-h\theta}\ \frac{1}{1-\theta h} = h\frac{1}{(1-\theta h)(1-h\theta)} = h\frac{1}{1-[\theta,h]}$$

$$[d-\theta, h] = 1-\theta h-h\theta = 1-[\theta,h]$$

and the calculation above becomes

$$\left[d-\theta,\ h\frac{1}{1-[\theta,h]}\right] = (1-[\theta,h])\frac{1}{1-[\theta,h]} - h\left[d-\theta, \frac{1}{1-[\theta,h]}\right]$$

$$= 1 + h\frac{1}{1-[\theta,h]}\underbrace{[d-\theta, 1-[\theta,h]]}_{[d-\theta,[d-\theta,h]]\ =\ 0}\frac{1}{1-[\theta,h]}$$

This last is [redacted] reminiscent of equivariant [redacted] differential forms:

$$\iota_X\alpha = 1 \qquad (d-\iota_X)\alpha = -1+d\alpha$$

$$(d-i_X)\left(-\alpha\frac{1}{1-d\alpha}\right) = 1$$

August 8, 1992

I have been looking at towers of complexes. These form a natural setting for HPT arguments. To fix the setting suppose we consider towers of $\mathbb{Z}/2$ graded complexes, i.e. inverse systems

$$\longrightarrow X^n \longrightarrow X^{n-1} \longrightarrow \cdots$$

bdd below: $X^n = 0$ $n \ll 0$. Let $\bar{X} = \oplus \bar{X}^n$, $\bar{X}^n = \mathrm{Ker}(X^n \to X^{n-1})$ be the associated graded complex.

The problem is to show that if $f: X \to Y$ is a map of towers such that $\bar{f}: \bar{X} \to \bar{Y}$ is a homotopy equivalence, then f is a homotopy equivalence of towers. We assume that the towers split if differentials are ignored. By a h.eq. $f: X \to Y$ of towers we mean a map such that $\exists$ a map $g: Y \to X$ and homotopies joining gf and fg to the identities. Thus $f \in Z_+ \mathrm{Hom}^0(X,Y)$, $g \in Z_+ \mathrm{Hom}^0(Y,X)$, and $1 - gf = [d, h_X]$, $1 - fg = [d, h_Y]$ with $h_X \in \mathrm{Hom}^0(X,X)_-$ and similarly for h_Y. The problem is to construct g, h_X, h_Y starting from $\bar{g}, \bar{h}_X, \bar{h}_Y$ on the associated graded complexes.

I think I need the proper higher homotopy definition of homotopy inverse to handle this problem in a completely satisfactory way. However, one can reduce to the lemma of HPT by replacing f by the dual mapping cylinder:

$$\begin{tikzcd}
 & X \times_Y Y^I \arrow[dr] & \\
X \arrow[ur, "i"] \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

The map j is the inclusion for a special deformation retraction. Thus we can assume $f = p$ is part of an exact sequence of towers

$$0 \longrightarrow Z \longrightarrow X \xrightarrow{f} Y \longrightarrow 0$$

Then we can choose SDR data

$$h \circlearrowright \bar{X} \underset{i}{\overset{p = \bar{f}}{\rightleftarrows}} \bar{Y}$$

and apply HPT to get $\tilde{h} \circlearrowright X \underset{\tilde{\imath}}{\overset{\tilde{p}}{\rightleftarrows}} Y$. It should be true that $\tilde{p} = f$ in which case we win.

<u>November 22, 1992</u>

Derived categories.

Let $\mathcal{C}$ be a category, S a set of arrows in $\mathcal{C}$, $\mathcal{C}[S^{-1}]$ the category obtained by formally inverting these arrows. Then we have an <u>isomorphism</u> of categories

$$(*) \quad \mathrm{Fun}(\mathcal{C}[S^{-1}], \mathcal{C}') \longrightarrow \begin{array}{l}\text{full subcat of } \mathrm{Fun}(\mathcal{C},\mathcal{C}') \\ \text{consisting of } F \text{ inverting } S\end{array}$$

for any category $\mathcal{C}'$. The isomorphism is clear for objects; this is just the universal property for $\mathcal{C}[S^{-1}]$. So we have to check the isomorphism for arrows. Let $F, G : \mathcal{C}[S^{-1}] \to \mathcal{C}'$ be functors and $\theta : Fj \to Gj$ a natural transf; here $j : \mathcal{C} \to \mathcal{C}[S^{-1}]$ is the canonical functor. We have to see that θ comes from a unique natural transf. $F \to G$. Uniqueness is clear since j is the identity on objects. We have to see that the maps $\theta_X : F(X) \longrightarrow G(X)$ $\forall X$ in $\mathcal{C}$ are compatible with the arrows in $\mathcal{C}[S^{-1}]$. But an arrow $X \to Y$ in $\mathcal{C}[S^{-1}]$ is a composition of arrows in $\mathcal{C}$ and inverses of arrows in S, e.g.

$$X \xrightarrow{f} Z \xleftarrow{s} Y$$

In this case we have

$$\begin{tikzcd} FX \arrow[r] \arrow[d] & FZ \arrow[d] & FY \arrow[l, "\sim"'] \arrow[d] \\ GX \arrow[r] & GZ & GY \arrow[l, "\sim"'] \end{tikzcd}$$

so it's all clear.

It seems that one wants in practice to replace the <u>isomorphism</u> (*) by an <u>equivalence</u> of categories.

Let's take an example.

Consider $\mathcal{C}_\Lambda$ complexes of Λ modules, Λ a ring, bounded below for the lower indexing, $\mathcal{C}_\Lambda^f$ full subcategory of free module complexes, $\operatorname{Ho}\mathcal{C}_\Lambda$, $\operatorname{Ho}\mathcal{C}_\Lambda^f$ the homotopy categories,

$$i : \operatorname{Ho}\mathcal{C}_\Lambda^f \hookrightarrow \operatorname{Ho}\mathcal{C}_\Lambda$$

the inclusion functor (which is fully faithful). The basic facts are [redacted]:

(i) L free, $M \to N$ quis $\Longrightarrow$ $[L, M] \xrightarrow{\sim} [L, N]$.

(ii) $\forall M$, $\exists L$ free together with a quis $L \to M$. (Call a quis $L \to M$ with L free a free resolution of M.)

Note that if $L_M \to M$ is a free resolution of M, then we have

$$[L', L_M] \xrightarrow{\sim} [L', M]$$

for all free L'. We can write this as

$$\operatorname{Hom}_{\operatorname{Ho}\mathcal{C}_\Lambda}(iL', M) = \operatorname{Hom}_{\operatorname{Ho}\mathcal{C}_\Lambda^f}(L', L_M).$$

This means that we have a right adjoint functor

$$i_* : \operatorname{Ho}\mathcal{C}_\Lambda \to \operatorname{Ho}\mathcal{C}_\Lambda^f \qquad i_*(M) = L_M$$

[redacted] to i. Properties of i_*:

1) the adjunction arrow $L \to \iota_* \iota L$ is an isomorphism $\forall L$ free

2) the adjunction arrow $\iota \iota_* M \to M$ is a quis for all M.

3) ι_* carries any quis to an isom.

Note that 1) is a formal consequence of adjointness and the fact that i is fully faithful:

$$\mathrm{Hom}_{\mathrm{Ho}\mathcal{C}_\Lambda^f}(L', \iota_* \iota L) = \mathrm{Hom}_{\mathrm{Ho}\,\mathcal{C}_\Lambda}(\iota L', \iota L)$$

$$= \mathrm{Hom}_{\mathrm{Ho}\,\mathcal{C}_\Lambda^f}(L', L)$$

Note that 3) follows from 2) and the fact that a quis of free complexes $L \to L'$ is an isomorphism by (i). In effect

$$\begin{array}{ccc} \iota_* M' & \xrightarrow{\iota_* f} & i_* M \\ \downarrow \text{quis} & & \downarrow \text{quis} \\ M' & \xrightarrow{f} & M \end{array}$$

so f quis $\Rightarrow$ $i_* f$ quis $\Longrightarrow$ $\iota_* f$ isom.

The upshot seems to be the following

Prop: Let $\iota : \mathcal{C}' \to \mathcal{C}$ be fully faithful, and assume there is a right adjoint $\iota_* : \mathcal{C} \to \mathcal{C}'$. Let S be the set of arrows inverted by ι_*. Then $\mathcal{C}' \longrightarrow \mathcal{C} \longrightarrow \mathcal{C}[S^{-1}]$ is an equivalence of categories.

Proof. Because ι_* inverts S we have

$$\begin{tikzcd} \mathcal{C}' \arrow[r, hook, "i"] & \mathcal{C} \arrow[r, "j"', "\text{(canon)}"] \arrow[rr, bend left, "\iota_*"] & \mathcal{C}[S^{-1}] \arrow[r, dashed, "k", "\exists"'] & \mathcal{C}' \end{tikzcd}$$

The composite $\iota_* i$ is isomorphic to the identity functor via the adjunction arrow $id \to \iota_* i$.
The two functors id, ~~[illegible]~~ $j\iota k : \mathcal{C}[S^{-1}] \to \mathcal{C}[S^{-1}]$ when composed with j are linked by an ~~[illegible]~~ arrow

$$j\iota k j = j\iota\iota_* \xrightarrow{j(\text{adj})} j$$

~~[illegible]~~ which is an isomorphism as the adjunction arrow is inverted by ι_* hence by j. This means by $\circledast$ that we have an isomorphism

$$j\iota k \xrightarrow{\sim} id$$

Alternative proof. Check that $\iota_* : \mathcal{C} \to \mathcal{C}'$ gives an equivalence for all categories $\mathcal{E}$

$$\operatorname{Fun}(\mathcal{C}', \mathcal{E}) \xrightarrow{(\iota_*)^t} \begin{array}{l}\text{full subcat of } \\ \operatorname{Fun}(\mathcal{C}, \mathcal{E}) \\ \text{consisting of } F \\ \text{inverting } S\end{array}$$

The point is that we have the functor

$$\operatorname{Fun}(\mathcal{C}, \mathcal{E}) \xrightarrow{\iota^t} \operatorname{Fun}(\mathcal{C}', \mathcal{E}) \quad , \quad F \mapsto F\iota$$

and

$$G \mapsto G\iota_* \longmapsto G\iota_* i \simeq G$$

$$F \mapsto F\iota \longmapsto \underbrace{F\iota\iota_* \longrightarrow F}_{}$$

$F(\text{adjunction})$ which is an isom when F inverts S.

Here is another idea which preceded the realization that free resolution is an adjoint functor. The goal is to define (introduce, construct) the derived category $D\mathcal{C}_\Lambda$, and there are two methods:

<u>Abstract</u>: Formally inverting quis in either $\mathcal{C}_\Lambda$ or $Ho\,\mathcal{C}_\Lambda$.

<u>Concrete</u>: Define Ext groups $\mathrm{Ext}^0_\Lambda(M,N)$ with composition products + identities and canonical maps $[M,N] \longrightarrow \mathrm{Ext}^0_\Lambda(M,N)$ compatible with this structure.

November 30, 1992

This past two weeks I have been struggling to find a clear introduction of the derived category DC_Λ of mixed complexes. ('clear' means clear enough for one to be able to write it down)

Let $\varepsilon_M : L_M \to M$ be a choice of free resolution for each mixed complex M. Such a choice is the same as a right adjoint $i_* : \mathrm{Ho}\,\mathcal{C}_\Lambda \to \mathrm{Ho}\,\mathcal{C}_\Lambda^f$ to the inclusion $i : \mathrm{Ho}\,\mathcal{C}_\Lambda^f \hookrightarrow \mathrm{Ho}\,\mathcal{C}_\Lambda$ because

$$[L, L_M] \xrightarrow{\sim} [L, M] \qquad \forall\ L \text{ free}$$

How do I introduce the derived category DC_Λ? The most naive way is to say what the objects, morphisms, and composition are. The problem is what are the morphisms. Some possibilities: A morphism $M \to N$ in DC_Λ is

an element of $HC^0(M, N)$

an element of $[L_M, L_N]$

____________ $[B(\Lambda)\otimes M, B(\Lambda)\otimes N]$

So there's a difficulty because $HC^0(M,N)$ is defined up to canonical isomorphism. It is necessary to fix a choice of groups $HC^0(M,N)$ for each M, N in order to obtain a definite category DC_Λ. Another choice leads to an isomorphic category.

Therefore the derived category is defined only up to canonical isomorphism. This fact suggests that it might be better to define it by means of its properties, as in the case of the tensor product. (I think I have wasted a lot of time by trying to say what the derived category *is* rather than how it *behaves*.)

At some point in the past two weeks I arrived at the following elementary picture of the derived category which does not use category language:

$\forall (M,N)$ a v. space $\mathrm{Ext}^0_\Lambda(M,N)$

$\forall (M,M',M'')$ products

$$\mathrm{Ext}^0_\Lambda(M',M'') \otimes \mathrm{Ext}^0_\Lambda(M,M') \longrightarrow \mathrm{Ext}^0_\Lambda(M,M'')$$

which are associative and such that $\exists$ identity element $1_M \in \mathrm{Ext}^0_\Lambda(M,M)$ $\forall M$.

$\forall (M,N)$ a map

$$j : [M,N] \longrightarrow \mathrm{Ext}^0_\Lambda(M,N)$$

compatible with products + identities

satisfying

j carry quis into isos

$j : [M,N] \xrightarrow{\sim} \mathrm{Ext}^0_\Lambda(M,N)$ if M free.

At first I did not translate this into category language **suitably**, but here is the good translation. One has

a category $D\mathcal{C}_\Lambda$ whose objects are all mixed complexes

a functor $j : \mathrm{Ho}\,\mathcal{C}_\Lambda \longrightarrow D\mathcal{C}_\Lambda$ which on objects is the identity map satisfying

j carries quis into isos.

$j : [M, N] \xrightarrow{\sim} \mathrm{Hom}_{D\mathcal{C}_\Lambda}(M, N)$ if M free

[redacted]

Suppose one is given such at pair $(D\mathcal{C}_\Lambda, j)$. [redacted] Let us check that this pair is universal with respect to inverting quis. We suppose given $F : \mathrm{Ho}\, \mathcal{C}_\Lambda \longrightarrow \mathcal{C}'$ inverting quis and wish to prove that $\exists!\, \tilde{F} : D\mathcal{C}_\Lambda \longrightarrow \mathcal{C}'$ such that $\tilde{F}j = F$.

Note that

$$\begin{array}{ccccc} [L_M, L_N] & \xrightarrow{\sim} & \mathrm{Hom}_{D\mathcal{C}_\Lambda}(L_M, L_N) & \xrightarrow{\sim} & \mathrm{Hom}_{D\mathcal{C}_\Lambda}(M, N) \\ u & \longmapsto & j(u) & \longmapsto & j(\varepsilon_N)\, j(u)\, j(\varepsilon_M)^{-1} \end{array}$$

If $\tilde{F}$ exists, then $\tilde{F}M = \tilde{F}jM = FM \quad \forall M$

and $\tilde{F}$ on $f \in$ [redacted] $\mathrm{Hom}_{D\mathcal{C}_\Lambda}(M,N)$ is given as follows. There's a unique $u \in [L_M, L_N] \ni f = j(\varepsilon_N)\, j(u)\, j(\varepsilon_M)^{-1}$, and then

$$\tilde{F}(f) = F(\varepsilon_N)\, F(u)\, F(\varepsilon_M)^{-1}$$

Conversely can use these formulas to define $\tilde{F}$.

The moral seems to be that a good path is

for ~~definiteness~~ take $HC^0_\Lambda(M, N) = \mathrm{Ext}^0_\Lambda(M, N)$ to be $[L_M, L_N]$

[redacted]

Recall that if C is a DG coalgebra and A is a DG algebra, then a twisting cochain $\tau : C \longrightarrow A$ is a map of degree -1 such that $[d,\tau] + \tau^2 = 0$, where τ^2 is

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{\tau \otimes \tau} A \otimes A \xrightarrow{m} A$$

Suppose we replace C by [redacted] its dual C^* and blur the distinction between the algebras $\mathrm{Hom}(C, A)$ (product like τ^2 above) and $C^* \otimes A$.
Let's write [redacted] B for C^*. Then $\tau \in (B \otimes A)_{-1}$ satisfies $d\tau + \tau^2 = 0$.

Suppose that M is an A-module and that P is a B-module (i.e. P a C-comodule). Then we have canon. isos.

$$\boxed{\mathrm{Hom}_A(A \otimes P, M) = \mathrm{Hom}(P, M) = \mathrm{Hom}_B(P, \mathrm{Hom}(B, M))}$$

Use τ to form a twisted differential on $\mathrm{Hom}(P, M)$

$$d_\tau(f) = [d, f] + \tau(f)$$

Here $\mathrm{Hom}(P, M)$ is regarded as a $B^{op} \otimes A$-module and $\tau(f)$ is the action of $\tau \in B \otimes A$ on f. Corresponding to this twisted differential on $\mathrm{Hom}(P, M)$ are twisted differentials on $A \otimes P$, $\mathrm{Hom}(B, M)$ so we get adjoint functors $A \otimes_\tau -$, $\mathrm{Hom}_\tau(B, -)$

Typical example arises with $A = \mathbb{C} \oplus \mathbb{C}B$
$B = \mathbb{C}[S]$ in cyclic homology.

Adjunction formula

$$\mathrm{Hom}_\Lambda(\Lambda \otimes_\Sigma P, M) \simeq \mathrm{Hom}_\Sigma(P, M) = \mathrm{Hom}_S(P, \mathrm{Hom}_\Sigma(\mathbb{C}[S], M))$$

holds with M a DG Λ-module and P a DG S-module; both can be unbounded below. It seems that the adjunction arrows

$$\eta_P : P \longrightarrow \mathrm{Hom}_\Sigma(\mathbb{C}[S], \Lambda \otimes_\Sigma P)$$

$$\varepsilon_M : \Lambda \otimes_\Sigma \mathrm{Hom}_\Sigma(\mathbb{C}[S], M) \longrightarrow M$$

are always quis. The functor $P \mapsto \Lambda \otimes_\Sigma P$ preserves quis, but $M \longmapsto \mathrm{Hom}_\Sigma(\mathbb{C}[S], M) = \mathcal{B}M$ does not. Note that

$$(\mathcal{B}M)_n = \prod_{p \geq 0} M_{n-2p} \quad , \qquad d = b + B$$

so the increasing column filtration (giving rise to the spectral sequence from HH to HC) doesn't converge. On the other hand the increasing row filtration does converge showing that

(*) $\quad BM \longrightarrow M/BM \quad$ quis if B is exact

~~[illegible]~~ This arrow should result from the adjunction $\Lambda \otimes_\Sigma \mathcal{B}M \longrightarrow M$, so in the case $M = \Lambda \otimes_\Sigma P$ we get a quis $\mathcal{B}(\Lambda \otimes_\Sigma P) \longrightarrow P$ which is retraction for the adjunction homomorphism. (Actually it should be a heq because in the case of $\Lambda \otimes_\Sigma P$ where we have a splitting of $0 \to BM \to M \to M/BM \to 0$ (*) should be a heq in an explicit way.)

Similarly for P divisible we have a quis ${}_SP \longrightarrow \Lambda \otimes_\Sigma P$ which in the case $P = \mathcal{B}M$ gives a quis $M \longrightarrow \Lambda \otimes_\Sigma \mathcal{B}M$ which should be a section of the adjunction map.

Note the counterexample to B preserving quis as follows. Start with $P = \mathbb{C}[S, S^{-1}]$ on which S is invertible. Then $\Lambda \otimes_{\tau} P \overset{\sim}{\text{quis}}\ {}_S P = 0$. But $P \ \tilde{\text{quis}}\ B(\Lambda \otimes_{\tau} P)$ is not $\tilde{\text{quis}}\ 0$.

December 8, 1992:

Some facts: The problem with $M \longmapsto \mathrm{Hom}_{\tau}(\mathbb{C}[S], M)$ is that it does not respect quis. However $M \longmapsto B(\Lambda) \otimes_{\tau} M$ does respect quis. I think it's true that the adjunction maps

$$\Lambda \otimes_{\tau} B(\Lambda) \otimes_{\tau} M \longrightarrow M$$

$$P \longrightarrow B(\Lambda) \otimes_{\tau} \Lambda \otimes_{\tau} P$$

are quis even when M, P are unbounded below. Here I am thinking of P as a $B(\Lambda)$-comodule, i.e. S on P is locally nilpotent. In this case P should be an inductive limit of things bounded below.

Sometime you should work this example out thoroughly with formulas. The idea is that there should canonical SDR formulas for the quis

$$B(\Lambda) \otimes_{\tau} M \longrightarrow M/BM \qquad M \text{ free}$$

in the case $M = \Lambda \otimes_{\tau} P$, more generally when there is a connection in a suitable sense.

December 12, 1992

Here's a technical point which I want to assert without giving the proof.

I want to prove that for divisible OG S-modules P, Q the canonical isomorphism

$$HC^k(P,Q) \simeq HC^k(X,Y)$$

where $X = \alpha P$, $Y = \alpha Q$ are the associated towers, is compatible with Connes exact sequences.

Recall that

$$HC^k(P,Q) = H^k(\mathrm{Hom}_S(P,Q))$$

$$HC^k(X,Y) = H_{k+2\mathbb{Z}}(\mathrm{Hom}^k(X,Y))$$

and that shifting $Q \mapsto Q[n]$ (together with the appropriate version for special towers) allows to take $k=0$ for many questions. Recall also that $\hat{P} = \hat{X}$, $\hat{Q} = \hat{Y}$ where $\hat{P}_+ = \varprojlim P_{2n}$, $\hat{P}_- = \varprojlim P_{2n-1}$, so that $\mathrm{Hom}^k_S(P,Q)$ and $\mathrm{Hom}^k(X,Y)$ can be identified with subspaces of the supercomplex $\mathrm{Hom}(\hat{P},\hat{Q}) = \mathrm{Hom}(\hat{X},\hat{Y})$.

Recall $X^n = P_n/d({}_SP_{n+1}) \rightleftarrows P_{n-1}$

also

$$Z^0\,\mathrm{Hom}_S(P,Q) = Z_+\,\mathrm{Hom}^0(X,Y)$$

$$\mathrm{Hom}^0(X,Y)_- = \mathrm{Hom}_S^{-1}(P,Q) + [d, \mathrm{Hom}^0(X,Y)_+]$$

We have the inclusions

$$\mathrm{Hom}_S^{-2}(P,Q) \subset \mathrm{Hom}^{-1}(X,Y)_+ \subset \mathrm{Hom}^0(X,Y)_+ \subset \mathrm{Hom}_S^0(P,Q) \subset \mathrm{Hom}^1(X,Y)_+$$

$$\subset \mathrm{Hom}^{-1}(X,Y)_- \subset \mathrm{Hom}_S^{-1}(P,Q) \subset \mathrm{Hom}^0(X,Y)_- \subset \mathrm{Hom}^1(X,Y)_- \subset \mathrm{Hom}_S^1(P,Q)$$

We want to prove commutativity of the squares

$$(*)\quad \begin{tikzcd} HC^{-2}(P,Q) \arrow[r] \arrow[d] & HC^{0}(P,Q) \arrow[r] \arrow[d] & HH^{0}(P,Q) \arrow[r] \arrow[d] & HC^{-1}(P,Q) \arrow[d] \\ HC^{-2}(X,Y) \arrow[r] & HC^{0}(X,Y) \arrow[r] & HH^{0}(X,Y) \arrow[r] & HC^{-1}(X,Y) \end{tikzcd}$$

The following is commutative with left exact rows

$$\begin{tikzcd} 0 \arrow[r] & Z^{-2}\mathrm{Hom}_S(P,Q) \arrow[r] \arrow[d] & Z^{0}\mathrm{Hom}_S(P,Q) \arrow[r] \arrow[d, "="] & Z^{0}\mathrm{Hom}({}_SP, {}_SQ) \arrow[d] \\ 0 \arrow[r] & Z_+\mathrm{Hom}^{-1}(X,Y) \arrow[r] & Z_+\mathrm{Hom}^{0}(X,Y) \arrow[r] & Z_+\mathrm{Hom}^{0}(\bar{X}, \bar{Y}) \end{tikzcd}$$

and this implies the first two squares in $(*)$ commute. (Recall that $H_+\mathrm{Hom}^{-1}(X,Y) = \mathrm{Im}\{HC^{-2}(X,Y) \to HC^{0}(X,Y)\}$.)

We have to compute the coboundary. Let us start with $\bar{f} \in Z^0\mathrm{Hom}({}_SP, {}_SQ)$ and lift it to $f \in \mathrm{Hom}^0_S(P,Q)$. I claim that f also lies in $\mathrm{Hom}^0(X,Y)_+$. To see this, examine

$$\begin{tikzcd} P_n \arrow[rr, "f_n"] \arrow[d] & & Q_n \arrow[d] \\ P_n/d({}_SP_{n+1}) \arrow[d] & & Q_n/d({}_SQ_{n+1}) \arrow[d] \\ P_{n-2} \arrow[rr, "f_{n-2}"] & & Q_{n-2} \end{tikzcd}$$

The issue is whether f_n carries $d({}_SP_{n+1})$ into $d({}_SQ_{n+1})$. But f_0 restricted to ${}_SP$ is the map $\bar{f} : {}_SP \to {}_SQ$ commuting with d. Thus $f_n d({}_SP_{n+1}) = d f_{n+1}({}_SP_{n+1}) \subset d({}_SQ_{n+1})$. Thus $f \in \mathrm{Hom}^0(X,Y)_+$.

Consider $[d,f] \in Z^{-2}\mathrm{Hom}_S(P,Q) = Z_+\mathrm{Hom}^{-2}(X,Y)$

On one hand we know this represents the image of $[\bar{f}]$ under the connecting homomorphism $HH^0(P,Q) \longrightarrow HC^{-1}(P,Q)$ associated to the exact sequence of cxs.

$$0 \longrightarrow \mathrm{Hom}_S(P,Q)[2] \xrightarrow{S} \mathrm{Hom}_S(P,Q) \longrightarrow \mathrm{Hom}({}_SP, {}_SQ) \longrightarrow 0$$

On the other hand it represents the image of $[\bar{f}]$ under $HH^0(X,Y) \longrightarrow HC^{-1}(X,Y)$ associated to the exact sequence of supercomplexes

$$0 \longrightarrow \mathrm{Hom}^{-1}(X,Y) \longrightarrow \mathrm{Hom}^0(X,Y) \longrightarrow \mathrm{Hom}(\tilde{X},\tilde{Y}) \longrightarrow 0$$

Thus we have compatibility of the Connes exact sequences.

December 21, 1992

In writing the material on towers of supercomplexes I noticed a similarity which seems worth recording. Namely when you define special tower, the condition:

$$H_\nu(\bar{X}^n) \neq 0 \implies \nu = n + 2\mathbb{Z}$$

is equivalent to the homology of $\bar{X} = \bigoplus_n \bar{X}^n$ being even for the total $\mathbb{Z}/2$-grading. This might relate to superconnections.

Another question in this area is whether towers of supercomplexes, not just special towers, might occur in dealing with cyclic homology theory for superalgebras.

December 25, 1992

Background: In discussing biv. cyc. coh. I found that I want to think of bcc. w cp (biv cyc. coh with cup product) as equivalent to a category with automorphism:

$$Q \longmapsto \Sigma Q \qquad \text{autom}: \quad [Q, Q'] \xrightarrow{\sim} [\Sigma Q, \Sigma Q']$$

$$HC^k(Q, Q') = [Q, \Sigma^k Q'] = [\Sigma^p Q, \Sigma^{p+k} Q']$$

At some point you should organize the following patterns or impressions:

mixing degrees in derived categories, localizing wrt a multiplicative system of homogeneous elements (cobordism), what Hopkins told you about Pic, Graeme's equivariant stable homotopy